THE SIMULTANEOUS EQUATIONS

A. **Simultaneous Linear equations with two unknowns**

Case I: A pair of linear equations in <u>two</u> variables

$$a_0x + b_0y + c_0 = 0 \ , \ a_1x + b_1y + c_1 = 0$$

NOTE: x and y are <u>two</u> variables and the unknowns.

Methods to be used

- Solution by substitution
- Solution by eliminating coefficients (Elimination method)
- Matrix method
- Crammer's rule method
- Determinant method
- Graphical method

The last four methods have not been discussed, but the first two are to be reviewed in the subsequent examples below.

Examples

Solve the pair of simultaneous equations

a) $5x + 2y = 14\,, 3x - 4y = 24$

b) $3x + 4y - 9 = 0\,, 2x + 3y = 8$

Solution

Solution by substitution

a) $5x + 2y = 14$..... (1)

$3x - 4y = 24$ ..... (2)

From eqn. (1), making x the subject

$\Rightarrow x = \frac{14-2y}{5}$ ..... (3)

Substituting eqn. (3) in to eqn. (2)

$\Rightarrow 3\left(\frac{14-2y}{5}\right) - 4y = 24$.....$(\times 5)$

$\Rightarrow 3(14 - 2y) - 20y = 120$

$\Rightarrow 42 - 6y - 20y = 120 \; or \; -26y = 78$

$\therefore y = -3$

Substituting $y = -3$ in two eqn. (3) (either of the original equations)

$\Rightarrow x = \frac{14-2\times-3}{5} = \frac{20}{5} = 4$

$\therefore x = 4\,, y = -3$

Solving by equating coefficients

$5x + 2y = 14$..... (1)

$3x - 4y = 24$ ..... (2)

Multiplying both sides of eqn. (1) by 3 (the coefficient of x in eqn. (2)) and multiplying both sides of eqn. (2) by 5 (the coefficient of y in eqn. (1))

$\Rightarrow 15x + 6y = 42$

$\Rightarrow 15x - 20y = 120$

Now subtracting the equations,

$26y = -78 \therefore y = -3$

Substituting $y = -3$ in either of the original equations

$5x + 2y = 14\,, y = -3$
$\Rightarrow 5x - 6 = 14 \therefore 5x = 20 \;\; or \; x = 4$
$\therefore x = 4\,, y = -3$

b) Left as an exercise. **ANSWER**: $\boldsymbol{x = -5\,, y = 6}$
Try also to solve simultaneously
$3x + 2y = 16\,, 4x - 3y - 10 = 0.$ **ANSWER**: $\boldsymbol{x = -5\,, y = 6}$

Case II: A pair of simultaneous equations in two variables, one Quadratic and one Linear

Example

Solve the pair of simultaneous equations

a) $4x - 3y = 1\,, 12xy + 13x^2 = 25$

b) $x^2 + y^2 - 6x + 4y - 13 = 0\,, 2x = 3y - 1$

Solution

a) $4x - 3y = 1$..... (1)
$12xy + 13x^2 = 25$ ..... (2)
From eqn. (1)
$y = \frac{4x-1}{3}$ ..... (3)
Substitute eqn. (3) in to eqn. (2)
$\Rightarrow 12x\left(\frac{4x-1}{3}\right) + 13x^2 = 25$ .....$(\times 3)$
$\Rightarrow 48x^2 - 12x + 39x^2 = 75$
$\therefore 87x^2 - 12x - 75 = 0$ .....$(\div 3)$
$\Rightarrow 29x^2 - 4x - 25 = 0$
$\Rightarrow (29x + 25)(x - 1) = 0$
$\therefore x = 1\,, -\frac{25}{29}$ ..... (4)
Eqn. (4) in to eqn. (3) yields
$y = \frac{4-1}{3} = 1$, for $x = 5$ and $y = \frac{-\frac{100}{29}-1}{3}$ for $x = -\frac{25}{29}$
$\therefore x = 1\,, y = 1\,; x = -\frac{25}{29}\,, y = -\frac{43}{29}$

b) $2x = 3y - 1 \Rightarrow x = \frac{3y-1}{2}$ ..... (1)
$x^2 + y^2 - 6x + 4y - 13 = 0$ ..... (2)
Substitute eqn. (1) in to eqn. (2)
$\Rightarrow \left(\frac{3y-1}{2}\right)^2 + y^2 - 6\left(\frac{3y-1}{2}\right) + 4y - 13 = 0$
$\Rightarrow \frac{9y^2-6y+1}{4} + y^2 + 3 - 9y + 4y - 13 = 0$ .....$(\times 4)$
$\Rightarrow 9y^2 - 6y + 1 + 4y^2 + 12 - 36y + 16y - 52 = 0$
$\Rightarrow 13y^2 - 26y - 39 = 0 \;\; or \; y^2 - 2y - 4 = 0$
$\Rightarrow (y - 3)(y + 1) = 0$
$\therefore y = -1\,, 3$
Substitute eqn. (3) in to eqn. (1)
For $y = -1\,, x = \frac{3\times-1-1}{2} = -2$
For $y = 3\,, x = \frac{3\times3-1}{2} = 4$
$\therefore x = -2\,, y = -1; x = 4\,, y = 3$

Other techniques of solving specific set of simultaneous equations

Solve:

a) $x^2 + y^2 = 185\,, x - y = 3$

b) $x^3 + y^3 = 4914\,, x + y = 18$

c) $x^3 - y^3 = 218\,, x - y = 2$

d) $2x + 3y = 5\,, xy = 1$

e) $5x - y = 3\ ,\ y^2 - 6x^2 = 25$

f) $3x - 2y = 7\ , xy = 20$

NOTE:

The above equations can also be solved using substitution as in the previous examples.

Solution

a) $x^2 + y^2 = 185$..... (1)

$x - y = 3$ ..... (2)

From eqn. (2), squaring both sides

$(x - y)^2 = 3^2$

$\Rightarrow x^2 + y^2 - 2xy = 9$

But from eqn. (1), $x^2 + y^2 = 185$

$\Rightarrow 185 - 2xy = 9\ or\ xy = 88$ ..... (3)

Again $(x + y)^2 = x^2 + y^2 + 2xy$

From eqn. (1) and eqn. (3)

$\Rightarrow (x + y)^2 = 185 + 2 \times 88$

$\Rightarrow (x + y)^2 = 361$

$\therefore x + y = \pm 19$ ..... (4)

Now solving eqn. (2) and eqn. (4) simultaneously

Adding the equations,

$\Rightarrow 2x = \pm 19 + 3$

$\therefore 2x = 19 + 3 = 22\ \ or\ x = 11$

$\therefore 2x = -19 + 3 = -16\ \ or\ x = -8$

From eqn. (2), $y = x - 3$

For $x = 11\,, y = 11 - 3 = 8$

For $x = -8\,, y = -8 - 3 = -11$

$\therefore x = 11\,, y = 8; x = -8\,, y = -11$

b) $x^3 + y^3 = 4914$ ..... (1)

$x + y = 18$ ..... (2)

From eqn. (2), cubing both sides

$(x + y)^3 = 18^3$

$\Rightarrow x^3 + y^3 + 3x^2y + 3xy^2 = 5832$

$\Rightarrow x^3 + y^3 + 3xy(x + y) = 5832$

But from eqn. (1) and eqn. (2), $x^3 + y^3 = 4914\,, x + y = 18$

$\Rightarrow 4914 + 3xy(18) = 5832$

$\therefore xy = 17$ ..... (3)

Squaring eqn. (2) on both sides

$\Rightarrow\ (x + y)^2 = 18^2$

$\Rightarrow x^2 + y^2 + 2xy = 324$ ..... (4)

Substitute eqn. (3) in to eqn. (4)

$\Rightarrow x^2 + y^2 + 34 = 324$

$\therefore x^2 + y^2 = 290$ ..... (5)
Also from $(x-y)^2 = x^2 + y^2 - 2xy$
$\Rightarrow (x-y)^2 = 290 - 34 = 256$
$\therefore x - y = \pm 16$ ..... (6)
Solving eqn. (2) and eqn. (6) simultaneously
$x + y = 18\,, x - y = \pm 16$
Adding the equations,
$\Rightarrow 2x = \pm 16 + 18$
$\therefore 2x = 16 + 18 = 34 \;\; or \; x = 17$
$\therefore 2x = -16 + 18 = 2 \;\; or \; x = 1$
Also subtracting the equations,
$\Rightarrow 2y = 18 - \pm 16$
$\therefore 2y = 18 - 16 = 2 \;\; or \; y = 1$
$\therefore 2y = 18 + 16 = 34 \;\; or \; y = 17$
$\therefore x = 1\,, y = 17; x = 17\,, y = 1$

NOTE:
The equations have been reduced to a solution depending on the quadratic equation.

c) $x^3 - y^3 = 218$ ..... (1)
$x - y = 2$ ..... (2)
From eqn. (2), cubing both sides
$(x-y)^3 = 2^3$
$\Rightarrow x^3 - y^3 - 3x^2y + 3xy^2 = 8$
$\Rightarrow x^3 - y^3 - 3xy(x-y) = 8$
But from eqn. (1) and eqn. (2), $x^3 - y^3 = 218\,, x - y = 2$
$\Rightarrow 218 - 3xy(2) = 8$
$\therefore xy = 35$ ..... (3)
Squaring eqn. (2) on both sides
$\Rightarrow (x-y)^2 = 2^2$
$\Rightarrow x^2 + y^2 - 2xy = 4$ ..... (4)
Substitute eqn. (3) in to eqn. (4)
$\Rightarrow x^2 + y^2 - 70 = 4$
$\therefore x^2 + y^2 = 74$ ..... (5)
Also from $(x+y)^2 = x^2 + y^2 + 2xy$
$\Rightarrow (x+y)^2 = 74 + 70 = 144$
$\therefore x + y = \pm 12$ ..... (6)
Solving eqn. (2) and eqn. (6) simultaneously
$x - y = 2\,, x + y = \pm 12$
Adding the equations,
$\Rightarrow 2x = \pm 12 + 2$
$\therefore 2x = 12 + 2 = 14 \;\; or \; x = 7$
$\therefore 2x = -12 + 2 = -10 \;\; or \; x = -5$
Also subtracting the equations,
$\Rightarrow 2y = \pm 12 - 2$
$\therefore 2y = 12 - 2 = 10 \;\; or \; y = 5$
$\therefore 2y = -12 - 2 = -14 \;\; or \; y = -7$
$\therefore x = 7\,, y = 5; x = -5\,, y = -7$

d) $2x + 3y = 5$ ..... (1)
$xy = 1$ ..... (2)
Re-writing eqn. (2) in terms of terms of eqn. (1)
$\Rightarrow 2x.3y = 6$ (No change in the meaning in eqn. (2))
Now squaring eqn. (1)
$(2x + 3y)^2 = 5^2$
$\Rightarrow 4x^2 + 2.2x.3y + 9y^2 = 25$, but $\Rightarrow 2x.3y = 6$..... (*)
$\therefore 4x^2 + 9y^2 = 25 - 12 = 13$ ..... (3)
Also from $(2x - 3y)^2 = 4x^2 + 9y^2 - 2.2x.3y$
But from eqn. (*) and eqn. (3), $2x.3y = 6\,, 4x^2 + 9y^2 = 13$
$\Rightarrow (2x - 3y)^2 = 13 - 12 = 1$
$2x - 3y = \pm 1$..... (4)
Solving eqn. (1) and eqn. (4) simultaneously
$2x + 3y = 5\,, 2x - 3y = \pm 1$
Adding the equations,
$\Rightarrow 4x = 5 \pm 1$
$\therefore 4x = 5 + 1 = 6 \ or \ x = \frac{3}{2}$
$\therefore 4x = 5 - 1 = 4 \ or \ x = 1$
Subtracting the equations,
$\Rightarrow 6y = 5 - \ \pm 1$
$\therefore 6y = 5 - 1 = 4 \ or \ y = \frac{3}{2}$
$\therefore 6y = 5 - \ -1 = 6 \ or \ y = 1$
$\therefore x = \frac{3}{2}\,, y = \frac{2}{3}; x = 1\,, y = 1$

e) Left as an exercise. **ANSWER**: $\boldsymbol{x = 2\,, y = 7}$; $\boldsymbol{x = -\frac{8}{9}\,, y = -\frac{97}{19}}$

f) Left as an exercise. **ANSWER**: $\boldsymbol{x = 5\,, y = 4}$; $\boldsymbol{x = -\frac{8}{3}\,, y = -\frac{15}{2}}$

Task

Solve: $3x - 5y = 2\,, xy = 8$ **ANSWER**: $\boldsymbol{x = 4\,, y = 2\,; x = -3\frac{1}{3}\,, y = -2\frac{2}{5}}$

HOMOGENEOUS EQUATIONS

Homogeneous equations are equations in which all terms have the same degree.

$e.g.\ ax^3 + bx^2y + cy^2x + dy^3 = f$, is a homogeneous of 3^0

$lx^2 + mxy + ny^2 = p$, is a homogeneous of 2^0

If the powers in every term in variables x and y are added, give a uniform degree.

Solving Homogeneous Equations

The equations to be considered are:

$$\left.\begin{array}{l} \boldsymbol{lx^2 + mxy + ny^2 = A} \\ \boldsymbol{px^2 + qxy + ry^2 = B} \end{array}\right\} \text{Both quadratic}$$

$$\begin{array}{l} \boldsymbol{lx^2 + mxy + ny^2 = A} \\ \boldsymbol{qxy = B} \end{array}$$

, where $l\,, m\,, n\,, p\,, q\,, r\,, A\,, B$ are constants.

Examples

1) *Solve the pair of simultaneous equations.*

$x^2 + xy + 4y^2 = 6$ ..... (1)

$3x^2 + 8y^2 = 14$ ..... (2)

Solution

Re-writing the equations by factoring

$\Rightarrow x^2\left[1 + \frac{y}{x} + 4\left(\frac{y}{x}\right)^2\right] = 6$ ..... (3)

$\Rightarrow x^2\left[3 + 8\left(\frac{y}{x}\right)^2\right] = 14$ ..... (4)

Eqn. (3) ÷ eqn. (4)

$\frac{x^2\left[1+\frac{y}{x}+4\left(\frac{y}{x}\right)^2\right]}{x^2\left[3+8\left(\frac{y}{x}\right)^2\right]} = \frac{6}{14}$; let $m = \frac{y}{x}$

$\Rightarrow \frac{1+m+4m^2}{3+8m^2} = \frac{3}{7}$ *or* $7 + 7m + 28m^2 = 9 + 24m^2$

$\therefore 4m^2 + 7m - 2 = 0$ *or* $(4m - 1)(m + 2) = 0$

$\therefore m = \frac{1}{4}, -2$

But $m = \frac{y}{x}$ *or* $y = mx$

$\Rightarrow y = \frac{1}{4}x$ ..... (5)

$\Rightarrow y = -2x$ ..... (6)

Using any of the equations (3) or (4)

From eqn. (4) $x^2\left[3 + 8\left(\frac{y}{x}\right)^2\right] = 14$

$\Rightarrow x^2[3 + 8m^2] = 14$

For $m = -2$,

$x^2[3 + 8(-2)^2] = 14$ *or* $x^2 = \frac{14}{35} = \frac{2}{5}$

$\therefore x = \pm\sqrt{\frac{2}{5}}$

From eqn. (6), $y = -2x = -2\left(\pm\sqrt{\frac{2}{5}}\right)$

$\therefore x = \sqrt{\frac{2}{5}}, y = -2\sqrt{\frac{2}{5}}$

$\therefore x = -\sqrt{\frac{2}{5}}, y = 2\sqrt{\frac{2}{5}}$

Now for $m = \frac{1}{4}$, equation $x^2[3 + 8m^2] = 14$ becomes;

$\Rightarrow x^2\left[3 + 8\left(\frac{1}{4}\right)^2\right] = 14$ *or* $x^2 = 4$

$\therefore x = \pm 2$

From eqn. (6), $y = -2x = -2(\pm 2)$

$\therefore x = 2, y = -4$

$\therefore x = -2, y = 4$

$\therefore x = \sqrt{\frac{2}{5}}, y = -2\sqrt{\frac{2}{5}}$; $x = -\sqrt{\frac{2}{5}}, y = 2\sqrt{\frac{2}{5}}$; $x = 2, y = -4$; $x = -2, y = 4$

2) *Solve the pair of simultaneous equations.*

$x^2 + 4xy + y^2 = 13$ ..... (1)

$2x^2 + 3xy = 8$ ..... (2)

Solution

Re-writing the equations by factoring

$\Rightarrow x^2\left[1+4\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\right]=13$ ..... (3)

$\Rightarrow x^2\left[2+3\left(\frac{y}{x}\right)\right]=8$ ..... (4)

Eqn. (3) ÷ eqn. (4)

$\frac{x^2\left[1+4\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\right]}{x^2\left[2+3\left(\frac{y}{x}\right)\right]}=\frac{13}{8}$; let $m=\frac{y}{x}$

$\Rightarrow \frac{1+4m+m^2}{2+3m}=\frac{13}{8}$ $or\ 1+4m+m^2=2+3m$

$\therefore 8m^2-7m-18=0$

$\therefore m=\frac{7\pm\sqrt{49+4(8)(18)}}{16}=\frac{7\pm25}{16}$

$\therefore m=2\,,-\frac{9}{8}$

From $m=\frac{y}{x}\Rightarrow y=mx$

$\Rightarrow y=2x$ ..... (5)

$\Rightarrow y=-\frac{9}{8}x$ ..... (6)

From equation (4)

$\Rightarrow x^2\left[2+3\left(\frac{y}{x}\right)\right]=8\therefore x^2[2+3m]=8$

For $m=2\,,x^2(2+6)=8$

$\therefore x^2=1$ $or\ x=\pm1$

But $y=2x$

$\therefore$ For $x=1\,,y=2$

$\therefore$ For $x=-1\,,y=-2$

$\therefore x=1\,,y=2\,;x=-1\,,y=-2$

Also for $m=-\frac{9}{8}$

From equation (4)

$\Rightarrow x^2\left[2+3\left(-\frac{9}{8}\right)\right]=8\therefore x^2=-\frac{64}{11}$, hence roots are complex.

Alternatively

Dealing with question (1)

$x^2+xy+4y^2=6$ ..... (1)

$3x^2+8y^2=14$ ..... (2)

Eliminating constants 6 and 14

$Eqn.(1)\times14-Eqn.(2)\times6$

$\Rightarrow 14x^2+14xy+56y^2=84$

$\Rightarrow \underline{18x^2+0xy+48y^2=84}$

$-4x^2+14xy+8y^2=0$

$or\ \Rightarrow 2x^2-7xy-4y^2=0$

Factorizing to linear factors gives

$(2x+y)(x-2y)=0$

$\therefore y=-2x$ $or\ y=\frac{1}{4}x$

Eqn. (3) in to Eqn. (2) or (1), the original equations

$3x^2+8y^2=14\,,y=-2x$, gives

$3x^2+32x^2=14\ \therefore x^2=\frac{2}{5}$ $or\ x=\pm\sqrt{\frac{2}{5}}$

For $x=\sqrt{\frac{2}{5}}\,,y=-\sqrt{\frac{2}{5}}$

For $x = -\sqrt{\frac{2}{5}}, y = \sqrt{\frac{2}{5}}$

Also $3x^2 + 8y^2 = 14, y = \frac{1}{4}x$ gives

$3x^2 + 8\left(\frac{1}{4}x\right)^2 = 14 \ or \ x = \pm 2$

Now for $x = 2, y = \frac{1}{4} \times 2 = \frac{1}{2}$

Also for $x = -2, y = -\frac{1}{4} \times 2 = -\frac{1}{2}$

Dealing with question (2)

$x^2 + 4xy + y^2 = 13$ ..... (1)

$2x^2 + 3xy = 8$ ..... (2)

Eliminating constants 13 and 8

$Eqn.(1) \times 8 - Eqn.(2) \times 13$

$8x^2 + 32xy + 8y^2 = 104$

$\underline{26x^2 + 39xy + 0y^2 = 104}$

$-18x^2 - 7xy + 8y^2 = 0 \ or \ 18x^2 + 7xy - 8y^2 = 0$

Factoring in to linear factors

$(2x - y)(9x + 8y) = 0$

$\therefore y = 2x \ or \ y = -\frac{9}{8}x$

Eqn. (3) in to Eqn. (2) or (1), the original equations

$2x^2 + 3xy = 8, y = 2x$

$\Rightarrow 2x^2 + 6x^2 = 8 \ \therefore x^2 = 1 \ or \ x = \pm 1$

$\therefore$ For $x = 1, y = 2 \ ; x = -1, y = -2$

Also $2x^2 + 3xy = 8, y = -\frac{9}{8}x$

$\Rightarrow 2x^2 + \frac{27}{8}x^2 = 8 \ \therefore x^2 = -\frac{64}{11} \ or \ x = complex$

NOTE:

1. The second alternative gives rise to a quadratic equation in terms of x and y which is factorisable to linear factors after eliminating the constant terms.
2. Substitution method can also be applicable to solve homogeneous simultaneous equations.

Task

Solve the pair of simultaneous equations

a) $2x^2 - xy - y^2 = 8, xy = 6$

b) $x^3 + y^3 = 35, x^2y + xy^2 = 30$

c) $x^2 - 2xy + 8y^2 = 8, 3xy - 2y^2 = 4$

ANSWERS: $\boldsymbol{a)\ x = \pm 3 ; y = \pm 1 \quad c)\ x = \pm 2 \ , y = \pm 1 ; x = \pm 1 , y = \pm 12}$

Simultaneous Linear Equations with three unknowns

$$\boldsymbol{a_1x + b_1y + c_1z = A}$$

$$\boldsymbol{a_2x + b_2y + c_2z = B}$$

$$\boldsymbol{a_3x + b_3y + c_3z = C}$$

Methods to be considered are;

- Elimination method
- Reducing to roe Echelon form

Elimination method of solving linear set of equations

$a_1x + b_1y + c_1z = A$ ..... (1)
$a_2x + b_2y + c_2z = B$ ..... (2)
$a_3x + b_3y + c_3z = C$ ..... (3)

a) Eliminating any variable ($i.e.\ x$ *or* y *or* z) from any pair of equations
Eliminating x from eqn. (1) and eqn. (2)
Multiply $\frac{a_2}{a_1}$ by eqn. (1) and subtract the result from eqn. (2)
The resulting equation is of the form

$$\boxed{\boldsymbol{Py + Qz = R}} \quad \text{..... (4)}$$

b) Eliminating the same variable $i.e.\ x$ from any other pair of equations
Eliminating x from eqn. (1) and eqn. (3)
Multiply $\frac{a_3}{a_1}$ by eqn. (1) and subtract the result from eqn. (3)
The resulting equation is of the form

$$\boxed{\boldsymbol{my + nz = r}} \quad \text{..... (5)}$$

Now solving eqn. (4) and eqn. (5) as discussed earlier (i.e. solving set of linear equations in two variables), this gives y and z values.
Now substituting y and z values in any of the original equations .$i.e.$ (1) or (2) or (3), the value of x can be obtained.

Examples

Solve the linear set of equations simultaneously

$x + 4y + 4z = 7$
$3x + 2y + 2z = 6$
$9x + 6y + 2z = 14$

Solution

$x + 4y + 4z = 7$ ..... (1)
$3x + 2y + 2z = 6$ ..... (2)
$9x + 6y + 2z = 14$ ..... (3)
Eliminating x from eqn. (1) and eqn. (2)
$Eqn.(1) \times \left(\frac{3}{1}\right) - eqn.(2)$
$3x + 12y + 12z = 21$
$3x + 2y + 2z = 6$
$10y + 10z = 15$ *or* $2y + 2z = 3$..... (4)
Eliminating x from eqn. (1) and eqn. (3)
$Eqn.(1) \times \left(\frac{9}{1}\right) - eqn.(3)$
$9x + 36y + 36z = 63$
$9x + 6y + 2z = 14$
$30y + 34z = 49$ ..... (5)
Now eliminating y from eqn. (4) and eqn. (5)
$Eqn.(4) \times \left(\frac{30}{2}\right) - eqn.(5)$
$30y + 30z = 45$
$30y + 34z = 49$
$-4z = -4 \ \therefore z = 1$
Put $z = 1$ in to eqn. (4) or (5)

$2y + 2z = 3\,, z = 1$

$\Rightarrow 2y + 2 = 3 \therefore y = \frac{1}{2}$

Now substitute $z = 1\,, y = \frac{1}{2}$ in to eqn. (1) or eqn. (2) or eqn. (3) the original equations

Using eqn. (1)

$x + 4y + 4z = 7 \;\; \therefore x + 2 + 4 = 7 \;\; or \; x = 1$

$\therefore x = 1\,, y = \frac{1}{2}\,, z = 1$

Task

Solve the simultaneous equations

a) $x + 2y - 3z = 3$
$2x - y - z = 11$
$3x + 2y + z = -5$
ANSWER: $\boldsymbol{x = 2\,, y = -4\,, z = -3}$

b) $3x + 2y - z = 19$
$4x - y + 2z = 4$
$2x + 4y - 5z = 32$
ANSWER: $\boldsymbol{x = 3\,, y = 4\,, z = -2}$

Solving Linear set of equations in 3 unknowns by reducing to Row echelon form

Consider the equations to be solved

$a_1x + b_1y + c_1z = A$ ..... (1)

$a_2x + b_2y + c_2z = B$ ..... (2)

$a_3x + b_3y + c_3z = C$ ..... (3)

Writing the equations in matrix form

$\underbrace{\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}}_{\text{Coefficient}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{Variable matrix}} = \underbrace{\begin{pmatrix} A \\ B \\ C \end{pmatrix}}_{\text{Constant matrix}}$, which may be written in short form $\mathbf{P.V = K}$

Writing elements of K in P matrix, the augmented matrix is formulated as;

$$Q = \left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & B \\ a_3 & b_3 & c_3 & C \end{array}\right)$$

Eliminating elements beow the major diagonal in the coefficient matrix gives rise to a triangular matrix

$\begin{pmatrix} a_1 & b_1 & c_1 & A \\ \boldsymbol{a_2} & b_2 & c_2 & B \\ \boldsymbol{a_3} & \boldsymbol{b_3} & c_3 & C \end{pmatrix}$, elements to be zeroed/ eliminated are $a_1\,, a_2\,, a_3$

To eliminate a_2, subtract $\frac{\boldsymbol{a_2}}{\boldsymbol{a_1}}$ times first row from the 2nd row, and to eliminate a_3, subtract $\frac{\boldsymbol{a_3}}{\boldsymbol{a_1}}$ times first row from 3rd row $.i.e.$

$$\begin{pmatrix} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & B \\ a_3 & b_3 & c_3 & C \end{pmatrix} \begin{matrix} Row\ 1 = R_1 \\ Row\ 2 = R_2 \\ Row\ 3 = R_3 \end{matrix}$$

Now eliminating using the formulae

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & B \\ a_3 & b_3 & c_3 & C \end{array}\right) \begin{array}{l} R_1 \longrightarrow R_1 \\ R_2 \longrightarrow R_2 - a_2/a_1 R_1 \\ R_3 \rightarrow R_3 - a_3/a_1 R_1 \end{array}$$

This gives

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & A \\ 0 & d & e & D \\ 0 & f & g & E \end{array}\right)$$

Now eliminating **f**, subtract $\frac{f}{d}$ times 2nd row from the 3rd row

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & A \\ a_2 & b_2 & c_2 & D \\ a_3 & b_3 & c_3 & H \end{array}\right) \begin{array}{l} R_1 \longrightarrow R_1 \\ R_2 \longrightarrow R_2 \\ R_3 \rightarrow R_3 \end{array}$$

, which is a triangular matrix

Finally detach the right hand column back to its original position

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d & e \\ 0 & 0 & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A \\ D \\ H \end{pmatrix}$$

Multiplying the matrices, the result will be;

$a_1 x + b_1 y + c_1 z = A$ …… (i)

$dy + ez = D$ …… (ii)

$hz = H$ …… (iii)

From equation (iii) and by back substitution, values of x, y and z can be evaluated.

Examples

Solve the linear set of equations by reducing to row echelon form

$x - 2y - 3z = 0$

$3x + 5y + 2z = 0$

$2x + 3y - z = 2$

Solution

$x - 2y - 3z = 0$

$3x + 5y + 2z = 0$

$2x + 3y - z = 2$

Writing the equations in matrix form

$$\begin{pmatrix} 1 & -2 & -3 \\ 3 & 5 & 2 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Formulating the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 3 & 5 & 2 & 0 \\ 2 & 3 & -1 & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Now eliminating 3 and 2 from column 1

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 3 & 5 & 2 & 0 \\ 2 & 3 & -1 & 2 \end{array}\right) \begin{array}{l} R_1 \rightarrow R_1 \\ R_2 \rightarrow R_2 - 3/1 R_1 \\ R_3 \rightarrow R_3 - 2/1 R_1 \end{array}$$

This gives

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 11 & 11 & 0 \\ 0 & 7 & 5 & 2 \end{array}\right) \begin{array}{l} R_1 \rightarrow R_1 \\ R_2 \rightarrow R_2 \\ R_3 \rightarrow R_3 - 7/11 R_2 \end{array}$$

Now eliminating 7 from 3rd row

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 11 & 11 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right)$$

Detaching the matrices back to their original positions

$$\begin{pmatrix} 1 & -2 & -3 \\ 0 & 11 & 11 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$x - 2y - 3z = 0$ ..... (i)

$11y + 11z = 0$ ..... (ii)

$-2z = 2$ ..... (iii)

From eqn. (iii), $z = 1$

Put $z = 1$ in to eqn. (ii)

$\Rightarrow 11y - 11 = 0 \ \therefore y = 1$

Put $z = 1, y = 1$ in to eqn. (i)

$x - 2 + 3 = 0 \ \therefore x = -1$

$\therefore x = -1, y = 1, z = -1$

Task

1. *Solve the equations simultaneously using row- echelon method*
   a) $2x + 3y + 4z = -4$
      $4x + 2y + 3z = -11$ **ANSWER**: $\boldsymbol{x = -3, y = 2, z = -1}$
      $3x + 4y + 2z = -3$
   b) $2a + b + 3c = 11$
      $a + 2b - 2c = 3$ **ANSWER**: $\boldsymbol{a = 1, b = 3, c = 2}$
      $4a + 3b + c = 15$
   c) $2p + 3q + 4r = 8$
      $3p - 2q - 3r = -2$ **ANSWER**: $\boldsymbol{p = 1, q = -2, r = 3}$
      $5p + 4q + 2r = 3$
2. *Simplify and solve the following set of simultaneous equations*
   a) $4(x + 3y) - 2(4x + 3z) - 3(x - 2y - 4z = 17)$
      $2(4x - 3y) + 5(x - 4z) + 4(x - 3y + 2z) = 23$
      $3(y + 4z) + 4(2x - y - z) + 2(x + 3y - 2z) = 5$
      ANSWER: $\boldsymbol{x = 1, y = 3, z = -5}$
   b) $5(x + 2y) - 4(3x + 4z) - 2(x + 3y - 5z = 16)$
      $2(3x - y) + 3(x - 4z) + 4(2x - 3y + z) = -16$
      $4(4 - 2z) + 2(2x - 4y - 3) - 3(x + 4y - 2z) = -62$
      ANSWER: $\boldsymbol{x = 2, y = 4, z = -3}$

Other 3 – unknown simultaneous equations

Solve the equations

i). $x + 3 = \frac{y-1}{2} = \frac{z-5}{-2}, 11x - 3y + 7z = 8$

ii). $\frac{4x-3y}{4} = \frac{2y-x}{3} = \frac{z+4y}{2}, 6x + 6y + 2z = 6$ ($\boldsymbol{Uneb\,2005}$)

Solution

i). Let $x + 3 = \frac{y-1}{2} = \frac{z-5}{-2} = \lambda$

$\left.\begin{array}{l} \Rightarrow x + 3 = \lambda \ \therefore x = \lambda - 3 \\ \Rightarrow \frac{y-1}{2} = \lambda \ \therefore y = 2\lambda + 1 \\ \Rightarrow \frac{z-5}{-2} = \lambda \ \therefore z = -2\lambda + 5 \end{array}\right\}$ ..... (1)

Substituting eqn. (1) in to the 2nd equation $11x - 3y + 7z = 8$

$\Rightarrow 11(\lambda - 3) - 3(2\lambda + 1) + 7(-2\lambda + 5) = 8$

$\Rightarrow 11\lambda - 6\lambda - 14\lambda - 33 - 3 + 35 = 8$

$\Rightarrow -9\lambda = 9 \therefore \lambda = -1$

Substituting $\lambda = -1$ I to eqn. (1) gives

$x = -4\,, y = -1\,, z = 7$

ii). Let $\frac{4x-3y}{4} = \frac{2y-x}{3} = \frac{z+4y}{2} = \lambda$

$\Rightarrow \frac{4x-3y}{4} = \lambda \ \therefore 4x - 3y = 4\lambda$ ..... (1)

$\Rightarrow \frac{2y-x}{3} = \lambda \ \therefore 2y - x = 3\lambda$ ..... (2)

$\Rightarrow \frac{z+4y}{2} = \lambda \ \therefore z + 4y = 2\lambda$ ..... (3)

Expressing $x\,, y\,, z$ in terms of λ

Eliminating x from eqn. (1) and (2)

$eqn.(1) + eqn.(2) \times 4$

$\Rightarrow 4x - 3y + 8y - 4x = 4\lambda + 12\lambda$

$\therefore 5y = 16\lambda \Rightarrow y = \frac{16\lambda}{5}$ ..... (4)

Eqn. (4) in to (2) to find x in terms of λ

$\Rightarrow 2y - x = 3\lambda \ \therefore x = 2y - 3\lambda \ \ or\ x = \frac{32\lambda}{5} - 3\lambda$

$\Rightarrow x = \frac{17\lambda}{5}$ ..... (5)

Eqn. (4) in to (3) to find z in terms of λ

$z + 4y = 2\lambda \ \Rightarrow z = 2\lambda - 4y$

$\Rightarrow z = 2\lambda - \frac{64\lambda}{5} = -\frac{54\lambda}{5}$ ..... (6)

Substituting eqn. (4), (5), (6) in to $6x + 6y + 2z = 6$

$\Rightarrow 6\left(\frac{17\lambda}{5}\right) + 6\left(\frac{16\lambda}{5}\right) + 2\left(-\frac{54\lambda}{5}\right) = 6$

$\Rightarrow 9\lambda = 3 \therefore \lambda = \frac{1}{3}$

Substituting $\lambda = \frac{1}{3}$ in to eqn. (4), (5) and (6) gives

$x = \frac{17}{5} \times \frac{1}{3} = \frac{17}{15}$

$y = \frac{16}{5} \times \frac{1}{3} = \frac{16}{15}$

$x = -\frac{54}{5} \times \frac{1}{3} = -\frac{18}{5}$

$\therefore x = \frac{17}{15}\,, y = \frac{16}{15}\,, z = -\frac{18}{5}$

Task

Solve the simultaneous equations

a) $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4}$ and $3x + 4y + 2z - 25 = 0$ **ANSWER**: $\boldsymbol{x = 5\,, y = 0\,,\ z = 5}$

b) $\frac{x+2y}{-3} = \frac{y+2z}{4} = \frac{2x+z}{5}$ and $x + y + z = 2$ **ANSWER**: $\boldsymbol{x = 1\,, y = -2\,,\ z = 3}$

c) $\frac{x+4z}{4} = \frac{y+z}{6} = \frac{3x+y}{5}$ and $4x + 2y + 5z = 15$ **ANSWER**: $\boldsymbol{x = 0\,, y = 5\,, z = 1}$

d) $\frac{x-y}{4} = \frac{z-y}{3} = 2z - x$ and $x + 3y + 2z = 4$ **ANSWER**: $\boldsymbol{x = 3, y = -1\,, z = 2}$

SURDS AND INDICES

Indices:

When a quantity $\boldsymbol{a}$ multiplies it self $n-$ times $.i.e. a \times a \times a \times \ldots\ldots\ldots\ldots n-$ times, the product is called *a power of* $\boldsymbol{a}$ $.i.e. a^n$

Definition:

Let $\boldsymbol{a}$, x and y be real numbers such that $\boldsymbol{a}^x = y$, then y is a power of $\boldsymbol{a}, x$ is called index and $\boldsymbol{a}$ is the base.

For example in 2^4 , $3^{\frac{1}{2}}$, $5^{\sqrt{2}}$, 2 , 3 and 5 are the bases and 4 , $\frac{1}{2}$ and $\sqrt{2}$ are indices.

Laws of indices:

If m and n are integers then

i) $a^m \times a^n = a^{m+n}$

ii) $(a^m)^n = a^{mn}$

iii) $\frac{a^m}{a^n} = a^m \div a^n = a^{m-n}$

iv) $(a.b)^m = a^m.b^m$

v) $a^{-n} = \frac{1}{a^n}$

vi) $a^0 = 1$

vii) If $a^{\frac{p}{q}} = b$, then $a^p = b^q$ for $q > 0$

viii) If $a^m = a^n$ then $m = n$

Proof of the laws:

i) $\boxed{a^m \times a^n = a^{m+n}}$

Several cases arise,

Case I: $m > 0$ and $n > 0$ $.i.e.$ both m and n are positive.

By definition,

$a^m = a \times a \times a \times \ldots\ldots\ldots\ldots m-$ times and

$a^n = a \times a \times a \times \ldots\ldots\ldots\ldots n-$ times

$\Rightarrow a^m \times a^n = (a \times a \times a \times \ldots\ldots\ldots\ldots m - times) \times (a \times a \times a \times \ldots\ldots\ldots\ldots n - times)$

$= a \times a \times a \times \ldots\ldots\ldots\ldots (m+n) - times$

$= a^{m+n}$

$\therefore a^m \times a^n = a^{m+n}$

Case II: $m < 0$ and $n < 0$ $.i.e.$ both m and n are negative

Let $m = -k$ and $n = -l$ where $k , n > 0$

$\Rightarrow a^m \times a^n = a^{-k} \times a^{-l}$

$= \frac{1}{a^k} \times \frac{1}{a^l}$

$= \frac{1}{a^{k+l}}$, by first case

$\therefore a^{-k} \times a^{-l} = a^{-(k+l)} = a^{-k-l} = \frac{1}{a^{k+l}}$

Case III: $m > 0$ and $n < 0$ $.i.e.$ both $m-$ positive and $n-$ negative

Let $n = -l$, $l > 0$

$\Rightarrow a^m \times a^n = a^m \times a^{-l}$

$= a^m \times \frac{1}{a^l}$

$\therefore a^m \times a^{-l} = a^m \times \frac{1}{a^l}$

Now if $m = l$, then

$a^m \times a^{-l} = a^l \times a^{-l} = a^l \times \frac{1}{a^l} = 1$

But by first case,

$a^l \times a^{-l} = a^0 = 1$

$$\therefore a^0 = 1$$

Also from $a^l \times a^{-l} = 1$, then

$$a^{-l} = \frac{1}{a^l}$$, showing that a^{-n} is the reciprocal of a^n

Further simplification of $a^m \times a^{-l} = a^m \times \frac{1}{a^l}$, for $m > l$

$$\Rightarrow a^m \times a^{-l} = \left(\left(a \times \frac{1}{a}\right) \times \left(a \times \frac{1}{a}\right) \times \ldots\ldots\ldots\ldots . l - times\right) \times a \times a \times \ldots\ldots\ldots\ldots . (m - l) - times$$

$$= (1 \times 1 \times \ldots\ldots\ldots\ldots l - times) \times a^{m-l} = a^{m-l}$$

$\therefore a^m \times a^{-l} = a^m \times \frac{1}{a^l} = a^{m-l}$

Now again for $m < l$, then

$$a^m \times a^{-l} = \left(\left(a \times \frac{1}{a}\right) \times \left(a \times \frac{1}{a}\right) \times \ldots\ldots\ldots\ldots . m - times\right) \times \left(\frac{1}{a} \times \frac{1}{a} \times \ldots\ldots\ldots\ldots . (l - m) - times\right)$$

$$= (1 \times 1 \times \ldots\ldots\ldots\ldots\ldots\ldots m - times) \times \frac{1}{a^{l-m}} = a^{-(l-m)} = a^{m-l}$$

$\therefore$ For any integer m and n, $a^m \times a^n = a^{m+n}$ holds.

ii) $$(a^m)^n = a^{m.n}$$

Case I: let $m, n > o$ *.i.e.* m and n are positive.

$$(a^m)^n = a^m \times a^m \times \ldots\ldots\ldots\ldots\ldots\ldots \times \; n - times$$

$$= (a \times a \times a \times \ldots\ldots\ldots . m - times) \times (a \times a \times a \times \ldots\ldots . m - times) \times \ldots\ldots . n - times$$

$$= a \times a \times a \times \ldots\ldots\ldots\ldots\ldots . \ldots . mn - times$$

$$= a^{m.n}$$

Now if $m > 0, n < 0$, put $n = -l$ where $l > 0$

$\Rightarrow (a^m)^n = (a^m)^{-l} = \frac{1}{(a^m)^l} = \frac{1}{a^{ml}} = a^{-ml}$

$\therefore (a^m)^{-l} = a^{m(-l)} = a^{m.n}$,for $n = -l$

Also if $m > o$ and $n = 0$, then

$(a^m)^n = (a^m)^0 = a^0 = 1$

iii) $$\frac{a^m}{a^n} = a^m \div a^n = a^{m-n}$$

$$\frac{a^m}{a^n} = \frac{a \times a \times a \times \ldots\ldots\ldots\ldots\ldots\ldots m - times}{a \times a \times a \times \ldots\ldots\ldots\ldots\ldots\ldots n - times}$$

$$= a.a.a \ldots\ldots\ldots\ldots\ldots\ldots\ldots . (m - n) - times$$

$$= a^{m-n} \quad \therefore \frac{a^m}{a^n} = a^{m-n}$$

iv) $$(a.b)^m = a^m . b^m$$

$(a.b)^m = ab \times ab \times ab \ldots\ldots\ldots\ldots\ldots\ldots m-times$

$= (a \times a \times a \times \ldots\ldots\ldots\ldots\ldots m-times) \times (b \times b \times b \times \ldots\ldots\ldots\ldots\ldots m-times)$

$\therefore (a.b)^m = a^m.b^m$

NOTE:

The above laws hold for rational/ fractional indices.

Consider a^r and a^s where $r = \frac{p}{q}$ and $s = \frac{m}{n}$ with p, m, q, n are integers and $p, q > 0$

Let $a^r = b$ and $a^s = c$

$\Rightarrow a^{\frac{p}{q}} = b \quad \therefore a^p = b^q$

$\Rightarrow a^{\frac{m}{n}} = c \quad \therefore a^m = c^n$

Making powers of b and c the same

$\Rightarrow (a^p)^n = (b^q)^n \quad \therefore a^{np} = b^{qn}$

Also $(a^m)^q = (c^n)^q \quad \therefore a^{mq} = c^{nq}$

Now multiplying

$\Rightarrow b^{qn}.c^{nq} = a^{np}.a^{mq}$

$\Rightarrow (b.c)^{nq} = a^{np+mq}$

$\therefore b.c = a^{\frac{np+mq}{nq}} = a^{\frac{m}{n}+\frac{p}{q}}$

$\Rightarrow a^r.a^s = a^{\frac{m}{n}+\frac{p}{q}} = a^{r+s}$

Illustrative examples

1. *Simplify*

a) $\frac{x^{m+2n}.x^{3m-8n}}{x^{5m-6n}}$

b) $b^{-3}.x^{-2} \div 4b^2x$

c) $\frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^n}$

d) $9^{-\left(\frac{1}{2}\right)n} \times 3^{n+2} \times 27^{n+1}$

e) $\left(\frac{a^{-1}b^2}{a^2b^{-4}}\right)^7 \div \left(\frac{a^3b^{-5}}{a^{-2}b^3}\right)^{-5}$

Solution

a) $\frac{x^{m+2n}.x^{3m-8n}}{x^{5m-6n}} = x^{m+2n+(m-8n)-(5m-6n)}$

$= x^{(m+3m-5m)+(2n-8n+6n)}$

$= x^{-m+0} = x^{-m} = \frac{1}{x^m}$

b) $b^{-3}.x^{-2} \div 4b^2x = \frac{b^{-3}.x^{-2}}{4b^2x}$

$= \frac{b^{-3}}{b^2}.\frac{x^{-2}}{x}.\frac{1}{4} = \frac{b^{-3-2}.x^{-2-1}}{4}$

$= \frac{b^{-5}.x^{-3}}{4} = \frac{1}{4b^5.x^3}$

c) $\frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^n} = \frac{3(2^n.2^1)-4(2^n.2^{-1})}{(2^n.2^1)-2^n}$, let $2^n = x$

$= \frac{3(2x)-4\left(\frac{x}{2}\right)}{2x-x} = \frac{6x-2x}{x} = \frac{4x}{x} = 4$

$\therefore \frac{3(2^{n+1})-4(2^{n-1})}{2^{n+1}-2^n} = 4$

d) $9^{-\left(\frac{1}{2}\right)n} \times 3^{n+2} \times 27^{n+1} = [3^2]^{-\frac{1}{2}n} \times 3^{n+2} \times (3^3)^{n+1}$

$= 3^{-2\times\frac{1}{2}n+n+2+3n+3} = 3^{3n+5}$

e) $\left(\frac{a^{-1}b^2}{a^2b^{-4}}\right)^7 \div \left(\frac{a^3b^{-5}}{a^{-2}b^3}\right)^{-5} = \frac{a^{-7}b^{14}}{a^{14}b^{-28}} \div \frac{a^{-15}b^{25}}{a^{10}b^{-15}}$

$= \left(\frac{a^{-7}}{a^{14}}\right).\left(\frac{b^{14}}{b^{-28}}\right) \div \left(\frac{a^{-15}}{a^{10}}\right).\left(\frac{b^{25}}{b^{-15}}\right)$

$= a^{-7-14}.b^{14-(-28)} \div a^{-15-10}.b^{25-(-15)}$

$= \frac{a^{-21}.b^{42}}{a^{-25}.b^{40}} = a^{-21-(-25)}.b^{42-40} = a^4.b^2$

2. *Simplify*

a) $\frac{x^2(x^2+1)^{-\frac{1}{2}}-(x^2+1)^{\frac{1}{2}}}{x^2}$

b) $\frac{(1+x)^{\frac{1}{3}}-\frac{1}{3}x(1+x)^{-\frac{2}{3}}}{(1+x)^{\frac{2}{3}}}$

c) $\frac{\sqrt{(1-x)}.\frac{1}{2}(1+x)^{-\frac{1}{2}}+\frac{1}{2}(1-x)^{-\frac{1}{2}}.\sqrt{(1+x)}}{1-x}$

d) $\frac{x^{\frac{3}{2}}+xy}{xy-y^3} - \frac{\sqrt{x}}{\sqrt{x}-y}$

e) $\frac{(2^{2n}-3.2^{2n-3})(3^n-2.3^{n-2})}{3^{n-4}(4^{n+8}-2^{2n})}$

Solution

a) $\frac{x^2(x^2+1)^{-\frac{1}{2}}-(x^2+1)^{\frac{1}{2}}}{x^2}$, let $x^2+1=a$

$\Rightarrow Expression = \frac{x^2.a^{-\frac{1}{2}}-a^{\frac{1}{2}}}{x^2} = \frac{a^{-\frac{1}{2}}(x^2-a^1)}{x^2}$ (Factoring term with smallest power)

$= \frac{a^{-\frac{1}{2}}[x^2-(x^2+1)]}{x^2} = \frac{(x^2+1)^{-\frac{1}{2}}.(-1)}{x^2}$

$\therefore \frac{x^2(x^2+1)^{-\frac{1}{2}}-(x^2+1)^{\frac{1}{2}}}{x^2} = -\frac{1}{x^2(x^2+1)^{-\frac{1}{2}}}$

b) $\frac{(1+x)^{\frac{1}{3}}-\frac{1}{3}x(1+x)^{-\frac{2}{3}}}{(1+x)^{\frac{2}{3}}}$, let $b=1+x$

$Expression = \frac{b^{\frac{1}{3}}-\frac{1}{3}xb^{-\frac{2}{3}}}{b^{\frac{2}{3}}} = \frac{b^{-\frac{2}{3}}\left(b-\frac{1}{3}x\right)}{b^{\frac{2}{3}}}$ (Factoring term with smallest power)

$= b^{-\frac{2}{3}-\frac{2}{3}}.\left(b-\frac{1}{3}x\right) = b^{-\frac{4}{3}}.\left(b-\frac{1}{3}x\right)$

$= (1+x)^{-\frac{4}{3}}.\left(1+x-\frac{1}{3}x\right) = (1+x)^{-\frac{4}{3}}.\left(1+\frac{2}{3}x\right)$

$\therefore \frac{(1+x)^{\frac{1}{3}}-\frac{1}{3}x(1+x)^{-\frac{2}{3}}}{(1+x)^{\frac{2}{3}}} = \frac{3+2x}{3(1+x)^{\frac{4}{3}}}$

c) $\frac{\sqrt{(1-x)}.\frac{1}{2}(1+x)^{-\frac{1}{2}}+\frac{1}{2}(1-x)^{-\frac{1}{2}}.\sqrt{(1+x)}}{1-x}$, let $1-x=a, b=1+x$

$Expression = \frac{\sqrt{a}.\frac{1}{2}b^{-\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}.\sqrt{b}}{a} = \frac{\frac{1}{2}.a^{\frac{1}{2}}.b^{-\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}.b^{\frac{1}{2}}}{a}$

$= \frac{a^{-\frac{1}{2}}.b^{-\frac{1}{2}}}{2}.\frac{(1+b)}{a} = a^{-\frac{1}{2}-1}.b^{-\frac{1}{2}}.(1+b)$

$= a^{-\frac{3}{2}}.b^{-\frac{1}{2}}.(1+b)$

$= (1-x)^{-\frac{3}{2}}.(1+x)^{-\frac{1}{2}}.(1+1+x)$

$= (1-x)^{-\frac{3}{2}}.(1+x)^{-\frac{1}{2}}.(2+x) = \frac{2+x}{(1-x)^{\frac{3}{2}}.(1+x)^{\frac{1}{2}}}$

$= \frac{2+x}{(1-x)(1-x)^{\frac{1}{2}}.(1+x)^{\frac{1}{2}}} = \frac{2+x}{(1-x)(1-x^2)^{\frac{1}{2}}}$

$\therefore \frac{\sqrt{(1-x)}.\frac{1}{2}(1+x)^{-\frac{1}{2}}+\frac{1}{2}(1-x)^{-\frac{1}{2}}.\sqrt{(1+x)}}{1-x} = \frac{2+x}{(1-x)\sqrt{1-x^2}}$

d) $\frac{x^{\frac{3}{2}}+xy}{xy-y^3} - \frac{\sqrt{x}}{\sqrt{x}-y} = \frac{x^{\frac{3}{2}}+xy}{xy-y^3} - \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}-y}$, let $x^{\frac{1}{2}} = a, x = a^2$

$Expression = \frac{a^3+a^2y}{a^2y-y^3} - \frac{a}{a-y} = \frac{a^2(a+y)}{y(a^2-y^2)} - \frac{a}{a-y}$

$= \frac{a^2(a+y)}{y(a+y)(a-y)} - \frac{a}{a-y}$

$= \frac{a^2}{y(a-y)} - \frac{a}{a-y} = \frac{a}{a-y}\left(\frac{a}{y}-1\right)$

$= \frac{a}{a-y}\frac{(a-y)}{y} = \frac{a}{y}$

$= \frac{x^{\frac{1}{2}}}{y} = \frac{\sqrt{x}}{y}$

$\therefore \frac{x^{\frac{3}{2}}+xy}{xy-y^3} - \frac{\sqrt{x}}{\sqrt{x}-y} = \frac{\sqrt{x}}{y}$

e) $\frac{(2^{2n}-3.2^{2n-3})(3^n-2.3^{n-2})}{3^{n-4}(4^{n+8}-2^{2n})} = \frac{[(2^n)^2-3((2^n)^2.2^{-3})][3^n-2.3^n.3^{-2}]}{3^n.3^{-4}.((2^2)^{n+2}-(2^n)^2)}$ let $2^n = a$, $3^n = b$

$= \frac{\left[a^2-3\left(a^2.\frac{1}{8}\right)\right]\left[b-2.b.\frac{1}{9}\right]}{b.\frac{1}{81}.((2^2)^{n+2}-a^2)} = \frac{\left(\frac{8a^2-3a^2}{8}\right).\left(\frac{9b-2b}{9}\right)}{\frac{b}{81}.((2^n)^2.2^4-a^2)}$

$= \frac{\frac{5a^2}{8}.\frac{7b}{9}}{\frac{b}{81}.(a^2.16-a^2)} = \frac{\frac{5a^2}{8}.\frac{7b}{9}}{\frac{b}{81}.15a^2}$

$= \frac{5a^2}{8}.\frac{7b}{9}.\frac{81}{15a^2b} = \frac{5\times7\times81}{8\times9\times15} = \frac{21}{8}$

$\therefore \frac{(2^{2n}-3.2^{2n-3})(3^n-2.3^{n-2})}{3^{n-4}(4^{n+8}-2^{2n})} = \frac{21}{8}$

3. *Simplify*

a) $\left(\frac{81}{256}\right)^{-\frac{5}{4}}$

b) $\frac{(0.3)^{\frac{1}{3}}.\left(\frac{1}{27}\right)^{\frac{1}{4}}.(9)^{\frac{1}{6}}.(0.81)^{\frac{2}{3}}}{(0.9)^{\frac{2}{3}}.(3)^{-\frac{1}{2}}.\left(\frac{1}{3}\right)^{-2}.(243)^{-\frac{1}{4}}}$

c) $\frac{3^{-2}\times6^2\times\sqrt{48}}{5^2\times\sqrt[3]{\frac{1}{25}}\times(15)^{-\frac{4}{3}}\times(3)^{\frac{1}{3}}}$

d) $\frac{8^{\frac{2}{3}}+4^{\frac{3}{2}}}{16^{\frac{3}{4}}}$

e) $\frac{(8)^{\frac{1}{6}}\times(4)^{\frac{1}{3}}}{(32)^{\frac{1}{6}}\times(16)^{\frac{1}{2}}}$

Solution

a) $\left(\frac{81}{256}\right)^{-\frac{5}{4}} = \left[\frac{3^4}{4^4}\right]^{-\frac{5}{4}} = \frac{3^{4\times-\frac{5}{4}}}{4^{4\times-\frac{5}{4}}} = \frac{3^{-5}}{4^{-5}} = \frac{4^5}{3^5} = \frac{1024}{243}$

b) $\frac{(0.3)^{\frac{1}{3}}.\left(\frac{1}{27}\right)^{\frac{1}{4}}.(9)^{\frac{1}{6}}.(0.81)^{\frac{2}{3}}}{(0.9)^{\frac{2}{3}}.(3)^{-\frac{1}{2}}.\left(\frac{1}{3}\right)^{-2}.(243)^{-\frac{1}{4}}} = \frac{\left(\frac{3}{10}\right)^{\frac{1}{3}}.\left(\frac{1}{3^3}\right)^{\frac{1}{4}}.(3^2)^{\frac{1}{6}}.\left(\frac{81}{100}\right)^{\frac{2}{3}}}{\left(\frac{9}{10}\right)^{\frac{2}{3}}.(3)^{-\frac{1}{2}}.(3^{-1})^{-2}.(3^5)^{-\frac{1}{4}}}$

$= \frac{(3)^{\frac{1}{3}}.(10)^{-\frac{1}{3}}.(3)^{-\frac{3}{4}}.(3)^{\frac{1}{3}}.\left(\frac{3^4}{10^2}\right)^{\frac{2}{3}}}{(3^2)^{\frac{2}{3}}.(10^{-1})^{\frac{2}{3}}.(3)^{-\frac{1}{2}}.(3^2).(3)^{-\frac{5}{4}}} = \frac{3^{\frac{1}{3}+-\frac{3}{4}+\frac{1}{3}+\frac{8}{3}}.10^{-\frac{1}{3}+-\frac{4}{3}}}{3^{\frac{4}{3}+-\frac{1}{2}+2+-\frac{5}{4}}.10^{-\frac{2}{3}}}$

$= \frac{3^{\frac{31}{12}}}{3^{\frac{19}{12}}} \times \frac{10^{-\frac{5}{3}}}{10^{-\frac{2}{3}}} = 3^{\frac{31}{12}-\frac{19}{12}} \times 10^{-\frac{5}{3}+\frac{2}{3}}$

$= 3^1 \times 10^{-1} = \frac{3}{10}$

c) $\frac{3^{-2}\times 6^2\times\sqrt{48}}{5^2\times\sqrt[3]{\frac{1}{25}}\times(15)^{-\frac{4}{3}}\times(3)^{\frac{1}{3}}}=\frac{3^{-2}\times(3\times2)^2\times(3\times16)^{\frac{1}{2}}}{5^2\times\sqrt[3]{\frac{1}{25}}\times(15)^{-\frac{4}{3}}\times(3)^{\frac{1}{3}}}$

$=\frac{3^{-2}\times(3\times2)^2\times(3\times16)^{\frac{1}{2}}}{5^2\times(5^{-2})^{\frac{1}{3}}\times(3\times5)^{-\frac{4}{3}}\times(3)^{\frac{1}{3}}}=\frac{3^{-2}\times3^2\times2^2\times3^{\frac{1}{2}}\times(2^4)^{\frac{1}{2}}}{5^2\times5^{-\frac{2}{3}}\times3^{-\frac{4}{3}}\times5^{-\frac{4}{3}}\times(3)^{\frac{1}{3}}}$

$=\frac{3^{-2+2+\frac{1}{2}}\times2^{2+2}}{5^{2-\frac{2}{3}-\frac{4}{3}}\times3^{-\frac{4}{3}+\frac{1}{3}}}=\frac{3^{\frac{1}{2}}}{3^{-1}}\times2^4\times\frac{1}{5^0}$

$=3^{\frac{1}{2}+1}\times16=16\sqrt{27}$

d) $\frac{8^{\frac{2}{3}}+4^{\frac{3}{2}}}{16^{\frac{3}{4}}}=\frac{(2^3)^{\frac{2}{3}}+(2^2)^{\frac{3}{2}}}{(2^4)^{\frac{3}{4}}}=\frac{2^2+2^3}{2^3}=\frac{4+8}{8}=\frac{12}{8}=\frac{3}{2}$

e) $\frac{(8)^{\frac{1}{6}}\times(4)^{\frac{1}{3}}}{(32)^{\frac{1}{6}}\times(16)^{\frac{1}{2}}}=\frac{(2^3)^{\frac{1}{6}}\times(2^2)^{\frac{1}{3}}}{(2^5)^{\frac{1}{6}}\times(2^4)^{\frac{1}{2}}}=\frac{2^{\frac{3}{6}}\times2^{\frac{2}{3}}}{2^{\frac{5}{6}}\times2^{\frac{1}{3}}}$

$=2^{\frac{1}{2}+\frac{2}{3}-\frac{5}{6}-\frac{1}{3}}=2^0=1$

PROOFS IN INDICES

1. *If* $a=2, b=3$ *, show that* $(a^3b^2c^3)^2\sqrt{a^{-3}b^{-4}c^6}=144\sqrt{2}.c^9$

Solution

$(a^3b^2c^3)^2\sqrt{a^{-3}b^{-4}c^6}=a^6b^4c^6.(a^{-3}b^{-4}c^6)^{\frac{1}{2}}$

$=a^6b^4c^6.a^{-\frac{3}{2}}b^{-2}c^3$

$=a^{6-\frac{3}{2}}.b^{4-2}.c^{3+6}$

$=a^{\frac{9}{2}}.b^2.c^9$

$\Rightarrow a^{\frac{9}{2}}.b^2.c^9=(2)^{\frac{9}{2}}.(3)^2.c^9=(2^9)^{\frac{1}{2}}.9c^9$

$=(2^8\times2)^{\frac{1}{2}}.9c^9=2^4\times2^{\frac{1}{2}}.9c^9=16\times9\sqrt{2}c^9=144\sqrt{2}.c^9$

2. *If* $x=\left[a+\sqrt{a^2+b^3}\right]^{\frac{1}{3}}+\left[a-\sqrt{a^2+b^3}\right]^{\frac{1}{3}}$*, show that* $x^3+3bx-2a=0$

Solution

Let $\sqrt{a^2+b^3}=B$

$\Rightarrow x=[a+B]^{\frac{1}{3}}+[a-B]^{\frac{1}{3}}$

Cubing both sides

$x^3=\left[(a+B)^{\frac{1}{3}}+(a-B)^{\frac{1}{3}}\right]^3$

$=\left[(a+B)^{\frac{1}{3}}\right]^3+3\left[(a+B)^{\frac{1}{3}}\right]^2(a-B)^{\frac{1}{3}}+3(a+B)^{\frac{1}{3}}.(a-B)^{\frac{2}{3}}+\left[(a-B)^{\frac{1}{3}}\right]^3$

$=a+B+3(a+B)^{\frac{2}{3}}.(a-B)^{\frac{1}{3}}+3(a+B)^{\frac{1}{3}}.(a-B)^{\frac{2}{3}}+a-B$

$=2a+3(a+B)^{\frac{1}{3}}.(a-B)^{\frac{1}{3}}\left[(a+B)^{\frac{1}{3}}+(a-B)^{\frac{1}{3}}\right]$

$=2a+3[(a+B)(a-B)]^{\frac{1}{3}}.x$

$=2a+3(a^2-B^2)^{\frac{1}{3}}.x$

$=2a+3\left(a^2-(a^2+b^3)\right)^{\frac{1}{3}}.x\ \ \left[\because\sqrt{a^2+b^3}=B\right]$

$=2a+3\left(-b^{3\times\frac{1}{3}}\right).x$

$=2a-3bx$

$\therefore x^3-2a+3bx=0$

3. *If* $x=3^{\frac{2}{3}}+3^{-\frac{2}{3}}$*, show that* $9x^3-27x=82$

Solution

$x = 3^{\frac{2}{3}} + 3^{-\frac{2}{3}}$, cubing both sides,

$\Rightarrow x^3 = \left(3^{\frac{2}{3}} + 3^{-\frac{2}{3}}\right)^3 = \left(3^{\frac{2}{3}}\right)^3 + \left(3^{-\frac{2}{3}}\right)^3 + 3.3^{\frac{2}{3}}.3^{-\frac{2}{3}}\left(3^{\frac{2}{3}} + 3^{-\frac{2}{3}}\right)$

$[\because (\alpha+\beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)]$

$\Rightarrow x^3 = 3^2 + 3^{-2} + 3x = 9 + \frac{1}{9} + 3x$

$\therefore x^3 = \frac{82}{9} + 3x \ \ or \ \ 9x^3 - 27x = 82$

4. *If* $2^x = 4^y = 8^z$ *and* $\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} = \frac{22}{7}$, *show that* $x = \frac{7}{16}$, $y = \frac{7}{32}$ *and* $z = \frac{7}{48}$

Solution

$2^x = 4^y \Rightarrow 2^x = 2^{2y} \ \therefore x = 2y$ ..... (1)

Also $2^x = 8^z \Rightarrow 2^x = 2^{3z} \ \therefore x = 3z$ ..... (2)

$Eqn.(1) = eqn.(2)$

$\Rightarrow 2y = 3z \ \ or \ \ z = \frac{2y}{3}$ ..... (3)

From $\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} = \frac{22}{7}$

$\Rightarrow \frac{1}{2x} + \frac{1}{4\left(\frac{x}{2}\right)} + \frac{1}{8\left(\frac{2y}{3}\right)} = \frac{22}{7}$

$\Rightarrow \frac{1}{2x} + \frac{1}{4\left(\frac{x}{2}\right)} + \frac{1}{8\left(\frac{2}{3}\times\frac{x}{2}\right)} = \frac{22}{7}$

$\Rightarrow \frac{1}{2x} + \frac{1}{2x} + \frac{3}{8x} = \frac{22}{7}$

$\Rightarrow \frac{11}{8x} = \frac{22}{7} \ \therefore x = \frac{7}{22} \times \frac{11}{8} = \frac{7}{16}$

From eqn. (1), $x = 2y$

$\Rightarrow y = \frac{x}{2} = \frac{7}{16} \times \frac{1}{2} = \frac{7}{32}$

From eqn. (2), $x = 3z$

$\Rightarrow z = \frac{x}{3} = \frac{7}{16} \times \frac{1}{3} = \frac{7}{48}$

$\therefore x = \frac{7}{16}$, $y = \frac{7}{32}$, $z = \frac{7}{48}$

5. *If* $\frac{x}{y} = \frac{z}{w}$, *show that* $\frac{x^m + y^m + z^m + w^m}{x^{-m} + y^{-m} + z^{-m} + w^{-m}} = (wxyz)^{\frac{m}{2}}$

Solution

Let $\frac{x}{y} = \frac{z}{w} = k$

$\Rightarrow x = yk \ , z = wk$

$\Rightarrow \frac{x^m + y^m + z^m + w^m}{x^{-m} + y^{-m} + z^{-m} + w^{-m}} = \frac{y^m k^m + y^m + w^m k^m + w^m}{y^{-m} k^{-m} + y^{-m} + w^{-m} k^{-m} + w^{-m}}$

$= \frac{y^m(k^m+1) + w^m(k^m+1)}{y^{-m}(k^{-m}+1) + w^{-m}(k^{-m}+1)}$

$= \frac{(k^m+1)(y^m + w^m)}{(k^{-m}+1)(y^{-m} + w^{-m})}$

$= \frac{(k^m+1)}{\left(\frac{1}{k^m}+1\right)} \cdot \frac{(y^m + w^m)}{\left(\frac{1}{y^m} + \frac{1}{w^m}\right)}$

$= \frac{k^m(k^m+1)}{(1+k^m)} \cdot \frac{(y^m + w^m)(y^m . w^m)}{(y^m + w^m)}$

$= k^m . y^m . w^m = (k.y.w)^m$

But from $\frac{x}{y} = \frac{z}{w} = k \Rightarrow \frac{x}{y} = k \ , \frac{z}{w} = k$, multiplying the equations

$\Rightarrow k^2 = \frac{xz}{yw} \ \ or \ k = \left(\frac{xz}{yw}\right)^{\frac{1}{2}}$

$\therefore (k.y.w)^m = \left(\left(\frac{xz}{yw}\right)^{\frac{1}{2}}.y.w\right)^m = \left((x.z.y^{-1}.w^{-1})^{\frac{1}{2}}.y.w\right)^m$

$= \left(x^{\frac{1}{2}}.z^{\frac{1}{2}}.y^{1-\frac{1}{2}}.w^{1-\frac{1}{2}}\right)^m = \left(x^{\frac{1}{2}}.z^{\frac{1}{2}}.y^{\frac{1}{2}}.w^{\frac{1}{2}}\right)^m = (wxyz)^{\frac{m}{2}}$

6. *If* $a^x = b^y = c^z = d^w$ *and* $ab = cd$ *, show that* $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}+\frac{1}{w}$

Solution

Let $a^x = b^y = c^z = d^w = k$

$\Rightarrow a^x = k \;\; or \;\; a = k^{\frac{1}{x}}$

$\Rightarrow b^y = k \;\; or \;\; b = k^{\frac{1}{y}}$

$\Rightarrow c^z = k \;\; or \;\; c = k^{\frac{1}{z}}$

$\Rightarrow d^w = k \;\; or \;\; d = k^{\frac{1}{w}}$

From $a.b = c.d$

$\Rightarrow k^{\frac{1}{x}}.k^{\frac{1}{y}} = k^{\frac{1}{z}}.k^{\frac{1}{w}} \;\; or \;\; k^{\frac{1}{x}+\frac{1}{y}} = k^{\frac{1}{w}+\frac{1}{z}}$

$\therefore \frac{1}{x}+\frac{1}{y}=\frac{1}{z}+\frac{1}{w}$

7. *If* $3^x = 5^y = 75^z$*, show that* $xz = z(2x + y)$

Solution

Let $3^x = 5^y = 75^z = k$

$\Rightarrow 3^x = k \;\; or \;\; 3 = k^{\frac{1}{x}}$

$\Rightarrow 5^y = k \;\; or \;\; 5 = k^{\frac{1}{y}}$

$\Rightarrow 75^z = k \;\; or \;\; 75 = k^{\frac{1}{z}}$

But $75 = 25 \times 3 = 5^2 \times 3$

$\Rightarrow k^{\frac{1}{z}} = \left(k^{\frac{1}{y}}\right)^2 \times k^{\frac{1}{x}}$

$\Rightarrow k^{\frac{1}{z}} = k^{\frac{1}{x}+\frac{2}{y}}$

$\therefore \frac{1}{z} = \frac{1}{x}+\frac{2}{y} \;\; or \;\; \frac{1}{z} = \frac{y+2}{xy}$

$\therefore xz = z(2x + y)$

8. *If* $\frac{x^{n+1}}{a^n} = \frac{y^{n+1}}{b^n} = \frac{z^{n+1}}{c^n} = a + b + c$*, prove that* $\left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}}\right)^{\frac{n}{n+1}} = a + b + c$

Solution

Let $a + b + c = k$

$\Rightarrow \frac{x^{n+1}}{a^n} = k \;\; or \;\; a^n = \frac{x^{n+1}}{k}$

$\therefore a = \left(\frac{x^{n+1}}{k}\right)^{\frac{1}{n}} = \frac{x^{\frac{n+1}{n}}}{k^{\frac{1}{n}}}$

Similarly

$\therefore b = \frac{y^{\frac{n+1}{n}}}{k^{\frac{1}{n}}}$, $c = \frac{z^{\frac{n+1}{n}}}{k^{\frac{1}{n}}}$

$\Rightarrow \left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}}\right)^{\frac{n}{n+1}} = \left(a.k^{\frac{1}{n}} + b.k^{\frac{1}{n}} + c.k^{\frac{1}{n}}\right)^{\frac{n}{n+1}}$

$= \left[(a + b + c).k^{\frac{1}{n}}\right]^{\frac{n}{n+1}} = (a + b + c)^{\frac{n}{n+1}}.k^{\frac{1}{n+1}}$

But $a + b + c = k$

$\therefore \left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}}\right)^{\frac{n}{n+1}} = (a + b + c)^{\frac{n}{n+1}}.(a + b + c)^{\frac{1}{n+1}}$

$= (a+b+c)^{\frac{n}{n+1}+\frac{1}{n+1}} = a+b+c$

9. *If* $a = xy^{p-1}$, $b = xy^{q-1}$, $c = xy^{r-1}$, *prove that* $a^{q-r} \times b^{r-p} \times c^{p-q} = 1$

Solution

$$a^{q-r} \times b^{r-p} \times c^{p-q} = (xy^{p-1})^{q-r} \times (xy^{q-1})^{r-p} \times (xy^{r-1})^{p-q}$$
$$= x^{(q-r)}.y^{(p-1)(q-r)} \times x^{(r-p)}.y^{(q-1)(r-p)} \times x^{(p-q)}.y^{(r-1)(p-q)}$$
$$= x^{q-r+r-p+p-q} \times y^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$$
$$= x^0.y^{pq-pr-q+r+qr-qp-r+p+rp-rq-p+q}$$
$$= x^0.y^0 = 1$$

Solving equations involving indices

1. *Find* x *if* $x^{x\sqrt{x}} = \left(x\sqrt{x}\right)^x$, *where* x *is a positive integer.*

Solution

$x^{x\sqrt{x}} = \left(x\sqrt{x}\right)^x \Rightarrow \ x^{x.x^{\frac{1}{2}}} = \left(x.x^{\frac{1}{2}}\right)^x$

$\Rightarrow x^{x^{\frac{3}{2}}} = \left(x^{\frac{3}{2}}\right)^x = x^{\frac{3}{2}x}$

$\Rightarrow x^{\frac{3}{2}} = \frac{3}{2}x$, squaring both sides

$\Rightarrow x^3 = \frac{9}{4}x^2 \ \ or \ \ 4x^3 - 9x^2 = 0$

$\Rightarrow x^2(4x-9) = 0 \ \ \therefore x = 0 \ , \frac{9}{4}$

But $x \neq 0 \ \ \therefore x = \frac{9}{4}$

2. *Solve* $4^x.2^y = 128$ *and* $3^{2x+2y} = 9^{xy}$

Solution

$4^x.2^y = 128 \Rightarrow 2^{2x+y} = 2^7$

$\therefore 2x + y = 7$..... (1)

Also $3^{2x+2y} = 9^{xy} \Rightarrow 3^{2x+2y} = 3^{2xy}$

$\therefore 2x + 2y = 2xy \ \ or \ \ x + y = xy$..... (2)

From eqn. (1), $y = 7 - 2x$..... (3)

Eqn. (3) in to eqn. (2)

$\Rightarrow x + 7 - 2x = x(7-2x)$

$\Rightarrow 7 - x = 7x - 2x^2 \ \ or \ 2x^2 - 8x + 7 = 0$

$\therefore x = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 2 \times 7}}{2 \times 2}$

$= \frac{8 \pm \sqrt{8}}{4} = \frac{8 \pm 2\sqrt{2}}{4} = \frac{4 \pm \sqrt{2}}{2}$

From eqn. (3) $\ y = 7 - 2x$

For $x = \frac{4 \pm \sqrt{2}}{2}$, $y = 7 - 2\left(\frac{4 \pm \sqrt{2}}{2}\right) = 7 - \left(4 \pm \sqrt{2}\right)$

$\therefore x = \frac{4+\sqrt{2}}{2}$, $y = 3 - \sqrt{2}$; $x = \frac{4-\sqrt{2}}{2}$, $y = 11 + \sqrt{2}$

3. *Solve*

i) $a^{x-3}.a^{y+2} = a^2.a^x$, $a^x.a^y = a$, *where* $a \neq 0$, 1

ii) $\left(\sqrt{a}\right)^{x+y} = \left(\sqrt[3]{a}\right)^{y+z-1}$, $\left(\sqrt{b}\right)^{x+z-2} = \left(\sqrt[5]{b}\right)^{y+z}$, $\left(\sqrt[4]{c}\right)^{y} = \left(\sqrt[7]{c}\right)^{x+y+z}$

iii) $2^{x+y+z} = 8^{x+z-y}$, $5^{3y+2} = 25^{x+z}$, $3^{2x+y+2z} = 9^{3x+y}$

Solution

i) $a^{x-3}.a^{y+2} = a^2.a^x$

$\Rightarrow a^{x-3+y+2} = a^{2+x}$

$\Rightarrow x + y - 1 = 2 + x \ \ \therefore y = 3$

Also $a^x.a^y = a \Rightarrow a^{x+y} = a$
$\Rightarrow x + y = 1$, but $y = 3$
$\Rightarrow x = 1 - 3 = -2$
$\therefore x = -2 , y = 3$
ii) $\left(\sqrt{a}\right)^{x+y} = \left(\sqrt[3]{a}\right)^{y+z-1}$
$\Rightarrow a^{\frac{x+y}{2}} = a^{\frac{y+z-1}{3}}$
$\Rightarrow \frac{x+y}{2} = \frac{y+z-1}{3}$ or $3x + 3y = 2y + 2z - 2$
$\therefore 3x + y - 2z = -2$ ..... (1)
$\left(\sqrt{b}\right)^{x+z-2} = \left(\sqrt[5]{b}\right)^{y+z}$
$\Rightarrow b^{\frac{x+z-2}{2}} = b^{\frac{y+z}{5}}$
$\Rightarrow \frac{x+z-2}{2} = \frac{y+z}{5}$ or $5x + 5z - 10 = 2y + 2z$
$\therefore 5x - 2y + 3z = 10$ ..... (2)
$\left(\sqrt[4]{c}\right)^{y} = \left(\sqrt[5]{c}\right)^{x+y+z}$
$\Rightarrow c^{\frac{y}{4}} = c^{\frac{x+y+z}{7}}$
$\Rightarrow \frac{y}{4} = \frac{x+y+z}{7}$ or $7y = 4x + 4y + 4z$
$\therefore 4x - 3y + 4z = 0$ ..... (3)
Eliminating y from eqn. (1) and eqn. (2)
$2 \times eqn, (1) + eqn. (2)$
$\Rightarrow 6x + 2y - 4z + 5x - 2y + 3z = -4 + 10$
$\therefore 11x - z = 6$ ..... (4)
Eliminating y from eqn. (1) and eqn. (3)
$3 \times eqn, (1) + eqn. (3)$
$\Rightarrow 9x + 3y - 6z + 4x - 3y + 4z = -6 + 0$
$\therefore 13x - 2z = -6$ ..... (5)
Eliminating z from eqn. (4) and eqn. (5)
$2 \times eqn, (4) - eqn. (5)$
$\Rightarrow 22x - 2z - (13x - 2z) = 12 - -6$
$\Rightarrow 9x = 18 \quad \therefore x = 2$
From eqn. (4), $z = 11x - 6$
$\Rightarrow z = 22 - 6 = 16$
From eqn. (1), $y = -2 + 2z - 3x$
$\Rightarrow y = -2 + 32 - 6 = 24$
$\therefore x = 2 , y = 24 , z = 16$
iii) $2^{x+y+z} = 8^{x+z-y}$
$\Rightarrow 2^{x+y+z} = 2^{3x+3z-3y}$ $or\ x + y + z = 3x + 3z - 3y$
$\Rightarrow 2x - 4y + 2z = 0$
$\therefore x - 2y + z = 0$ ..... (1)
$5^{3y+2} = 25^{x+z}$
$\Rightarrow 5^{3y+2} = 5^{2x+2z}$ or $3y + 2 = 2x + 2z$
$\therefore 2x - 3y + 2z = 2$ ..... (2)
$3^{2x+y+2z} = 9^{3x+y}$
$\Rightarrow 3^{2x+y+2z} = 3^{6x+2y}$ or $2x + y + 2z = 6x + 2y$
$\therefore 4x + y - 2z = 0$ ..... (3)
Eliminating z from eqn. (1) and eqn. (2)
$2 \times eqn, (1) - eqn. (2)$

$\Rightarrow 2x - 4y + 2z - (2x - 3y + 2z) = 0 - 2$
$\Rightarrow -y = -2 \ \ \therefore y = 2$
Eliminating z from eqn. (2) and eqn. (3)
$eqn,(2) + eqn.(3)$
$\Rightarrow 6x - 2y = 2$, but $y = 2$
$\Rightarrow 6x - 4 = 2 \ \ \therefore x = 1$
From eqn. (1), $z = 2y - x$
$\Rightarrow z = 4 - 1 = 3$
$\therefore x = 1 , y = 2 , z = 3$

Task

1. *Show that*

i) $\frac{\sqrt{x^3}+xy}{xy-y^3} - \frac{\sqrt{x}}{\sqrt{x}-y} = \sqrt{\left(\frac{x}{y}\right)}$

ii) *If* $x = \sqrt[3]{p+q} + \sqrt[3]{p-q}$ *and* $p^2 - q^2 = r$, *then* $x^3 - 3rx - 2p = 0$

iii) *If* $(2.381)^x = (0.2381)^y = 10^z$, *then* $\frac{1}{x} = \frac{1}{y} + \frac{1}{z}$

iv) *If* $x = 3^{\frac{1}{4}} + 3^{-\frac{1}{4}}, y = 3^{\frac{1}{4}} - 3^{-\frac{1}{4}}$, *then* $3(x^2 + y^2)^2 = 64$

v) *If* $2^x = 3^y = 6^{-z}$, *then* $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

2. *Prove that*

a) $\frac{16(32)^m - 2^{3m-2}.4^{m+1}}{15.(2^{m-1})(16)^m} - \frac{5^m}{\sqrt{5^{2m}}} = 1$

b) $\frac{\left(9^{n+\frac{1}{4}}\right).\sqrt{3\times 3^n}}{3\sqrt{3^{-n}}} = 27$

c) *If* $pqr = 1$, *then* $\frac{1}{1+p+q^{-1}} + \frac{1}{1+q+r^{-1}} + \frac{1}{1+r+p^{-1}} = 1$

d) $\frac{(81)^n.3^5 - 3^{4n-1}.(243)}{9^{2n}.3^2} - \frac{4.3^n}{3^{n+1}-3^n} = 4$

e) *If* $x = 3^{\frac{2}{3}} + 3^{-\frac{2}{3}}$, *then* $9x^3 - 27x = 82$

3. *Simplify*

a) $\sqrt[3]{a^6b^{-2}c^{-4}} \times \sqrt[4]{a^{-6}b^4c^3}$ **ANSWER**: $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{3}{2}}$

b) $\left(\frac{5^{-1}.7^2}{5^2.7^{-4}}\right)^{\frac{7}{2}} \div \left(\frac{5^3.7^{-5}}{5^{-2}.7^3}\right)^{\frac{7}{2}}$ **ANSWER**: 175

LOGARITHMS

Definition:

The logarithm of a *positive* quantity N to any given base a is defined as the index of the power to which the base a must be raised to make it equal to the given quantity $N.i.e.$ if $\boxed{a^x = N}$, then x is called the logarithm of N to the base a and is written as

$$\boxed{\log_a N = x}$$

NOTE:

1. We notice that a logarithm is just a power, thus it can also be called an *index* or *an exponent*. This implies that in the derivation of laws of logarithms, laws of indices come in to use.
2. Logarithms are defined for only *positive numbers*.

PROPERTIES/LAWS/THEOREMS OF LOGARITHMS

1. *The logarithm of unity to any base is zero.*

   $\boxed{\log_a 1 = 0}$

   Proof:

   From $a^0 = 1$ $[Seting\ x = 0\ for\ a^x = N]$, then

   $0 = \log_a 1$ $[From\ defination]$

   $\therefore \log_a 1 = 0$

2. *Logarithm of any number to the same base is 1*

   $\boxed{\log_a a = 1}$

   Proof:

   From $a^1 = a$ $[Seting\ x = 1\ for\ a^x = N]$, then

   $\log_a a = 1$ $[From\ defination\ x = \log_a N]$

   $\therefore \log_a a = 1$

3. *A number a power logarithm of another number N to base of a number a is N*

   $\boxed{a^{\log_a N} = N}$

   Proof:

   Let $\log_a N = x \Rightarrow a^x = N$

   $\therefore a^{\log_a N} = N$

   In general,

   $\boxed{1.\, a^{\log_a b\ \log_b N} = N}$

   Proof:

   Let $a^{\log_a b} = y \Rightarrow y = b$

   Now $y^{\log_b N} = b^{\log_b N} = N$

   $1.\, a^{\log_a b\ \log_b N} = N$

4. *The logarithm of a product is equal to the sum of logarithms of its factors. i.e.*

   $\boxed{\log_a(M.N) = \log_a M + \log_a N}$

Proof:

From **LHS** $= \log_a(M.N)$

Let $\log_a(M.N) = b \Rightarrow a^b = M.N$..... (1)

Let $\log_a M = x \Rightarrow a^x = M$..... (2)

Let $\log_a M = y \Rightarrow a^y = N$..... (3)

Eqn. (2) , (3) in to eqn. (1)

$\Rightarrow a^b = a^x.a^y = a^{x+y}$

$\Rightarrow b = x + y$

$\therefore \log_a(M.N) = \log_a M + \log_a N$

NOTE:

- $\log_a(M + N) \neq \log_a M + \log_a N$
- $\log_a(M.N) = \log_a M \times \log_a N$

5. *The logarithm of a fraction is equal to the logarithm of numerator diminished by the logarithm of the denominator .i.e.*

$$\boxed{\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N}$$

Let $\log_a(M.N) = b \Rightarrow a^b = \frac{M}{N}$..... (1)

Let $\log_a M = x \Rightarrow a^x = M$..... (2)

Let $\log_a M = y \Rightarrow a^y = N$..... (3)

Eqn. (2), (3) in to eqn. (1)

$\Rightarrow a^b = a^x \div a^y = a^{x-y}$

$\Rightarrow b = x - y$

$\therefore \log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

NOTE:

- $\log_a\left(\frac{M}{N}\right) \neq \log_a M \div \log_a N$
- $\log_a(M - N) = \log_a M - \log_a N$

6. *The logarithm of the pth power of a positive number is p times the logarithm of that number.i.e.*

$$\boxed{\log_a M^p = p \log_a M}$$

Proof:

Let $\log_a M = x \Rightarrow a^x = M$

Now powering by p on both sides

$(a^x)^p = M^p \Rightarrow a^{xp} = M^p$

$\Rightarrow \log_a M^p = xp = p \log_a M$

$\therefore \log_a M^p = p \log_a M$

Similarly if $p = \frac{1}{r}$, then $\boxed{\log_a M^{\frac{1}{r}} = \frac{1}{r}\log_a M}$

7. *To transform a system of logarithms having a given base to a new system with a different base. i. e.*

$$\boxed{\log_{mn} N = \frac{\log_n N}{1+\log_n M}}$$

Proof:

Let $\log_a N = x$ be given and required $\log_b N = y \Rightarrow b^y = N$

$\Rightarrow \log_a N = x$, becomes

$\log_a b^y = x = \log_a N$

$\Rightarrow y \log_a b = \log_a N \quad or \;\; y = \frac{1}{\log_a b} \times \log_a N$

$\therefore \log_b N = \frac{\log_a N}{\log_a b} = \frac{1}{\log_a b} \times \log_a N$

NOTE:

1. To transform a logarithm from base a to base b we have only to multiply it by $\frac{1}{\log_a b}$. This is known as *Base changing formula*
2. From the above base change of bases, if $N = a$,

$\log_b N = \frac{\log_a N}{\log_a b} = \frac{1}{\log_a b} \times \log_a N$, becomes

$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \times \log_a a = \frac{1}{\log_a b}$

$$\boxed{\log_b a = \frac{1}{\log_a b}}$$

Proof:

Let $\log_b a = x \Rightarrow b^x = a$

$\Rightarrow b = a^{\frac{1}{x}}$, introducing logarithm to base a on both sides

$\Rightarrow \log_a b = \log_a a^{\frac{1}{x}} = \frac{1}{x}\log_a a = \frac{1}{x}$

$\therefore \log_b a = \frac{1}{\log_a b} \;\; or \;\; \log_a b = \frac{1}{\log_b a}$

8. *A base and its reciprocal inverse as a base*

$$\boxed{\log_b N = -\log_{\frac{1}{b}} N}$$

Proof:

Let $\log_b N = x \Rightarrow b^x = N \;\; or \;\; b = N^{\frac{1}{x}}$

Introducing logarithm to base $\frac{1}{b}$

$\Rightarrow \log_{\frac{1}{b}} b = \log_{\frac{1}{b}} N^{\frac{1}{x}} = \frac{1}{x}\log_{\frac{1}{b}} N$

$\Rightarrow x \log_{\frac{1}{b}} b = \log_{\frac{1}{b}} N \;\; or \;\; x = \frac{\log_{\frac{1}{b}} N}{\log_{\frac{1}{b}} b}$

From change of base formula

$\Rightarrow \log_{\frac{1}{b}} b = \frac{\log_b b}{\log_b \frac{1}{b}} = \frac{1}{\log_b b^{-1}} = \frac{1}{-\log_b b} = -1$

$\Rightarrow x = \frac{\log_{\frac{1}{b}} N}{\log_{\frac{1}{b}} b} = \frac{\log_{\frac{1}{b}} N}{-1} = -\log_{\frac{1}{b}} N$

$\therefore \log_b N = -\log_{\frac{1}{b}} N$

9. *Logarithm of a number with a composite base*

$$\log_b N = \frac{\log_a N}{\log_a b} = \frac{1}{\log_a b} \times \log_a N$$

Proof:

Let $\log_{mn} N = x \Rightarrow (mn)^x = N \;\; or \;\; mn = N^{\frac{1}{x}}$

Introducing logarithm to base n on both sides

$\Rightarrow \log_n (mn) = \log_n N^{\frac{1}{x}} = \frac{1}{x}\log_n N$

$\Rightarrow x = \frac{\log_n N}{\log_n (mn)} = \frac{\log_n N}{\log_n m + \log_n n} = \frac{\log_n N}{\log_n m + 1}$

$\therefore \log_{mn} N = \frac{\log_n N}{1 + \log_n M}$

Application of laws of logarithms

A. Solving equations

1. *Solve the equations:*
   - i) $\log(x^2 + 2x) = 0.9031$
   - ii) $\log_a(5x - 6) + \log_a(2x + 3) = \log_a(10x^2 - 3x - 6)$
   - iii) $2\log_a(x - 2) = \log_a(2x - 5)$
   - iv) $\log_x 3 + \log_x 9 + 729 = 9$

Solution

i) $\log(x^2 + 2x) = 0.9031$

$\Rightarrow \log_{10}(x^2 + 2x) = 0.9031$

$\Rightarrow 10^{0.9031} = x^2 + 2x$ (From the definition)

$\Rightarrow x^2 + 2x - 8 = 0 \;\; or \;\; (x + 4)(x - 2) = 0$

$\therefore x = -4, 2$

ii) $\log_a(5x - 6) + \log_a(2x + 3) = \log_a(10x^2 - 3x - 6)$

$\Rightarrow \log_a(5x - 6)(2x + 3) = \log_a(10x^2 - 3x - 6)$

Since the bases are the same, then also the numbers must be equal.

$\Rightarrow (5x - 6)(2x + 3) = 10x^2 - 3x - 6$

$\Rightarrow 10x^2 + 3x - 18 = 10x^2 - 3x - 6$

$\Rightarrow 6x = 12 \;\; \therefore x = 2$

iii) $\log_a(x - 2) = \log_a(2x - 5)$

$\Rightarrow \log_a(x - 2)^2 = \log_a(2x - 5)$

$\Rightarrow (x - 2)^2 = 2x - 5$

$\Rightarrow x^2 - 4x + 3 = 2x - 5$

$\Rightarrow x^2 - 6x + 9 = 0 \;\; or \;\; (x - 3)^2 = 0$

$\therefore x = 3$

iv) $\log_x 3 + \log_x 9 + 729 = 9$

$\Rightarrow \log_x(3 \times 9 \times 729) = 9$

$\Rightarrow x^9 = 19683$

$\Rightarrow (x^9)^{\frac{1}{9}} = (19683)^{\frac{1}{9}}$

$\therefore x = 3$

Task

1. *Solve the equations*
   - i) $\ln(x^2 - 1) - \ln(x - 1) = \ln 5 \qquad (x > 0)$
   - ii) $\ln 4 + 2\ln x - \ln(x^2 + x) = \ln 2 \qquad (x > 0)$

ANSWER: i) $\frac{3}{2}$ ii) $\frac{3}{2}$

NOTE: $\log_e N = ln\, N$

2. *Solve that equations*

i) $3^{\log_3 2} + 5^{\log_5 7} = 6^{\log_6 x}$

ii) $11^{\log_x 7} = 7$

iii) $\log_{10} x - 2 + \log_x 10 = 0$

iv) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

v) $\log_2 x - \log_x 8 = 2$ (***Uneb* 2007**)

vi) $\log_4(6 - x) = \log_2 x$ (***Uneb* 2003**)

vii) $\log_n 4 + \log_4 n^2 = 3$

viii) $\log_2 x + \log_x 64 = 5$

ix) $\log_5 x - \log_x 5 = \frac{3}{2}$

Solution

i) $3^{\log_3 2} + 5^{\log_5 7} = 6^{\log_6 x}$

Using $a^{\log_a x} = x$,

$\Rightarrow 2 + 7 = x \ \therefore x = 9$

ii) $11^{\log_x 7} = 7$

Let $\log_x 7 = y \ \Rightarrow 11^y = 7$

Introducing logarithm to base 11

$\Rightarrow \log_{11} 11^y = \log_{11} 7 \ \ or\ y = \log_{11} 7$

But $\log_x 7 = y \ \ \therefore \log_x 7 = \log_{11} 7$

Since numbers are the same, then $x = 11$

Alternatively

From $a^{\log_a x} = x$, since $11^{\log_x 7} = 7$, then $x = 11$

$\Rightarrow a^{\log_a 7} = 7$, also a must be 11

$\therefore 11^{\log_{11} 7} = 7 \Rightarrow x = 11$

iii) $\log_{10} x - 2 + \log_x 10 = 0$

$\Rightarrow \log_{10} x + \log_x 10 = 2$

Changing to base 10

$\Rightarrow \log_{10} x + \frac{\log_{10} 10}{\log_{10} x} = 2$

$\Rightarrow \log_{10} x + \frac{1}{\log_{10} x} = 2$; let $\log_{10} x = y$

$\Rightarrow y + \frac{1}{y} = 2 \ \ or\ y^2 - 2y + 1 = 0 \ \ \therefore y = 1$

$\therefore \log_{10} x = 1 \ \ or \ \ x = 10$

iv) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

Since 4 and 16 are multiples of 2, changing the base to 2

$\Rightarrow \log_2 x + \frac{\log_2 x}{\log_2 4} + \frac{\log_2 x}{\log_2 16} = \frac{21}{4}$

$\Rightarrow \log_2 x + \frac{\log_2 x}{\log_2 2^2} + \frac{\log_2 x}{\log_2 2^4} = \frac{21}{4}$

$\Rightarrow \log_2 x + \frac{\log_2 x}{2} + \frac{\log_2 x}{4} = \frac{21}{4}$

$\Rightarrow \left(1 + \frac{1}{2} + \frac{1}{4}\right)\log_2 x = \frac{21}{16}$

$\Rightarrow \frac{7}{4}\log_2 x = \frac{21}{4} \ \ or \ \ \log_2 x = 3$

$\therefore x = 2^3 = 8$

v) $\log_2 x - \log_x 8 = 2$, changing to base 2

$\Rightarrow \log_2 x - \frac{\log_2 8}{\log_2 x} = 2$

$\Rightarrow \log_2 x - \frac{3\log_2 2}{\log_2 x} = 2$

$\Rightarrow \log_2 x - \frac{3}{\log_2 x} = 2$, let $\log_2 x = y$

$\Rightarrow y - \frac{3}{y} = 2 \ or \ y^2 - 2y - 3 = 0$

$\Rightarrow (y+1)(y-3) = 0 \ or \ y = -1, 3$

$\Rightarrow \log_2 x = -1 \ or \ 2^{-1} = \frac{1}{2} = x$

$\Rightarrow \log_2 x = 3 \ or \ 2^3 = 8 = x$

$\therefore x = \frac{1}{2}, 3$

vi) Left as an exercise (**Hint:** Change to base 2)
ANSWER: x = 2 , −3

vii) $\log_n 4 + \log_4 n^2 = 3$

$\Rightarrow \log_n 4 + 2\log_4 n = 3$

$\Rightarrow \frac{\log_4 4}{\log_4 n} + 2\log_4 n = 3$, let $\log_4 n = x$

$\Rightarrow \frac{1}{x} + 2x = 3 \ or \ 2x^2 - 3x + 1 = 0$

$\Rightarrow (2x-1)(x-1) = 0 \ \therefore x = \frac{1}{2}, 1$

$\Rightarrow \log_4 n = \frac{1}{2} \ or \ 4^{1/2} = 2 = n$

$\Rightarrow \log_4 n = 1 \ or \ 4^1 = 4 = n$

$\therefore n = 2, 4$

viii) Left as an exercise. **ANSWER**: 4 , 8

ix) Left as an exercise. **ANSWER**: $x = \frac{\sqrt{5}}{5}, 25$

Task

a) Find x if $\log_x 8 - \log_{x^2} 16 = 1$ **ANSWER**: $x = 2$ [$HINT$: $Change\ to\ base\ 2\ or\ x$]

b) Find x if $\log_x 3 + \log_3 x = 2.5$ **ANSWER**: $x = \sqrt{3}, 9$

c) Find x if $\log_x 3 + \log_3 x = \frac{10}{3}$ **ANSWER**: $x = 27, \sqrt[3]{3}$

d) Solve for x: $\log_{10}\left(\frac{x^2+24}{x}\right) = 1$ **ANSWER**: $x = 4, 6$

B. In proof numbers.

1. Show that $\log_{10}\frac{1}{n}$ is negative where $n \geq 2$

Solution

$\log_{10}\frac{1}{n} = \log_{10} 1 - \log_{10} n = -\log_{10} n$, which is negative for $n \geq 2$

2. *If* $a^2 + b^2 = 7ab$, *prove that* $\log\left[\frac{1}{3}(a+b)\right] = \frac{1}{2}(\log a + \log b)$

Solution

$a^2 + b^2 = 7ab \Rightarrow (a+b)^2 - 2ab = 7ab$

$\therefore (a+b)^2 = 9ab \ or \ \left(\frac{a+b}{3}\right)^2 = ab$

$\Rightarrow \log\left[\frac{1}{3}(a+b)\right]^2 = \log a.b = \log a + \log b$

$\Rightarrow 2\log\left[\frac{1}{3}(a+b)\right] = \log a + \log b$

$\therefore \log\left[\frac{1}{3}(a+b)\right] = \frac{1}{2}(\log a + \log b)$

3. *Prove that* $\log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = 1$ *(Assuming that the base of each logarithm is 10)*

Solution

L.H.S:

$\log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = \log 2 + 16(\log 16 - \log 15) + 12(\log 25 - \log 24)$

$+7(\log 81 - \log 80)$

$= \log 2 + 16\log 2^4 - 16\log(3\times 5) + 12\log 5^2 - 12\log(3\times 2^3) + 7\log 3^4 - 7\log(2^4\times 5)$

$= \log 2 + 64\log 2 - 16\log 3 - 16\log 5 + 24\log 5 - 12\log 3 - 36\log 2 + 28\log 3 - 28\log 2 - 7\log 5$

$= (1 + 64 - 36 - 28)\log 2 + (-16 - 12 + 28)\log 3 + (-16 + 24 - 7)\log 5$

$= \log 2 + \log 5$

$= \log 2 + \log 5 = \log(2\times 5)$

$= \log 10 = 1$

4. *Prove that* $\frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = 1$

Solution

Let $\log_a(ab) = x \Rightarrow ab = a^x \;\; or \;\; a = (ab)^{1/x}$

$\log_b(ab) = y \Rightarrow ab = a^y \;\; or \;\; b = (ab)^{1/y}$

Now $(ab)^1 = (ab)^{1/x}.(ab)^{1/y} = (ab)^{\frac{1}{x}+\frac{1}{y}}$

$\therefore 1 = \frac{1}{x} + \frac{1}{y}$

$\Rightarrow \frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)} = 1$

5. *Prove that* $\frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_b(abc)} = 1$

Solution

Proceeding as in the previous example,

Let $\log_a(abc) = x \Rightarrow abc = a^x \;\; or \;\; a = (abc)^{1/x}$

$\log_b(abc) = y \Rightarrow abc = a^y \;\; or \;\; b = (abc)^{1/y}$

$\log_b(abc) = z \Rightarrow abc = a^z \;\; or \; c = (abc)^{1/z}$

Now $(abc)^1 = (abc)^{1/x}.(ab)^{1/y}.(abc)^{1/z} = (ab)^{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$

$\therefore 1 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

$\Rightarrow \frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_b(abc)} = 1$

6. *Prove that*

   i) $\log_a b \times \log_b c \times \log_c d = \log_a d$

   ii) $\log_a b \times \log_b c \times \log_c d \times \log_d a = 1$

   iii) $\log_a b \times \log_b a = 1$

Solution

Changing to another base, say k

i) **L.H.S:** $\log_a b \times \log_b c \times \log_c d = \frac{\log_k b}{\log_k a} \times \frac{\log_k c}{\log_k b} \times \frac{\log_k d}{\log_k c} = \frac{\log_k d}{\log_k a}$

Now changing to base a

$\Rightarrow \frac{\log_k d}{\log_k a} = \left(\frac{\log_a d}{\log_a k}\right) \times \left(\frac{\log_a a}{\log_a k}\right) = \log_a d$

$\therefore \log_a b \times \log_b c \times \log_c d = \log_a d$

ii) **L.H.S:** $\log_a b \times \log_b c \times \log_c d \times \log_d a = \frac{\log_k b}{\log_k a} \times \frac{\log_k c}{\log_k b} \times \frac{\log_k d}{\log_k c} \times \frac{\log_k a}{\log_k d} = 1$

iii) **L.H.S:** $\log_a b \times \log_b a = \frac{\log_k b}{\log_k a} \times \frac{\log_k a}{\log_k b} = 1$

Task

Prove that:

a) $\log_a x \times \log_b y = \log_b x \times \log_a y$

b) $\log_b a \times \log_a b \times \log_a c = \frac{1}{\log_c a}$

c) $\log_2 3 \times \log_3 8 = 1$

More conditional proofs

a) *If* $\log(x^3.y^3) = a$ *and* $\log\frac{x}{y} = b$, *show that* $\log x = \frac{1}{6}(3b + a)$ *and* $\log y = \frac{1}{6}(a - 3b)$

b) *If* $a^3 + b^3 = 0$, *prove that* $\log(a + b) = \frac{1}{2}(\log a + \log b + \log 3)$ *given that* $a + b \neq 0$.

c) *If* $a^x = b^y = c^z$ *and* $b^2 = ac$, *prove that* $y = \frac{2xz}{x+z}$

d) *If* $\log_{2a} a = x$, $\log_{3a} 2a = y$, $\log_{4a} 3a = z$, *show that* $xyz + 1 = 2yz$

e) *If* $a = \log_{24} 12$, $b = \log_{36} 24$, $c = \log_{48} 36$, *prove that* $1 + abc = 2bc$

f) *If* $z = \log_r(mn)$, $x = \log_m(rn)$ *and* $y = \log_n(mr)$, *prove that* $x + y + z = xyz - 2$

g) *If* $\log_a N = x$ *and* $\log_b N = y$, *prove that* $\log_{ab} N = \frac{xy}{x+y}$

Solution

a) $\log(x^3.y^3) = a$, $\log\frac{x}{y} = b$, to show that: $\log x = \frac{1}{6}(3b + a)$ *and* $\log y = \frac{1}{6}(a - 3b)$

From $\log(x^3.y^3) = a \Rightarrow \log(xy)^3 = a \;\; \therefore \log(xy) = \frac{a}{3}$

$\Rightarrow \log x + \log y = \frac{a}{3}$ ..... (1)

Also $\log\frac{x}{y} = b$ gives

$\Rightarrow \log x - \log y = b$ ..... (2)

$eqn.(1) + eqn.(2)$

$\Rightarrow 2\log x = \frac{a}{3} + b = \frac{a+3b}{3}$

$\therefore \log x = \frac{1}{6}(3b + a)$

$eqn.(1) - eqn.(2)$

$\Rightarrow 2\log y = \frac{a}{3} - b = \frac{a-3b}{3}$

$\therefore \log y = \frac{1}{6}(a - 3b)$

b) $a^3 + b^3 = 0$

But $(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$\Rightarrow a^3 + b^3 = (a + b)^3 - 3ab(a + b)$

$\Rightarrow (a + b)^3 - 3ab(a + b) = 0$

$\Rightarrow (a + b)[(a + b)^2 - 3ab] = 0$

$\therefore (a + b)^2 - 3ab = 0 \;\; or \;\; (a + b)^2 = 3ab$

Introducing logarithms

$\Rightarrow \log(a + b)^2 = \log(3ab)$

$\Rightarrow 2\log(a + b) = \log 3 + \log a + \log b$

$\therefore \log(a + b) = \frac{1}{2}(\log a + \log b + \log 3)$

c) Let $a^x = b^y = c^z = k$

$\Rightarrow a^x = k \;\; or \;\; a = k^{1/x}$

$\Rightarrow b^y = k \;\; or \;\; b = k^{1/y}$

$\Rightarrow c^z = k \;\; or \;\; c = k^{1/z}$

Given: $b^2 = ac$

$\Rightarrow \left(k^{\frac{1}{y}}\right)^2 = k^{\frac{1}{x}}.\, k^{\frac{1}{z}}$

$\Rightarrow k^{\frac{2}{y}} = k^{\frac{1}{x}+\frac{1}{z}}$

$\therefore \frac{2}{y} = \frac{1}{x} + \frac{1}{z} = \frac{x+z}{xz}$

$\therefore y = \frac{2xz}{x+z}$

Note: the number portrays the use of laws of indices.

d) $\log_{2a} a = x \Rightarrow (2a)^x = a \;\; or \;\; 2a = a^{\frac{1}{x}}$

$\log_{3a} 2a = y \Rightarrow (3a)^y = 2a \;\; or \;\; 3a = (2a)^{\frac{1}{y}}$

$\log_{4a} 3a = z \Rightarrow (4a)^z = 3a \;\; or \;\; 4a = (3a)^{\frac{1}{z}}$

From $2a = a^{\frac{1}{x}} \Rightarrow 2 = a^{\frac{1}{x}-1}$ ..... (1)

From $3a = (2a)^{\frac{1}{y}} \Rightarrow 3a = \left(a^{\frac{1}{x}}\right)^{\frac{1}{y}} = a^{\frac{1}{xy}}$

$\therefore 3 = a^{\frac{1}{xy}-1}$ ..... (2)

From $4a = (3a)^{\frac{1}{z}} \Rightarrow 2^2.a = \left(a^{\frac{1}{xy}}\right)^{\frac{1}{z}} = a^{\frac{1}{xyz}}$

$\therefore \left(a^{\frac{1}{x}-1}\right)^2.a = a^{\frac{1}{xyz}}$

$\Rightarrow a^{\left(\frac{2}{x}-2+1\right)} = a^{\frac{1}{xyz}}$

$\therefore \frac{2}{x} - 1 = \frac{1}{xyz} \;\; or \;\; 2yz - xyz = 1$

$\therefore xyz + 1 = 2yz$

e) $a = \log_{24} 12$, $b = \log_{36} 24$, $c = \log_{48} 36$, to prove that $1 + abc = 2bc$

$a = \log_{24} 12 \Rightarrow (24)^a = 12 \;\; or \;\; 24 = (12)^{\frac{1}{a}}$ ..... (1)

$b = \log_{36} 24 \Rightarrow (36)^b = 24 \;\; or \;\; 36 = (24)^{\frac{1}{b}}$ ..... (2)

$c = \log_{48} 36 \Rightarrow (48)^c = 36 \;\; or \;\; 48 = (36)^{\frac{1}{c}}$ ..... (3)

From eqn. (1)

$24 = (12)^{\frac{1}{a}} \Rightarrow 2 = \frac{(12)^{\frac{1}{a}}}{12} = (12)^{\frac{1}{a}-1}$ ..... (4)

From eqn. (2)

$36 = (24)^{\frac{1}{b}} \Rightarrow 3 = \frac{(24)^{\frac{1}{b}}}{12} = \frac{(12)^{\frac{1}{b}}.(2)^{\frac{1}{b}}}{12} = (12)^{\frac{1}{b}-1}.(2)^{\frac{1}{b}}$ ..... (5)

$eqn.(4) in\ to\ eqn.(5)$

$\Rightarrow 3 = (12)^{\frac{1}{b}-1}.\left((12)^{\frac{1}{a}-1}\right)^{\frac{1}{b}}$

$\Rightarrow 3 = (12)^{\frac{1}{ab}-\frac{1}{b}}.(12)^{\frac{1}{b}-1} = (12)^{\frac{1}{ab}-\frac{1}{b}+\frac{1}{b}-1}$

$\therefore 3 = (12)^{\frac{1}{ab}-1}$ ..... (6)

$eqn.(4),(6) in\ to\ eqn.(3)$

$48 = (36)^{\frac{1}{c}} \Rightarrow 2^2 \times 12 = [3(12)]^{\frac{1}{c}}$

$\Rightarrow \left[(12)^{\frac{1}{a}-1}\right]^2.(12) = \left[(12)^{\frac{1}{ab}-1}.(12)\right]^{\frac{1}{c}}$

$\Rightarrow (12)^{\frac{2}{a}-2+1} = \left[(12)^{\frac{1}{ab}-1+1}\right]^{\frac{1}{c}}$

$\therefore \frac{2}{a} - 1 = \frac{1}{abc} \quad or \;\; 2bc - abc = 1$

$\therefore 1 + abc = 2bc$

f) Given: $z = \log_r(mn)$, $x = \log_m(rn)$, $y = \log_n(mr)$ to prove: $x + y + z = xyz - 2$

$z = \log_r(mn) \Rightarrow r^z = mn \;\; or \;\; r = (mn)^{\frac{1}{z}}$ ..... (1)

$x = \log_m(rn) \Rightarrow m^x = rn \;\; or \;\; m = (nr)^{\frac{1}{x}}$ ..... (2)

$y = \log_n(mr) \Rightarrow n^y = mr \;\; or \;\; n = (mr)^{\frac{1}{y}}$ ..... (3)

$eqn.(1) \div eqn.(2)$, to eliminate n

$\Rightarrow \frac{r^z}{m^x} = \frac{m}{r} \Rightarrow r^{z+1} = m^{x+1}$

$\therefore m = r^{\left(\frac{z+1}{x+1}\right)}$ ..... (4)

$eqn.(2) \div eqn.(3)$, to eliminate r

$\Rightarrow \frac{m^x}{n^y} = \frac{n}{m} \Rightarrow m^{x+1} = n^{y+1}$

$\therefore n = m^{\left(\frac{x+1}{y+1}\right)}$ ..... (5)

$eqn.(4) in\ to\ eqn.(5)$

$\Rightarrow n = \left[r^{\left(\frac{z+1}{x+1}\right)}\right]^{\left(\frac{x+1}{y+1}\right)} = r^{\left(\frac{z+1}{y+1}\right)}$ ..... (6)

Now eqn. (4), (6) in to eqn. (1)

$r^z = mn$, becomes

$\Rightarrow r^z = r^{\left(\frac{z+1}{x+1}\right)} . r^{\left(\frac{z+1}{y+1}\right)}$

$\Rightarrow z = \left(\frac{z+1}{x+1}\right) + \left(\frac{z+1}{y+1}\right)$

$\therefore z = \frac{yz+y+z+xz+x+z+1}{xy+x+y+1}$

$\Rightarrow xyz + xz + yz + z = yz + y + z + 1 + xz + x + z + 1$

$\therefore xyz + z = x + y + 2z + 2$

$\therefore x + y + z = xyz - 2$

g) $\log_a N = x \Rightarrow a^x = N \;\; or \;\; a = N^{\frac{1}{x}}$..... (1)

$\log_b N = y \Rightarrow b^y = N \;\; or \;\; b = N^{\frac{1}{y}}$..... (2)

Multiplying the equations

$\Rightarrow a.b = N^{\frac{1}{x}} . N^{\frac{1}{y}} = N^{\left(\frac{1}{x}+\frac{1}{y}\right)}$

Introducing logarithm to base ab

$\Rightarrow \log_{ab} ab = \log_{ab} N^{\left(\frac{1}{x}+\frac{1}{y}\right)}$

$\Rightarrow 1 = \left(\frac{1}{x} + \frac{1}{y}\right) \log_{ab} N \quad or \quad 1 = \left(\frac{x+y}{xy}\right) \log_{ab} N$

$\therefore \log_{ab} N = \frac{xy}{x+y}$

Task

1. *Prove that*
   i) $\log_{b^2} a \times \log_{x^2} b = \frac{1}{4}\log_x a$
   ii) If $\log_a\left(1+\frac{1}{8}\right) = l$, $\log_a\left(1+\frac{1}{15}\right) = m$ and $\log_a\left(1+\frac{1}{24}\right) = n$, then $\log_a\left(1+\frac{1}{80}\right) = l - m - n$
   iii) If $\log_a n = x$ and $\log_c n = y$ where $n \neq 1$ then $\frac{x-y}{x+y} = \frac{\log_b c - \log_b a}{\log_b c + \log_b a}$
2. a) *Given that* $\log_3 x = p$ *and* $\log_{18} x = q$, *show that* $\log_6 3 = \frac{q}{p-q}$ (***Uneb* 2002**).
   b) *Show that* $\log_8 x = \frac{2}{3}\log_4 x$. *Hence without using tables/calculator, evaluate* $\log_8 6$ *to* 3 *d.p, if* $\log_4 3 = 0.7925$. (***Uneb* 2000**)
   c) *Prove that* $\log_6 x = \frac{\log_3 x}{1+\log_3 2}$. *Hence given that* $\log_3 2 = 0.631$, *find without using tables/calculator* $\log_6 4$ *correct to* 3 *s.f.* (***Uneb* 1998 *March***)
   d) *If* x *and* y *are positive numbers and each is greater than 1,* $x^{x+y} = y^{12}$ and $y^{x+y} = x^3$
   Prove that $\log x = 2\log y$. *Hence find* x *and* y
   ANSWER:

a) $\log_3 x = p \Rightarrow 3^p = x$ *or* $3 = x^{\frac{1}{p}}$ ..... (1)

$\log_{18} x = q \Rightarrow 18^q = x$ *or* $18 = x^{\frac{1}{q}}$ ..... (2)

eqn. (2) ÷ *eqn.* (1)

$\Rightarrow 6 = x^{\frac{1}{q}} \div x^{\frac{1}{p}} = x^{\frac{1}{q}-\frac{1}{p}} = x^{\left(\frac{p-q}{pq}\right)}$

Now introducing logarithm to base 6

$\Rightarrow \log_6 6 = \log_6 x^{\left(\frac{p-q}{pq}\right)} = \left(\frac{p-q}{pq}\right)\log_6 x$

$\Rightarrow 1 = \left(\frac{p-q}{pq}\right)\log_6 x \quad \therefore \log_6 x = \frac{pq}{p-q}$, dividing through by p

$\Rightarrow \frac{\log_6 x}{p} = \frac{q}{p-q}$

$\Rightarrow \frac{q}{p-q} = \frac{\log_6 x}{\log_3 x} = \frac{\log_6 x}{\frac{\log_6 x}{\log_6 3}}$

$\therefore \log_6 3 = \frac{q}{p-q}$

Alternatively

From $\log_{18} x = q$

$q = \frac{\log_3 x}{\log_3 18} = \frac{\log_3 x}{\log_3 (3\times 6)}$

$q = \frac{\log_3 x}{\log_3 3 + \log_3 6} = \frac{p}{1+\log_3 6}$

$\Rightarrow q(1+\log_3 6) = p$

$\Rightarrow q + q\log_3 6 = p$ *or* $p - q = q\log_3 6$

$\Rightarrow \log_3 6 = \frac{p-q}{q}$, but $\log_3 6 = \frac{\log_6 6}{\log_6 3} = \frac{1}{\log_6 3}$

$\therefore \frac{p-q}{q} = \frac{1}{\log_6 3}$ *or* $\log_6 3 = \frac{q}{p-q}$

b) To show that $\log_8 x = \frac{2}{3}\log_4 x$

LHS: $\log_8 x = \frac{\log_4 x}{\log_4 8} = \log_4 x . \frac{1}{\log_4 8}$

But $\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3\log_2 2}{2\log_2 2} = \frac{3}{2}$

$\therefore \log_8 x = \log_4 x . \frac{1}{\log_4 8} = \frac{2}{3}\log_4 x$

Putting $x = 6$ in to the proof,

$\Rightarrow \log_8 6 = \frac{2}{3}\log_4 6 = \frac{2}{3}\log_4(2 \times 3)$

$= \frac{2}{3}[\log_4 2 + \log_4 3]$

$= \frac{2}{3}\left[\frac{\log_2 2}{\log_2 4} + \log_4 3\right] = \frac{2}{3}\left[\frac{1}{2\log_2 2} + \log_4 3\right] = \frac{2}{3}[0.5 + 0.7925] = 0.862$

c) To prove $\log_6 x = \frac{\log_3 x}{1+\log_3 2}$

Let $\log_6 x = y \Rightarrow 6^y = x \;\; or \;\; 6 = x^{\frac{1}{y}}$

Introducing logarithm to base 3

$\Rightarrow \log_3 6 = \log_3 x^{\frac{1}{y}} = \frac{1}{y}\log_3 x$

$\Rightarrow \log_3 x = y\log_3 6 \;\; or \;\; y = \frac{\log_3 x}{\log_3 6} = \frac{\log_3 x}{\log_3(2\times 3)}$

$\Rightarrow y = \frac{\log_3 x}{\log_3 2 + \log_3 3} = \frac{\log_3 x}{1+\log_3 2}$

$\therefore \log_6 x = \frac{\log_3 x}{1+\log_3 2}$

Now setting $x = 4$

$\Rightarrow \log_6 4 = \frac{\log_3 4}{1+\log_3 2} = \frac{2\log_3 2}{1+\log_3 2}$

$\therefore \log_6 4 = \frac{2\times 0.631}{1+0.631} = \frac{1.262}{1.631} = 0.774$

d) $x^{x+y} = y^{12} \Rightarrow \log x^{x+y} = \log y^{12}$

$\Rightarrow (x+y)\log x = 12\log y$ …… (1)

$y^{x+y} = x^3 \Rightarrow \log y^{x+y} = \log x^3$

$\Rightarrow (x+y)\log y = 3\log x$ …… (2)

$eqn.(1) \div eqn.(2)$

$\Rightarrow \frac{(x+y)\log x}{(x+y)\log y} = \frac{12\log y}{3\log x} \;\; or \;\; \frac{\log x}{\log y} = \frac{4\log x}{\log y}$, let $\log x = a, \log y = b$

$\Rightarrow \frac{a}{b} = \frac{4b}{a} \;\; or \; a^2 = 4b^2$

$\therefore a = 2b$

$\therefore \log x = 2\log y$

Now from $\log x = 2\log y \Rightarrow x = y^2$…… (3)

$x^{x+y} = y^{12}$, becomes

$\Rightarrow (y^2)^{(y^2+y)} = y^{12} \;\; or \; y^{2y^2+2y} = y^{12}$

$\Rightarrow 2y^2 + 2y = 12 \;\; or \; y^2 + y - 6 = 0$

$\Rightarrow (y+3)(y-2) = 0 \;\; \therefore y = -3, 2$

From eqn. (3), $x = y^2$

For $y = -3$, $x = (-3)^2 = 9$

$y = 2$, $x = (2)^2 = 4$

$\therefore x = 9, y = -3; x = 4, y = 2$

Task

1. *Prove that* $abc = 1$ *if* $a = \log_b c$, $b = \log_a b$, $\log_a c$
2. *Show that* $\frac{\log\sqrt{27}+\log\sqrt{8}-\log\sqrt{125}}{\log 6 - \log 5} = \frac{3}{2}$ *without using tables/calculator.*
3. *Prove that* $\log(1+2+3) = \log 1 + \log 2 + \log 3$
4. *Prove that* $\log_4 2 . \log_2 6 . \log_6 4 = 1$
5. *Prove that* $\log_2[\log_2\{\log_3(\log 27^3)\}] = 0$

SIMULTANEOUS EQUATIONS INVOLVING LOGARITHMS

1) Solve the following simultaneous equations

a) $2\log_x y = 1$
$xy = 64$

b) $6\log_3 x + 6\log_{27} y = 7$
$4\log_9 x + 4\log_3 y = 9$

c) $y\log_2 8 = x$
$2^x + 8^y = 8192$

d) $\log x - \log 2 = 2\log y$
$x - 5y + 2 = 0$

e) $\log_2 x + 2\log_4 y = 4$
$x + 12y = 52$

f) $\ln 6 + \ln(x-3) = 2\ln y$
$2y - x = 3$

g) $2 + \log_2(2x+1) = 2\log_2 y$
$x = 22 - y$

Solution

a) $2\log_x y = 1 \Rightarrow \log_x y^2 = 1 \quad \therefore y^2 = x$ ..... (1)
$xy = 64$ ..... (2)
eqn. (1) in to eqn. (2)
$\Rightarrow y.y^2 = 64$ *or* $y^3 = 4^3 \quad \therefore y = 4$ ..... (3)
From eqn. (1), $x = 4^2 = 16$
$\therefore x = 16\,,\ y = 4$

b) $6\log_3 x + 6\log_{27} y = 7$
Changing to the same base
$\Rightarrow 6\log_3 x + 6\frac{\log_3 y}{\log_3 27} = 7$
$\Rightarrow 6\log_3 x + 6\frac{\log_3 y}{\log_3 3^3} = 7$ or $\Rightarrow 6\log_3 x + 6\frac{\log_3 y}{3} = 7$
$\therefore 6\log_3 x + 2\log_3 y = 7$ ..... (1)
$4\log_9 x + 4\log_3 y = 9 \Rightarrow 4\frac{\log_3 x}{\log_3 9} + 4\log_3 y = 9$
$\therefore 2\log_3 x + 4\log_3 y = 9$ ..... (2)
Eliminating $\log_3 x$ from eqn. (1) and eqn. (2)
eqn. (1) $-$ 3 $\times$ eqn. (2)
$\Rightarrow 6\log_3 x + 2\log_3 y = 7$
$\underline{6\log_3 x + 12\log_3 y = 27}$
$-10\log_3 y = -20 \Rightarrow \log_3 y = 2$
$\Rightarrow y = 3^2 = 9$ ..... (3)
eqn. (3) in to eqn. (2)
$\Rightarrow 2\log_3 x + 4\log_3 9 = 9$
$\Rightarrow \quad 2\log_3 x + 8 = 9$
$\Rightarrow 2\log_3 x = 1$ *or* $\log_3 x^2 = 1$
$\Rightarrow x^2 = 3 \quad \therefore x = +\sqrt{3}$ ..... (4)
$\therefore x = +\sqrt{3}\,; y = 9$

c) $y\log_2 8 = x \Rightarrow y\log_2 2^3 = x \quad or \quad x = 3y$..... (1)
$2^x + 8^y = 8192$ ..... (2)
eqn. (1)in to eqn. (2)
$\Rightarrow 2^{3y} + 8^y = 8192$
$\Rightarrow \quad 8^y + 8^y = 8192$
$\Rightarrow 2(8^y) = 8192 \quad or \quad 8^y = 4096 = 8^4$
$\therefore y = 4$ ..... (3)
eqn. (3)in to eqn. (1)
$\Rightarrow x = 3 \times 4 = 12$
$\therefore x = 12\,,\ y = 4$

d) Left as an exercise **ANSWER:** $\boldsymbol{x = \frac{1}{2}\,,\ y = \frac{1}{4}\,; x = 8\,, y = 2}$

e) Left as an exercise **ANSWER:** $\boldsymbol{x = 4\,,\ y = 4\,; x = 48\,, y = \frac{1}{3}}$

f) Left as an exercise **ANSWER:** $\boldsymbol{x = 9\,,\ y = 6}$

g) Left as an exercise **ANSWER:** $\boldsymbol{x = 12\,,\ y = 10}$

SURDS

Definition:

A surd is an irrational root of a rational number.

A rational number is a number with exact n^{th} root . $i.e. \sqrt[n]{b} = Real\ number$

$e.g.\ \sqrt{4} = 2\ , \sqrt[4]{16} = 2\ , \sqrt[3]{8} = 2$, these are rational numbers.

If $a = \sqrt[n]{b}$ for some rational number b such that a itself is an irrational number, then $a = \sqrt[n]{b}$ is a surd.

When $a = \sqrt[n]{b}$ is a surd, n is called *order* of a and b is called *a radicand* of a. Surds of $2^{nd}\ , 3^{rd}\ , 4^{th}$ and higher orders are respectively called *quadratic, cubic, quartic* and *quintic*.

Compound, Simple and Similar surds

A compound surd is a surd with coefficient different from $\pm 1.\ e.g. 2\sqrt{2}\ , 3\sqrt{5}\ , -3\sqrt{7}\ etc$

A simple surd is a surd with coefficient $\pm 1.\ e.g. \sqrt{2}\ , \sqrt[4]{\frac{1}{4}}\ , \sqrt[3]{7}\ etc$

Similar surds are surds if in reduced form (simplified form) contain same surds. $e.g\ \sqrt{45}$ and $\sqrt{80}$ are similar surds, since $\sqrt{45} = 3\sqrt{5}$ and $\sqrt{80} = 4\sqrt{5}$. Similar surds can be added or subtracted easily after being reduced to their simplest forms.

$e.g. \sqrt{75} + \sqrt{27} = 5\sqrt{3} + 3\sqrt{3} = 8\sqrt{3}$ and $2\sqrt{128} - \sqrt{162} = 16\sqrt{2} - 9\sqrt{2} = 7\sqrt{2}$.

Rules of simplifying surds

i) $\sqrt[a]{a} \times \sqrt[n]{b} = \sqrt[n]{a.b}$; surds of the same order can be multiplied or divided

ii) $\sqrt[a]{a} \div \sqrt[n]{b} = \sqrt[n]{\frac{a}{b}}$

iii) $m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}$

iv) $\left(m\sqrt{a} + n\sqrt{b}\right)\left(m\sqrt{a} - n\sqrt{b}\right) = \left(m\sqrt{a}\right)^2 - \left(n\sqrt{b}\right)^2$

v) $N\sqrt[a]{a} \times M\sqrt[n]{b} = MN\sqrt[n]{a.b}$

vi) If $a + \sqrt{b} = 0$ if and only if $a = b = 0$

vii) If $a + \sqrt{b} = \sqrt{c}$ if and only if $a = 0\ , b = c$

viii) If $a + \sqrt{b} = c + \sqrt{d}$ if and only if $a = c\ , b = d$

ix) If $\sqrt{a + \sqrt{b}} = \sqrt{x} - \sqrt{y}$ then $\sqrt{a - \sqrt{b}} = \pm\left(\sqrt{x} - \sqrt{y}\right)$

Rules /properties from (viii) to (ix) represent quadratic mixed surds properties.

RATIONALIZATION

If the product of two surds is a rational number and if one of them is multiplied by the other, the process is called *Rationalization*.

Thus $\sqrt{3} \times \sqrt{3} = 3$, so multiplication by $\sqrt{3}$ is rationalization of $\sqrt{3}$.

Similarly $\sqrt{5} - \sqrt{3}$ then multiplied by $\sqrt{5} + \sqrt{3}$ is rationalized to 2 as $\left(\sqrt{5} + \sqrt{3}\right)\left(\sqrt{5} - \sqrt{3}\right) = 5 - 3 = 2$

NOTE:

Rationalization can be in the numerator/denominator of a given surd.

Rationalization of Denominator

Case I: *Two quadratic surds Denominator.* $\frac{a}{\sqrt{b}+\sqrt{c}}$

$$\frac{a}{\sqrt{b}+\sqrt{c}}=\frac{a(\sqrt{b}-\sqrt{c})}{(\sqrt{b}+\sqrt{c})(\sqrt{b}-\sqrt{c})}=\frac{a(\sqrt{b}-\sqrt{c})}{(\sqrt{b})^2-(\sqrt{c})^2}$$
$$=\frac{a(\sqrt{b}-\sqrt{c})}{b-c}=\frac{a}{b-c}\sqrt{b}-\frac{a}{b-c}\sqrt{c}$$

Note:

$\sqrt{b}-\sqrt{c}$ is a surd conjugate of $\sqrt{b}+\sqrt{c}$

Case II: *Three quadratic surds Denominator.* $\frac{a}{\sqrt{b}+\sqrt{c}+\sqrt{d}}$

First multiply both numerator and denominator by $(\sqrt{b}+\sqrt{c})-\sqrt{d}$

Denominator becomes $(\sqrt{b}+\sqrt{c})^2-(\sqrt{d})^2$ *or* $b+c-d+2\sqrt{bc}$

Then multiply both numerator and denominator by $(b+c-d)-2\sqrt{bc}$

Denominator becomes $(b+c-d)^2-(2\sqrt{bc})^2=(b+c-d)^2-4bc$ which is a rational quantity.

Example

1. *Express as equivalent fraction with a rational denominator*
   a) $\frac{1}{\sqrt{2}+\sqrt{3}}$
   b) $\frac{7+4\sqrt{3}}{7-4\sqrt{3}}$
   c) $\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{2\sqrt{3}-\sqrt{2}}$

Solution

a) $\frac{1}{\sqrt{2}+\sqrt{3}}=\frac{\sqrt{2}-\sqrt{3}}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}=\frac{\sqrt{2}-\sqrt{3}}{(\sqrt{2})^2-(\sqrt{3})^2}=\frac{\sqrt{2}-\sqrt{3}}{-1}=\sqrt{3}-\sqrt{2}$

b) $\frac{7+4\sqrt{3}}{7-4\sqrt{3}}=\frac{(7+4\sqrt{3})(7+4\sqrt{3})}{(7-4\sqrt{3})(7+4\sqrt{3})}=\frac{49+28\sqrt{3}+28\sqrt{3}+16\sqrt{9}}{7^2-(4\sqrt{3})^2}$

$$=\frac{49+48+56\sqrt{3}}{49-48}=74+56\sqrt{3}$$

c) $\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{2\sqrt{3}-\sqrt{2}}=\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}+\frac{2\sqrt{3}+\sqrt{2}}{(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})}$

$$=\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}+\frac{2\sqrt{3}+\sqrt{2}}{(2\sqrt{3})^2-(\sqrt{2})^2}=\sqrt{3}-\sqrt{2}+\frac{2\sqrt{3}+\sqrt{2}}{12-2}$$
$$=\frac{10\sqrt{3}-10\sqrt{2}+2\sqrt{3}+\sqrt{2}}{10}=\frac{12\sqrt{3}-9\sqrt{2}}{10}$$
$$=\frac{3}{10}(4\sqrt{3}-3\sqrt{2})$$

2. *Express as equivalent fraction with a rational denominator*
   a) $\frac{1}{1+\sqrt{2}-\sqrt{3}}$
   b) $\frac{1}{2+\sqrt{3}+\sqrt{5}}$

Solution

a) $\frac{1}{1+\sqrt{2}-\sqrt{3}}=\frac{1}{(1+\sqrt{2})-\sqrt{3}}=\frac{(1+\sqrt{2})+\sqrt{3}}{[(1+\sqrt{2})-\sqrt{3}](1+\sqrt{2})+\sqrt{3}}$

$$=\frac{1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})^2-(\sqrt{3})^2}=\frac{1+\sqrt{2}+\sqrt{3}}{1+2\sqrt{2}+2-3}$$
$$=\frac{1+\sqrt{2}+\sqrt{3}}{2\sqrt{2}}=\frac{(1+\sqrt{2}+\sqrt{3})\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}$$
$$=\frac{\sqrt{2}(1+\sqrt{2}+\sqrt{3})}{4}=\frac{\sqrt{2}+2+\sqrt{6}}{4}$$

b) $\frac{1}{2+\sqrt{3}+\sqrt{5}}=\frac{1}{(2+\sqrt{3})+\sqrt{5}}=\frac{(2+\sqrt{3})-\sqrt{5}}{[(2+\sqrt{3})+\sqrt{5}][(2+\sqrt{3})-\sqrt{5}}$

$=\frac{2+\sqrt{3}-\sqrt{5}}{(2+\sqrt{3})^2-(\sqrt{5})^2}=\frac{2+\sqrt{3}-\sqrt{5}}{4+4\sqrt{3}+3-5}$

$=\frac{2+\sqrt{3}-\sqrt{5}}{2+4\sqrt{3}}=\frac{(2+\sqrt{3}-\sqrt{5})(2-4\sqrt{3})}{(2+4\sqrt{3})(2-4\sqrt{3})}$

$=\frac{4-8\sqrt{3}+2\sqrt{3}-4\sqrt{9}-2\sqrt{5}+4\sqrt{15}}{2^2-(4\sqrt{3})^2}=\frac{-8-6\sqrt{3}-2\sqrt{5}+4\sqrt{15}}{4-48}$

$=\frac{1}{44}\left(8+6\sqrt{3}+2\sqrt{5}-4\sqrt{15}\right)$

Task

Express as equivalent fractions with rational denominators

a) $\frac{12}{3+\sqrt{5}-2\sqrt{2}}$ **ANSWER**: $1+\sqrt{5}+\sqrt{10}-\sqrt{2}$

b) $\frac{1}{2+\sqrt{3}-\sqrt{5}}$ **ANSWER**: $\frac{1}{44}\left(8+6\sqrt{3}+2\sqrt{5}-4\sqrt{15}\right)$

3. *Express as equivalent fractions with rational denominators*

a) $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$

b) $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$

Solution

a) $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}=\frac{\sqrt{2}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}}=\frac{\sqrt{2}[(\sqrt{2}+\sqrt{3})+\sqrt{5}]}{[(\sqrt{2}+\sqrt{3})-\sqrt{5}]\left((\sqrt{2}+\sqrt{3})+\sqrt{5}\right)}$

$=\frac{\sqrt{4}+\sqrt{6}+\sqrt{10}}{(\sqrt{2}+\sqrt{3})^2-(\sqrt{5})^2}=\frac{2+\sqrt{6}+\sqrt{10}}{2+2\sqrt{6}+3-5}$

$=\frac{2+\sqrt{6}+\sqrt{10}}{2\sqrt{6}}=\frac{\sqrt{6}(2+\sqrt{6}+\sqrt{10})}{2\sqrt{6}\times\sqrt{6}}$

$=\frac{2\sqrt{6}+6+\sqrt{60}}{12}=\frac{6+2\sqrt{6}+2\sqrt{15}}{12}$

$=\frac{3+\sqrt{6}+\sqrt{15}}{6}$

b) $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}=\frac{1}{(\sqrt{2}+\sqrt{3})+\sqrt{5}}=\frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{[(\sqrt{2}+\sqrt{3})+\sqrt{5}][(\sqrt{2}+\sqrt{3})-\sqrt{5}}$

$=\frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{(\sqrt{2}+\sqrt{3})^2-(\sqrt{5})^2}=\frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2+2\sqrt{6}+3-5}$

$=\frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2\sqrt{6}}=\frac{\sqrt{6}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{2\sqrt{6}\times\sqrt{6}}$

$=\frac{\sqrt{12}+\sqrt{18}-\sqrt{30}}{12}=\frac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}$

Task

Express as equivalent fractions with rational denominators

$\frac{\sqrt{10}-\sqrt{5}-\sqrt{3}}{\sqrt{3}-\sqrt{10}-\sqrt{5}}$ **ANSWER**: $\frac{3\sqrt{30}+5\sqrt{15}-10\sqrt{2}-12}{7}$

4. *If* $x=\frac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}}$ *, find the value of;*

i) $\frac{x+\sqrt{8}}{x-\sqrt{8}}$

ii) $\frac{x+\sqrt{12}}{x-\sqrt{12}}$

iii) $\frac{x+\sqrt{8}}{x-\sqrt{8}}+\frac{x+\sqrt{12}}{x-\sqrt{12}}$

Solution

$x = \frac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}} = \frac{2\sqrt{6\times4}}{\sqrt{2}+\sqrt{3}} = \frac{4\sqrt{6}}{\sqrt{2}+\sqrt{3}}$

$= \frac{4\sqrt{6}(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = \frac{4\sqrt{12}-4\sqrt{18}}{(\sqrt{2})^2-(\sqrt{3})^2}$

$= \frac{4(2\sqrt{3})-4(3\sqrt{2})}{2-3} = 12\sqrt{2}-8\sqrt{3}$

i) Now $\frac{x+\sqrt{8}}{x-\sqrt{8}} = \frac{12\sqrt{2}-8\sqrt{3}+\sqrt{8}}{12\sqrt{2}-8\sqrt{3}-\sqrt{8}} = \frac{12\sqrt{2}-8\sqrt{3}+2\sqrt{2}}{12\sqrt{2}-8\sqrt{3}-2\sqrt{2}}$

$= \frac{14\sqrt{2}-8\sqrt{3}}{10\sqrt{2}-8\sqrt{3}} = \frac{7\sqrt{2}-4\sqrt{3}}{5\sqrt{2}-4\sqrt{3}}$

$= \frac{(7\sqrt{2}-4\sqrt{3})(5\sqrt{2}+4\sqrt{3})}{(5\sqrt{2}-4\sqrt{3})(5\sqrt{2}+4\sqrt{3})}$

$= \frac{70-48+18\sqrt{6}}{(5\sqrt{2})^2-(4\sqrt{3})^2} = \frac{22+8\sqrt{6}}{50-48} = 11+4\sqrt{6}$

ii) $\frac{x+\sqrt{12}}{x-\sqrt{12}} = \frac{12\sqrt{2}-8\sqrt{3}+\sqrt{12}}{12\sqrt{2}-8\sqrt{3}-\sqrt{12}} = \frac{12\sqrt{2}-8\sqrt{3}+2\sqrt{3}}{12\sqrt{2}-8\sqrt{3}-2\sqrt{3}}$

$= \frac{12\sqrt{2}-6\sqrt{3}}{12\sqrt{2}-10\sqrt{3}} = \frac{6\sqrt{2}-3\sqrt{3}}{6\sqrt{2}-5\sqrt{3}}$

$= \frac{(6\sqrt{2}-3\sqrt{3})(6\sqrt{2}+5\sqrt{3})}{(6\sqrt{2}-5\sqrt{3})(6\sqrt{2}+5\sqrt{3})}$

$= \frac{72-45+30\sqrt{6}-18\sqrt{6}}{(6\sqrt{2})^2-(5\sqrt{3})^2} = \frac{27+12\sqrt{6}}{72-75}$

$= \frac{27+12\sqrt{6}}{-3} = -9-4\sqrt{6}$

iii) $\frac{x+\sqrt{8}}{x-\sqrt{8}} + \frac{x+\sqrt{12}}{x-\sqrt{12}} = 11+4\sqrt{6}+\ -9-4\sqrt{6}\ = (11-9)+\left(4\sqrt{6}-4\sqrt{6}\right) = 2$

5. *If* $x = \sqrt{3}+\frac{1}{\sqrt{3}}$, *calculate;*

   i) $x - \frac{\sqrt{126}}{\sqrt{42}}$

   ii) $x - \frac{1}{x-\frac{2\sqrt{3}}{3}}$

   iii) $\left(x-\frac{\sqrt{126}}{\sqrt{42}}\right).\left(x-\frac{1}{x-\frac{2\sqrt{3}}{3}}\right)$, correct to 2 $d.p$

Solution

$x = \sqrt{3}+\frac{1}{\sqrt{3}} = \frac{\sqrt{9}+1}{\sqrt{3}} = \frac{3+1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$

i) $x - \frac{\sqrt{126}}{\sqrt{42}} = x - \sqrt{\left(\frac{126}{42}\right)} = x-\sqrt{3}$

$\therefore\ x - \frac{\sqrt{126}}{\sqrt{42}} = \frac{4}{\sqrt{3}} - \sqrt{3} = \frac{4-\sqrt{9}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

ii) $x - \frac{1}{x-\frac{2\sqrt{3}}{3}} = \frac{4}{\sqrt{3}} - \frac{1}{\frac{4}{\sqrt{3}}-\frac{2\sqrt{3}}{3}}$

$= \frac{4\sqrt{3}}{3} - \frac{1}{\frac{4\sqrt{3}}{3}-\frac{2\sqrt{3}}{3}} = \frac{4\sqrt{3}}{3} - \frac{3}{2\sqrt{3}}$

$= \frac{8\sqrt{9}-9}{6\sqrt{3}} = \frac{15}{6\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6}$

iii) $\left(x-\frac{\sqrt{126}}{\sqrt{42}}\right).\left(x-\frac{1}{x-\frac{2\sqrt{3}}{3}}\right) = \frac{1}{\sqrt{3}} \times \frac{5\sqrt{3}}{6} = \frac{5}{6} = 0.83$

Proofs in surds

1. *If* $x = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}, y = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, *show that* $3x^2-5xy+3y^2 = 289$

Solution

$x = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{(\sqrt{3}-\sqrt{2})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = \frac{3+2-2\sqrt{6}}{(\sqrt{3})^2-(\sqrt{2})^2}$

$= 5 - 2\sqrt{6}$

$x^2 = \left(5-2\sqrt{6}\right)^2 = 25 - 20\sqrt{6} + 24$

$= 49 - 20\sqrt{6}$

$x.y = \left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right) = 1$

$y = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{1}{x} = \frac{1}{5-2\sqrt{6}}$

$= \frac{5+2\sqrt{6}}{(5-2\sqrt{6})(5+2\sqrt{6})} = \frac{5+2\sqrt{6}}{5^2-(2\sqrt{6})^2}$

$= 5 + 2\sqrt{6}$

$y^2 = \left(5+2\sqrt{6}\right)^2 = 25 + 20\sqrt{6} + 24$

$= 49 + 20\sqrt{6}$

$\therefore 3x^2 - 5xy + 3y^2 = 3\left(49 - 20\sqrt{6}\right) - 5(1) + 3\left(49 + 20\sqrt{6}\right)$

$= 147 - 60\sqrt{6} - 5 + 147 + 60\sqrt{6} = 289$

2. *If* $x = \frac{5-\sqrt{21}}{2}$, *prove that* $x^3 + \frac{1}{x^3} - 5\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 0$

Solution

Let $u = x + \frac{1}{x} = \frac{5-\sqrt{21}}{2} + \frac{2}{5-\sqrt{21}}$

$= \frac{(5-\sqrt{21})^2+2}{2(5-\sqrt{21})} = \frac{25-10\sqrt{21}+21+4}{2(5-\sqrt{21})}$

$= \frac{50-10\sqrt{21}}{2(5-\sqrt{21})} = \frac{10(5-\sqrt{21})}{2(5-\sqrt{21})} = \frac{10}{2} = 5$

Now $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2.x.\frac{1}{x} = u^2 - 2$

$= 5^2 - 2 = 23$

$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$

$= 5^3 - 3 \times 5 = 125$

$\therefore x^3 + \frac{1}{x^3} - 5\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 110 - 5(23) + 5 = 0$

3. *If* $x = \frac{-3+\sqrt{5}}{2}$, *show that* $\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 = 0$

Solution

$x = \frac{-3+\sqrt{5}}{2} \Rightarrow \frac{1}{x} = \frac{2}{-3+\sqrt{5}} = \frac{2(-3-\sqrt{5})}{(-3+\sqrt{5})(-3-\sqrt{5})}$

$\frac{1}{x} = \frac{-6-2\sqrt{5}}{(-3)^2-(\sqrt{5})^2} = \frac{-2(3+\sqrt{5})}{9-5} = -\frac{3+\sqrt{5}}{2}$

Now $x + \frac{1}{x} = \frac{-3+\sqrt{5}}{2} - \frac{3+\sqrt{5}}{2} = -3$

$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$

$\therefore x^2 + \frac{1}{x^2} = (-3)^2 - 2 = 7$

$\therefore \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 = 7 - 3 - 4 = 0$

Task

1. *If* $x = 7 - \sqrt{48}$, *prove that* $x^3 + \frac{1}{x^3} - 15\left(x^2 + \frac{1}{x^2}\right) + 20\left(x + \frac{1}{x}\right) - 72 = 0$
2. *If* $x = 7 - \sqrt{5}$, *prove that* $x^4 - x^3 - 20x^2 - 16x + 39 = 15$
3. *It is given that* $\sqrt{3} = 1.732$ *and* $\sqrt{5} = 2.236$, *evaluate* $\frac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}}$

   HINT: $\frac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}} = \frac{(\sqrt{5}+\sqrt{3})(4-\sqrt{15})}{(4+\sqrt{15})(4-\sqrt{15})} = \frac{4\sqrt{5}+4\sqrt{3}-\sqrt{75}-\sqrt{45}}{16-15}$

   $= 4\sqrt{5} + 4\sqrt{3} - 5\sqrt{3} - 3\sqrt{5} = \sqrt{5} - \sqrt{3}$

   $= 2.236 - 1.732 = \mathbf{0.504}$

4. *If* $2\sqrt{54} + 5\sqrt{294} + \frac{19}{30}\sqrt{6} - \sqrt{\frac{27}{50}} - \sqrt{\frac{2}{3}} = a\sqrt{b}$, *find a.*

   ANSWER: $a = 41$

TO FIND THE SQUARE ROOT OF A MIXED QUADRATIC SURD

A surd of the form $a + \sqrt{b}$ is a mixed quadratic surd. To find $\sqrt{\boldsymbol{a} + \sqrt{\boldsymbol{b}}}$, proceed as follows.

Let $\sqrt{\boldsymbol{a} + \sqrt{\boldsymbol{b}}} = \sqrt{x} + \sqrt{y}$, squaring both sides

$\left(\sqrt{a + \sqrt{b}}\right)^2 = \left(\sqrt{x} + \sqrt{y}\right)^2$

$\Rightarrow a + \sqrt{b} = x + y + 2\sqrt{xy}$

$\Rightarrow a = x + y$ ..... (1)

$\Rightarrow \sqrt{b} = 2\sqrt{xy}$ *or* $b = 4xy$

$\Rightarrow b = 4xy$ ..... (2)

From $(x - y)^2 = x^2 + y^2 - 2xy$

$\Rightarrow (x - y)^2 = (x + y)^2 - 2xy - 2xy$

$\Rightarrow (x - y)^2 = (x + y)^2 - 4xy$

$\therefore x - y = \pm\sqrt{a^2 - b}$ ..... (3)

Now taking the positive sign in the equation

$\Rightarrow x - y = +\sqrt{a^2 - b}$ ..... (4)

Adding eqn. (1) and (4)

$\Rightarrow x = \frac{1}{2}\left(a + \sqrt{a^2 - b}\right)$

$\Rightarrow y = \frac{1}{2}\left(a - \sqrt{a^2 - b}\right)$

Hence $\sqrt{a + \sqrt{b}}$ can be determined

NOTE:

If we take a $-ve$ sign from eqn. (3), the value of x and y will be interchanged and we shall get a second square root of $\sqrt{a + \sqrt{b}}$

Example

1. *Find the square root of*

   $31 + 4\sqrt{21}$

   Solution

   Let $\sqrt{31 + 4\sqrt{21}} = \sqrt{x} + \sqrt{y}$, squaring both sides

   $\Rightarrow 31 + 4\sqrt{21} = x + y + 2\sqrt{xy}$

   $\Rightarrow x + y = 31$ ..... (1)

$2\sqrt{xy} = 4\sqrt{21}$,squaring
$\Rightarrow 4xy = 336$ *or* $xy = 84$ ..... (2)
But $(x-y)^2 = (x+y)^2 - 4xy$
$\Rightarrow$ x – y $= \sqrt{31^2 - 336} = \pm 25$
Taking the $+ve$ sign
$\Rightarrow x - y = 25$ ..... (3)
Adding eqn. (1) and (3)
$2x = 56 \Rightarrow x = 28$
From eqn. (1) $y = 31 - x = 31 - 28 = 3$
$\therefore \sqrt{31 + 4\sqrt{21}} = \pm\left(\sqrt{28} + \sqrt{3}\right) = \pm\left(2\sqrt{7} + \sqrt{3}\right)$

2. *Find the square root of* $\sqrt{32} - \sqrt{24}$
Solution
$\sqrt{32} - \sqrt{24} = 4\sqrt{2} - 2\sqrt{6} = \sqrt{2}\left(4 - 2\sqrt{3}\right)$
Then $\sqrt{\sqrt{32} - \sqrt{24}} = \sqrt{\sqrt{2}\left(4 - 2\sqrt{3}\right)} = 2^{\frac{1}{4}}\sqrt{4 - 2\sqrt{3}}$
Evaluating $\sqrt{4 - 2\sqrt{3}}$
Let $\sqrt{4 - 2\sqrt{3}} = \sqrt{x} - \sqrt{y}$, squaring both sides
$\Rightarrow 4 - 2\sqrt{3} = x + y - 2\sqrt{xy}$
$\Rightarrow x + y = 4$ ..... (1)
$2\sqrt{xy} = 2\sqrt{3}$ *or* $\sqrt{xy} = \sqrt{3}$, squaring
$\Rightarrow xy = 3$..... (2)
But $(x-y)^2 = (x+y)^2 - 4xy$
$\Rightarrow$ x – y $= \sqrt{4^2 - 3} = \pm 2$
Taking the $+ve$ sign
$\Rightarrow x - y = 2$ ..... (3)
Adding eqn. (1) and (3)
$2x = 6 \Rightarrow x = 3$
From eqn. (1) $y = 4 - x = 4 - 3 = 1$
$\therefore \sqrt{\sqrt{32} - \sqrt{24}} = 2^{\frac{1}{4}}\sqrt{4 - 2\sqrt{3}} = \pm 2^{\frac{1}{4}}\left(\sqrt{3} - 1\right)$

3. *If* $x = \sqrt{\left(\frac{7+4\sqrt{3}}{7-4\sqrt{3}}\right)}$, *show that* $x^2(x-14)^2 = 1$
Solution
$x = \sqrt{\left(\frac{7+4\sqrt{3}}{7-4\sqrt{3}}\right)} \Rightarrow x = \sqrt{\frac{(7+4\sqrt{3})(7+4\sqrt{3})}{(7-4\sqrt{3})(7+4\sqrt{3})}}$
$\Rightarrow x = \frac{7+4\sqrt{3}}{49-48} = 7 + 4\sqrt{3}$
Now $(x-14)^2 = \left(7 + 4\sqrt{3} - 14\right)^2 = \left(-7 + 4\sqrt{3}\right)^2$
$\Rightarrow (x-14)^2 = 49 - 56\sqrt{3} + 48$
$= 97 - 56\sqrt{3}$
Also $x^2 = \left(7 + 4\sqrt{3}\right)^2$
$= 49 + 56\sqrt{3} + 48 = 97 + 56\sqrt{3}$
$\therefore x^2(x-14)^2 = \left(97 + 56\sqrt{3}\right)\left(97 - 56\sqrt{3}\right) = 97^2 - \left(56\sqrt{3}\right)^2 = 1$

4. *If* $2x = \sqrt{a} + \frac{1}{\sqrt{a}}$, *show that* $\frac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}} = \frac{a-1}{2}$

Solution

$2x = \sqrt{a} + \frac{1}{\sqrt{a}} \Rightarrow x = \frac{a+1}{2\sqrt{a}}$

Now $x^2 = \frac{(a+1)^2}{4a} = \frac{a^2+2a+1}{4a}$

Also $x^2 - 1 = \frac{a^2+2a+1}{4a} - 1 = \frac{a^2-2a+1}{4a} = \frac{(a-1)^2}{4a}$

$$\Rightarrow \frac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}} = \frac{\sqrt{\left(\frac{(a-1)^2}{4a}\right)}}{\frac{a+1}{2\sqrt{a}} - \sqrt{\left(\frac{(a-1)^2}{4a}\right)}}$$

$$= \frac{\frac{a-1}{2\sqrt{a}}}{\frac{a+1}{2\sqrt{a}} - \frac{a-1}{2\sqrt{a}}} = \frac{a-1}{2\sqrt{a}} \div \frac{2a}{2\sqrt{a}} = \frac{a-1}{2}$$

Task

1. Simplify $\frac{\sqrt{3}-2}{2\sqrt{3}+3}$ (***Uneb* 2007**)
2. *Express as an equivalent fraction with a rational denominator*

   $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ (***Uneb* 1996**) **ANSWER**: $\frac{6+2\sqrt{6}+2\sqrt{15}}{12}$
3. *Express* $\frac{1+\sqrt{3}}{(\sqrt{3}-1)^3}$ *in the form* $a + b\sqrt{c}$ **ANSWER**: $\frac{7}{2} + 2\sqrt{3}$
4. *Find the square root of* $14 + 6\sqrt{5}$ **ANSWER**: $\pm(3 + \sqrt{5})$
5. *Simplify* $\frac{2\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}$
6. *Find the rational numbers a and b such that*

   $3 + \sqrt{2} = (a + b\sqrt{2})(6 - 2\sqrt{3})$

   ANSWER: $a + b\sqrt{2} = \frac{13}{14} + \frac{3}{7}\sqrt{2} \Rightarrow a = \frac{13}{14}, b = \frac{3}{7}, c = 2$

THE THEORY OF QUADRATIC EQUATIONS

An equations of degree two is a quadratic equation and generally takes the form $\boxed{ax^2+bx+c=0}$, where a, b, c are rational numbers.

Types of Quadratic Equations

Are of two types . $i.e.$

a) **Pure Quadratic Equation**

Is an equation of the form $\boxed{ax^2+c=0}$, where $b=0$ in $ax^2+bx+c=0$

b) **Affected Quadratic Equation**

Is an equation which is not a pure quadratic equation, and takes on the form, with $ab \neq 0 \Leftrightarrow a \neq 0, b \neq 0$ $\boxed{ax^2+bx+c=0}$

Methods of solving a Quadratic Equation $ax^2+bx+c=0$

I) **Method of Factorization**

If the expression ax^2+bx+c can be factored in to linear factors, then each of the factors put to zero, provides us with a root of the quadratic equation. Thus if

$$\boxed{ax^2+bx+c=a(x-\alpha)(x-\beta)}$$

, then the roots of the equation $ax^2+bx+c=0$ are $\boxed{x=\alpha \text{ and } x=\beta}$

Example

Solve:

a) $x^2-5x+6=0$

b) $2x^2-x-3=0$

c) $4x^2-x-3=0$

Solution

a) $x^2-5x+6=0$

Sum of roots= -5 , (coefficient of middle term)

Product = 6 , (product of the constant term and coefficient of x^2)

Numbers = $(-2,-3)$

$\therefore x^2-2x-3x+6=0$

$\Rightarrow x(x-2)-3(x-2)=0$

$\Rightarrow (x-3)(x-2)=0$

$\therefore x-3=0 \; or \; x=3$

$\therefore x-2=0 \; or \; x=2$

Hence roots are 3 and 2

b) $2x^2-x-3=0$

Sum of roots= -5

Product = $2 \times -3 = -6$

Numbers = $(-3,2)$

$\therefore 2x^2-3x+2x-3=0$

$\Rightarrow x(2x-3)+1(2x-3)=0$

$\Rightarrow (x+1)(2x-3)=0$

$\therefore x+1=0 \; or \; x=-1$

$\therefore 2x-3=0 \; or \; x=\frac{3}{2}$

Hence roots are -1 and $\frac{3}{2}$

c) Left as an exercise: **ANSWER**: x = 1 ; $-\frac{3}{4}$

II) **Method of Perfect Square**

Let $\boldsymbol{ax^2+bx+c=0}$ **be given equation.**

Step 1: Divide both sides by a

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Step 2: Transpose the constant term (i.e. term independent of x) on the **R.H.S**

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Step 3: Add $\frac{b^2}{4a^2}$ to both sides

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$or \; \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

NOTE: $\frac{b^2}{4a^2}=\left(\frac{b}{a}\div 2\right)^2=\left(\frac{coeff\ .of\ x}{2}\right)^2$

Step 4: Find the square root on both sides

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

Now solving for x

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. This is a quadratic formula for solving quadratic equations.

$$\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}\text{, for } \boldsymbol{ax^2+bx+c=0}$$

Example

Solve:

a) $x^2-5x+6=0$

b) $2x^2-x-3=0$

Solution

By quadratic formula

a) $x^2-5x+6=0 \Rightarrow a=1, b=-5\,, c=6$

$\therefore x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(6)}}{2(1)}=\frac{5\pm1}{2}$

$\therefore x=\frac{5+1}{2}=3$

$\therefore x=\frac{5-1}{2}=2$

$\therefore$ Roots are 3 and 2

b) $2x^2-x-3=0 \Rightarrow a=2\,, b=-1\,, c=-3$

$\therefore x=\frac{-(-1)\pm\sqrt{(-1)^2-4(2)(-3)}}{2(2)}=\frac{1\pm5}{2}$

$\therefore x=\frac{1+5}{4}=\frac{3}{2}$

$\therefore x=\frac{1-5}{2}=-2$

$\therefore$ Roots are $\frac{3}{2}$ and -2

Nature of Roots of a Quadratic Equation

The equation $\boldsymbol{ax^2 + bx + c = 0}$ has roots $\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

Nature of roots depend on $\boldsymbol{(b^2 - 4ac)}$ inside the radical, called

DISCRIMINANT

Let the roots be α and β

Let $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$

Deduction

Case I:

If $b^2 - 4ac$ is *positive*. $i.e. b^2 - 4ac > 0$, α and β are *real and unequal or real and distinct*

Case II:

If $b^2 - 4ac$ is zero. $i.e. b^2 - 4ac = 0$, α and β are *real and equal* each reducing in this case to $-\frac{b}{2a}$ or *roots are coincident/ double/ repeated.*

Case III:

If $b^2 - 4ac$ is *negative* . $i.e. b^2 - 4ac < 0$, α and β are *imaginary and unequal*

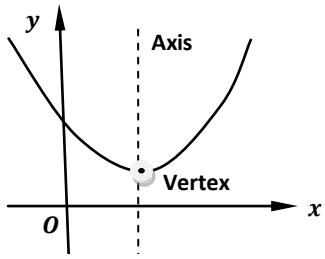
Case IV:

If $b^2 - 4ac$ is a *perfect square,* α and β are *rational and unequal*

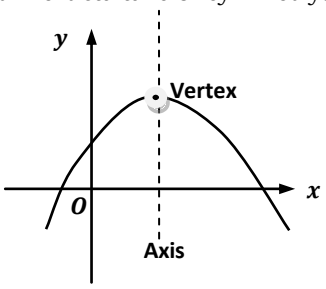
By applying these tests the nature of roots of any quadratic equation may be determined without solving.

Graphical Interpretation of $\boldsymbol{y = ax^2 + bx + c}$

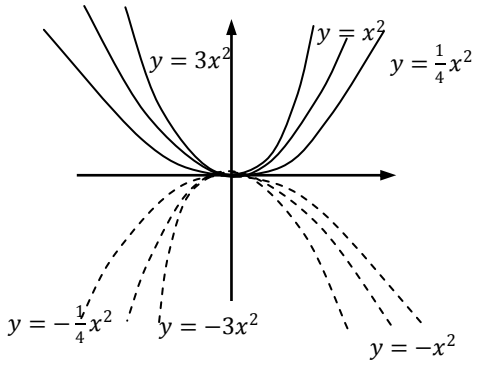
The graph $y = ax^2 + bx + c$ is a parabola with vertical axis of symmetry.



$Fig. y = ax^2 + bx + c, a > 0$
(Opens upwards and has a minimum values)



$Fig. y = ax^2 + bx + c, a < 0$
(Opens down wards and has a minimum values)



Note: The bigger $|a|$ the narrower the parabola.

For an equation $ax^2 + bx + c = 0$

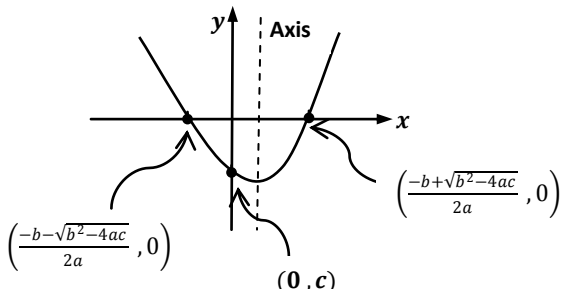


Fig I. $b^2 - 4ac > 0$ There are two intercepts

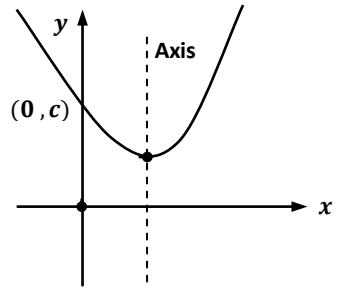


Fig II. $b^2 - 4ac < 0$. No x − intercepts

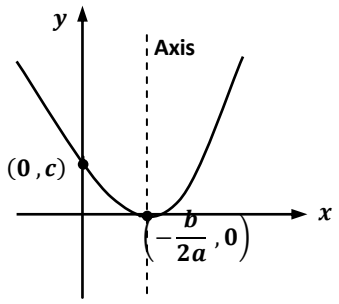


Fig III. $b^2 - 4ac = 0$. There is one intercept. (at the point where the parabola meets the x −axis and $x - axis$ in a

Illustrative examples on deduced cases on roots of a quadratic equation $ax^2 + bx + c = 0$

1. *Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real value of x.*

   Solution

   From discriminate, $b^2 - 4ac, a = 2, b = -6, c = 7$

   $\Rightarrow (-6)^2 - 4(2)(7) = -20$. Thus roots are imaginary.

2. *If the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find k.*

   Solution

   For equal roots $b^2 - 4ac = 0, a = 1, b = 2(k+2), c = 9k$

   $\Rightarrow [2(k+2)]^2 - 4(1)(9k) = 0$ ……÷ (4)

   $\Rightarrow (k+2)^2 - 9k = 0$

   $\Rightarrow k^2 + 4k + 4 - 9k = 0$ *or* $k^2 - 5k + 4 = 0$

   $\Rightarrow (k-1)(k-4) = 0 \therefore k = 1, 4$

3. *Show that the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational.*

   Solution

   For rational roots, $b^2 - 4ac$ is a perfect square

   $a = 1, b = -2p, c = p^2 - q^2 + 2qr - r^2$

   $\Rightarrow b^2 - 4ac = (-2p)^2 - 4(1)(p^2 - q^2 + 2qr - r^2)$

   $= 4p^2 - 4p^2 + 4(q^2 + 2qr + r^2) = 4(q^2 + 2qr + r^2)$

   $= 4(q-r)^2 = [2(q-r)]^2$, hence roots are rational.

4. *Prove that the roots of the equation* $x^2 - 2ax + a^2 - b^2 - c^2 = 0$ *are real.*

Solution

For real roots, $b^2 - 4ac \geq 0$

Note: for real and distinct roots, $b^2 - 4ac > 0$

$\Rightarrow (-2a)^2 - 4(1)(a^2 - b^2 - c^2) \geq 0$

$\Rightarrow 4a^2 - 4a^2 + 4(b^2 + c^2) \geq 0$

$\Rightarrow 4(b^2 + c^2) \geq 0$

R.H.S will always be true for any value of b and c.

5. *Prove that the roots of the equation*$(a - b - c)x^2 + ax + b + c = 0$ *are real if* a, b, c *are real*

Solution

For real roots $B^2 - 4AC \geq 0$

$\Rightarrow a^2 - 4(a - b - c)(b + c) \geq 0$

Let $b + c = n$

$\Rightarrow a^2 - 4(a - n)n \geq 0$

$\Rightarrow a^2 - 4an + 4n^2 \geq 0$

$\Rightarrow (a - 2n)^2 \geq 0$

$\therefore [a - 2(b + c)]^2 \geq 0$

This inequality will be satisfied if a , b and c are real, hence a , b and c are real.

Task

1. *Prove that the roots of the equations are real rational*
   a) $(a + c - b)x^2 + 2cx + (b + c - a = 0)$
   b) $abc^2x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0$
2. *Prove that the roots of the equation* $(a - b + c)x^2 + 4(a - b)x + (a - b - c) = 0$ *are real*
3. *For what values of m will the equations*
   a) $x^2 - 15 - m(2x - 8) = 0$
   b) $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$, have equal roots?

ANSWER: *a)* **3 , 5** *b)* $\mathbf{2}, -\frac{\mathbf{10}}{\mathbf{9}}$

Relationship between roots of a quadratic equation $\boldsymbol{.i.e.\alpha}$ **and** $\boldsymbol{\beta}$ **and its coefficients in** $\boldsymbol{ax^2 + bx + c = 0}$

From $ax^2 + bx + c = 0$, with roots α and β,

$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$ $\left[\because x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$

Then $\alpha + \beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}$

$$\boxed{\boldsymbol{\alpha + \beta = -\frac{b}{a}}}$$

Also $\alpha.\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}$

$= \frac{(-b+\sqrt{b^2-4ac}).(-b-\sqrt{b^2-4ac})}{4a^2}$

$= \frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2}$

$= \frac{4ac}{4a^2} = \frac{c}{a}$

$$\boxed{\boldsymbol{\alpha.\beta = \frac{c}{a}}}$$

From $ax^2 + bx + c = 0$, re writing in the form

$$\boxed{x^2 + \frac{b}{a}x + \frac{c}{a} = 0}$$

From

$\alpha + \beta = -\frac{b}{a}$ or $\frac{b}{a} = -(\alpha + \beta)$ and $\alpha.\beta = \frac{c}{a}$

$\therefore$ the quadratic equation becomes

$$\boxed{x^2 - x(\alpha + \beta) + \alpha.\beta = 0}$$

In general, if a quadratic equation $ax^2 + bx + c = 0$ can be written with coefficient of the first term as unity $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then

a) Sum of roots = coefficient of x with its sign changed
b) Product of roots = the third term

Note:

Re writing $x^2 - x(\alpha + \beta) + \alpha.\beta = 0$ by factorization,

$\boxed{(x - \alpha).(x - \beta) = 0}$, is a quadratic equation with coefficient of x^2 being unity.

In general, the quadratic equation with given roots is

$$\boxed{x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0}$$

Illustrative examples

1. *Form the equation whose roots are;*
   a) 3 and -2
   b) $2 + \sqrt{3}$ and $2 - \sqrt{3}$

Solution

Required equation is $x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$

a) Sum of roots $= 3 + -2 = 1$
Product of roots $= 2(-3) = -6$
$\therefore$ Required equations is $x^2 - x - 6 = 0$

b) Sum of roots $= (2 + \sqrt{3}) + (2 - \sqrt{3}) = 4$
Product of roots $= (2 + \sqrt{3}).(2 - \sqrt{3}) = 2^2 - (\sqrt{3})^2 = 1$
$\therefore$ Required equations is $x^2 - 4x + 1 = 0$

2. *If α and β are the roots of the equation $x^2 - px + q = 0$, find;*
   i) Values of $\alpha^2 + \beta^2$
   ii) Values of $\alpha^3 + \beta^3$
   iii) Values of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
   iv) Value of $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$

Solution

$x^2 - px + q = 0$, α , β are roots
$\Rightarrow \alpha + \beta = -(-p) = p$ $[i.e.\ sign\ of\ coeff.of\ x\ changed]$
$\Rightarrow \alpha.\beta = q$

i) From $(\alpha+\beta)^2 = \alpha^2+\beta^2+2\,\alpha.\beta$

$\therefore \alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\,\alpha.\beta$

$= p^2 - 2q$

ii) From $(\alpha+\beta)^3 = \alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$

$\therefore \alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\,\alpha.\beta(\alpha+\beta)$

$= p^3 - 3pq$

iii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2.\beta^2} = \frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{p^2-2q}{q^2}$

iv) $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2 = \frac{\alpha^2}{\beta^2} - 2.\frac{\alpha}{\beta}.\frac{\beta}{\alpha} + \frac{\beta^2}{\alpha^2}$

$= \frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2} - 2 = \frac{\alpha^4+\beta^4}{(\alpha\beta)^2} - 2$

From $(\alpha+\beta)^4 = \alpha^4+4\alpha^3\beta+6\alpha^2\beta^2+4\alpha\beta^3+\beta^4$

$\therefore \alpha^4+\beta^4 = (\alpha+\beta)^4 - (4\alpha^3\beta+6\alpha^2\beta^2+4\alpha\beta^3)$

$= (\alpha+\beta)^4 - [4\alpha.\beta(\alpha^2+\beta^2)+6(\alpha\beta)^2]$

$= p^4 - [4q(p^2-2q)+6q^2]$

$= p^4 - 4q(p^2-2q) - 6q^2$

$= p^4 - 4qp^2 + 8q^2 - 6q^2$

$= p^4 - 4qp^2 + 2q^2$

3. *If α and β are the roots of the equation $x^2-7x+2=0$, evaluate;*

   i) $\alpha^2+\beta^2$

   ii) $\frac{2-\alpha}{3+\beta}+\frac{2-\beta}{3+\alpha}$

Solution

$x^2-7x+2=0$, with roots α and β, then $\alpha+\beta=7$, $\alpha.\beta=2$

i) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\,\alpha.\beta$

$= 7^2 - 2(2) = 45$

ii) $\frac{2-\alpha}{3+\beta}+\frac{2-\beta}{3+\alpha} = \frac{(2-\alpha)(3+\beta)+(2-\beta)(3+\beta)}{(3+\beta)(3+\alpha)}$

$= \frac{6+2\alpha-3\alpha-\alpha^2+6+2\beta-3\beta-\beta^2}{9+3\alpha+3\beta+\alpha\beta}$

$= \frac{12+2(\alpha+\beta)-3(\alpha+\beta)-(\alpha^2+\beta^2)}{9+3(\alpha+\beta)+\alpha\beta}$

$= \frac{12+2(\alpha+\beta)-3(\alpha+\beta)-[(\alpha+\beta)^2-2\,\alpha.\beta\,]}{9+3(\alpha+\beta)+\alpha\beta}$

$= \frac{12+14-21-(7^2-2(2))}{9+21+2} = -\frac{28}{32} = -\frac{7}{8}$

4. *If α and β are the roots of the equation $x^2-px+q=0$, form an equation whose roots are $\alpha\beta+\alpha+\beta$ and $\alpha\beta-\alpha-\beta$*

Solution

Required equation is $\boldsymbol{x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0}$

From $x^2-px+q=0$ with roots α and β,

$\alpha+\beta = -(-p) = p$, $\alpha.\beta = q$

Sum of roots = $\alpha\beta+\alpha+\beta+\alpha\beta-\alpha-\beta = 2\alpha\beta = 2q$

Product of roots = $(\alpha\beta+\alpha+\beta).(\alpha\beta-\alpha-\beta)$

$= (q+p)+(q-p) = q^2-p^2$

$\therefore$ Equation is $x^2-2qx+q^2-p^2=0$

5. *If α and β are the roots of the equation $2x^2-4x+1=0$, form an equation whose roots are $\alpha^2+\beta$ and $\beta^2+\alpha$*

Solution

From $2x^2 - 4x + 1 = 0$ with roots α *and* β,

$\Rightarrow \alpha + \beta = -\left(-\frac{4}{2}\right) = 2\,,\ \alpha.\beta = \frac{1}{2}$

Required equation is $\boldsymbol{x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0}$

Sum of roots $= \alpha^2 + \beta + \beta^2 + \alpha = \alpha^2 + \beta^2 + \alpha + \beta$

$= (\alpha + \beta)^2 - 2\,\alpha.\beta + \alpha + \beta$

$= 2^2 - 2.\frac{1}{2} + 2 = 4 - 1 + 2 = 5$

Product of roots $= (\alpha^2 + \beta).(\beta^2 + \alpha)$

$= \alpha^2\beta^2 + \alpha^3 + \beta^3 + \alpha\beta$

$= (\alpha\beta)^2 + \alpha\beta + (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= \frac{1}{4} + 2 + 8 - 3.\frac{1}{2}.2 = \frac{23}{4}$

$\therefore x^2 - 5x + \frac{23}{4} = 0$ *or* $4x^2 - 20x + 23 = 0$

6. *If* α *and* β *are the roots of the equation* $ax^2 + bx + c = 0$, *find an equation whose roots are* $a\alpha + b\beta$ *and* $b\alpha + a\beta$

Solution

Sum of roots $= a\alpha + b\beta + b\alpha + a\beta$

$= \alpha(a + b) + \beta(a + b) = (\alpha + \beta)(a + b)$

$= -\frac{b}{a} \times (a + b) = \frac{-b(a+b)}{a}$

Product of roots $= (a\alpha + b\beta).(b\alpha + a\beta)$

$= ab\alpha^2 + a^2\beta\alpha + b^2\alpha\beta + ab\beta^2$

$= ab(\alpha^2 + \beta^2) + \alpha\beta(a^2 + b^2)$

$= ab[(\alpha + \beta)^2 - 2\,\alpha.\beta] + \alpha\beta(a^2 + b^2)$

$= ab\left(\frac{b^2}{a^2} - \frac{2c}{a}\right) + \frac{c(a^2+b^2)}{a}$

$= \frac{b(b^2-2ac)+c(a^2+b^2)}{a}$

Hence the required equation is

$x^2 - \frac{b(a+b)}{a}x + \frac{b(b^2-2ac)+c(a^2+b^2)}{a} = 0$

or $ax^2 - b(a + b)x + b(b^2 - 2ac) + c(a^2 + b^2) = 0$

or $ax^2 - bax - b^2x + b^3 - 2abc + a^2c + b^2c = 0$

or $ax(x - b) - b^2(x - b) + c(a^2 - 2ab + b^2) = 0$

or $(x - b)(ax - b^2) + c(a - b)^2 = 0$

7. *The roots of a quadratic equation* $x^2 - px + q = 0$ *are* α *and* β. *Determine the equation whose roots are* $\alpha^2 + \beta^{-2}$ *and* $\beta^2 + \alpha^{-2}$ *expressing the coefficients in terms of p and q. prove further that if p and q are both real, then this equation can have roots if* $p = 0$ *or* $p^2 = 4q$.

Solution

Required equation: $x^2 - (A + B)x + A.B = 0$

Let $A = \alpha^2 + \beta^{-2} = \alpha^2 + \frac{1}{\beta^2}$ and $B = \beta^2 + \alpha^{-2} = \beta^2 + \frac{1}{\alpha^2}$

$A + B = \alpha^2 + \frac{1}{\beta^2} + \beta^2 + \frac{1}{\alpha^2}$

$= (\alpha^2 + \beta^2) + \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$= (\alpha+\beta)^2 - 2\alpha\beta + \frac{\alpha^2+\beta^2}{\alpha^2.\beta^2=(\alpha+\beta)^2-2\alpha\beta+\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha.\beta)^2}}$

But from the equation $x^2 - px + q = 0$, α *and* β are roots then $\alpha+\beta = p, q = \alpha\beta$

$\therefore A + B = p^2 - 2q + \frac{p^2-2q}{q^2}$

$= (p^2-2q)\left(1+\frac{1}{q^2}\right) = \frac{(p^2-2q)(q^2+1)}{q^2}$

$A.B = (\alpha^2+\beta^2).\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)$

$= \alpha^2.\beta^2 + 1 + 1 + \frac{1}{\alpha^2.\beta^2}$

$= (\alpha.\beta)^2 + 2 + \frac{1}{(\alpha.\beta)^2}$

$= q^2 + 2 + \frac{1}{q^2} = \frac{q^4+2q^2+1}{q^2}$

$= \frac{(q^2+1)^2}{q^2}$

Required equation is; $x^2 - \frac{(p^2-2q)(q^2+1)}{q^2}x + \frac{(q^2+1)^2}{q^2} = 0$

If p and q are real then x must also be real $.i.e.$

$\left[-\frac{(p^2-2q)(q^2+1)}{q^2}\right]^2 \geq 4\frac{(q^2+1)^2}{q^2} \Rightarrow \frac{(p^2-2q)^2(q^2+1)^2}{q^4} \geq 4\frac{(q^2+1)^2}{q^2}$

Dividing both sides by $\frac{(q^2+1)^2}{q^4}$

$\Rightarrow (p^2-2q)^2 \geq 4q^2 \;\; or \;\; (p^2-2q)^2 - 4q^2 \geq 0$

By factoring using a difference of two squares identity,

$\Rightarrow (p^2-2q+2q)(p^2-2q-2q) \geq 0 \;\; or \; p^2(p^2-4q) \geq 0$

$\Rightarrow p^2 \geq 0 \;\; or \; p^2 - 4q \geq 0$

$\therefore p = 0 \;\; or \;\; p = 4q$

8. *If the roots of the equation* $ax^2 + bx + c = 0$ *are in the ratio* $p:q$*, prove that* $ac(p+q)^2 = b^2pq$

Solution

Let the roots of the equation be α *and* β

$\Rightarrow \alpha+\beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$ ..... (1)

But $\alpha:\beta = \frac{p}{q} \;\; or \;\; \frac{\alpha}{\beta} = \frac{p}{q}$

$\therefore \alpha = \frac{p}{q}\beta$ ..... (2)

Eqn. (2) in to eqn. (1)

$\Rightarrow \frac{p}{q}\beta + \beta = -\frac{b}{a}$, also $\beta.\frac{p}{q}\beta = \frac{c}{a}$

$\therefore \beta\left(\frac{p}{q}+1\right) = -\frac{b}{a} \;\; or \; \beta = -\frac{bq}{a(p+q)}, \beta^2 = \frac{qc}{ap}$

Solving these equations

$\Rightarrow \left[-\frac{bq}{a(p+q)}\right]^2 = \frac{qc}{ap} \;\; or \;\; \frac{b^2q^2}{a^2(p+q)^2} = \frac{qc}{ap}$

$\therefore b^2q^2.ap = a^2(p+q)^2.qc$, simplifying

$\Rightarrow ac(p+q)^2 = b^2pq$

Task

a) *If* α *and* β *are the roots of the equation* $ax^2 + bx + c = 0$*, and* r *is the ratio of the roots, show that* $ac(r+q)^2 = b^2r$

b) *If the roots of the equation* $lx^2 + nx + n = 0$ *are in the ratio* $p:q$ *prove that* $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$

9. *If the roots of the equation* $ax^2 + 2bx + c = 0$ *are* α *and* β *and those of the equation* $Ax^2 + 2Bx + C = 0$ *are* $\alpha + \delta$ *and* $\beta + \delta$, *prove that* $\frac{b^2-ac}{a^2} = \frac{B^2-AC}{A^2}$

Solution

Given: $ax^2 + 2bx + c = 0$, α , β are roots, then

$$\alpha + \beta = -\frac{2b}{a} \ , \alpha\beta = \frac{c}{a} \quad \text{..... (1)}$$

Given: $Ax^2 + 2Bx + C = 0$, $\alpha + \delta$, $\beta + \delta$ are roots, then

$$\alpha + \delta + \beta + \delta = -\frac{2B}{A}$$

$$\Rightarrow \alpha + \beta + 2\delta = -\frac{2B}{A}$$

$Or\ 2\delta = -\frac{2B}{A} - (\alpha + \beta)$, from eqn. (1)

$\Rightarrow 2\delta = -\frac{2B}{A} - \left(-\frac{2b}{a}\right)$

$\therefore \delta = \frac{b}{a} - \frac{B}{A}$ ..... (2)

Also $(\alpha + \delta)(\beta + \delta) = \frac{C}{A}$

$\Rightarrow \alpha\beta + \alpha\delta + \beta\delta + \delta^2 = \frac{C}{A} \quad or \quad \alpha\beta + \delta(\alpha + \beta) + \delta^2 = \frac{C}{A}$

$\Rightarrow \frac{c}{a} + \delta\left(-\frac{2b}{a}\right) + \delta^2 = \frac{C}{A}$

$\therefore \delta\left(-\frac{2b}{a}\right) + \delta^2 = \frac{C}{A} - \frac{c}{a}$ ..... (3)

Eqn. (2) in to eqn. (3)

$\Rightarrow \left(\frac{b}{a} - \frac{B}{A}\right)\left(-\frac{2b}{a}\right) + \left(\frac{b}{a} - \frac{B}{A}\right)^2 = \frac{C}{A} - \frac{c}{a}$

Let $\frac{b}{a} = p \ \frac{B}{A} = q$

$\Rightarrow (p - q)(-2p) + (p - q)^2 = \frac{C}{A} - \frac{c}{a}$

$\Rightarrow -2p^2 + 2pq + p^2 - 2pq + q^2 = \frac{C}{A} - \frac{c}{a}$

$\Rightarrow q^2 - p^2 = \frac{C}{A} - \frac{c}{a}$

$\Rightarrow \frac{B^2}{A^2} - \frac{b^2}{a^2} = \frac{C}{A} - \frac{c}{a} \quad or \quad \frac{B^2}{A^2} - \frac{C}{A} = \frac{b^2}{a^2} - \frac{c}{a}$

$\therefore \frac{b^2-ac}{a^2} = \frac{B^2-AC}{A^2}$ as required.

To find the condition that the roots of the equation $ax^2 + bx + c = 0$ **should be;**

i) Equal in magnitude and opposite in sign
ii) Reciprocals
iii) With one of the roots as zero
iv) Both positive

i) The roots will be *equal in magnitude* and *opposite in sign* if their *sum is zero*. Hence the required condition is:

$\alpha + \beta = -\frac{b}{a} = 0 \ \ or$ $\boxed{\boldsymbol{b = 0}}$, where α *and* β are roots of the equation.

ii) The roots will be *reciprocals* if their *product is unity*. Hence the required condition is:

$\alpha\beta = \frac{c}{a} = 1 \ or$ $\boxed{\boldsymbol{c = a}}$

iii) When one of the roots is zero then the product must be zero. Hence the required condition is:

$\alpha\beta = \frac{c}{a} = 0$ *or* $\boxed{\boldsymbol{c = 0}}$

iv) The roots will be both positive if;

a) The *product* of the roots is *positive*. $i.e. \alpha\beta = \frac{c}{a} = +ve$

This is only true if c and a are of *like signs*

b) The *sum* of the roots is also *positive* $.i.e. \alpha + \beta = -\frac{b}{a} = +ve$

This is possible if b and a are of *unlike signs*. Thus the required condition is that the signs of a and c are alike and opposite to that of b.

10. *The roots of the equation* $x^2 + ax + b = 0$ *are* α *and* β. *Find the equation whose roots are* $p\alpha + q\beta$ *and* $p\beta + q\alpha$.

If the original equation is $x^2 - 4x - 5 = 0$, *find the values of* $\frac{p}{q}$ *in order that the new equation shall have one zero root.*

Solution

Required equation: $x^2 - (A + B)x + A.B = 0$

Let $A = p\alpha + q\beta$ and $B = p\beta + q\alpha$

$A + B = p\alpha + q\beta + p\beta + q\alpha$

$= \alpha(p + q) + \beta(p + q) = (p + q)(\alpha + \beta)$

But $\alpha + \beta = -a$, $\alpha\beta = b$

$\therefore A + B = -a(p + q)$

$A.B = (p\alpha + q\beta)(p\beta + q\alpha)$

$= p^2\alpha\beta + pq\alpha^2 + pq\beta^2 + q^2\alpha\beta$

$= (p^2 + q^2)\alpha\beta + pq(\alpha^2 + \beta^2)$

$= (p^2 + q^2).b + pq[(\alpha + \beta)^2 - 2\alpha\beta]$

$= (p^2 + q^2).b + pq(a^2 - 2b)$

$\therefore$ Required equation is: $x^2 + a(p + q)x + (p^2 + q^2).b + pq(a^2 - 2b) = 0$

Now to find the values of $\frac{p}{q}$ in original equation $x^2 - 4x - 5 = 0$, so that the new equation shall have one zero root,

Comparing the equations $x^2 - 4x - 5 = 0$ and $x^2 + ax + b = 0$,

$\Rightarrow a = -4, b = -5$

Substituting these values in the obtained equation gives;

$x^2 - 4(p + q)x - 5(p^2 + q^2) + pq((-4)^2 - 2 \times -5) = 0$

$\Rightarrow x^2 - 4(p + q)x - 5(p^2 + q^2) + 26pq = 0$

This equation will have one zero root if the product is zero

$\Rightarrow -5(p^2 + q^2) + 26pq = 0$ *or* $5p^2 - 26pq + 5q^2 = 0$

Factoring

$\Rightarrow 5p^2 - 25pq - pq + 5q^2 = 0$

$\Rightarrow 5p(p - 5q) - q(p - 5q) = 0$

$\Rightarrow (p - 5q)(5p - q) = 0$

$\therefore p - 5q = 0$ *or* $\frac{p}{q} = 5$

$\therefore 5p - q = 0$ *or* $\frac{p}{q} = \frac{1}{5}$

$\therefore \frac{p}{q} = 5, \frac{1}{5}$

Other examples on quadratic equation

11. *If α and β are roots of the equation $ax^2+2bx+c=0$ while δ and γ are the roots of the equation $a_1x^2+2b_1x+c_1=0$, show that if $(\alpha-\gamma)(\beta-\delta)+(\alpha-\delta)(\beta-\gamma)=0$, then $ac_1+a_1c=2bb_1$.*

Solution

From $ax^2+2bx+c=0$, $\alpha+\beta=-\frac{2b}{a}$, $\alpha\beta=\frac{c}{a}$

From $a_1x^2+2b_1x+c_1=0$, $\gamma+\delta=-\frac{2b_1}{a_1}$, $\gamma\delta=\frac{c_1}{a_1}$

$(\alpha-\gamma)(\beta-\delta)+(\alpha-\delta)(\beta-\gamma)=0$

$\Rightarrow \alpha\beta-\alpha\delta-\gamma\beta+\gamma\delta+\alpha\beta-\alpha\gamma-\delta\beta+\gamma\delta=0$

$\Rightarrow 2\alpha\beta+2\gamma\delta-\delta(\alpha+\beta)-\gamma(\delta+\gamma)=0$

$\Rightarrow \frac{2c}{a}+\frac{2c_1}{a_1}-(\alpha+\beta)(\delta+\gamma)=0$

$\Rightarrow \frac{2c}{a}+\frac{2c_1}{a_1}+\frac{2b}{a}.-\frac{2b_1}{a_1}=0$

$\Rightarrow \frac{2c}{a}+\frac{2c_1}{a_1}-\frac{4bb_1}{aa_1}=0$ ……$\left(\times\frac{aa_1}{2}\right)$

$\Rightarrow a_1c+ac_1-2bb_1=0$

$\therefore ac_1+a_1c=2bb_1$

12. *If x is a real number, prove that the expression $\frac{x^2+2x-11}{2(x-3)}$ doe not lie strictly between 2 and 6.*

Solution

Let $y=\frac{x^2+2x-11}{2(x-3)}$

$\Rightarrow 2yx-6y=x^2+2x-11$ *or* $x^2+(2-2y)x+6y-11=0$

$\Rightarrow x=\frac{-(2-2y)\pm\sqrt{[-(2-2y)]^2-4(6y-11)}}{2}$

$=\frac{-(2-2y)\pm\sqrt{4-8y+4y^2-24+44}}{2}$

$=\frac{-(2-2y)\pm\sqrt{4y^2-32y+48}}{2}$

x will be real if $4y^2-32y+48\geq 0$ *or* $y^2-8y+12\geq 0$

$\Rightarrow (y-2)(y-6)\geq 0$

Between 2 and 6 *.i.e.* $2<y<6$, one of the factors on L.H.S will be $+ve$ and the other $-ve$, so the product will be negative. Hence $\frac{x^2+2x-11}{2(x-3)}$ will not lie between 2 and 6

13. *If x is a real number, prove that the expression $\frac{x}{x^2-5x+9}$ lies between $-\frac{1}{11}$ and 1.*

Solution

Let $y=\frac{x}{x^2-5x+9}$ *or* $yx^2+(-5y-1)x+9y=0$

$\Rightarrow x=\frac{(5y+1)\pm\sqrt{(5y+1)^2-36y^2}}{2y}$

For real values of x, $(5y+1)^2-36y^2\geq 0$

$\Rightarrow 25y^2+10y+1-36y^2\geq 0$

$\Rightarrow -11y^2+10y+1\geq 0$ *or* $11y^2-10y-1\leq 0$

$\Rightarrow 11y^2-11y+y-1\leq 0$

$\Rightarrow 11y(y-1)+(y-1)\leq 0$

$\Rightarrow (11y+1)(y-1\leq 0)$

Between $-\frac{1}{11} < y < 1$, the inequality is true for any value of y. Hence $\frac{x}{x^2-5x+9}$ lies between $-\frac{1}{11} and\ 1$.

14. *For real x, $f(x) = \frac{x^2-k}{x-2}$ can take any real value. Find the range of values of k can take.*

Solution

Let $\frac{x^2-k}{x-2} = y \ \ or\ \ yx - 2y = x^2 - k$

$\therefore x^2 - yx + 2y - k = 0$

$\Rightarrow x = \frac{y \pm \sqrt{y^2-4(2y-k)}}{2}$

For real value of x, $y^2 - 4(2y - k) \geq 0$

$\Rightarrow y^2 - 8y + 4k \geq 0$

$\Rightarrow y^2 - 8y \geq -4k$, completing squares,

$\Rightarrow y^2 - 8y + 16 \geq -4k + 16$

$\therefore (y-4)^2 \geq 16 - 4k$

Since L.H.S is always $+ve$, taking its smallest value say $0\ .i.e.\ y = 4$, then

$0 \geq 16 - 4k \ \Rightarrow k \geq 4$

$\therefore k \geq 4$

15. *Find the range of values of k can take given that for real $x, f(x) = \frac{x^2+3x}{x+k}$ can take any real value.*

Solution

Let $\frac{x^2+3x}{x+k} = y \ \ or\ \ yx + ky = x^2 + 3x$

$\therefore x^2 + (3-y)x - ky = 0$

$\Rightarrow x = \frac{-(3-y) \pm \sqrt{(3-y)^2+4ky}}{2}$

$= \frac{(y-3) \pm \sqrt{(3-y)^2+4ky}}{2}$

For real values of x, $(3-y)^2 + 4ky \geq 0$

$\Rightarrow 9 - 6y + y^2 + 4ky \geq 0$

$\Rightarrow y^2 + (4k-6)y + 9 \geq 0$

$\Rightarrow y^2 + (4k-6)y \geq -9$, completing squares

$\Rightarrow y^2 + (4k-6)y + (2k-3)^2 \geq -9 + (2k-3)^2$

$\therefore \big(y + (2k-3)\big)^2 \geq -9 + (2k-3)^2$

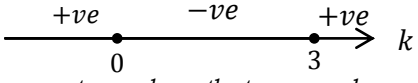
Since y can take on any real value, let its smallest value be $-(2k-3)^2$

$\Rightarrow 0 \geq -9 + (2k-3)^2$

$\Rightarrow 4k^2 - 12k + 9 - 9 \leq 0$

$\therefore 4k^2 - 12k \leq 0 \ \ or\ \ k^2 - 3k \leq 0$

For $0 \leq k \leq 3$, the inequality is true.



16. *Given that $y = \frac{3x+k}{x^2-1}$, where x is real and k is a constant, show that y can take all real values if $|k| < 3$.*

Solution

$y = \frac{3x+k}{x^2-1} \ \ or\ \ yx^2 - y = 3x + k$

$\therefore yx^2 - 3x - y - k = 0$

For real values of x, $(-3)^2 \geq 4y(-y-k)$

$\Rightarrow 9 + 4y^2 + 4ky \geq 0$

$\Rightarrow 4y^2 + 4ky \geq -9 \quad or \quad y^2 + ky \geq -\frac{9}{4}$, completing squares,

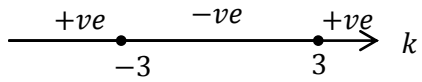
$\Rightarrow y^2 + ky + \frac{k^2}{4} \geq -\frac{9}{4} + \frac{k^2}{4}$

$\therefore \left(y + \frac{k}{2}\right)^2 \geq \frac{k^2-9}{4}$

Since y can take on any real value, let its smallest value be $-\frac{k}{2}$

$\Rightarrow 0 \geq \frac{k^2-9}{4} \; or \; k^2 - 9 \leq 0$

$\therefore (k+3)(k-3) \leq 0$



For interval $-3 \leq k \leq 3$, the inequality holds. Thus $|k| < 3$ is the required interval.

Task

1. *If α and β are the roots of the equation $ax^2 + bx + c = 0$, find the value of;*
   a) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ **ANSWER**: $\frac{b^2-2ac}{ac}$
   b) $\alpha^4 + \beta^4$ **ANSWER**: $\frac{b^4-4ab^2c+2a^2c^2}{a^4}$
   c) $(\alpha^2 - \beta)^2 + (\beta^2 - \alpha)^2$ **ANSWER**: $\frac{b^4-4ab^2c+2a^2c^2+a^2b^2-2a^3c+2a^2bc}{a^4}$
   d) $\alpha^4\beta^7 + \alpha^7\beta^4$ **ANSWER**: $\frac{\boldsymbol{bc^4(3ac-b^2)}}{\boldsymbol{a^7}}$
2. *The roots of the equation are α and β*
   i) Show that if $\alpha - \beta = 1$, then $a^2 = 4(b^2 - ac)$
   ii) Find the quadratic equation whose roots are $\alpha + \alpha\beta$ and β
3. *If α and β are the roots of the equation $x^2 - px + q = 0$, show that* $\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} = \frac{p^4}{q^2} - \frac{4p^2}{q} + 2$
4. *If p, q are the roots of the equation $3x^2 + 6x + 2 = 0$, form an equation whose roots are $-\frac{p^2}{q}$ and $-\frac{q^2}{p}$.* **ANSWER**: $3x^2 - 18x + 2 = 0$
5. *If α and β are the roots of the equation $ax^2 + bx + c = 0$, form an equation whose roots are;*
   i) $\frac{1-\alpha}{1+\alpha}$, $\frac{1-\beta}{1+\beta}$ **ANSWER**: $(a - b + c)x^2 - 2x(a - c) + a + b + c = 0$
   ii) $\alpha^2 + \beta$, $\alpha^{-2} + \beta^{-2}$
   ANSWER: $a^2c^2x^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0$
6. *If the roots of the equation $ax^2 + 8(b - a)x + 4(4a - 8b + c) = 0$ are $(4 - 2\alpha)$ and $(4 - 2\beta)$, find the equation whose roots are α and β.*
   ANSWER: $x^2 - \frac{4b}{a}x + \frac{c}{a} = 0$
7. *If $(a + b + c)x^2 + x(b + 2c) + c = 0$ has roots α and β, find the equation whose roots are $\frac{\alpha}{\alpha+1}$, $\frac{\beta}{\beta+1}$ expressing the coefficients in terms of a, b and c.*
   ANSWER: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
8. *Given that the roots of the equation $x + \frac{1}{x} = 4$ are α and β, evaluate*
   i) $(\alpha - \beta)^2$
   ii) $\frac{1}{\alpha} + \frac{1}{\beta}$

iii) $\alpha^3 + \beta^3$

iv) $2\alpha^2 + 2\beta^2 + 5\alpha\beta$

9. *Given that the roots of the equation* $ax^2 + bx + c = 0$ *are* β *and* $n\beta$*, show that* $(n+1)^2 ac = nb^2$

10. *Given that* α *and* β *are the roots of the equation* $x^2 - px + q = 0$*, obtain the quadratic equation whose roots are* α^{-2}, β^{-2},
ANSWER: $q^2x^2 - (p^2 - 2p)x + 1 = 0$

11. *If the roots of a quadratic equation* $x^2 + 2bx + c = 0$ *differ by 4, show that* $8a = -c \pm \sqrt{(c^2 + 16b^2)}$ *. Hence find two values of a for which the equation* $ax^2 + 2x + 3 = 0$ *has roots differing by 4.* **ANSWER**: $a = -1, \frac{1}{4}$

12. *The roots of the equation* $9x^2 + 6x + 1 = 4kx$ *where k is a real constant, are denoted as* α *and* β*.*
    a) *Show that the equation whose roots are* α^{-1} *and* β^{-1} *is* $x^2 + 6x + 9 = 4kx$
    b) *Find the set of values of k for which* α *and* β *are real.*
    ANSWER: $k \leq 0, k \geq 3$
    c) *Find also the set of values of k for which* α *and* β *are real and positive.* **ANSWER**: $k \geq 3$

13. *If* α *and* β *are the roots of the equation* $x^2 + bx + c = 0$*, express* $(\alpha - \beta^2)(\alpha^2 - \beta)$ *in terms of b and c. hence or otherwise show that if one root of the equation is the square of the other, then* $b^3 + c^3 + c = 3bc$

14. *Given that the roots of the equation* $x^2 + px + q = 0$ *has roots* α *and* β*, express p and q in terms of* α *and* β*. Find the equation, the coefficients expressed in terms of p and q, whose roots are* $3\alpha - \beta$ *and* $3\beta - \alpha$*.*
ANSWER: $p = -\alpha - \beta$, $q = \alpha\beta$, $x^2 + 2px + 16q - 3q^2 = 0$

15. *Given that the roots of the equation* $x^2 + px + q = 0$ *are* α *and* β*, express* $(\alpha - 2\beta)(\beta - 2\alpha)$ *in terms of p and q. hence/otherwise show that the condition for one root of the equation to be double the other is* $2p^2 = 9q$*.*
ANSWER: $9q - 2p^2$

16. *Given that* α *and* β *are the roots of the equation* $2x^2 - 7x - 17 = 0$*, show that the equation with roots* $(\alpha - 4), (\beta - 4)$ *is* $2x^2 + 9x - 13 = 0$*.*

17. *Given that* α *and* β *are the roots of the equation* $x^2 - px + q = 0$*, prove that* $\alpha + \beta = p$, $\alpha\beta = q$*.Prove also that;*
    a) $\alpha^{2n} + \beta^{2n} = (\alpha^n + \beta^n)^2 - 2q^n$ [***Hint***: *Let* $\alpha^n = A$, $\beta^n = B$]
    b) $\alpha^4 + \beta^4 = p^4 - 4p^2q + 2q^2$

    Hence/otherwise form a quadratic equation whose roots are the fourth powers of those of the equation $x^2 - 3x + 1 = 0$

18. *The roots of the equation* $x^2 + 6x + c = 0$ *differ by* $2n$*, where n is real and non zero, show that* $n^2 = 9 - c$*. Given that the roots also have opposite signs, find set of possible values of n.*
ANSWER: $n > 3$, $n < -3$

19. *If the roots of the equation* $x^2 - px + q = 0$ *are* α *and* β*, find the value of;*
    i) $\alpha^2(\alpha^2\beta^{-1} - \beta) + \beta^2(\beta^2\alpha^{-1} - \alpha)$ **ANSWER**: $\frac{p(p^2-4q)(p^2-q)}{q}$
    ii) $(\alpha - p)^{-4} + (\beta - p)^{-4}$ **ANSWER**: $\frac{p^4-4p^2q+2q^2}{q^4}$

20. a) *For what values of m will the equation* $\frac{x^2-bx}{ax-c}=\frac{m-1}{m+1}$ *have roots equal in magnitude but opposite in sign?* **ANSWER**: $m=\frac{a-b}{a+b}$

b) *If x is real, prove that* $\frac{x^2+34x-71}{x^2+2x-7}$ *cannot lie strictly between 5 and 9*

c) *Show that for real* $x, -\frac{1}{4}\leq\frac{x}{x^2+4}\leq\frac{1}{4}$

d) *The roots of a quadratic equation* $x^2+px+q=0$ *are* α *and* β*. Show that the equation whose roots are* $\alpha^2-q\alpha$ *and* $\beta^2-q\beta$ *is given by* $x^2-(p^2+pq-2q)x+q^2(p+q+1)=0$ ***Ubeb* 2011**

CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE A COMMON ROOT

Let $ax^2+bx+c=0$ and

$lx^2+mx+n=0$

, be the two equations with a common root $x=\alpha\neq 0$

Substituting $x=\alpha$ in to the equations

$\Rightarrow a\alpha^2+b\alpha+c=0$ ..... (1)

$\Rightarrow l\alpha^2+m\alpha+n=0$ ..... (2)

Eliminating α^2 from eqn. (1) and eqn. (2)

$l\times eqn.(1)-m\times eqn.(2)$, gives

$\alpha(lb-am)+lc-an=0$

$\therefore \alpha=\frac{an-lc}{bl-am}$ ..... (3)

Now eliminating constants c and n from eqn. (1) and eqn. (2)

$n\times eqn.(1)-c\times eqn.(2)$, gives

$\alpha^2(an-lc)+\alpha(bn-mc)=0$

$\Rightarrow \alpha[\alpha(an-lc)+(bn-mc)]=0$

Either $\alpha=0$, but $\alpha\neq 0$

, or $\alpha(an-lc)+(bn-mc)$

$\Rightarrow \alpha=\frac{mc-bn}{an-lc}$ ..... (4)

Equating equations (3) and (4)

$\Rightarrow \frac{an-lc}{bl-am}=\frac{mc-bn}{an-lc}$

$\therefore (an-lc)^2=(mc-bn)(bl-am)$

Illustrative examples

1) *Given that* $x^2+px+q=0$ *and* $x^2+rx+s=0$ *share a common root* $\alpha\neq 0$*, prove that* $\frac{ps-qr}{q-s}=\frac{q-s}{r-p}$

Solution

$x^2+px+q=0, x=\alpha, \alpha^2+p\alpha+q=0$ ..... (1)

$x^2+rx+s=0, x=\alpha, \alpha^2+r\alpha+s=0$ ..... (2)

Eliminating α^2 from eqn. (1) and eqn. (2)

$eqn.(1)-eqn.(2)$, gives

$\Rightarrow (p-r)\alpha+q-s=0$

$\therefore \alpha=\frac{s-q}{p-r}=\frac{q-s}{r-p}$ ..... (3)

Now eliminating constants q and s from eqn. (1) and eqn. (2)

$q\times eqn.(2)-s\times eqn.(1)$, gives

$\Rightarrow \alpha^2(s-q)+\alpha(ps-qr)=0$

$\Rightarrow \alpha[\alpha(s-q)+(ps-qr)]=0$

Either $\alpha=0$, but $\alpha\neq 0$

, or $\alpha(s-q)+(ps-qr)$

$\Rightarrow \alpha=\frac{qr-ps}{s-q}=\frac{ps-qr}{q-s}$ ..... (4)

Equating equations (3) and (4)

$\Rightarrow \frac{ps-qr}{q-s}=\frac{q-s}{r-p}$, as required

2) *Given that* $ax^2+bx+c=0$ *and* $bx^2+cx+a=0$ *have a common root, then* $a^3+b^3+c^3=3abc$

Solution

Let $x=\alpha\neq 0$ be the common root.

$\Rightarrow a\alpha^2+b\alpha+c=0$ ..... (1)

$\Rightarrow b\alpha^2+c\alpha+a=0$ ..... (2)

Eliminating α^2 from eqn. (1) and eqn. (2)

$b\times eqn.(1)-a\times eqn.(2)$, gives

$\Rightarrow (b^2-ac)\alpha+bc-a^2=0$

$\therefore \alpha=\frac{a^2-bc}{b^2-ac}$ ..... (3)

Now eliminating constants a and c from eqn. (1) and eqn. (2)

$a\times eqn.(1)-c\times eqn.(2)$, gives

$\Rightarrow \alpha^2(a^2-bc)+\alpha(ab-c^2)=0$

$\Rightarrow \alpha[\alpha(a^2-bc)+(ab-c^2)]=0$

Either $\alpha=0$, but $\alpha\neq 0$

, or $\alpha(a^2-bc)+(ab-c^2)=0$

$\Rightarrow \alpha=\frac{c^2-ab}{a^2-bc}$ ..... (4)

Equating equations (3) and (4)

$\Rightarrow \frac{a^2-bc}{b^2-ac}=\frac{c^2-ab}{a^2-bc}$

$\Rightarrow (a^2-bc)^2=(b^2-ac)(c^2-ab)$

$\Rightarrow a^4-2a^2bc+b^2c^2=b^2c^2-ac^3-ab^3+a^2bc$

$\Rightarrow a^4-2a^2bc=-ac^3-ab^3+a^2bc$ .....$(\div a)$

$\Rightarrow a^3-2abc=-c^3-b^3+abc$

$\therefore a^3+b^3+c^3=3abc$

3) *If* $x^2-px+q=0$ *and* $x^2-qx+p=0$ *have a common root, prove that* $p+q+1=0$

Solution

Let $x=\alpha\neq 0$ be the common root.

$\Rightarrow \alpha^2-p\alpha+q=0$ ..... (1)

$\Rightarrow \alpha^2-q\alpha+p=0$ ..... (2)

Eliminating α^2 from eqn. (1) and eqn. (2)

$eqn.(1)-eqn.(2)$, gives

$(-p+q)\alpha+q-p=0$

$\Rightarrow \alpha=-\frac{q-p}{q-p}=-1$

$\therefore \alpha=-1$ ..... (3)

$eqn.(3) in to\ eqn.(1)\ or\ (2)$

$\Rightarrow (-1)^2-p(-1)+q=0$

$\therefore 1+p+q=0$. As required

4) *Find p such that* $x^2-5x+6=0$ *and* $x^2+px-10=0$ *have a common root.*

Solution

Let $x = \alpha \neq 0$ be the common root.

$\Rightarrow \alpha^2 - 5\alpha + 6 = 0$ ..... (1)

$\Rightarrow \alpha^2 + p\alpha - 10 = 0$ ..... (2)

Eliminating α^2 from eqn. (1) and eqn. (2)

$eqn.(1) - eqn.(2)$, gives

$(-5 - p)\alpha + 16 = 0$

$\Rightarrow \alpha = \frac{-16}{-5-p} = \frac{16}{p+5}$ ..... (3)

$eqn.(3)\ in\ to\ (1)$

$\Rightarrow \left(\frac{16}{p+5}\right)^2 - 5\left(\frac{16}{p+5}\right) + 6 = 0$

$\Rightarrow 256 - 80(p+5) + 6(p+5)^2 = 0$

$\Rightarrow 256 - 80p - 400 + 6p^2 + 60p + 150 = 0$

$\Rightarrow 6 - 20p + 6p^2 = 0$ ..... $(\div 2)$

$\therefore 3p^2 - 10p + 3 = 0$

$\Rightarrow 3p^2 - 9p - p + 3 = 0$

$\Rightarrow 3p(p-3) - (p-3) = 0$

$\Rightarrow (3p-1)(p-3) = 0$

$\therefore p = 3, \frac{1}{3}$

Alternatively

Eliminating constants 6 and -10 from eqn. (1) and eqn. (2)

$10 \times eqn.(1) + 6 \times eqn.(2)$, gives

$16\alpha^2 + (6p - 50)\alpha = 0$

$\Rightarrow \alpha = \frac{50-6p}{16} = \frac{25-3p}{8}$ ..... (4)

Equating equations (3) and (4)

$\Rightarrow \frac{16}{p+5} = \frac{25-3p}{8}$, simplifying the equation gives

$3p^2 - 10p + 3 = 0$, solving the equation

$\therefore p = 3, \frac{1}{3}$

5) *Prove that if the equations* $x^2 + ax + b = 0$ *and* $cx^2 + 2ax - 3b = 0$ *have a common root and nether a nor b is zero, then* $b = \frac{5a^2(c-2)}{(c+3)^2}$

Solution

Let $x = \alpha \neq 0$ be the common root.

$\Rightarrow \alpha^2 + a\alpha + b = 0$ ..... (1)

$\Rightarrow c\alpha^2 + 2a\alpha - 3b = 0$ ..... (2)

Eliminating α^2 from eqn. (1) and eqn. (2)

$c \times eqn.(1) - eqn.(2)$, gives

$\alpha(ac - 2a) + bc + 3b = 0$

$\therefore \alpha = -\frac{(bc+3b)}{(ac-2a)} = \frac{-b(c+3)}{a(c-2)}$ ..... (3)

Eliminating constants b , $-3b$ from eqn. (1) and eqn. (2)

$3 \times eqn.(1) + eqn.(2)$, gives

$\alpha^2(3 + c) + 5a\alpha = 0$

$\Rightarrow \alpha[\alpha(3 + c) + 5a] = 0$

$\therefore \alpha = -\frac{5a}{c+3}$ ..... (4)

Equating equations (3) and (4)

$\Rightarrow \frac{-b(c+3)}{a(c-2)} = -\frac{5a}{c+3}$

$\Rightarrow b(c+3)^2 = 5a^2(c-2)$

$\therefore b = \frac{5a^2(c-2)}{(c+3)^2}$, as required.

Task

1. *Show that if the equations* $x^2 + px + q = 0$ *and* $x^2 + 2Px + Q = 0$ *have a common root then* $(q-Q)^2 + 4(P-p)(Pp-pQ) = 0$
2. *If the equations* $a_1x^2 + b_1x + c_1 = 0$ *and* $a_2x^2 + b_2x + c_2 = 0$ *have a common root, then* $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (a_1c_2 - a_2c_1)^2$
3. *Show that if the equations* $x^2 + bx + c = 0$, $x^2 + px + q = 0$ *have a common root, then* $(c-q)^2 = (b-p)(cp-bq)$
4. *Show that if the equations* $x^2 + ax + 1 = 0$ *and* $x^2 + x + b = 0$ *have a common root, then* $(b-1)^2 = (a-1)(1-ab)$

COMMON FACTORS

If two functions $f(x)$ and $g(x)$ have a common factor $(x-a)$, then

$f(x) = (x-a).F(x)$ ..... (1)

$g(x) = (x-a).G(x)$ ..... (2)

For any constant k,

$f(x) + kg(x) = (x-a).F(x) + k(x-a).G(x)$

$$\boxed{f(x) + kg(x) = (x-a)[F(x) + kG(x)]}$$

Thus for any constant k, $(x-a)$ is a common factor of $f(x) + kg(x)$

NOTE:

Equations (1) and (2) are identities. Thus ,

$$\boxed{f(a) = 0\,,\ g(a) = 0}$$, which gives a pair of simultaneous equations which provides another approach to problems involving a common factor.

Examples

1. *Find constants p and q such that* $(x-2)$ *is a common factor of* $x^3 - x^2 - 2px + 3q$ *and* $qx^3 - px^2 + x + 2$

Solution

Let $f(x) = x^3 - x^2 - 2px + 3q$

$g(x) = qx^3 - px^2 + x + 2$

Since $(x-2)$ is a factor to both $f(x)$ and $g(x)$, then $f(2) = 0$ $g(2) = 0$

$\Rightarrow 8 - 4 - 4p + 3q = 0$ *or* $-4p + 3q = -4$ ..... (1)

$\Rightarrow 8q - 4p + 2 + 2 = 0$ *or* $-4p + 8q = -4$ ..... (2)

Solving eqn. (1) and eqn. (2) simultaneously,

$eqn.(1) - eqn.(2)$

$\Rightarrow -5q = 0$ $\therefore q = 0$

From eqn. (1), for $q = 0$, $p = 1$

$\therefore p = 1$, $q = 0$

2. *Find constants p and q such that* $(x-1)$ *is a common factor of* $x^4 + x^2 - q$ *and* $x^4 - 2px^2 + 2$

Solution

Let $f(x) = x^4 + x^2 - q$, $f(1) = 1 + 1 - q = 0 \quad \therefore q = 2$

$g(x) = qx^3 - px^2 + x + 2$, $g(1) = 1 - 2p + 2 = 0 \quad \therefore p = \frac{3}{2}$

$\therefore p = \frac{3}{2}$, $q = 2$

3. *Show that if* $(x + 1)$ *is a common factor of* $x^3 - ax^2 + b$ *and* $x^4 - ax^3 + bx^2 + c$, *then* $2a = 2b - 2 = -2 - c$.

Solution

Let $f(x) = x^3 - ax^2 + b$

$g(x) = x^4 - ax^3 + bx^2 + c$

For any constant k, $(x + 1)$ is a factor of $f(x) + kg(x)$

$\Rightarrow f(-1) + kg(-1) = 0$, for any value of k.

$\therefore -1 - a - b + k(1 + a + b + c) = 0$

For $k = 0$, $a - b = -1$ ..... (1)

For $k = 1$, $2b + c = 0$ ..... (2)

From eqn. (1), multiplying through by 2

$\Rightarrow 2a - 2b = -2$

From eqn. (2), $2b + c = 0$ *or* $b = -\frac{c}{2}$..... (3)

Eqn. (3) in to eqn. (1)

$a + \frac{c}{2} = -1$ *or* $2a + c = -2$ *or* $2a = -2 - c$

$\therefore 2a = 2b - 2 = -2 - c$

4. *Find constant a for which the function* $f(x) = ax^2 + 2x - 1$ *and* $g(x) = x^2 + 4x + a$ *have a common factor.*

Solution

Let $x - \alpha$ be a common factor

$\Rightarrow f(x) + kg(x)$ has a factor of $x - \alpha$

$\Rightarrow f(\alpha) + kg(\alpha) = 0$

$\Rightarrow a\alpha^2 + 2\alpha - 1 + k(\alpha^2 + 4\alpha + a) = 0$

For $k = -a$, to eliminate α^2

$\Rightarrow a\alpha^2 + 2\alpha - 1 - a\alpha^2 - 4a\alpha - a^2 = 0$

$\Rightarrow \alpha(2 - 4a) - 1 - a^2 = 0$

$\therefore \alpha = \frac{a^2+1}{2-4a}$ ..... (1)

For $k = \frac{1}{a}$, to eliminate constants

$\Rightarrow a\alpha^2 + 2\alpha - 1 + \frac{\alpha^2}{a} + \frac{4\alpha}{a} + 1 = 0$

$\Rightarrow \alpha^2\left(a + \frac{1}{a}\right) + \alpha\left(2 + \frac{4}{a}\right) = 0$

$\Rightarrow \alpha\left[\alpha\left(a + \frac{1}{a}\right) + \left(2 + \frac{4}{a}\right)\right] = 0$

$\Rightarrow \alpha\left(a + \frac{1}{a}\right) + \left(2 + \frac{4}{a}\right) = 0$

$\therefore \alpha = -\frac{\left(2+\frac{4}{a}\right)}{\left(a+\frac{1}{a}\right)} = \frac{-(2a+4)}{a^2+1}$ ..... (2)

Equating equations (1) and (2)

$\Rightarrow \frac{a^2+1}{2-4a} = \frac{-(2a+4)}{a^2+1}$

$\Rightarrow a^4 + 2a^2 + 1 = -(2a + 4)(2 - 4a)$

$\Rightarrow a^4 + 2a^2 + 1 = -(4a - 8a^2 + 8 - 16a) = 12a + 8a^2 - 8$

$\therefore a^4 - 6a^2 - 12a + 9 = 0$

Let $h(a) = a^4 - 6a^2 - 12a + 9$
For $a = 3$, $h(3) = 0$ $\therefore a - 3$ is a factor of $h(a)$

$$
\begin{array}{r|l}
 & a^3 + 3a^2 + 3a - 3 \\
\hline
a - 3 & a^4 + 0a^3 - 6a^2 - 12a + 9 \\
 & \underline{a^4 - 3a^3} \\
 & \quad 3a^3 - 6a^2 - 12a + 9 \\
 & \quad \underline{3a^3 - 9a^2} \\
 & \quad\quad 3a^2 - 12a + 9 \\
 & \quad\quad \underline{3a^2 - 9a} \\
 & \quad\quad\quad 3a + 9 \\
 & \quad\quad\quad \underline{3a + 9} \\
 & \quad\quad\quad 0
\end{array}
$$

$\therefore h(a) = (a - 3)(a^3 + 3a^2 + 3a - 3) = 0$
$\Rightarrow a = 3,\ 0.26$
$\therefore a = 3$

5. *Determine the value of m such that the equations* $x^3 + mx - 1 = 0$ *and* $x^3 - 3x + m = 0$ *have a common root.*

Solution

Let common root be $x = \alpha \neq 0 \Rightarrow (x - \alpha)$ is a common factor.
Let $f(x) = x^3 + mx - 1$, $g(x) = x^3 - 3x + m$
$\Rightarrow f(x) + kg(x)$, has a common factor $(x - \alpha)$
$\Rightarrow f(\alpha) + kg(\alpha) = 0$
$\Rightarrow \alpha^3 + m\alpha - 1 + k(\alpha^3 - 3\alpha + m) = 0$
For $k = -1$ to eliminate α^3
$\Rightarrow \alpha^3 + m\alpha - 1 - \alpha^3 + 3\alpha - m = 0$
$\Rightarrow \alpha(m + 3) = m + 1 \therefore \alpha = \frac{m+1}{m+3}$
Now from the equation $x^3 + mx - 1 = 0$ with $x = \alpha$ as a root, then $\alpha^3 + m\alpha - 1 = 0$ must have the root $\alpha = \frac{m+1}{m+3}$
$\Rightarrow \left(\frac{m+1}{m+3}\right)^3 + m\left(\frac{m+1}{m+3}\right) - 1 = 0$ ……$\times (m + 3)^3$
$\Rightarrow (m + 1)^3 + (m^2 + m)(m + 3)^2 - (m + 3)^3 = 0$
$\Rightarrow m^3 + 3m^2 + 3m + 1 + (m^2 + m)(m^2 + 6m + 9) - (m^3 + 9m^2 + 27m + 27) = 0$
$\Rightarrow m^3 + 3m^2 + 3m + 1 + m^4 + 6m^3 + 9m^2 + m^3 + 6m^2 + 9m - m^3 - 9m^2 - 27m - 27 = 0$
$\Rightarrow m^4 + 7m^3 + 9m^2 - 15m - 26 = 0$
Solving by factor theorem,
Let $f(m) = m^4 + 7m^3 + 9m^2 - 15m - 26$
For $m = -2$, $f(-2) = (-2)^4 + 7(-2)^3 + 9(-2)^2 - 15(-2) - 26 = 0$
$\Rightarrow m + 2$, is a factor of $f(m)$

$$
\begin{array}{r|l}
 & m^3 + 5m^2 - m - 13 \\
\hline
m + 2 & m^4 + 7m^3 + 9m^2 - 15m - 26 \\
 & \underline{m^4 + 2m^3} \\
 & \quad 5m^3 + 9m^2 - 15m - 26 \\
 & \quad \underline{5m^3 + 10m^2} \\
 & \quad\quad -m^2 - 15m - 26 \\
 & \quad\quad \underline{-m^2 - 2m} \\
 & \quad\quad\quad -13m - 26 \\
 & \quad\quad\quad \underline{-13m - 26} \\
 & \quad\quad\quad 0
\end{array}
$$

$\therefore (m+2)(m^3+5m^2-m-13)=0$
$\therefore m=-2\,,1.49\,,-4.60\,,-1.89$

6. *Show that If the cubic functions* x^3+ax^2+b *and* ax^3+bx^2+x-a *have a common factor, it is also a common factor of the quadratic function* $(b-a^2)x^2+x-a(1+b)=0$

Solution

Let the common factor be $(x-\alpha)$
Let $f(x)=x^3+ax^2+b\,,g(x)=ax^3+bx^2+x-a$
$\Rightarrow f(x)+kg(x)$, has a common factor $(x-\alpha)$
$\Rightarrow f(\alpha)+kg(\alpha)=0$
$\Rightarrow \alpha^3+a\alpha^2+b+k(a\alpha^3+b\alpha^2+\alpha-a)=0$
For $k=-\frac{1}{a}$ to eliminate α^3
$\Rightarrow \alpha^3+a\alpha^2+b-\frac{1}{a}(a\alpha^3+b\alpha^2+\alpha-a)=0$
$\Rightarrow a\alpha^3+a^2\alpha^2+ab-a\alpha^3-b\alpha^2-\alpha+a=0$
$\Rightarrow \alpha^2(a^2-b)-\alpha+a(1+b)=0$
$\therefore (b-a^2)\alpha^2+\alpha-a(1+b)=0$. this equation is as a result of $x=\alpha$ being a true root of the quadratic equation $(b-a^2)x^2+x-a(1+b)=0$. Hence the common factor of the above two equations is also a common factor of the quadratic equation.

Task

Show that if $(x+\alpha)$ *is a common factor of* x^3+px^2+q *and* ax^3+bx+c*, then it is also a factor of* $apx^2-bx+aq-c$*.*

REPEATED FACTORS

Let $(x-a)$ be a repeated factor of $f(x)$
$\Rightarrow f(x)=(x-a)^2.g(x)$ $.i.e.$ repeated twice
Differentiating with respect to x
$f'(x)=g(x).2(x-a)+(x-a)^2.g'(x)$
Thus if $(x-a)$ is a repeated factor of $f(x)$, then it is also a factor of $f'(x)$

NOTE:

$(x-a)$ is a repeated factor of $f(x)$ if $f'(a)=0$

Examples

1. *Determine whether* $f(x)\equiv 3x^4-8x^3-6x^2+24x-13$ *has any repeated factors, and if so find them.*

Solution

$f'(x)\equiv 12x^3-24x^2-12x+24$
$\equiv 12(x^3-2x^2-x+2)$
$\equiv 12[x^2(x-2)-(x-2)]$
$\equiv 12(x-2)(x^2-1)\equiv 12(x-1)(x+1)(x-2)$
Now $f'(x)=0$ if $x=\pm 1\,,2$
Checking the value of $f(x)$ for these values of x
$\Rightarrow f(1)=0\,,f(-1)\neq 0\,,f(2)\neq 0$
So $(x+1)$ and $(x-2)$ are not factors of $f(x)$
Hence $(x-1)$ is the only repeated factor of $f(x)$

2. *If the equation* $3x^4+2x^3-6x^2-6x+p=0$ *has equal roots, find the possible values of p.*

Solution

Let $f(x) = 3x^4 + 2x^3 - 6x^2 - 6x + p$

The equation $f(x) = 0$ has equal roots if $f(x)$ has repeated factors.

Any linear factor of $f'(x)$ is a possible repeated factor of $f(x)$

$\Rightarrow f'(x) \equiv 12x^3 + 6x^2 - 12x - 6$

$\equiv 6(2x^3 + x^2 - 2x - 1)$

$\equiv 6[x^2(2x + 1) - (2x + 1)]$

$\equiv 6(2x + 1)(x^2 - 1)$

$\equiv 6(x - 1)(x + 1)(2x + 1)$

If $(x - 1)$ is a repeated factor of $f(x)$, then $f(1) = 0$

$\Rightarrow 3 + 2 - 6 - 6 + p = 0 \ \therefore p = 7$

Similarly $(x + 1)$ is a repeated factor of $f(x)$, then $f(-1) = 0$

$\Rightarrow 3 - 2 - 6 + 6 + p = 0 \ \therefore p = -1$

Also $(2x + 1)$ is a repeated factor of $f(x)$, then $f\left(-\frac{1}{2}\right) = 0$

$\Rightarrow 3\left(-\frac{1}{2}\right)^4 + 2\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right)^2 - 6\left(-\frac{1}{2}\right) + p = 0 \ \therefore p = -\frac{23}{16}$

3. *If the equation $2x^3 - 9x^2 + 12x + p = 0$ has two equal roots, find the possible values of p.*

Solution

Let $f(x) = 2x^3 - 9x^2 + 12x + p = 0$

Any linear factor of $f'(x)$ is a possible repeated factor of $f(x)$

$\Rightarrow f'(x) \equiv 6x^2 - 18x + 12$

$\equiv 6(x^2 - 3x + 2)$

$\equiv 6(x - 2)(x - 1)$

If $(x - 2)$ is a repeated factor of $f(x)$, then $f(2) = 0$

$\Rightarrow 16 - 36 + 24 + p = 0 \ \therefore p = -4$

Similarly $(x - 1)$ is a repeated factor of $f(x)$, then $f(1) = 0$

$\Rightarrow 2 - 9 + 12 + p = 0 \ \therefore p = -5$

$\therefore p = -4, -5$

Task

1. *Find the possible values of a for which the equation $2x^3 - 3x^2 - 12x + a = 0$ has repeated roots.* **ANSWER**: $-7, 20$
2. *Find the value of a for which the function $2x^3 - ax^2 - 12x - 7$ has a repeated factor.* **ANSWER**: 3

MISCELLANIOUS EQUATIONS

Special quadratic equations

Some equations do not appear quadratic but they can be transformed in to quadratic equations using a *suitable substitution.*

TYPE I: *Equation of the form* $\frac{u}{c}+\frac{a}{u}=b$ where a and b are constants and u is a suitable substitution.

Example

Solve the equations:

a) $x^2+\frac{9}{x^2}=10$

b) $x^{4/3}+16x^{-4/3}=17$

c) $x^2+2x=34+\frac{35}{x^2+2x}$

d) $y(y+1)+\frac{12}{y(y+1)}=8$

e) $9x^{2/3}+16x^{-2/3}=37$

f) $x^2+2x+\frac{12}{x^2+2x}=7$

Solution

a) $x^2+\frac{9}{x^2}=10$, let $x^2=u$

$\Rightarrow u+\frac{9}{u}=10$ *(Multiplying both sides by u and re-arranging)*

$\Rightarrow \mathrm{u}^2-10\mathrm{u}+9=0$ *or* $(u-9)(u-1)=0$

$\therefore u=9\,,1$

But $x^2=u\Rightarrow x^2=9\,,x^2=1$

$\therefore x=\pm1\,,\pm3$

Note:

The values of x are *four* because the equation is of degree $4.i.e.4^0$

b) $x^{4/3}+16x^{-4/3}=17$ *or* $x^{4/3}+\frac{16}{x^{4/3}}=17$, let $x^{4/3}=u$

$\Rightarrow u+\frac{16}{u}=17$ *or* $\mathrm{u}^2-17\mathrm{u}+16=0$

$\Rightarrow (u-16)(u-1)=0\ \therefore u=16\,,1$

But $x^{4/3}=u\Rightarrow x^{4/3}=16\,,x^{4/3}=1$

Multiplying the powers by $\frac{3}{4}$

$x=16^{3/4}=8\,;x=(1)^{3/4}=1$

$\therefore x=1\,,8$

c) $x^2+2x=34+\frac{35}{x^2+2x}$, let $x^2+2x=u$

$\Rightarrow u=34+\frac{35}{u}$ *or* $\mathrm{u}^2-34\mathrm{u}-35=0$

$\Rightarrow (u-35)(u+1)=0\ \therefore u=35\,,-1$

But $x^2+2x=u$,

$\Rightarrow x^2+2x=35$ *or* $x^2+2x-35=0$

$\Rightarrow (x+7)(x-5)=0\ \therefore x=5\,,-7$

Also $x^2+2x=-1$ *or* $x^2+2x+1=0$

$\Rightarrow (x+1)^2=0\ \therefore x=-1\,,-1$

$\therefore x=5\,,-1\,,-1\,,-7$

d) $y(y+1)+\frac{12}{y(y+1)}=8$, let $y(y+1)=u$

$\Rightarrow u+\frac{12}{u}=8$ *or* $\mathrm{u}^2-8\mathrm{u}+12=0$

$\Rightarrow (u-6)(u-2)=0\ \therefore u=6\,,2$

$\Rightarrow y(y+1)=2 \ or\ y^2+y-2=0$
$\Rightarrow (y+2)(y-1)=0 \ \therefore y=1,-2$
Also $y(y+1)=6 \ or\ y^2+y-6=0$
$\Rightarrow (y+3)(y-2)=0 \ \therefore y=2,-3$
$\therefore y=1,\pm 2,-3$

e) $9x^{2/3}+16x^{-2/3}=37 \ \ or\ 9x^{2/3}+\frac{16}{x^{2/3}}=37$, let $x^{2/3}=u$
$\Rightarrow 9u+\frac{16}{4}=37 \ \ or\ 9\mathrm{u}^2-37\mathrm{u}+4=0$
$\Rightarrow 9\mathrm{u}^2-36\mathrm{u}-\mathrm{u}+4=0$
$\Rightarrow 9u(u-4)-(u-4)=0$
$\Rightarrow (9u-1)(u-4)=0 \ \therefore u=\frac{1}{9},4$
Now $x^{2/3}=u$
$\Rightarrow x^{2/3}=\frac{1}{9} \ \ or\ x=\left(\frac{1}{9}\right)^{3/2}=\frac{1}{27}$
$\Rightarrow x^{2/3}=4 \ \ or\ x=(4)^{3/2}=8$
$\therefore x=8,\frac{1}{27}$

f) Left as an exercise. **ANSWER**: $\pm 1\ -3,\pm\sqrt{5}$

Task

Solve the equations:

a) $\sqrt{3^x}+\frac{3}{\sqrt{3^x}}=4$ ***Answer***: $0,2$

b) $(x^2-2x)^2+24=11(x^2-2x)$ **ANSWER**: $-2,-1,3,4$

Note: the equation can be expressed in the form $x^2-2x+\frac{24}{x^2-2x}=11$

c) $\frac{x^2+4x}{3}+\frac{84}{x^2+4x}=11$ **ANSWER**: $-7,-6,2,3$

d) $(3x^2+2x)^2+8=9(3x^2+2x)$ **ANSWER**: $-2,-1,\frac{1}{3},\frac{4}{3}$

TYPE II: *Indicial equations reducible to quadratic equation .i.e.* $\boldsymbol{ax^2+bx+c=0}$

Example

1. *Solve the equations:*

i) $3^{2x}+9=10.3^x$
ii) $4^x-3.2^{x+3}=-128$
iii) $5(5^x+5^{-x})=26$
iv) $5^x+5^{2-x}=26$
v) $2-5e^{-x}+2e^{-2x}=0$
vi) $3^{2(x+1)}-10\times 3^x+1=0$

Solution

i) $3^{2x}+9=10.3^x$
$\Rightarrow (3^x)^2+9=10\times 3^x$, let $3^x=u$
$\Rightarrow u^2+9=10u \ or\ u^2-10u+9=0$
$\Rightarrow (u-9)(u-1)=0 \ \therefore u=9,1$
But $3^x=u$
$\Rightarrow 3^x=9=3^2 \ \therefore x=2$
$\Rightarrow 3^x=1=3^0 \ \therefore x=0$
$\therefore x=2,0$

ii) $4^x - 3.2^{x+3} = -128$

$\Rightarrow (2^x)^2 - 3 \times 2^x \times 2^3 = -128$ *or* $(2^x)^2 - 24 \times 2^x = -128$, let $2^x = u$

$\Rightarrow u^2 - 24u = -128$ *or* $u^2 - 24u + 128 = 0$

$\Rightarrow u = \frac{24 \pm \sqrt{24^2 - 4 \times 128}}{2} = \frac{24 \pm 8}{2}$

$\therefore u = 16\,, 8$

But $2^x = u$

$\Rightarrow 2^x = 16 = 2^4 \ \therefore x = 4$

$\Rightarrow 2^x = 8 = 2^3 \ \therefore x = 3$

$\therefore x = 3\,, 4$

iii) $5(5^x + 5^{-x}) = 26$ *or* $5\left(5^x + \frac{1}{5^x}\right) = 26$, let $5^x = u$

$\Rightarrow 5\left(u + \frac{1}{u}\right) = 26$ *or* $5u^2 - 26u + 5 = 0$

$\Rightarrow 5u^2 - 25u - u + 5 = 0$

$\Rightarrow (5u - 1)(u - 5) = 0 \ \therefore u = 5\,, \frac{1}{5}$

Now $5^x = u$

$\Rightarrow 5^x = 5 \ \therefore x = 1$

$\Rightarrow 5^x = \frac{1}{5} = 5^{-1} \ \therefore x = -1$

$\therefore x = \pm 1$

v) $2 - 5e^{-x} + 2e^{-2x} = 0$ *or* $2 - 5e^{-x} + 2(e^{-x})^2 = 0$, let $e^{-x} = u$

$\Rightarrow 2 - 5u + 2u^2 = 0$

$\Rightarrow (2u - 1)(u - 2) = 0 \ \therefore u = 2\,, \frac{1}{2}$

Now $e^{-x} = u$

$\Rightarrow e^{-x} = 2$, introducing logarithm to base e

$\Rightarrow \ln e^{-x} = \ln 2 \ \therefore x = -\ln 2$

Also $e^{-x} = \frac{1}{2}$

$\Rightarrow \ln e^{-x} = \ln\left(\frac{1}{2}\right) \ \therefore x = +\ln 2$

$\therefore x = \pm \ln 2$

vi) $3^{2(x+1)} - 10 \times 3^x + 1 = 0$ *or* $3^{2x}.3^2 - 10 \times 3^x + 1 = 0$, let $3^x = u$

$\Rightarrow 9u^2 - 10u + 1 = 0$

$\Rightarrow (9u - 1)(u - 1) = 0 \ \therefore u = 1\,, \frac{1}{9}$

But $3^x = u$

$\Rightarrow 3^x = 1 = 3^0 \ \therefore x = 0$

$\Rightarrow 3^x = \frac{1}{9} = 3^{-2} \ \therefore x = -2$

$\therefore x = 0\,, -2$

Task

Solve: i) $4^x - 5.2^x + 4 = 0$ **ANSWER**: $0\,, 2$

ii) $3^{2x+1} - 3^{x+1} - 3^x + 1 = 0$ **ANSWER**: $-1\,, 0$

iii) $5^{2x} - 5^{x+1} + 4 = 0 \quad 0$ **ANSWER**: $0\,, 0.861$

iv) $2^{2x+8} - 32(2^x) + 1 = 0$ **ANSWER**: -4

v) $2^{2x+3} - 57 = 65(2^x - 1) = 0$ **ANSWER**: ± 3

2. *Solve the equation*

a) $t - 1.324\sqrt{t} - 2.896 = 0$

b) $x + \sqrt{x} = \frac{6}{25}$

Solution

a) $t - 1.324\sqrt{t} - 2.896 = 0$, let $\sqrt{t} = u \Rightarrow u^2 = t$

$\therefore u^2 - 1.324u - 2.896 = 0$

$\therefore u = \frac{1.324 \pm \sqrt{(-1.324)^2 + 4 \times 2.896}}{2} = \frac{1.324 \pm 3.652}{2}$

$\therefore u = 2.488 , -1.164$

$\Rightarrow t = (2.488)^2 = 6.19 ; (-1.164)^2 = 1.35$

$\therefore t = 6.19 , 1.35$

b) $x + \sqrt{x} = \frac{6}{25}$, let $\sqrt{x} = u \Rightarrow x = u^2$

$\Rightarrow u^2 + u = \frac{6}{25}$ *or* $25u^2 + 25u - 6 = 0$

$\Rightarrow u = \frac{-25 \pm \sqrt{25^2 + 4 \times 6 \times 25}}{2 \times 25} = \frac{-25 \pm 5}{50}$

$\therefore u = -\frac{2}{5} , -\frac{3}{5}$

$\therefore x = \frac{4}{25} , \frac{9}{25}$

Task

Solve the equations:

a) $6\sqrt{x} = 5x^{-\frac{1}{2}} - 13$; **ANSWER**: $\frac{1}{9} , \frac{25}{4}$

b) $\sqrt{2^x} + \frac{1}{\sqrt{2^x}} = 2$; **ANSWER**: 0

TYPE III: *Equation of the form* $\boldsymbol{Ax^4 + Bx^2 + C = 0}$**.** *This equation contains only* $3 -$ *terms*

Solve the equation:

i) $x^4 - 13x^2 + 36 = 0$

ii) $2x^4 - x^2 = 6$

Solution

i) $x^4 - 13x^2 + 36 = 0$, let $x^2 = u$

$\Rightarrow u^2 - 13u + 36 = 0$ *or* $(u - 9)(u - 4) = 0$

$\therefore u = 9 , 4$

But $x^2 = u$

$\Rightarrow x^2 = 9 \ \therefore x = \pm 3$

$\Rightarrow x^2 = 4 \ \therefore x = \pm 2$

$\therefore x = \pm 2 , \pm 3$

ii) $2x^4 - x^2 = 6,$ let $x^2 = u$

$\Rightarrow 2u^2 - u - 6 = 0$ *or* $(2u + 3)(u - 2) = 0$

$\therefore u = -\frac{3}{2} , 2$

But $x^2 = u$

$\Rightarrow x^2 = 2 \ \therefore x = \sqrt{2}$

$\Rightarrow x^2 = -\frac{3}{2} \ \therefore x = \sqrt{\left(-\frac{3}{2}\right)}$, no real values of x

$\therefore x = \pm\sqrt{2}$

TYPE IV: *Equation of the form* $(x+a)(x+b)(x+c)(x+d)=n$*, where* $a+b=c+d$ *and* a,b,c,d *and* n *are constants*

Example:

Solve the equations:

i) $(x+1)(x+3)(x+4)(x+6)=72$

ii) $(x-7)(x-3)(x+5)(x+1)=1680$

iii) $(x-5)(x-7)(x+6)(x+4)=504$

Solution

i) $(x+1)(x+3)(x+4)(x+6)=72$

Since $1+6=3+4=7$, then re-grouping

$\Rightarrow (x+1)(x+6).(x+3)(x+4)=72$

$\Rightarrow (x^2+7x+6).(x^2+7x+12)=72$

Now let $x^2+7x=u$

$\Rightarrow (u+6)(u+12)=72$

$\Rightarrow u^2+18u+72=72 \quad or\ u(u+18)=0$

$\therefore u=0,-18$

But $x^2+7x=u$

$\Rightarrow x^2+7x=0 \ \therefore x=0,-7$

$\Rightarrow x^2+7x=-18 \ or \ x^2+7x+18=0$

$\therefore x=\frac{-7\pm\sqrt{7^2-4\times18}}{2}=\frac{-7\pm\sqrt{-23}}{2}=\frac{-7\pm i\sqrt{23}}{2}$

$\therefore x=0,-7,\frac{-7\pm i\sqrt{23}}{2}$

ii) $(x-7)(x-3)(x+5)(x+1)=1680$

Since $-7+5=-3+1=-2$, then re-grouping

$\Rightarrow (x-7)(x+5).(x-3)(x+1)=1680$

$\Rightarrow (x^2-2x-35).(x^2-2x-3)=1680$

Now let $x^2-2x=u$

$\Rightarrow (u-35)(u-3)=1680$

$\Rightarrow u^2-38u+105=1680 \quad or\ u^2-38u-1575=0$

$\Rightarrow u=\frac{38\pm\sqrt{38^2+4\times1575}}{2}=\frac{38\pm88}{2}$

$\therefore u=63,-25$

But $x^2-2x=u$

$\Rightarrow x^2-2x=63 \ \ or \ x^2-2x-63=0$

$\Rightarrow (x-9)(x+7)=0$

$\therefore x=9,-7$

$\Rightarrow x^2-2x=-25 \ or \ x^2-2x+25=0$

$\therefore x=\frac{2\pm\sqrt{(-2)^2-4\times25}}{2}=\frac{2\pm\sqrt{-96}}{2}=\frac{2\pm i\sqrt{96}}{2}=\frac{2\pm i4\sqrt{6}}{2}$

$\therefore x=9,-7,1\pm i2\sqrt{6}$

iii) Left as an exercise. **ANSWER**: $3,-2,8,-7$

Task

Solve the equations:

i) $x(2x+1)(x-2)(2x-3)=63$ **ANSWER**: $3,-\frac{3}{2},\frac{3\pm\sqrt{-47}}{4}$

ii) $16x(x+1)(x+2)(x+3)=9$ **ANSWER**: $-\frac{3}{2},-\frac{3}{2},\frac{3\pm\sqrt{10}}{2}$

TYPE IV: e*quation of the form* $\boldsymbol{ax^4 \pm bx^3 \pm cx^2 \pm bx + a = 0}$*, where coefficients of terms equidistant from the beginning and end are equal, can be made to depend on the solution of the quadratic.*

Equations of this form are known as *Reciprocal Equations*, and so are named because they are not altered when x is changed in to its $\frac{1}{x}$.

Example:

Solve the equations:

a) $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$
b) $x^4 + 2x^3 - x^2 + 2x + 1 = 0$
c) $4x^4 + 17x^3 + 8x^2 + 17x + 4 = 0$
d) $x^4 + x^3 - 4x^2 + x + 1 = 0$
e) $x^4 + 1 - 3(x^3 + x) = 2x^2$

Solution

a) $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$

Dividing through by x^2

$\Rightarrow 12x^2 - 56x + 89 - \frac{56}{x} + \frac{12}{x^2} = 0$

$\Rightarrow 12\left(x^2 + \frac{1}{x^2}\right) - 56\left(x + \frac{1}{x}\right) + 89 = 0$

Now let $\left(x + \frac{1}{x}\right) = u$

$\Rightarrow u^2 = x^2 + \frac{1}{x^2} + 2 \;\; or \;\; x^2 + \frac{1}{x^2} = u^2 - 2$

The substitutions that solve reciprocal equations are;

$$\boxed{\boldsymbol{x + \frac{1}{x} = u \;\; , \;\; x^2 + \frac{1}{x^2} = u^2 - 2}}$$

$\Rightarrow 12(u^2 - 2) - 56u + 89 = 0$

$\Rightarrow 12u^2 - 56u + 65 = 0$

$\Rightarrow u = \frac{56 \pm \sqrt{(56)^2 - 4 \times 12 \times 65}}{2 \times 12} = \frac{56 \pm 4}{48}$

$\therefore u = \frac{5}{2}, \frac{13}{6}$ (**Note:** left as fractions)

But $\left(x + \frac{1}{x}\right) = u$

$\Rightarrow x + \frac{1}{x} = \frac{13}{6} \;\; or \;\; 6x^2 - 13x + 6 = 0$

$\Rightarrow (3x - 2)(2x - 3) = 0 \;\; \therefore x = \frac{3}{2}, \frac{2}{3}$

Also $x + \frac{1}{x} = \frac{5}{2} \;\; or \;\; 2x^2 - 5x + 2 = 0$

$\Rightarrow (2x - 1)(x - 2) = 0 \;\; \therefore x = 2, \frac{1}{2}$

$\therefore x = 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}$

b) $x^4 + 2x^3 - x^2 + 2x + 1 = 0$

Dividing through by x^2

$\Rightarrow x^2 + 2x - 1 + \frac{2}{x} + \frac{1}{x^2} = 0$

$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) + 2\left(x + \frac{1}{x}\right) - 1 = 0$

Now let $x + \frac{1}{x} = u, x^2 + \frac{1}{x^2} = u^2 - 2$ (as derived as before)

$\Rightarrow u^2 - 2 + 2u - 1 = 0$

$\Rightarrow u^2 + 2u - 3 = 0$

$\Rightarrow (u+3)(u-1)=0 \quad \therefore u=-1,-3$

But $x+\frac{1}{x}=u$

$\Rightarrow x+\frac{1}{x}=-1 \ or \ x^2+x+1=0$

$\Rightarrow x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}$

Also $x+\frac{1}{x}=-3 \ or \ x^2+3x+1=0$

$\Rightarrow x=\frac{-3\pm\sqrt{9-4}}{2}=\frac{-1\pm\sqrt{5}}{2}$

$\therefore x=\frac{-1\pm i\sqrt{3}}{2},\frac{-1\pm\sqrt{5}}{2}$

c) Left as an exercise. **ANSWER**: $\frac{1}{3},\frac{1}{2},2,3$

d) Left as an exercise. **ANSWER**: $-\frac{1}{4},-4$

e) Left as an exercise. **ANSWER**: $1,\frac{-3\pm\sqrt{5}}{2}$

f) $x^4+1-3(x^3+x)=2x^2$

Re arranging the equation,

$\Rightarrow x^4-3x^3-2x^2-3x+1=0$

Dividing through by x^2

$\Rightarrow x^2-3x-2-\frac{3}{x}+\frac{1}{x^2}=0$

$\Rightarrow \left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-2=0$

Now let $x+\frac{1}{x}=u, x^2+\frac{1}{x^2}=u^2-2$ (as derived as before)

$\Rightarrow u^2-2-3u-2=0$

$\Rightarrow u^2-3u-4=0$

$\Rightarrow (u+1)(u-4)=0 \quad \therefore u=-1,4$

But $x+\frac{1}{x}=u$

$\Rightarrow x+\frac{1}{x}=-1 \ or \ x^2+x+1=0$

$\Rightarrow x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}$

Also $x+\frac{1}{x}=4 \ or \ x^2-4x+1=0$

$\Rightarrow x=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm\sqrt{12}}{2}=\frac{4\pm 2\sqrt{3}}{2}=2\pm\sqrt{3}$

$\therefore x=\frac{-1\pm i\sqrt{3}}{2},2\pm\sqrt{3}$

Solving equations which are not reciprocal but solved in a similar manner as reciprocal

Example

Solve the equation:

a) $6x^4-25x^3+12x^2+25x+6=0$

b) $x^2+\frac{9}{x^2}-4\left(x-\frac{3}{x}\right)-2=0$

Solution

a) $6x^4-25x^3+12x^2+25x+6=0$

Dividing through by x^2

$\Rightarrow 6x^2-25x+12+\frac{25}{x}+\frac{6}{x^2}=0$

$\Rightarrow 6\left(x^2+\frac{1}{x^2}\right)-25\left(x-\frac{1}{x}\right)+12=0$

Now let $x-\frac{1}{x}=u$

$u^2=x^2+\frac{1}{x^2}-2 \Rightarrow u^2+2=x^2+\frac{1}{x^2} \Rightarrow u^2-2-3u-2=0$

$\Rightarrow 6(u^2+2)-25u+12=0 \;\; or \;\; 6u^2-25u+24=0$

$\Rightarrow u=\frac{25\pm\sqrt{25^2-4\times6\times24}}{2\times6}=\frac{25\pm7}{12} \;\; \therefore u=\frac{32}{12}=\frac{8}{3}\;,\frac{18}{12}=\frac{3}{2}$

Now $x-\frac{1}{x}=\frac{8}{3} \;\; or \;\; 3x^2-8x-3=0$

$\Rightarrow 3x^2-9x+x-3=0 \;\; or \; 3x(x-3)+1(x-3)=0$

$\Rightarrow (3x+1)(x-3)=0 \;\; \therefore x=-\frac{1}{3}\,,3$

Also $x-\frac{1}{x}=\frac{3}{2} \;\; or \;\; 2x^2-3x-2=0$

$\Rightarrow 2x^2-4x+x-2=0 \;\; or \; 2x(x-2)+1(x-2)=0$

$\Rightarrow (2x+1)(x-2)=0 \;\; \therefore x=-\frac{1}{2}\,,2$

$\therefore x=-\frac{1}{3}\,,3\,,2\,,-\frac{1}{2}$

b) $x^2+\frac{9}{x^2}-4\left(x-\frac{3}{x}\right)-2=0$

Let $x-\frac{3}{x}=u\,, u^2=x^2+\frac{9}{x^2}-6 \Rightarrow x^2+\frac{9}{x^2}=u^2+6$

$\Rightarrow u^2+6-4u-2=0 \; or \; u^2-4u+4=0$

$\Rightarrow (u-2)^2=0 \;\; \therefore u=2$

$\Rightarrow x-\frac{3}{x}=2 \;\; or \;\; x^2-2x-3=0$

$\Rightarrow (x+1)(x-3)=0 \;\; \therefore x=-1\,,3$

Task

Solve the equation: $x^2+\frac{9}{x^2}-4\left(x-\frac{3}{x}\right)-6=0$

EQUATIONS INVOLVING SQUARE ROOTS

Steps followed:

1. Isolate one square root on either L.H.S or R.H.S of the equation and square both sides
2. If there is another remaining square root that persists, isolate it again and square both sides
3. Continue to square both sides until there is no more square root, then solve the resulting equation.

NOTE:

These equations yield *extraneous* roots which do not satisfy the equation being solved. Thus for correct roots, test the roots in the original equation, that which satisfy is the correct root.

TYPE A: $\boldsymbol{P\sqrt{bx+c}+Q\sqrt{mx+n}+R\sqrt{\alpha x+\beta}=k}$**,** where $P\,, Q\,, R$ are constants.

Case I: For $R=0\,.i.e.$ $\boldsymbol{P\sqrt{bx+c}+Q\sqrt{mx+n}=k}$

Example:

Solve the equations:

a) $\sqrt{x+1}+\sqrt{x-2}=3$

b) $\sqrt{3x-3}-\sqrt{x}=1$

c) $\sqrt{x-5}+\sqrt{x}=5$

d) $\sqrt{x-1}+2\sqrt{x-4}=4$

Solution

a) $\sqrt{x+1}+\sqrt{x-2}=3$; let $A=x+1\,,B=x-2$

$\Rightarrow \sqrt{A}+\sqrt{B}=3$

Isolating $\sqrt{A}$ and squaring both sides

$\Rightarrow \left(\sqrt{A}\right)^2=\left(3-\sqrt{B}\right)^2$

$\Rightarrow A=9-6\sqrt{B}+B$

Isolating $\sqrt{B}$ and squaring both sides

$\Rightarrow 6\sqrt{B}=9+B-A$

$\Rightarrow 6\sqrt{x-2}=9+x-2-x-1$

$\Rightarrow 6\sqrt{x-2}=6 \;\; or \; \sqrt{x-2}=1$

$\therefore \left(\sqrt{x-2}\right)^2=1\Rightarrow x-2=1$

$\therefore x=3$

Testing $x=3$ in to the original equation,

$\sqrt{3+1}+\sqrt{3-2}=3=R.H.S$

$\therefore x=3$ is a root.

b) $\sqrt{3x-3}-\sqrt{x}=1$; let $A=3x+3$

$\Rightarrow \sqrt{A}+\sqrt{x}=1$

Isolating $\sqrt{A}$ and squaring both sides

$\Rightarrow \left(\sqrt{A}\right)^2=\left(1+\sqrt{x}\right)^2$

$\Rightarrow A=1+2\sqrt{x}+x$

$\Rightarrow 3x-3=1+2\sqrt{x}+x$

Now $2\sqrt{x}=2x-4 \;\; or \; \sqrt{x}=x-2$

Squaring both sides again,

$\Rightarrow x=(x-2)^2=x^2-4x+4$

$\Rightarrow x^2-5x+4=0 \;\; or \;\; (x-1)(x-4)=0$

$\Rightarrow x=1\,,4$

Testing the values,

For $x=1\,,\sqrt{3-3}-\sqrt{1}=0\neq 1$

Hence $x=1$ is not a correct root. $i.e.$ it is an extraneous root.

For $x=4\,,\sqrt{12-3}-\sqrt{4}=1=\text{R.H.S.}$

Only $x=4$ is the correct root.

c) $\sqrt{x-5}+\sqrt{x}=5$, let $B=x-5$

$\Rightarrow \sqrt{B}+\sqrt{x}=5$

Isolating $\sqrt{B}$ and squaring both sides

$\Rightarrow \left(\sqrt{B}\right)^2=\left(5-\sqrt{x}\right)^2$

$\Rightarrow B=25-10\sqrt{x}+x$

$\Rightarrow x-5=25-10\sqrt{x}+x \;\; or \;\; 10\sqrt{x}=30$

$\Rightarrow \sqrt{x}=3$, squaring both sides,

$\Rightarrow x=9$

Testing $x=9,\sqrt{9-4}+\sqrt{9}=5=\text{R.H.S}$

$\therefore x=9$ is the correct root.

d) $\sqrt{x-1}+2\sqrt{x-4}=4$, let $x-1=A\,,x-4=B$

$\Rightarrow \sqrt{A}+2\sqrt{B}=4$

Isolating $2\sqrt{B}$ and squaring both sides,

$\Rightarrow \left(2\sqrt{B}\right)^2 = \left(4-\sqrt{A}\right)^2$

$\Rightarrow 4B = 16 - 8\sqrt{A} + A$

Isolating $8\sqrt{A}$ and squaring both sides,

$\Rightarrow \left(8\sqrt{A}\right)^2 = (A - 4B + 16)^2$

$\Rightarrow 64A = (x - 1 - 4(x-4) + 16)^2$

$\Rightarrow 64(x-1) = (x - 1 - 4x + 16 + 16)^2 = (-3x + 31)^2$

$\Rightarrow 64x - 64 = 9x^2 - 186x + 961 \quad or \quad 9x^2 - 250x + 1025 = 0$

$\Rightarrow x = \frac{250 \pm \sqrt{(-250)^2 - 4\times 9 \times 1025}}{18} = \frac{250 \pm 160}{18}$

$\Rightarrow x = \frac{205}{9}, 5$

Testing the values,

For $x = 5$, $\sqrt{5-1} + 2\sqrt{5-4} = 4 = \text{R.H.S}$

Hence $x = 5$ is root.

For $x = \frac{205}{9}$, $\sqrt{\left(\frac{205}{9} - 1\right)} + 2\sqrt{\left(\frac{205}{9} - 4\right)} = \frac{40}{3} \neq \text{R.H.S.}$

Only $x = 5$ is the correct root.

Task

Solve the equations:

a) $2\sqrt{2x-12} - \sqrt{2x-3} = 3$ **ANSWER**: $14\ not\ 6$

b) $2\sqrt{x+4} - \sqrt{x-1} = 4$ **ANSWER**: $5\ not\ 1\frac{4}{9}$

c) $\sqrt{x-5} + 2 = \sqrt{x+7}$ **ANSWER**: 9

d) $\sqrt{3x+4} - \sqrt{x-3} = 4$ **ANSWER**: 4 , 7

Case II: For $R \neq 0$ *.i.e.* $\boldsymbol{P\sqrt{bx+c} + Q\sqrt{mx+n} + R\sqrt{\alpha x + \beta}} = \boldsymbol{k}$

Example:

Solve the equations:

a) $2\sqrt{x+1} - 3\sqrt{2x-5} = \sqrt{x-2}$

b) $\sqrt{3-x} - \sqrt{7+x} = \sqrt{16+2x}$

c) $\sqrt{x+6} - \sqrt{x+3} = \sqrt{2x+5}$

Solution

a) $2\sqrt{x+1} - 3\sqrt{2x-5} = \sqrt{x-2}$

Let $x + 1 = A$, $2x - 5 = B$, $x - 2 = C$

$\Rightarrow 2\sqrt{A} - 3\sqrt{B} = \sqrt{C}$, squaring both sides,

$\Rightarrow \left(2\sqrt{A} - 3\sqrt{B}\right)^2 = C$

$\Rightarrow 4A + 9B - 12\sqrt{AB} = C$

Isolating $12\sqrt{AB}$ and squaring both sides,

$\Rightarrow \left(12\sqrt{AB}\right)^2 = (4A + 9B - C)^2$

$\Rightarrow 144AB = (4A + 9B - C)^2$, substituting for A, B and C

$\Rightarrow 144(x+1)(2x-5) = \left(4(x+1) + 9(2x-5) - (x-2)\right)^2$

$\Rightarrow 144(2x^2 - 3x - 5) = \left(4x + 4 + 18x - 45 — x + 2\right)^2$

$\Rightarrow 288x^2 - 432x - 720 = (21x - 39)^2 = 441x^2 - 1638x + 1521$

$\Rightarrow 153x^2 - 1206x + 2241 = 0 \quad or \quad 17x^2 - 134x + 249 = 0$

$\Rightarrow x = \frac{134 \pm \sqrt{(134)^2 - 4\times17\times249}}{34} = \frac{134 \pm 32}{34}$

$\Rightarrow x = 3, \frac{83}{17}$

Testing the values,

For $x = 3$, $2\sqrt{3+1} - 3\sqrt{6-5} = \sqrt{3-2} = 1$, hence $x = 3$ is a root

For $x = \frac{83}{17}$, $2\sqrt{\frac{83}{17}+1} - 3\sqrt{\frac{166}{17} - 5} \neq \sqrt{\frac{83}{17} - 2} = 1$, hence $x = \frac{83}{17}$ is not a root

$\therefore x = 3$ is the only correct root.

b) $\sqrt{3-x} - \sqrt{7+x} = \sqrt{16+2x}$

Let $3 - x = A, 7 + x = B, 16 + 2x = C$

$\Rightarrow \sqrt{A} - \sqrt{B} = \sqrt{C}$, squaring both sides,

$\Rightarrow \left(\sqrt{A} - \sqrt{B}\right)^2 = C$

$\Rightarrow A + B - 2\sqrt{AB} = C$

Isolating $2\sqrt{AB}$ and squaring both sides,

$= (A + B - C)^2 = 4AB$, substituting for A, B and C

$\Rightarrow 4(3-x)(7+x) = \left(3 - x + 7 + x - (16 + 2x)\right)^2$

$\Rightarrow 4(21 - 4x - x^2) = (-6 - 2x)^2$

$\Rightarrow 84 - 16x - 4x^2 = 36 + 24x + 4x^2$

$\Rightarrow 8x^2 + 40x - 48 = 0 \quad or\ x^2 + 5x - 6 = 0$

$\Rightarrow (x+6)(x-1) = 0 \Rightarrow x = 1, -6$

Testing the values,

For $x = 1$, $\sqrt{3-1} - \sqrt{7+1} \neq \sqrt{16+2}$, hence $x = 1$ is not a root.

For $x = -6$, $\sqrt{3+6} - \sqrt{7-6} = \sqrt{16-12}$, hence $x = 2$ is a root.

$\therefore x = 2$ is the correct root.

c) Left as an exercise. **ANSWER**: $x = -2$

TYPE B: Any equation which can be thrown in to the form

$\boldsymbol{ax^2 + bx + c + p\sqrt{ax^2 + bx + c} = q}$

This is solved by putting $y = \sqrt{ax^2 + bx + c}$

$\Rightarrow y^2 = ax^2 + bx + c$

$\therefore$ the equation being solved is $y^2 + py - q = 0$

Let α and β be roots of this equation.

$\Rightarrow \alpha = \sqrt{ax^2 + bx + c}$, $\beta = \sqrt{ax^2 + bx + c}$

From these equations we obtain *four* values of x.

NOTE:

If *no sign* is prefixed to *a radical* it is usually understood that it is to be taken as a *positive*. Hence if both α and β are both *positive*, all the four values of x satisfy the original equation.

If however, α or β is *negative*, the roots found from the resulting quadratic equation will satisfy the equation $\boldsymbol{ax^2 + bx + c - p\sqrt{ax^2 + bx + c} = q}$ but not the original equation.

Example

Solve the equations:

a) $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$

b) $x^2 + 2\sqrt{x^2 + 6x} = 24 - 6x$

c) $x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$

d) $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$

e) $3x^2 - 18 + \sqrt{3x^2 - 4x + 6} = 4x$

Solution

a) $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$

Adding 3 on both sides

$\Rightarrow x^2 - 5x + 3 + 2\sqrt{x^2 - 5x + 3} = 15$

Now let $y = \sqrt{x^2 - 5x + 3} \Rightarrow y^2 = x^2 - 5x + 3$

$\Rightarrow y^2 + 2y = 15 \;\; or\; y^2 + 2y - 15 = 0$

$\Rightarrow (y + 5)(y - 3) = 0 \therefore y = -5\,, 3$

But $y = \sqrt{x^2 - 5x + 3}$

$\Rightarrow \sqrt{x^2 - 5x + 3} = -5$ ..... (1)

$\Rightarrow \sqrt{x^2 - 5x + 3} = 3$ ..... (2)

From eqn. (1), squaring both sides,

$\Rightarrow x^2 - 5x + 3 = 25 \;\; or\; x^2 - 5x - 22 = 0$

$\therefore x = \frac{5 \pm \sqrt{25 + 88}}{2} = \frac{5 \pm \sqrt{113}}{2}$

From eqn. (2), squaring both sides,

$\Rightarrow x^2 - 5x + 3 = 9 \;\; or\; x^2 - 5x - 6 = 0$

$\Rightarrow (x - 6)(x + 1) = 0 \;\; \therefore x = -1\,, 6$

From the discussion above,

$x = -1\,, 6$ satisfy the equation $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12$, because the radical $2\sqrt{x^2 - 5x + 3}$ is *positive* and $y = +3$

But for $x = \frac{5 \pm \sqrt{113}}{2}$ satisfy the equation $x^2 - 5x - 2\sqrt{x^2 - 5x + 3} = 12$, because the radical $-2\sqrt{x^2 - 5x + 3}$ is *negative* and $y - 5$

b) $x^2 + 2\sqrt{x^2 + 6x} = 24 - 6x$, re arranging

$\Rightarrow x^2 + 6x + 2\sqrt{x^2 + 6x} = 24$

Let $y = \sqrt{x^2 + 6x} \Rightarrow y^2 = x^2 + 6x$

$\Rightarrow y^2 + 2y = 24 \;\; or\; y^2 + 2y - 24 = 0$

$\Rightarrow (y + 6)(y - 4) = 0 \therefore y = -6\,, 4$

But $y = \sqrt{x^2 - 5x + 3}$

$\Rightarrow \sqrt{x^2 - 5x + 3} = 4$ ..... (1)

$\Rightarrow \sqrt{x^2 - 5x + 3} = -6$ ..... (2)

From eqn. (1), squaring both sides,

$\Rightarrow x^2 + 6x = 16 \;\; or\; x^2 + 6x - 16 = 0$

$\Rightarrow (x + 8)(x - 2) = 0 \;\; \therefore x = -8\,, 2$

From eqn. (2), squaring both sides,

$\Rightarrow x^2 + 6x = 36 \;\; or\; x^2 + 6x - 36 = 0$

$\therefore x = \frac{-6 \pm \sqrt{36 + 4 \times 36}}{2} = -3 \pm 3\sqrt{5}$

$\therefore$ Roots to $x^2 + 2\sqrt{x^2 + 6x} = 24 - 6x$ are $\therefore x = -8\,, 2$ for $y = +4$ and roots of

$x^2 - 2\sqrt{x^2+6x} = 24 - 6x$ are $-3 \pm 3\sqrt{5}$ for $y = -4$. The roots of the latter equation are extraneous.

c) $x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$

Subtracting 3 from both sides,

$\Rightarrow x^2 - 6x + 6 = 4\sqrt{x^2 - 6x + 6} - 3$

Now let $y = \sqrt{x^2 - 6x + 6} \Rightarrow y^2 = x^2 - 6x + 6$

$\Rightarrow y^2 = 4y - 3 \;\; or\; y^2 - 4y + 3 = 0$

$\Rightarrow (y -)(y - 3) = 0 \therefore y = 1, 3$

But $y = \sqrt{x^2 - 6x + 6}$

$\Rightarrow \sqrt{x^2 - 6x + 6} = 1$ ..... (1)

$\Rightarrow \sqrt{x^2 - 6x + 6} = 3$ ..... (2)

From eqn. (1), squaring both sides,

$\Rightarrow x^2 - 6x + 6 = 1 \;\; or\; x^2 - 6x + 5 = 0$

$\Rightarrow (x - 5)(x - 1) = 0 \;\; \therefore x = 1, 5$

From eqn. (2), squaring both sides,

$\Rightarrow x^2 - 6x + 6 = 9 \;\; or\; x^2 - 6x - 3 = 0$

$\therefore x = \frac{6 \pm \sqrt{36+4\times3}}{2} = 3 \pm 2\sqrt{3}$

$\therefore x = 1, 5, 3 \pm 2\sqrt{3}$ satisfy the original equation.

d) $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$, re arranging the equation,

$x^2 + 3x - 40 + 10\sqrt{x^2 + 3x + 16} = 0$

Adding 56 on both sides,

$\Rightarrow x^2 + 3x + 16 + 10\sqrt{x^2 + 3x + 16} = 56$

Let $y = \sqrt{x^2 + 3x + 16} \Rightarrow y^2 = x^2 + 3x + 16$

$\Rightarrow y^2 + 10y = 56 \;\; or\; y^2 + 10y - 56 = 0$

$\Rightarrow (y + 14)(y - 4) = 0 \therefore y = 4, -14$

But $y = \sqrt{x^2 + 3x + 16}$

$\Rightarrow \sqrt{x^2 + 3x + 16} = 4$ ..... (1)

$\Rightarrow \sqrt{x^2 + 3x + 16} = -14$ ..... (2)

From eqn. (1), squaring both sides,

$\Rightarrow x^2 + 3x + 16 = 16 \;\; or\; x^2 + 3x = 0$

$\Rightarrow x(x + 3) = 0 \;\; \therefore x = 0, -3$

From eqn. (2), squaring both sides,

$\Rightarrow x^2 + 3x + 16 = 196 \;\; or\; x^2 + 3x - 180 = 0$

$\therefore x = \frac{-3 \pm \sqrt{9+4\times180}}{2} = \frac{3 \pm 27}{2} \Rightarrow x = -15, 12$

Since $y = +4$ and $10\sqrt{x^2 + 3x + 16}$ is also $+ve$, then $x = 0, -3$ satisfy the equation $x^2 + x + 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$.

Since $y = -14$ and $10\sqrt{x^2 + 3x + 16}$ is $+ve$, then $x = -15, 12$ satisfy the equation $x^2 + x - 10\sqrt{x^2 + 3x + 16} = 20(20 - x)$. These are extraneous roots.

e) Left as an exercise.

ANSWER: $x = 3, -\frac{5}{3}\; and\; \frac{2 \pm \sqrt{70}}{3}$

NOTE:

- In general some equations of degree 4^0 can be converted in to a quadratic
- Before clearing an equation of a radical, it is advisable whether any common factor can be removed by division.

Example:

Solve: $2\sqrt{x^2-x-2}-3\sqrt{2x^2-9x+10}=x-2$

$\Rightarrow 2\sqrt{(x+1)(x-2)}-3\sqrt{(2x-5)(x-2)}=x-2$

Dividing through by $\sqrt{x-2}$

$\Rightarrow 2\sqrt{x+1}-3\sqrt{2x-5}=\sqrt{x-2}$, which is as case II for $R\neq 0$

Solving the equation, $x=3$.

NOTE:

When one root of a quadratic equation is obvious by inspection, the other root may often be readily obtained by making use of the properties of the roots of a quadratic equation as illustrated below.

Example:

Solve the equation:

i) $(1-a^2)(x+a)-2a(1-x^2)=0$

ii) $\frac{(1-x)^2}{2-x^2}=\frac{(1-a)^2}{2-a^2}$

Solution

i) $(1-a^2)(x+a)-2a(1-x^2)=0$

By inspection, clearly $x=a$ is a root to the equation.

Re writing the equation,

$\Rightarrow x+a-a^2x-a^3-2a+2ax^2=0$

$\therefore 2ax^2+(1-a^2)x-a(1+a^2)=0$

Let α and β be roots of this equation, then

$\Rightarrow \alpha\beta=-\frac{a(1+a^2)}{2a}=-\frac{(1+a^2)}{2}$

Since one root of the equation is $a\,.i.e.$ say $\alpha=a$, then the other root β is;

$\beta==-\frac{(1+a^2)}{2a}=-\frac{(1+a^2)}{2a}$

ii) Left as an exercise.

Task

Solve the equation:

a) $\sqrt{4x^2-7x-15}-\sqrt{x^2-3x}=\sqrt{x^2-9}$ **ANSWER**: $1\,,3$

b) $\sqrt{2x^2-9x+4}+3\sqrt{2x-1}=\sqrt{2x^2+21x-11}$ **ANSWER**: $5\,,\frac{1}{2}$

c) $\sqrt{x^2-7ax+10a^2}-\sqrt{x^2+ax-6a^2}=x-2a$ **ANSWER**: $2a\,,-\frac{10a}{3}$

SERRIES
Arithmetic and Geometric Progressions
(A. ps and G. ps)

Basic concepts:

a) A *Sequence* is a set of quantities/numbers $a_1, a_2, a_3 \ldots\ldots\ldots$, stated in a definite order and each term formed according to a fixed pattern.
$i.e. a_r = f(r)$.

Example

- $1, 3, 5, 7 \ldots\ldots\ldots\ldots$, is a sequence (the next term would be 9)
- $2, 6, 18, 54 \ldots\ldots\ldots$, is a sequence (the next term would be 162)
- $1^2, -2^2, 3^2, -4^2 \ldots\ldots\ldots$, is a sequence (the next term would be $+5^2$)

i) A *finite sequence* contains only a finite number of terms.

ii) An *infinite sequence* is unending.

NOTE:
The numbers in a sequence are called *terms.*

b) A *series* is formed by the *sum* of the terms of a sequence.

Example
For $1, 3, 5, 7 \ldots\ldots\ldots\ldots$ is a sequence
$1 + 3 + 5 + 7 \ldots\ldots\ldots\ldots$ is a series.

c) **The *Arithmetic Progression (A.P)***
A series in which each term is obtained from the proceeding one by adding or subtracting a constant quantity is called the *Arithmetic Progression.*

Example

- $1 + 3 + 5 + 7 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ *is an A.P*
- $3 - 2 - 7 - \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ *is an A.P*
- $a + (a + d) + (a + 2d) + \cdots \ldots\ldots\ldots$ *is an A.P*

The constant quantity is called *a common difference, d.*
Consider the A.P below;

$$a_1 + a_2 + a_3 + \cdots \ldots\ldots\ldots + a_{n-1} + a_n$$

This is an A.P if the common difference

$$d = a_n - a_{n-1}, \text{ for all } n > 1$$

From the above examples of series,

- $1 + 3 + 5 + 7 \ldots\ldots\ldots\ldots, d = 3 - 1 = 5 - 3 = 7 - 5 = 2$
- $3 - 2 - 7 - \cdots \ldots\ldots\ldots\ldots, d = -2 - 3 = -7 - -2 = -5$
- $a + (a + d) + (a + 2d) + \cdots \ldots\ldots\ldots$,
$common\ difference = (a + d) - a = (a + 2d) - (a + d) = d$

To find General term$(n^{th} - term)$ of an A.P.

Let $a_1 + a_2 + a_3 + \cdots \ldots\ldots\ldots$, be an A.P.
Let $d = common\ difference$, then
$d = a_n - a_{n-1}$, for all $n > 1$
$\Rightarrow a_n = d + a_{n-1}$
Now for $n = 2, a_2 = d + a_1$
$n = 3, \ a_3 = d + a_2 = d + (d + a_1) = a_1 + 2d$

$$n = 4\,, a_4 = d + a_3 = d + (2d + a_1) = a_1 + 3d$$

Observation of the terms;

$a_2 = a_1 + d = a_1 + (2-1)d$
$a_3 = a_2 + d = a_1 + (3-1)d$
$a_4 = a_3 + d = a_1 + (4-1)d$

$a_n = a_{n-1} + d = a_1 + (n-1)d$

Thus the general term of an A.P is

$$\boldsymbol{a_n = a + (n-1)d}$$

From the formula,
$a = 1^{st}\ term$
$n = no.of\ terms$
$a_n = last\ term$
$d = common\ difference$

NOTE:

1. The general term of an A.P is $\boldsymbol{a_n = a_1 + (n-1)d}$**,** has four terms. If the *three* are given then the 4th can be evaluated.
2. From the general term of an A.P, the standard form of an A.P is stated as;

$$\boldsymbol{a + (a+d) + (a+2d) + \cdots \ldots\ldots\ldots + [a + (n-1)d]}$$

3. *The most convenient terms in an A.P with given sum and product:*
   a) If **three terms** are in an A.P, then use

$$\boldsymbol{(a-d)\,, a\,, (a+d)}$$

   b) If **four terms** are in an A.P, then use

$$\boldsymbol{(a-3d)\,,\ (a-d)\,, (a+d)\,,\ (a-3d)}$$

NOTICE:

a) If you sum up the terms, the common difference**, d,** vanishes, and the sum is in terms of only the first term**, a.**
b) In case of **odd number** of terms, the common difference is taken as **d** and the middle term as **a.**
c) In case of **even number** of terms, the common difference is taken as **2d** and the two middle terms are$(\boldsymbol{a-d})$ *and* $(\boldsymbol{a+d})$**.**

Properties of an A.P

a) If the constant is *added* to or *subtracted* from every term in an A.P, the resulting series is an A.P.
b) If every term is *multiplied* or *divided* by a constant, the resulting series is also in an A.P.

Examples

1. *Find the* $17^{th} - term$ *of the series*
   a) $4 + 6 + 8 + \cdots \ldots \ldots \ldots$
   b) $-8 - 7 - 6 - 5 - \cdots \ldots \ldots$

Solution

a) $4 + 6 + 8 + \cdots \ldots \ldots \ldots$
Since $d = 6 - 4 = 8 - 6 = 2$, the series is an A.P
Using $a_n = a + (n-1)d$
Given: $a = 4\,, n = 17\,, d = 2$
Required: a_n
$\Rightarrow a_n = 4 + (17-1) \times 2 = 36$
$\therefore$ *the* $17^{th} - term\ is$ 36

b) $-8 - 7 - 6 - 5 - \cdots \ldots \ldots$
$d = -7 - -8 = -6 - -7 = 1$
Given: $a = -8\,, n = 17\,, d = 1$
Required: a_{17}
$\Rightarrow a_{17} = -8 + (17-1) \times 1 = 8$
$\therefore$ *the* $17^{th} - term\ is$ 8

2. *State the* $20^{th} - term$ *and the number of terms of the following series;*
   a) $2 + 4 + 6 + \cdots \ldots \ldots \ldots . . + 100$
   b) $-8 - 6 - 4 - \cdots \ldots . + 200$

Solution

a) $2 + 4 + 6 + \cdots \ldots \ldots \ldots . . + 100$
Given: $a = 2\,, n = ?\,, d = 4 - 2 = 2\,, a_n = 100$
Required: $a_{20} = ?\,, n = ?$
Using $a_n = a + (n-1)d$
$100 = 2 + (n-1) \times 2$ *or* $50 = 1 + n - 1$
$\therefore n = 50$
Now $a_{20} = a + (20-1)d = a + 19d$
$a_{20} = 2 + 19 \times 2 = 40$
$\therefore$ *No. of terms is* $50\,, 20^{th} - term\ is$ 40

b) $-8 - 6 - 4 - \cdots \ldots . + 200$
Given: $a = -8\,, n = ?\,, d = -6 - -8 = 2\,, a_n = 200$
Required: $a_{20} = ?\,, n = ?$
Using $a_n = a + (n-1)d$
$200 = -8 + (n-1) \times 2$ *or* $100 = -4 + n - 1$
$\therefore n = 105$
Now $a_{20} = a + (20-1)d = a + 19d$
$a_{20} = -8 + 19 \times 2 = 30$
$\therefore$ *No. of terms is* $105\,, 20^{th} - term\ is$ 30

3. *The* $3^{rd} - term$ *of an A.P is 18 and the* $7^{th} - term$ *is 30. Find the* $17^{th} - term.$

Solution

Using $a_n = a + (n-1)d$
Given: $a_3 = 18\,,\ a_7 = 30$
Required: $a_{17} = ?$

$\Rightarrow a_3 = a + (3-1)d = a + 2d = 18$…… (1)
$\Rightarrow a_7 = a + (7-1)d = a + 6d = 30$ …… (2)
Solving the equations simultaneously
$Eqn.(1) - eqn.(2)$
$\Rightarrow -4d = -12\,, d = 3$
$Subst.\ d = 3$ in to eqn. (1) or (2)
For $a + 2d = 18 \Rightarrow a = 18 - 2 \times 3 = 12$
Now $a_{17} = a + (17-1)d = a + 16d = 12 + 16 \times 3 = 60$
$\therefore the\ 17^{th} - term\ is\ 60$

4. *The* 3rd *and* 13th *term of an A.P are respectively equal to* -40 *and* 0. *Find the A.P and its* $20^{th} - term.$

Solution

Using $a_n = a + (n-1)d$
Given: $a_3 = -40\,,\ a_{13} = 0$
Required: $a_{20} = ?$
$\Rightarrow a_3 = a + (3-1)d = a + 2d = -40$…… (1)
$\Rightarrow a_{13} = a + (13-1)d = a + 12d = 0$ …… (2)
Solving the equations simultaneously
$Eqn.(1) - eqn.(2)$
$\Rightarrow -10d = -40\,,\ d = 4$
$Subst.\ d = 4$ in to eqn. (1) or (2)
For $a + 2d = -40 \Rightarrow a = -40 - 2 \times 4 = -48$
Now $a_{20} = a + (20-1)d = a + 19d = -48 + 19 \times 4 = -12$
$\therefore the\ 20^{th} - term\ is - 12$
The A.P is $a\,, (a+d)\,, (a+2d)\,, \ldots\ldots\ldots$
$\therefore the\ A.P\ is\ -48\,, -44\,, -40\,, \ldots\ldots\ldots\ldots, 28$

5. *In an A.P* $u_1 + u_2 + \cdots\ldots\ldots + u_4 = 15$ *and* $u_{11} = -3$. *Find the greatest integer* N *such that* $u_N \geq 0$

Solution

Using $a_n = a + (n-1)d$
$u_1 = a\,, u_2 = a + d\,, u_3 = a + 2d\,, u_4 = a + 3d$
$\Rightarrow a + a + d + a + 2d + a + 3d = 15$
$\therefore 4a + 6d = 15$ …… (1)
Given: $u_{16} = -3 \Rightarrow a + 15d = -3$ …… (2)
Solving (1) and (2) simultaneously,
From (1) $a = -3 - 15d$ …… (3)
Eqn. (3) in to (1)
$\Rightarrow 4[-3 - 15d] + 6d = 15$
$\Rightarrow -12 - 60d + 6d = 15\ \ or\ d = -\frac{27}{54} = -\frac{1}{2}$
$Subst.\ d = -\frac{1}{2}\ in\ to\ (3)$
$\Rightarrow a = -3 - 15\left(-\frac{1}{2}\right) = \frac{-6+15}{2} = \frac{9}{2}$
Now $u_N = a + (N-1)d \geq 0$
$\Rightarrow \frac{9}{2} + (N-1) \times \left(-\frac{1}{2}\right) \geq 0$……$(\times 2)$
$\Rightarrow 9 + (1 - N) \geq 0\ or\ 10 \geq N$
$\therefore the\ greatest\ integer\ is\ 10$

6. *The nth term of a series is $U_n = a\,3^n + b\,n + c$. Given that $U_1 = 4, U_2 = 13$ and $U_3 = 46$. Find the values of a, b and c.*

Solution

Given: $U_n = a\,3^n + b\,n + c$

For $U_1 = 4, n = 1 \Rightarrow 4 = 3a + b + c$ ..... (1)

For $U_2 = 13, n = 2 \Rightarrow 13 = 9a + 2b + c$ ..... (2)

For $U_3 = 46, n = 3 \Rightarrow 46 = 27a + 3b + c$ ..... (3)

Solving (1), (2) and (3) simultaneously

$eqn.(1) - eqn.(2)$, to eliminate c

$\Rightarrow -9 = -6a - b \ or\ 6a + b = 9$ ..... (4)

$eqn.(1) - eqn.(3)$, to eliminate c

$\Rightarrow -42 = -24a - 2b \ or\ 12a + b = 21$ ..... (5)

Now eliminating b from (4) and (5)

$eqn.(4) - eqn.(5)$, to eliminate b

$\Rightarrow -6a = -12 \ \therefore a = 2$

$Subst.\ a = 2\ in\ to\ eqn.(5)\ or\ (4)$

$6a + b = 9, a = 2 \Rightarrow b = 2 - 6 \times 2 = -3$

$Subst.\ a = 2, b = -3\ in\ to\ (1) or\ (2) or\ (3)$

From $\Rightarrow c = 4 - (3a + b) = 4 - (6 - 3) = 1$

$\therefore a = 2, b = -3, c = 1$

Examples on sum and product of a certain number of terms in an A.P

1. *Find three terms in an A.P such that;*
   a) Their sum is 21 and the product is 315
   b) Their sum is 33 and the sum of their squares is 563

Solution

Let the numbers be $a - d, a\ a + d$

a) *Given: sum* $= 21,\ a - d + a + a + d = 21$

$\Rightarrow 3a = 21 \ \therefore a = 7$

$Product = 315, a\,(a - d)(a + d) = 315$

$\Rightarrow a(a^2 - d^2) = 315$

$\Rightarrow 7(7^2 - d^2) = 315 \ or\ 49 - d^2 = 45$

$\therefore d = \sqrt{4} = \pm 2$

The numbers are: $[7 - (\pm 2)], 2, [7 + (\pm 2)]$

$\therefore 5, 7, 9, \ for\ d = +2$

$\therefore 9, 7, 5, \ for\ d = -2$

b. $sum = 33,\ a - d + a + a + d = 33$

$\Rightarrow 3a = 33 \ \therefore a = 11$

$sum\ of\ squares = 563, (a - d)^2 + a^2 + (a + d)^2 = 563$

$\Rightarrow (11 - d)^2 + 11^2 + (11 + d)^2 = 563$

$\Rightarrow 121 - 22d + d^2 + 121 + 121 + 22d + d^2 = 563$

$\Rightarrow 2d^2 + 363 = 563, d^2 = 100 \ \therefore d = \pm 10$

Now numbers are:

For $d = 10, a = 11 \Rightarrow (11 - 10), 11, (11 + 10)\ or\ 1, 11, 21$

For $d = -10, a = 11 \Rightarrow (11 - -10), 11, (11 - 10)\ or\ 21, 11, 1$

2. *Find four numbers in an A.P such that their sum is 20 and sum of the squares is 120.*

Solution

Let the numbers be: $(a-3d)\,,(a-d)\,,(a+d)\,,(a+3d)$

$sum = 20 \Rightarrow a-3d+a-d+a+d+a+3d = 20$

$\Rightarrow 4a = 20 \;\therefore a = 5$

$sum\ of\ squares = 120\,,(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2 = 120$

$\Rightarrow (5-3d)^2+(5-d)^2+(5+d)^2+(5+3d)^2 = 120$

$\Rightarrow 25-30d+9d^2+25+30d+9d^2+25-10d+d^2+25+10d+d^2 = 120$

$\Rightarrow 100+20d^2 = 120 \;\therefore d = \pm 1$

The numbers are:

For $d = 1\,,a = 5\,,(5-3)\,,(5-1)\,,(5+1)\,,(5+3)$

$\therefore 2\,,4\,,6\,,8$

For $d = -1\,,a = 5\,,(5+3)\,,(5+1)\,,(5-1)\,,(5-3)$

$\therefore 8\,,6\,,4\,,2$

3. *Find five numbers in an A.P such that;*
   a) Their sum is 20 and product of the first and last term is 15
   b) Their sum is 25 and sum of their squares is 135

Solution

a) Let the numbers be: $a-2d\,,a-d\,,a\,,a+d\,,a+2d$

Now sum= $20 = a-2d+\ a-d+\ a+a+d+a+2d$

$\Rightarrow 20 = 5a \;\therefore a = 4$

Product of the first and last= $(a-2d)\times(a+2d\,) = 15$

$\Rightarrow (4-2d)\times(4+2d\,) = 15\ \ or\ 16-4d^2 = 15$

$\Rightarrow d = \pm\frac{1}{2}$

$\therefore numbers\ are: 3\,,\ 3\frac{1}{2}\,,4\,,\ 4\frac{1}{2}\ and\ 5$

b) $Sum = 5a = 25 \;\therefore a = 5$

$Sum\ of\ squares = 135 = (a-2d)^2+\ (a-d)^2+\ a^2+(a+d)^2+(a+2d)^2$

$135 = a^2-4ad+4d^2+a^2-2ad+d^2+\ a^2+a^2+2ad+d^2+a^2+4ad+4d^2$

$\Rightarrow 135 = 5a^2+10d^2$, but $a = 5$

$\Rightarrow 135 = 5\times 5^2+10d^2\ or\ 135 = 125+10d^2$

$\therefore d = \pm 1$

$\therefore numbers\ are: 3\,,4\,,5\,,6\ and\ 7$

Task

1. *Find three numbers in an A.P such that*
   a) Sum is 27 and product is 648
   b) Sum is 27 and sum of their squares is 275
   c) Sum is 12 and sum of their cubes is 408
   d) Sum is 15 and sum of squares of its first and third term is 58
   e) Sum is 9 and the sum of their squares is 77
   f) Sum is 6 and their product is -90

ANSWER: $(\boldsymbol{a})\ \mathbf{6\,,9\,,12}\,,\boldsymbol{d} = \pm\mathbf{3}\,,\boldsymbol{a} = \mathbf{9}\ (\boldsymbol{b})\ \mathbf{5\,,9\,,13}\,,\boldsymbol{d} = \pm\mathbf{4}\,,\boldsymbol{a} = \mathbf{9}\ (\boldsymbol{c})\ \mathbf{1\,.4\,,7}\,,\boldsymbol{d} = \pm\mathbf{3}\,,\boldsymbol{a} = \mathbf{4}$

$(\boldsymbol{d})\ \mathbf{3\,,5\,,7}\ (\boldsymbol{e})-\mathbf{2\,,3\,,8}\ \ (\boldsymbol{f})-\mathbf{5\,,2\,,9}$

2. *The sum of four integers of an A.P is 24 and their product is 945. Find the integers.*

ANSWER: $\mathbf{3\,,5\,,7\,,9}$ **Hint**: $Use\ (a-3d)\,,(a-d)\,,(a+d)\,,(a+3d)$

PROOFS IN A.PS

1) *If a, b and c are respectively the pth, q^{th} and r^{th} terms of an A.P, prove that* $a(q-r)+b(r-p)+c(p-q)=0$

Solution

Using $U_N = A+(N-1)D$ for an A.P

Given: $U_p = a, U_q = b, U_r = c$

$\Rightarrow a = A+(p-1)D$ ..... (1)

$\Rightarrow b = A+(q-1)D$ ..... (2)

$\Rightarrow c = A+(r-1)D$ ..... (3)

Note:

A and D are the unknowns, hence solving for them,

$eqn.(1)-eqn.(2)$

$\Rightarrow a-b=[p-1-(q-1)]D=(p-q)D$

$\Rightarrow D=\frac{a-b}{p-q}$ ..... (4)

$eqn.(4) in\ to\ (1) to\ find\ A$

From (1), $A = a-(p-1)D$

$\Rightarrow A = a-(p-1)\left(\frac{a-b}{p-q}\right)=\frac{ap-aq-(ap-pb-a+b)}{p-q}$

$\therefore A=\frac{-aq+pb+a-b}{p-q}$ ..... (5)

$Subst.(4) and\ (5) in\ to\ (3)$

From $c = A+(r-1)D$

$\Rightarrow c=\frac{-aq+pb+a-b}{p-q}+(r-1)\left(\frac{a-b}{p-q}\right)$

$\Rightarrow c(p-q) = -aq+pb+a-b+ar-br-a+b$

$= -aq+pb+ar-br$

$\therefore c(p-q)=a(r-q)+b(p-r)\ or\ a(q-r)+b(r-p)+c(p-q)=0$

2) *Prove that the nth term is $p+q-n$ when the pth term of an A.P is q and the qth term is p.*

Solution

Using $U_N = A+(N-1)D$ for an A.P

Given: $U_p = q, U_q = p$

$\Rightarrow p = A+(p-1)D$ ..... (1)

$\Rightarrow q = A+(q-1)D$ ..... (2)

Required: $U_n = p+q-n$ ..... (3)

Now solving for A and D

$eqn.(1)-eqn.(2)$, to eliminate A

$p-q=(q-p)D=-D(p-q)$

$\therefore D=-1$

From (1) and $D=-1$

$A=p-(q-1)D=p+q-1$

Required: $U_n = A+(n-1)D$

$U_n = p+q-1+(n-1)\times -1 = p+q-1-n+1 = p+q-n$

$\therefore n^{th}\ term = p+q-n$

3. *If p times the pth term of an A.P is equal to q times the qth, prove that the $(p+q)^{th}$ term is zero.*

Solution

Using $U_N = A + (N-1)D$ for an A.P

Given: $p \times U_p = q \times U_q$

Required: to prove $U_{p+q} = 0$

$U_p = A + (p-1)D$, $U_q = A + (q-1)D$

$\Rightarrow$ $p \times U_p = q \times U_q$ *becomes*

$p \times [A + (p-1)D] = q \times [A + (q-1)D]$ ..... (1)

Now $U_{p+q} = A + (p+q-1)D$..... (2)

Expressing eqn. (2) in terms of only D

From (1), $pA + (p^2 - p)D = Aq + (q^2 - q)D$

$\Rightarrow A(p-q) = (q^2 - q)D - (p^2 - p)D$

$A(p-q) = (q^2 - p^2 - q + p)D$

$= -[p^2 - q^2 - (p-q)]D$

$= -[(p-q)(p+q) - (p-q)]D$ ..... $[\div (p-q)]$

$A = -[p+q-1]D$

$\therefore A = [1-(p+q)]D$ ..... (3)

Subst. (3)*in to* (2) , must give zero for the proof to have been proved as required.

$U_{p+q} = A + (p+q-1)D$

$= [1-(p+q)]D + (p+q-1)D$

$= D[1-p-q+p+q-1] = 0$, as required.

Task

1. *The* m^{th} *term of an A.P is* $\frac{1}{n}$ *and the* n^{th} *term is* $\frac{1}{m}$ *, prove that the* $(mn)^{th}$ *term is unity.i.e.* $U_{mn} = 1$
2. *If* p , q , r , s *are any four consecutive terms of an A.P, show that* $p^2 - 3q^2 + 3r^2 - s^2 = 0$.
   [**Hint**: *Let* $p = \alpha - 3\beta, q = \alpha - \beta, r = \alpha + \beta$, $s = \alpha + 3\beta$]
3. *The* 8th *term of an A.P is double the* 13th *term. Prove that the* 2nd *term is double the* 10th *term.*

Other forms of proofs in A.Ps

1. *If* a , b *and* c *are in an A.P, prove that;*
   a) $a^3 + c^3 + 6abc = 8b^3$
   b) $a^2 + 4ac + c^2 = 2(ab + bc + ca)$

Solution

For a, b and c to be in A.P, then $b - a = c - b = d$ $(common\ difference)$

$\Rightarrow b = a + d$, $c = b + d = a + d + d = a + 2d$, $2b = a + c$

a) $\mathbf{L.H.S} = a^3 + c^3 + 6abc = a^3 + (a+2d)^3 + 6a(a+d)(a+2d)$

$= a^3 + a^3 + 6a^2d + 12ad^2 + 8d^3 + 6a(a^2 + 2ad + ad + 2d^2)$

$= 8a^3 + 6a^2d + 12ad^2 + 8d^3 + 18a^2d + 12ad^2$

$= 8a^3 + 24a^2d + 24ad^2 + 8d^3 = 8(a^3 + 3a^2d + 3ad^2 + d^3) = 8(a+d)^3 = 8b^3$

b) $\mathbf{L.H.S} = a^2 + 4ac + c^2 = a^2 + c^2 + 4ac$

$= (a + c)^2 - 2ac + 4ac = (a + c)^2 + 2ac$

$= (a + c)(a + c) + 2ac$

$= (2b)(a + c) + 2ac = 2(ab + bc + ac)$

2. *If* a , b *and* c *are in an A.P, show that*

i) $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in an A.P

ii) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in an A.P

Solution

i) if a, b, c are in an A.P, then

$b - a = c - b$ ..... (1)

Now $\frac{1}{ab}, \frac{1}{ac}, \frac{1}{ab}$ are in an A.P if $\frac{1}{ac} - \frac{1}{bc} = \frac{1}{ab} - \frac{1}{ac}$

$\Rightarrow \frac{1}{c}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{a}\left(\frac{1}{b} - \frac{1}{c}\right)$

$\Rightarrow \frac{b-a}{abc} = \frac{c-b}{abc}$

$\therefore b - a = c - b \Leftrightarrow a, b, c$ *are in an A.P* .hence the result follows.

ii) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in an A.P if $\frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{\sqrt{a}+\sqrt{b}} - \frac{1}{\sqrt{c}+\sqrt{a}}$

$\Rightarrow \frac{\sqrt{b}+\sqrt{c}-\sqrt{c}-\sqrt{a}}{(\sqrt{b}+\sqrt{c})(\sqrt{c}+\sqrt{a})} = \frac{\sqrt{c}+\sqrt{a}-\sqrt{a}-\sqrt{b}}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

$\Rightarrow \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}} = \frac{\sqrt{c}-\sqrt{b}}{\sqrt{b}+\sqrt{a}}$ $\therefore b - a = c - b \Leftrightarrow a, b, c$ *are in an A.P*. Hence the result.

3. *If* a^2, b^2 *and* c^2 *are in an A.P, show that* $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$

Solution

If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$, are in an A.P, then $\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

$\Rightarrow \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$

$\Rightarrow \frac{b-a}{b+c} = \frac{c-b}{a+b}$ $\therefore b^2 - a^2 = c^2 - b^2$, hence a^2, b^2, c^2 are in an A.P

Task:

a. If $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in an A.P, show that a^2, b^2, c^2 are also in an A.P

b. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in an A.P, show that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in an A.P

TO FIND THE SUM OF A FINITE NUMBER OF QUANTITIES IN AN A.P

Let $a_1, a_2, a_3, \ldots\ldots\ldots\ldots\ldots, a_n$ be $n-$ quantities in an A.P, and let the last term a_n be denoted as l. If their common difference, then

$a_n = a_1 + (n-1)d = l$

Now summing up the n- terms of an A.P

$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots\ldots\ldots\ldots + a_1 + (n-1)d$

$= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots\ldots\ldots\ldots + (l - d) + l$ ..... (1)

Re-writing this sum in a reverse order

$\Rightarrow S_n = l + (l - d) + (l - 2d) + \cdots\ldots\ldots + (a_1 + d) + a_1$ ..... (2)

Adding the equations (1) and (2)

$\Rightarrow 2S_n = (a_1 + l) + (a_1 + l) + (a_1 + l) + \cdots\ldots\ldots + (a_1 + l), (n - times)$

$\Rightarrow 2S_n = n(a_1 + l)$

$\therefore S_n = \frac{n}{2}(a_1 + l) = \frac{n}{2}[a_1 + a_1 + (n-1)d]$

$= \frac{n}{2}[2a_1 + (n-1)d]$

Consequently, $\boxed{\boldsymbol{S_n = \frac{n}{2}[2a_1 + (n-1)d]}}$

In general, $\boxed{\boldsymbol{S_n = \frac{n}{2}[1^{st}\ term + last\ term]}}$

NOTE:

The above formula contains four quantities. $i.e. S_n, a, n, d$. If three of them are known the fourth can be evaluated

Examples

1. *Find the sum of the series*
   a) $3 + 8 + 13 + \cdots \ldots \ldots \ldots + 12^{th}\ term$
   b) $\frac{3}{4} + \frac{2}{3} + \frac{7}{12} + \cdots \ldots \ldots \ldots . . + 19^{th}\ term$

Solution

a) $3 + 8 + 13 + \cdots \ldots \ldots \ldots + 12^{th}\ term$

Given: $a_1 = 3, d = 8 - 3 = 13 - 8 = 5, n = 12, a_{12} = ?$

Using $a_n = a_1 + (n-1)d$

$a_{12} = 3 + (12 - 1) \times 5 = 58$

Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow S_{12} = \frac{12}{2}[2 \times 3 + (12 - 1) \times 5] = 366$

Alternatively

Using $S_n = \frac{n}{2}[1^{st}\ term + last\ term]$

$\Rightarrow S_{12} = \frac{12}{2}[3 + 58] = 366$

b) $\frac{3}{4} + \frac{2}{3} + \frac{7}{12} + \cdots \ldots \ldots \ldots . . + 19^{th}\ term$

Given: $a_1 = \frac{3}{4}, d = \frac{2}{3} - \frac{3}{4} = \frac{7}{12} - \frac{2}{3} = -\frac{1}{12}, n = 19, a_{19} = ?$

$a_{19} = \frac{3}{4} + (19 - 1) \times \left(-\frac{1}{12}\right) = -\frac{3}{4}$

Now $S_{19} = \frac{19}{2}\left[2 \times \frac{3}{4} + (19 - 1) \times \left(-\frac{1}{12}\right)\right] = \frac{19}{2}\left[\frac{3}{2} - \frac{18}{12}\right] = 0$

Alternatively

Using $S_n = \frac{n}{2}[1^{st}\ term + last\ term]$

$\Rightarrow S_{19} = \frac{19}{2}\left[\frac{3}{4} + -\frac{3}{4}\right] = 0$

2. *How many terms of the following series may be taken to;*
   i) *Their sum is 66 if* $-9, -6, 3 \ldots \ldots \ldots \ldots \ldots . \ldots \ldots \ldots \ldots \ldots \ldots \ldots ?$
   ii) *Their sum is* -80 *if* $15 + 13 + 11 + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ?$
   iii) *Their sum is* $n(2n + 4)$ *if* $6 + 10 + 14 + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ?$

Solution

i) $-9, -6, 3 \ldots \ldots \ldots \ldots \ldots . \ldots \ldots$

Let $S_n = 66, a_1 = -9, d = -6 — 9 = 3$

Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow 66 = \frac{n}{2}[2 \times -9 + (n - 1) \times 3]$

$132 = -18n + 3(n^2 - n)$

$\therefore 3n^2 - 21n - 132 = 0\ or\ n^2 - 7n - 44 = 0$

$\Rightarrow (n - 11)(n + 4) = 0\ \ \therefore n = 11, -4$

Since n is a positive integer, $n = -4$ is rejected.

$\therefore number\ of\ terms\ is\ 11$

ii) $15 + 13 + 11 + \cdots \ldots \ldots \ldots$

Let $S_n = -80, a_1 = 15, d = 13 - 15 = -2$

Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow -80 = \frac{n}{2}[2 \times 15 + (n-1) \times -2]$

$-160 = 30n - 2(n^2 - n)$

$\therefore 2n^2 - 32n - 160 = 0 \; or \; n^2 - 16n - 80 = 0$

$\Rightarrow (n-20)(n+4) = 0 \;\; \therefore n = 20, -4$

Since n is a positive integer, $n = -4$ is rejected.

$\therefore number\ of\ terms\ is\ 20$

iii) $6 + 10 + 14 + \cdots \ldots \ldots \ldots$

Let $S_N = n(2n+4), a_1 = 6, d = 10 - 6 = 4$

Using $S_N = \frac{N}{2}[2a_1 + (N-1)d]$

$\Rightarrow n(2n+4) = \frac{N}{2}[2 \times 6 + \; (N-1) \times 4]$

$\Rightarrow 4n^2 + 8n = 12N + 4N^2 - 4N \; or \;\; n^2 + 2n = 3N + N^2 - N$

$\therefore N^2 + 2N - (n^2 + 2n) = 0$, solving for N

$\Rightarrow N = \frac{-2 \pm \sqrt{2^2 + 4(n^2+2n)}}{2}$

$= \frac{-2 \pm 2\sqrt{1+n^2+2n}}{2}$

$= -1 \pm \sqrt{(n+1)^2} = -1 \pm n + 1$

$\therefore N = -1 + n + 1 = n \;\; or \;\; N = -1 - n - 1 = -2 - n \; (Reject)$

$\therefore number\ of\ terms\ is\ n$

3. *The sum of the series is 525. Its last term is 39 and the first term is 3. If the series is an A.P, find its common difference.*

Solution

Given: $S_n = 525, a_n = 39, a_1 = 3$

Using $S_n = \frac{n}{2}[1^{st}\ term + last\ term] = \frac{n}{2}[a_1 + a_n]$

$\Rightarrow 525 = \frac{n}{2}[3 + 39] = 21n \;\; \therefore n = 25$

From $a_n = a_1 + (n-1)d \; or \; from \; S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$From\ S_n = \frac{n}{2}[2a_1 + (n-1)d]$

$\Rightarrow 525 = \frac{25}{2}[2 \times 3 + (25-1)d]$

$\Rightarrow 1050 = 150 + 600d \;\; \therefore d = \frac{3}{2}$

$\therefore Common\ difference\ is \frac{3}{2}$

4. *The first and last terms of an A.P are -4 and 146 respectively. The sum of the A.P is 7171. Find the number of terms and sum of the first 20 terms.*

Solution

Given: $S_n = 7171, a_n = 146, a_1 = -4$

Required: $n = ?, d = ?$

Using $S_n = \frac{n}{2}[1^{st}\ term + last\ term] = \frac{n}{2}[a_1 + a_n]$

$\Rightarrow 7171 = \frac{n}{2}[-4 + 146] = 71n$

$\Rightarrow 7171 = 71n \;\; \therefore n = 101$

$\therefore No.of\ terms\ is\ 101$

Now finding d using either $a_n = a_1 + (n-1)d \;\; or \;\; S_n = \frac{n}{2}[2a_1 + (n-1)d]$

This time let us use $a_n = a_1 + (n-1)d$

$\Rightarrow 146 = -4 + (101-1)d$

$\Rightarrow 150 = 100d \;\; \therefore d = \frac{3}{2}$

$S_{20} = \frac{20}{2}\left[2 \times -4 + (20-1) \times \frac{3}{2}\right] = 205$

$\therefore S_{20} = 205$

5. *The sum of a certain number of terms in an A.P is 5500. The firs and the last terms are 100 and 1000 respectively. Find the number of terms and the sum of the last 5 terms of the progression.*

Solution

Given: $S_n = 5500\,,\ a_1 = 100\,, a_n = 1000$

Required: $n = ?\,, d = ?\,, S_{last\ 5-terms} = ?$

Using $S_n = \frac{n}{2}[1^{st}\ term + last\ term] = \frac{n}{2}[a_1 + a_n]$

$\Rightarrow 5500 = \frac{n}{2}[100 + 1000] = \frac{1100}{2}n$

$\Rightarrow 5500 = \frac{1100}{2}n\ \ \therefore n = 10$

$\therefore No.of\ terms\ is\ 10$

Now finding d using $a_n = a_1 + (n-1)d$

$\Rightarrow 1000 = 100 + (10-1)d$

$\therefore d = 100$

The A.P can be stated as; $100\,, 200\,, 300\,, 400, \ldots\ldots\ldots\ldots\ldots\ldots 900\,, 1000$

$Sum\ of\ last\ 5-terms = Sum\ of\ all\ the\ terms - Sum\ of\ the\ 1^{st}\ 4-terms$

$\Rightarrow S_{last\ 5-terms} = S_{10} - S_{1^{st}\ five}$

$\Rightarrow S_{last\ 5-terms} = 5500 - \left[\frac{5}{2}(2 \times 100 + (5-1) \times 100)\right]$

$= 5500 - [500 + 1000] = 4000$

Note:

The last five terms are; $600, 700\,, 800\,, 900\,, 1000$

6. *The sum of the last three terms of an A.P having $n-$ terms is 150 times the sum of the first three terms of the progressions. If the third term and the second term are 15 and 5. Find the last term and the number of terms of the progression.*

Solution

Given: $S_{3(last)} = 150 \times S_{3(first)}\,,\ a_3 = 15\,, a_2 = 5$

Required: $n = ?\,, d = ?$

For an A.P, the last term is $a_n = a_1 + (n-1)d$

For the $2^{nd}\ last\ tem = a_n - d$

$= a_1 + (n-1)d - d = a_1 + (n-2)d$

For the $3^{rd}\ last\ tem = a_n - 2d$

$= a_1 + (n-1)d - 2d = a_1 + (n-3)d$

Now $S_{3(last)} = [a_1 + (n-1)d] + [a_1 + (n-2)d] + [a_1 + (n-3)d]$

$\therefore S_{3(last)} = 3a_1 + (3n-6)d$

Also $S_{3(first)} = \frac{3}{2}[2a_1 + (3-1)d] = 3a_1 + 3d$

From $S_{3(last)} = 150 \times S_{3(first)}$

$\Rightarrow 3a_1 + (3n-6)d = 150(3a_1 + 3d)$

$\Rightarrow a_1 + (n-2)d = 150(a_1 + d)$

$\therefore 149a_1 + (152-n)d = 0$ ..... (1)

Given: $a_3 = a_1 + (3-1)d = a_1 + 2d$

$\therefore 15 = a_1 + 2d$ ..... (2)

$a_2 = a_1 + d = 5$ ..... (3)

$Eqn.(2) - eqn.(3)$

$\Rightarrow 10 = d$
From $a_1 + d = 5 \Rightarrow a_1 = 5 - 10 = -5$
Now $subst. d = 10, a_1 = -5 \; in \; to \; eqn. (1)$
$\Rightarrow 149a_1 + (152 - n)d = 0 \; yields$
$149(-5) + (152 - n) \times 10 = 0$
$\Rightarrow 152 - n = \frac{149 \times 5}{10} = 74.5$
$\therefore No. of \; terms \; is \; 78$
$a_{78} = -5 + (78 - 1) \times 10 = 760$

7. *The sum of the first 13-terms of an A.P is 21 and the sum of the first 21 terms is 13. Find the sum of the first 34 – terms.*

Solution

Given: $S_{13} = 21, S_{21} = 13$
Required: $S_{34} = ?, d = ?, a = ?$
Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$
$\Rightarrow S_{13} = \frac{13}{2}[2a + (13-1)d] = 21$ ..... (1)
$\Rightarrow S_{21} = \frac{21}{2}[2a + (21-1)d] = 13$ ..... (2)
From (1), $2a + 12d = \frac{42}{13}$ ..... (3)
From (2), $2a + 20d = \frac{26}{21}$ ..... (4)
$Eqn. (3) - eqn. (4) gives$
$-8d = \frac{544}{273} \; \therefore d = -\frac{68}{273}$
Substitute $d = -\frac{68}{273}$ in to eqn. (3) to find a
$\Rightarrow 2a = \frac{42}{13} - 2 \times \left(-\frac{68}{237}\right) \; \therefore a = \frac{566}{91}$
$\therefore S_{34} = \frac{34}{2}\left[2 \times \frac{566}{91} + (34-1)\left(-\frac{68}{237}\right)\right] = -34$

8. *The fifth term of an A.P is 12 and the sum of the first five terms is 80. Determine the first term and the common difference.* (***Uneb* 2007**)

Solution

Given: $a_5 = 12, S_5 = 80$
Required: $a_1 = ?, d = ?$
Using $a_n = a_1 + (n-1)d$
$\Rightarrow a_5 = 12 = a_1 + 4d$ ..... (1)
Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$
$\Rightarrow 80 = \frac{5}{2}[2a_1 + 4d]$
$\Rightarrow 80 = 5a_1 + 10d \; or \; 16 = a_1 + 2d$ ..... (2)
$Eqn. (1) - eqn. (2) gives$
$-4 = 2d \; \therefore d = -2$
Substitute $d = 2$ in to eqn. (2) to find a_1
$a_1 = 16 - 2d = 16 + 4 = 20$
$1^{st} - term \; is \; 20, common \; difference = -2$

9. *The* $10^{th} - term$ *of an A.P is 29 and the* $15^{th} - term$ *is 44. Find the value of the common difference and the first term. Hence find the sum of the first 60 terms.*(***Uneb* 2002**)

Solution

Given: $a_{10} = 29\,, a_{15} = 44$

Required: $a_1 = ?\,, d = ?\,, S_{60} = ?$

Using $a_n = a_1 + (n-1)d$

$\Rightarrow a_{10} = 29 = a_1 + 9d$ ..... (1)

$\Rightarrow a_{15} = 44 = a_1 + 14d$ ..... (2)

$Eqn.(1) - eqn.(2) gives$

$-15 = -5d \;\; \therefore d = 3$

Substitute $d = 3$ in to eqn. (1) to find a_1

$a_1 = 29 - 9d = 29 - 27 = 2$

Now $S_{60} = \frac{60}{2}[2 \times 2 + (60-1) \times 3] = 5\,430$

$\therefore 1^{st} - term\ is\ 2\,, common\ difference = 3\,, S_{60} = 5\,430$

Other examples on summation of the terms in an A.P

1. a) *Find the sum of all natural numbers between 500 and 1000 which are divisible by 13*

   b) Obtain the sum of all natural numbers up to 1000 which are:

      i) Divisible by 5 but not divisible by 2

      ii) Not divisible by 2 and not divisible by 5

Solution

a) By inspection; $\frac{500}{13} \approx 38\,; \frac{1000}{13} \approx 76.9$

Now multiples of 13 after 38 are;

$507\,, 520\,, 533\,, \ldots\ldots\ldots\ldots\ldots, 988$

$\boldsymbol{Note}: 13 \times 76 = 988\,, 13 \times 77 = 1001$

This is an A.P with $a_1 = 507\,, a_n = 988\,, d = 13$

Now finding n;

Using $a_n = a_1 + (n-1)d$

$\Rightarrow 988 = 507 + (n-1) \times 13 \;\; \therefore n = 38$

Now $S_{38} = \frac{38}{2}[2 \times 507 + (38-1) \times 13] = 28\,405$

b) (i) Natural numbers divisible by 5 and not by 2

$\Rightarrow 5\,, 15\,, 25\,, \ldots\ldots\ldots\ldots\ldots\ldots, 955$

These form an A.P. finding how many terms they are,

$a_1 = 5\,, a_n = 955\,, d = 10\,, n = ?$

Using $a_n = a_1 + (n-1)d$

$\Rightarrow 955 = 5 + (n-1) \times 10 \;\; \therefore n = 100$

$\therefore S_{100} = \frac{100}{2}[2 \times 5 + (100-1) \times 10] = 50\,000$

ii) Natural numbers not divisible by 2 are;

$\Rightarrow 1\,, 3\,, 5\,, 9\,, \ldots\ldots\ldots\ldots\ldots\ldots, 999$

Using $a_n = a_1 + (n-1)d$

$999 = 1 + (n-1) \times 2 \;\; \therefore n = 500$

Using $S_n = \frac{n}{2}[a_1 + a_n]$

$\therefore S_{500} = \frac{500}{2}[1 + 999] = 250\,000$

Sum of natural numbers divisible by 5 is 50 000 as in above.

Natural numbers not divisible by 2 and not divisible by 5 is;

$S_{500} - S_{100} = 250\,000 - 50\,000 = 200\,000$

2. a) Find the sum of odd numbers between 100 and 200
   b) Find the sum of the even and odd numbers divisible by 3 lying between 400 and 500.

Solution

a) $101, 103, 105, \ldots\ldots\ldots\ldots\ldots\ldots, 199$ are odd numbers. Finding how many they are'

$a_n = 199, a_1 = 101, d = 2, \ n = ?$

Using $a_n = a_1 + (n-1)d$

$\Rightarrow 199 = 101 + (n-1) \times 2 \ \therefore n = 50$

Now $S_{50} = \frac{50}{2}[101 + 199] = 7\,500$

b) Multiples of 3 after 400 and before 500 are:

$402, 405, 408, \ldots\ldots\ldots\ldots\ldots., 498$

$\left[\boldsymbol{HINT}: \frac{400}{3} \approx 133, 134 \times 3 = 402; \frac{500}{3} \approx 167, 166 \times 3 = 498\right]$

Finding how many they are:

$a_n = 498, a_1 = 402, d = 3, \ n = ?$

Using $a_n = a_1 + (n-1)d$

$\Rightarrow 498 = 402 + (n-1) \times 3 \ \therefore n = 33$

Now $S_{33} = \frac{33}{2}[402 + 498] = 14\,850$

Also even multiples of 3 are:

$402, 408, 414, \ldots\ldots\ldots\ldots\ldots., 498$

Finding how many they are:

$a_n = 498, a_1 = 402, d = 6, \ n = ?$

Using $a_n = a_1 + (n-1)d$

$\Rightarrow 498 = 402 + (n-1) \times 6 \ \therefore n = 17$

Now $S_{17} = \frac{17}{2}[402 + 498] = 7\,650$

Now sum of odd numbers divisible by 3 is;

$S_{33} - S_{17} = 14\,850 - 7\,650 = 7\,200$

Alternatively, odd numbers are;

$405, 411, \ldots\ldots\ldots\ldots\ldots, 495$

Finding how many they are:

$a_n = 495, a_1 = 405, d = 6, \ n = ?$

Using $a_n = a_1 + (n-1)d$

$\Rightarrow 495 = 405 + (n-1) \times 6 \ \therefore n = 16$

Now $S_{16} = \frac{16}{2}[405 + 495] = 7\,200$

Task

1. *Find the sum of all integers lying between 200 and 800 which are divisible by 9.* **ANSWER: 32 967**
2. *Find the sum of all numbers between 200 and 400 which are divisible by 7.* **ANSWER: 8 729**
3. *In an A.*P consisting of $15-$*terms, the middle term is 20. Find the sum of all the terms.* **ANSWER: 300**
4. *An A.P consists of 12 terms. The sum of the two middle terms is 24 and the sum of the first 3-terms is 9. Find the A.P.* **ANSWER: 1, 3, 5, ……, 23**

PROOFS INVOLVING SUMMATION OF A.P FORMULAE,

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$ *or* $S_n = \frac{n}{2}[a_1 + a_n]$

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1. *a) The sum of the first $m-$terms of an A.P is n and the sum of the first $n-$terms is m. find the sum of the first $(m+n)-$terms.*
   b) The sum of the first $m-$terms of an A.P is the same as the sum of the first $n-$terms where $m \neq n$, show that the sum of the first $(m+n)-$terms is zero.

Solution

a) *Given:* $S_m = n$, $S_n = m$

Required: $S_{(m+n)} = ?$

$S_m = n = \frac{m}{2}[2a + (m-1)d]$

$\Rightarrow \frac{2n}{m} = 2a + (m-1)d$ ..... (1)

$S_n = m = \frac{m}{2}[2a + (n-1)d]$

$\Rightarrow \frac{2m}{n} = 2a + (n-1)d$ ..... (2)

In the above equations a and d are unknowns. Calculating for d,

$Eqn.(1) - eqn.(2) gives$

$\Rightarrow \frac{2n}{m} - \frac{2m}{n} = (m-n)d$

$\Rightarrow \frac{2(n^2-m^2)}{mn} = (m-n)d$

$\Rightarrow -\frac{2(m^2-n^2)}{mn} = (m-n)d \ \ or \ \ -\frac{2(m+n)(m-n)}{mn} = (m-n)d$

$\therefore d = -\frac{2(m+n)}{mn}$

$Subst.\ d = -\frac{2(m+n)}{mn}\ in\ to\ (1)\ or\ (2)$

From eqn. (1), $\frac{2n}{m} = 2a + (m-1)d$

$\Rightarrow 2a = \frac{2n}{m} - (m-1)d$

$\Rightarrow 2a = \frac{2n}{m} - \frac{(m-1)[-2(m+n)]}{mn}$

$\Rightarrow 2a = \frac{2n}{m} + \frac{2(m-1)(m+n)}{mn}$

$= \frac{2n^2+2(m^2+mn-m-n)}{mn}$

Now $S_{(m+n)} = \frac{m+n}{2}[2a + (m+n-1)d]$

$S_{(m+n)} = \frac{m+n}{2}\left[\frac{2n^2+2(m^2+mn-m-n)}{mn} + (m+n-1) \times -\frac{2(m+n)}{mn}\right]$

$= \frac{m+n}{mn}\left[n^2 + m^2 + mn - m - n - [(m+n)^2 - m - n]\right]$

$= \frac{m+n}{mn}[n^2 + m^2 + mn - m - n - m^2 - 2mn - n^2 + m + n]$

$= m + n$

b) *Given:* $S_m = S_n \ ; m \neq n$

Required: $S_{(m+n)} = 0$

$S_m = \frac{m}{2}[2a + (m-1)d] = \frac{2n}{m}[2a + (m-1)d] = S_n$

$\Rightarrow m[2a + (m-1)d] = n[2a + (n-1)d]$

$\Rightarrow 2am + (m^2 - m)d = 2an + (n^2 - n)d$

$\Rightarrow 2a(m-n) = (n^2 - n)d - (m^2 - m)d$

$= [-(m^2 - n^2) + (m-n)]d$

$= [-(m+n)(m-n) + (m-n)]d$

$\Rightarrow 2a = [-(m+n)+1]d$

$\therefore 2a = -(m+n-1)d$

But $S_{(m+n)} = \frac{m+n}{2}[2a + (m+n-1)d]$

$\Rightarrow S_{(m+n)} = \frac{m+n}{2}[2a - 2a] = 0$, *as required.*

2. a) *If the first, second and the last term of an A.P are respectively a, b and x, show that the sum of the* $n-$*terms is* $\frac{x+a}{2} + \frac{x^2-a^2}{2(b-a)}$.

b) *If the first ant last terms of an A.P are* a *and* L *respectively and if* S *is the sum of all the* $n-$*terms, show that the common difference is* $\frac{L^2-a^2}{2S-(L+a)}$.

Solution

a) *Given:* $a_1 = a\,, a_2 = b\,, a_n = x$

Required: $S_n = ?$

$S_n = \frac{n}{2}[1^{st} + last\ term]$

$S_n = \frac{n}{2}[a + x]$ ..... (1)

But $a_2 = a + d = b$ ..... (2)

Also $a_n = a + (n-1)d = x$ ..... (3)

In equations (2) and (3) d is the unknown.

From (2), $d = b - a$ ..... (4)

Equation (4) in to (3)

$\Rightarrow a + (n-1)d = x$, becomes

$n = \frac{x-a}{d} + 1$ ..... (5)

Equation (4) in to (5)

$n = \frac{x-a}{d} + 1 = n = \frac{x-a+b-a}{b-a}$

$\therefore n = \frac{x+b-2a}{b-a}$ ..... (6)

Now substitute equation (4) in to (1)

$\Rightarrow S_n = \frac{n}{2}(a + x)$, becomes

$\therefore S_n = \frac{\left[\frac{x-a}{b-a}+1\right]}{2} \times (a + x)$

$= \frac{(x-a)(x+a)}{2(b-a)} + \frac{a+x}{2}$

$\therefore S_n = \frac{x+a}{2} + \frac{x^2-a^2}{2(b-a)}$, as required.

b) *Given:* $a_1 = a\,, a_n = L\,, S_n = S$

Required: $d = ?$

From $a_n = a + (n-1)d = L$..... (1)

$S_n = \frac{n}{2}[2a + (n-1)d] = S$ ..... (2)

From equation (1) and (2), d is the unknown.

From equation (1) , $d = \frac{L-a}{n-1}$ ..... (3)

Now solving for n in equation (3)

$\Rightarrow \frac{n}{2}[2a + (n-1)d] = S$, becomes

$\therefore \frac{n}{2}\left[2a + (n-1) \times \frac{(L-a)}{n-1}\right] = S$

$\Rightarrow n[2a + L - a] = 2S$

$\therefore n = \frac{2S}{a+L}$ ..... (4)

Equation (4) in to (3)

$\therefore d = \frac{L-a}{\frac{2S}{a+L}-1} = \frac{L-a}{\frac{2S-(a+L)}{(a+L)}}$

$= \frac{(L-a)(L+a)}{2S-(a+L)} = \frac{L^2-a^2}{2S-(a+L)}$

3. *If p , q and r are sums $n, 2n$ and $3n$ terms of an A.P, prove that $r = 3(q-p)$*

Solution

Given: $S_n = p$, $S_{2n} = q$, $S_{3n} = r$

From $\mathbf{R.H.S} = 3(q-p)$

$\Rightarrow p = \frac{n}{2}[2a + (n-1)d]$ ..... (2)

$\Rightarrow q = \frac{2n}{2}[2a + (2n-1)d]$ ..... (3)

$\Rightarrow r = \frac{3n}{2}[2a + (3n-1)d]$ ..... (4)

Now $3(q-p) = 3\left[\frac{2n}{2}[2a + (2n-1)d] - \frac{n}{2}[2a + (n-1)d]\right]$

$\Rightarrow 3(q-p) = \frac{3n}{2}\left[2[2a + (2n-1)d] - [2a + (n-1)d]\right]$

$\Rightarrow 3(q-p) = \frac{3n}{2}[2a + (3n-1)d] = r$, as required.

Deducing *last term* of an A.P from summation of $n-$ *terms* of an A.P

In general if S_n is the sum of n-terms of an A.P and S_{n-1} is the summation of $(n-1)-$ *terms* of an A.P, then;

$$\textbf{Last term} = \mathbf{a_n} = \mathbf{S_n} - \mathbf{S_{n-1}}$$

Example

1. *The sum of n-terms of a series is $n^2 + 2n$, prove that this series is an A.P, and find its first term and common difference.*

Solution

Given: $S_n = n^2 + 2n$

Also $S_{n-1} = (n-1)^2 + 2(n-1)$

$= n^2 - 2n + 1 + 2n - 2$

$\therefore S_{n-1} = n^2 - 1$

Now $a_n = S_n - S_{n-1}$

$\Rightarrow a_n = n^2 + 2n - (n^2 - 1) = 2n + 1$

$\therefore$ *Last term* , $a_n = 2n + 1$

Now for $n = 1$, $a_1 = 2 + 1 = 3$

$n = 2$, $a_2 = 4 + 1 = 5$

$n = 3$, $a_2 = 3 + 1 = 7$

The terms are: $3, 5, 7, \ldots\ldots\ldots\ldots\ldots\ldots\ldots$; which form an A.P of first term 3 and common difference 2.

2. *If the sum up to n-terms of an A.P is given by $\frac{n}{2}(3n+5)$, show that the last term is $3n+1$, and that the series is an A.P.*

Solution

Given: $S_n = \frac{n}{2}(3n+5)$

Also $S_{n-1} = \frac{(n-1)}{2}[3(n-1)+5] = \frac{(n-1)(3n+2)}{2}$

Now $a_n = S_n - S_{n-1}$

$\Rightarrow a_n = \frac{n}{2}(3n+5) - \frac{(n-1)(3n+2)}{2}$

$= \frac{3n^2+5n}{2} - \frac{3n^2+2n-3n-2}{2}$

$= \frac{1}{2}(6n + 2) = 3n + 1$

$\therefore Last\ term, a_n = 3n + 1$, as required.

By putting $n = 1, 2, 3, 4, \ldots\ldots\ldots$, we get a series of;

$4, 7, 10, \ldots\ldots\ldots\ldots\ldots\ldots$, which is an A.P.

Task

1. *The sum of the first n-terms of a certain series is* $n^2 + 5n$ *for all integral values of n. find the first three terms and prove that the series is an A.P.* **ANSWER**: $\mathbf{6, 8, 10}$
2. *The sum of the first n-terms of an A.P is* $3n^2 + 2n$*. Find the* $r^{th} - term$*.* **ANSWER**: $\boldsymbol{6r - 1}$
3. *The second term of an A.*P is -4 and the sixth term is -24. Find the fifteenth term and sum of the first $15 - terms$. **ANSWER**: $\mathbf{-69, 540}$
4. *The eighth term of an A.P is twice the third term, and the sum of the first 8-terms is 39. Find the first three terms of the progression, and show that the sum of n-terms is* $\frac{3}{8}n(n + 5)$. **ANSWER**: $\mathbf{\frac{9}{4}, 3, \frac{15}{4}}$
5. *The sum of n-terms of a series* $2, 5, 8, \ldots\ldots\ldots\ldots$ *is 950. Find n.* **ANSWER**: $\mathbf{25}$
6. *The number of terms in an A.P is even; the sum of the odd terms is 24, of the even terms 30 and the last term exceeds the first by* $10\frac{1}{2}$*. Find the number of terms.* **ANSWER**: $\mathbf{8}$
7. *If* $p^{th} - term$ *of an A.P is* $\frac{1}{q}$ *and the* $q^{th} - term$ *is* $\frac{1}{p}$ *, show that the sum of* $pq - terms$ *is* $\frac{1}{2}(pq + 1)$
8. *Find the sum of all natural numbers from 100 up to 300 which are divisible by 4.* **ANSWER**: $\mathbf{10200}$
9. *In an A.P the sum of the first five terms is 30, and the third term is equal to the sum of the first two. Write down the first five terms of the progression.* **ANSWER**: $\mathbf{2, 4, 6, 8, 10}$
10. *The sum of the first three terms of an A.P is 3 and the sum of the first five terms is 20. Find the first five terms of the progression.* **ANSWER**: $\mathbf{-2, 1, 4, 4, 7, 10}$
11. *The sum of the first six terms of an A.P is 21 and the seventh term is three times the sum of the third term and fourth term. Find the first term and the common difference.* **ANSWER**: $\mathbf{-9, 5}$
12. *Sum of n-terms of three A.P's are* S_1 *,* S_2 *and* S_3 *. The first term of each of them is 1 and the common differences are 1, 2 and 3 respectively. Show that* S_1 *,* S_2 *,* S_3 *are in an A.P.*

$$\left[\begin{array}{l} \boldsymbol{HINT}: S_1 = \frac{n}{2}[2 + n - 1] = \frac{n(n+1)}{2}\ ,\ S_2 = \frac{n}{2}[2 + (n-1) \times 2 = n^2] \\ S_3 = \frac{n}{2}\left[2 + (3n-1) \times 2 = \frac{n(3n-1)}{2}; now\ show\ that\ S_2 - S_1 = S_3 - S_2\right] \end{array}\right]$$

WORD PROBLEMS IN A.P'S

1. *A firm produced 1000 sets of T.V during its first year. The total sum of the firm's production at the end of 10 years operation is 14 500 sets.*

i) *Estimate how many units production increased by each year if the increase each year is uniform.*

ii) *Fore cast based on the estimate of annual increments in production the level of output for the* 15th *year.*

Solution

i) *Given:* $a = 1000\,, S = 14\,500\,, n = 10$

Required: $d = ?$

Using $\Rightarrow S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow 14\,500 = \frac{10}{2}[2 \times 1000 + (10-1)d]$

$\Rightarrow 14\,500 = 5[2000 + 9d] \;\therefore d = 100$

Hence 100 units is the increase per annum.

ii) *Required:* $a_{15} = ?$

$a_{15} = a + (15-1)d = a + 14d$

$= 1000 + 14 \times 100 = 2\,400$

2. *Two posts were offered to a man. In one the starting salary was* £120 *per month and the annual increment was* £8.

In the other post the salary commenced at £85 *per month but the annual increment was* £12. *The man decided to accept the post which will give him more earnings in the first twenty years of the service. Which post was acceptable to him? Justify your answer.*

Solution

The total earnings of the man in the first twenty years for the first job are $S_{20} = ?$:

Given: $a =$ £$120\,, d = 8\,, n = 20$

$\Rightarrow S_{20} = \frac{20}{2}[2 \times 120 + (20-1) \times 8] \times 12 =$ £$47\,040$

His total earnings in the first twenty years for the second job are $S_{20} = ?$:

Given: $a =$ £$85\,, d = 12\,, n = 20$

$\Rightarrow S_{20} = \frac{20}{2}[2 \times 85 + (20-1) \times 12] \times 12 =$ £$47\,760$

Total earnings for the second job are greater than the first job hence the second job was accepted by the man.

3. *The annual salary of an employee is in an A.P over the first* $7 - years$*, during which the annual increment is* £200. *Over the next* $8 - years$*, the salary is in another A.P during which the annual increment is* £300. *After that the salary is constant.*

If the starting salary is £14 000*, calculate the employee's total earnings in the first 25 years of service.*

Solution

Case I

Given: $a = 14\,000\,, d = 200\,, n = 7$

Required: $S_7 = ?$

$\Rightarrow S_7 = \frac{7}{2}[2 \times 14\,000 + (7-1) \times 200] =$ £$102\,200$

Case II

At the end of the $7^{th} - year$, the annual salary is given from $a_n = a + (n-1)d$

$\Rightarrow a_7 = 14\,000 + (7-1) \times 200 = £15\,200$

Now the next $8 - years$, the sum of all his earnings is given by:

$S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow S_8 = \frac{8}{2}[2 \times 15\,200 + (8-1) \times 300] = £130\,000$

Now at the end of the$15 - years. i.e. 1^{st}\ 7 - years + next\ 8 - years$, the total earnings are given by;

$£130\,000 + £102\,200 = £232\,200$

Remaining years $= 25 - 17 = 10\ years$

The amount in the$15^{th} - year$ is given by $a_n = a + (n-1)d\,; a = 15\,200\,, n = 8\,, d = 300$

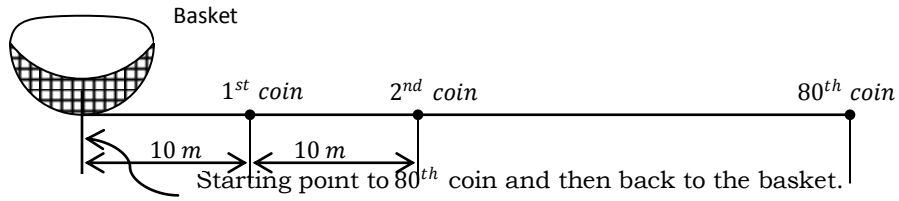
$\Rightarrow a_{15} = 15\,200 + (8-1) \times 300 = £17\,300$

Now for the remaining 10 years, total amount $= 17\,300 \times 10 = £173\,000$

$\therefore$ Total earnings for 25 years is$= 173\,000 + 232\,200 = £405\,200$

4. *Eighty coins are placed in a line on the ground. The distance between any two consecutive coins is 10 meters. How far must a person travel to bring them one by one to a basket placed 10 meters behind the first coin?*

Solution



Given: $a = 10\,, d = 10\ (distance\ between\ 2\ successive\ coins)\,, n = 80\ (No. of\ coins)$

Required: $S_{80} = \frac{80}{2}[2 \times 10 + (80-1) \times 10] = 32\,400\ m$

Total distance$= 32\,400\ (to\ 80^{th}\ coin) + 32\,400\ (back\ to\ basket) = 64\,800\ m$

5. *A man saved £16 500 in 10 years. In each year after the first he saved £100 more than he did in the preceding year. How much did he save in the first year?*

Solution

Given: $S_{10} = £16\,500\,, d = £100\,, n = 10$

Let $a = Amount\ saved\ in\ 1^{st}\ year$

From $S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow 16\,500 = \frac{10}{2}[2a + (10-1) \times 100]$

$\Rightarrow 3\,300 = 2a + 900$

$\therefore d = £1\,200$

6. *Miss Sikyomu Jane takes a loan of £2 000 from Anita and agrees to repay in a number of installment, each installment (beginning with the second) exceeding the previous by £10. If the first installment is £5, find how many installments will be necessary to wipe out the loan completely?*

Solution

Given: $S_{10} = £2000\,, d = £10\,, n = ?\,, a = £5$

Using $S_n = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow 2\,000 = \frac{n}{2}[2 \times 5 + (n-1) \times 10]$

$\Rightarrow 2\,000 = 5n + 5n^2 - 5$
$\Rightarrow n^2 + n - 401 = 0$
$\Rightarrow n = \frac{-1 \pm \sqrt{1+4\times401}}{2}$
$\therefore n \approx 20$, for a negative n, it is rejected
$\therefore$ Number of installments will be 20.

7. *The monthly salary of a person was £320 for each of the first three years. He next got annual increments of £40 per month for each of the following successive 12 years. His salary remained stationary till retirement when he found that his average monthly salary during the service period was £698. Find the period of his service.*

Solution

Let $n = total\ no.of\ years\ of\ the\ person's\ service$
Given: Average monthly salary $= \text{£}698 = \frac{Amount\ \ for\ n-years}{No.of\ years}$
$\Rightarrow 698 \times 12 = \frac{Amount\ \ for\ n-years}{n}$
$\therefore Amount\ for\ n-years = 698 \times 12n$
Total salary in the first $3-years = \text{£}320 \times 3 \times 12 = \text{£}960 \times 12$
In the $4^{th}\ year$, his monthly salary= $\text{£}(320+40) = \text{£}360$
In the $5^{th}\ year$, his monthly salary= $\text{£}(360+40) = \text{£}400$
For the next $12-years$, his total salary is;
$= \text{£}12 \times [360 + 400 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots up\ to\ 12^{th}-term]$
$= 12 \times \left[\frac{12}{2}[2\times360 + (12-1)\times40]\right]$
$= \text{£}6\,960 \times 12$
At the end of following 12 years, his monthly salary was;
$= \text{£}[360 + (12-1)\times40] = \text{£}800$
Now total salary for all the 15 years so far is
$= \text{£}6\,960 \times 12 + \text{£}960 \times 12 = \text{£}7\,920 \times 12$
Amount received after 15 years $.i.e.\ (n-15)\ years$ was constant for each year $.i.e.$ amount in the $15^{th}\ year = \text{£}800$
$\Rightarrow Total\ amount = \text{£}(n-15) \times 800 \times 12$
Total amount throughout his service
$= (n-15) \times 800 \times 12 + 7\,920 \times 12 = 698 \times 12n$
$\Rightarrow 800n - 12\,000 + 7\,920 = 698n$
$\therefore n = 40$

8. *Mr. Peter arranges to pay off a debt of £9 600 in 48 annual installments which form an arithmetical series. When 40 of these installments are paid, Mr. Peter becomes insolvent and his creditor finds that £2 400 still remains unpaid. Find the value of each of the first three installments of Mr. Peter. Ignore the interest.*

Solution

Let $a\ , a+d\ , a+2d\ , a+3d\ , \ldots\ldots\ldots$, be the annual installments
Given: $S_{48} = \text{£}9\,600$
$\Rightarrow 9\,600 = \frac{48}{2}[2a + (48-1)d]\ \ or\ 9\,600 = 24[2a + 47d]$
$\Rightarrow 2a + 47d = 400$ …… (1)
After 40 installments are paid, the balance is £2 400 $.i.e.$ the amount paid in 40 installments is;

$= 9\,600 - 2\,400 = £7\,200$
$\Rightarrow$ Sum of 40 terms of an A.P is $£7\,200$
$\Rightarrow 7\,200 = \frac{40}{2}[2a + (20-1)d]$ *or* $7\,200 = 20[2a + 39d]$
$\Rightarrow 2a + 39d = 360$ …… (2)
Subtracting the equations
$\Rightarrow 8d = 40 \ \therefore d = 5$
From (2) $2a + 39d = 360$, $a = 82.5$
Hence the first installment of Mr. Peter is $£82.5$, the second installment is $£(82.5 + 5)\,.i.e.£87.5$ *and the third installments is* $£(87.5 + 5)\,.i.e.£92.5$

Task

1. *Phionah secures an interest free loan of* $£14\,500$ *from a friend and agrees to repay it in 10 installments. She pays £ 1 000 as first installment and then increases each installment by equal amount over the preceding installment. What will be her last installment?*
   [**HINT**: $s_{10} = 14\,500 = 5(2a + 9d)$, $a = 1000 \ \therefore d = 100$
   Required: $a_{10} = a + (10-1)d = 1000 + 9 \times 100 = £\,\mathbf{1\,900}$]

2. *An enterprise produced 600 units in the third year of its existence and 700 units in the seventh year. What was the initial production in the first year?*
   ANSWER: £ **550** [**HINT**: $a_3 = 600 + 2d$, $a_7 = a + 6d$, *solving the equations* $a = 550$, $d = 45 \ \therefore 1^{st}$ *production is* £ **550**]
3. *A piece of equipment costs£ 600 000. It depreciates in value, 15% the first year, 13 ½ % the next year, 12% the third year and so on. What will be the value at the end of 10 years all percentages applying to the original cost?* **ANSWER** £ **105 000**
4. *A money lender lends £ 1000 and charges an overall interest of £ 140. He recovers the loan and interest by 12 monthly installments each less by £ 10 than the preceding one. Find the amount of the first installment.* **ANSWER**: £ **150**
5. *A man saves £ 20 in the first month, £ 30 in the second month, £ 40 in the third month, and so on. How much has he saved at the end of 5 years?* **ANSWER**: £ **18 900**
6. *A man saves £ 8 250 in 10 years. In each year after the first he saved £ 50 more than he did in previous year. How much did he save the first year?* **ANSWER**: £ **600**
7. *A man borrows £ 5 000 and agrees to repay with a total interest as £ 1 000 in 12 installments, each installment being less than the preceding by £ 50. What should be the first installment?* **ANSWER** : £ **775**
8. *A man borrows £ 4 500 and promises to repay back in 30 installments, each of value £ 10 more than the previous one. Find the first and last installment.* **ANSWER**: £ **5** , £ **295**
9. *A man of 60 years old has 9 children born at equal intervals. The sum of the ages of father and the nine children is known to be 222 years. Calculate the age of the youngest san if the eldest one is 30 years.* **ANSWER**: **6 years**

THE GEOMETRIC PROGRESSION (G.P)

A series in which non-zero quantities $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ each term of which is equal to the product of the preceding term and a constant number is a **Geometric Progression.** *i.e.* Each term in a G.P is obtained from the preceding one by multiplying or dividing by a constant number. *e.g.*

- $1, 2, 4, 8, 16, \ldots\ldots\ldots\ldots\ldots$ is a G.P
- $3, -1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, \ldots\ldots\ldots\ldots\ldots\ldots$ is a G.P
- $1, \sqrt{2}, 2, 2\sqrt{2}, \ldots\ldots\ldots\ldots\ldots\ldots$ is a G.P
- $1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \ldots\ldots\ldots\ldots\ldots\ldots$ is a G.P
- $a, ar, ar^2, ar^3, ar^4, \ldots\ldots\ldots.,$ is a G.P

The constant term is called a **common ratio, r** of a G.P

Consider the G.P below;

$$a_1 + a_2 + a_3 + \cdots\ldots\ldots\ldots\ldots\ldots + a_{n-1} + a_n$$

This will be a G.P if:

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots\ldots\ldots\ldots\ldots\frac{a_n}{a_{n-1}}, \text{ for all integers } n > 1$$

From the above examples of G.P series,

- $1, 2, 4, 8, 16, \ldots\ldots\ldots\ldots\ldots$, has common ratio $r = \frac{2}{1} = \frac{4}{2} = 2$
- $3, -1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, \ldots\ldots\ldots\ldots\ldots\ldots$, has common ratio $r = -\frac{1}{3} = \frac{\frac{1}{3}}{1} = \frac{1}{3}$
- $1, \sqrt{2}, 2, 2\sqrt{2}, \ldots\ldots\ldots\ldots\ldots\ldots.,$ has common ratio $r = \frac{\sqrt{2}}{1} = \frac{2}{\sqrt{2}}$
- $1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \ldots\ldots\ldots\ldots\ldots\ldots$, has common ratio $r = \frac{\frac{1}{5}}{1} = \frac{\frac{1}{25}}{\frac{1}{5}} = \frac{1}{5}$
- $a, ar, ar^2, ar^3, ar^4, \ldots\ldots\ldots.,$ has common ratio $r = \frac{ar}{a} = \frac{ar^2}{ar} = r$

To find the $n^{th} - term$ of a G.P

($General\ term\ of\ a\ G.P$)

Let $a_1 + a_2 + a_3 + \cdots\ldots\ldots\ldots\ldots\ldots$

Let $r = common\ ratio$

By definition,

$r = \frac{a_2}{a_1} \Rightarrow a_2 = a_1 r$

$r = \frac{a_3}{a_2} \Rightarrow a_3 = a_2 r = a_1 r.r = a_1 r^2$

$r = \frac{a_4}{a_3} \Rightarrow a_4 = a_3 r = a_1 r^2.r = a_1 r^3$

Observation of terms

$a_2 = a_1 r = a_1 r^{2-1}$

$a_3 = a_1 r^2 = a_1 r^{3-1}$

$a_4 = a_1 r^3 = a_1 r^{2-1}$

$\vdots \qquad\qquad \vdots$

$a_n = a_1 r^{n-1}$

Thus the $n^{th} - term$ of a G.P is

$\boldsymbol{a_n = ar^{n-1}}$, where $a = 1^{st} - term, n = no.of\ terms,\ r = common\ ratio$

NOTE:

1. The $n^{th} - term/$ general term of a G.P is $\boldsymbol{a_n = ar^{n-1}}$
   It contains $4 - terms$, is $3 - terms$ are given, the $4^{th} - term$ can be evaluated.
2. From the general term of a G.P, the standard form of a G.P series can be stated as;

$a + ar + ar^2 + ar^3 + ar^4, \ldots\ldots\ldots + ar^{n-1}$

3. **The most convenient terms of a G.P given their product are:**
   a) If *three terms* are in a G.P, then use

$\boldsymbol{\frac{a}{r}, a, ar}$

   b) If *four terms* are in a G.P, then use

$\boldsymbol{\frac{a}{r^3}, \frac{a}{r}, ar, ar^3}$

NOTICE:

1. In both cases, when terms are multiplied, the common ratio vanishes.
2. In the case of *odd number terms*, the middle terms $\boldsymbol{a}$ and the common ratio is $\boldsymbol{r}$.
3. In the case of *even number terms*, the middle terms are $\boldsymbol{\frac{a}{r}, ar}$ and the common ratio is $\boldsymbol{r^2}$.

Properties of a G.P

1. If every term of a G.P is multiplied or divided by a constant, the resulting series is a G.P.
2. If the terms of a G.P are raised to same power, the resulting series is a G.P.
3. If the reciprocals of terms in a G.P are got, the resulting series is a G.P.

Examples

1. *Write down the terms indicated in each of the following series.*
   a) $2 + 4 + 8 + \cdots\ldots\ldots\ldots\ldots\ldots 11^{th} - term$
   b) $\frac{2}{3} + \frac{3}{4} + \frac{27}{32} + \cdots\ldots\ldots\ldots\ldots\ldots 12^{th} - term$
   c) $\frac{1}{\sqrt{2}}, -2, \frac{8}{\sqrt{2}}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots 8^{th} - term$

Solution

a) $2 + 4 + 8 + \cdots\ldots\ldots\ldots\ldots\ldots 11^{th} - term$
This is not an A.P because $4 - 2 \neq 8 - 4$
This is a G.P because $\frac{4}{2} = \frac{8}{4} = 2$
Using $\boldsymbol{a_n = ar^{n-1}}$
Given: $a_n = ?, n = 11, r = 2, a = 2$
$\therefore a_{11} = 11 \times (2)^{11-1} = 11 \times 2^{10} = 11264$

b) $\frac{2}{3} + \frac{3}{4} + \frac{27}{32} + \cdots\ldots\ldots\ldots\ldots\ldots 12^{th} - term$

Given: $a_n = ?, n = 12, r = \frac{\frac{3}{4}}{\frac{2}{3}} = \frac{9}{8}, a = \frac{2}{3}$

Required: $a_{12} = ?$

$\therefore a_{12} = \left(\frac{2}{3}\right) \times \left(\frac{9}{8}\right)^{12-1} = \frac{2}{3} \times \left(\frac{9}{8}\right)^{11} = 11264$

c) $\frac{1}{\sqrt{2}}, -2, \frac{8}{\sqrt{2}}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots 8^{th} - term$

Given: $a_n = ?, n = 8, r = \frac{-2}{\frac{1}{\sqrt{2}}} = \frac{\frac{8}{\sqrt{2}}}{-2} = -\frac{4}{\sqrt{2}}, a = \frac{1}{\sqrt{2}}$

Required: $a_8 = ?$

$\therefore a_8 = \left(\frac{1}{\sqrt{2}}\right) \times \left(-\frac{4}{\sqrt{2}}\right)^{8-1} = \frac{1}{\sqrt{2}} \times \left(-\frac{4}{\sqrt{2}}\right)^7 = \left(\frac{1}{\sqrt{2}}\right)^8 \times (-4)^7 = -1024$

2. *Find the number of terms in the G.P's below.*
   a) $0.03 + 0.06 + 0.12 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 1.92$
   b) $81 + 27 + 9 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{27}$
   c) $a + ar + ar^2 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + ar^{n-1}$

Solution

a) $0.03 + 0.06 + 0.12 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 1.92$

Given: $a = 0.03, r = \frac{0.06}{0.03} = 2, a_n = 1.92$

Required: $n = ?$

Using $a_n = ar^{n-1}$

$\Rightarrow 1.92 = 0.03 \times (2)^{n-1}$

$\Rightarrow \frac{1.92}{0.03} = 2^{n-1}$ *or* $2^6 = 2^{n-1}$

$\therefore n - 1 = 6$ *or* $n = 7$

$\therefore No.of\ terms\ is\ 7$

b) $81 + 27 + 9 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{27}$

Given: $a = 81, r = \frac{27}{81} = \frac{1}{3}, a_n = \frac{1}{27}$

Required: $n = ?$

Using $a_n = ar^{n-1}$

$\Rightarrow \frac{1}{27} = 81 \times \left(\frac{1}{3}\right)^{n-1}$

$\Rightarrow \frac{1}{27 \times 81} = \frac{1}{3^{n-1}}$ *or* $3^3 \times 3^4 = 3^{n-1}$

$\Rightarrow 7 = n - 1 \ \therefore n = 8$

c) $a + ar + ar^2 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + ar^{n-1}$

Using $A_N = A.R^{N-1}$

Given: $A = a, R = \frac{ar}{a} = r, A_N = ar^{n-1}$

$\Rightarrow a.r^{n-1} = a.r^{N-1}$

$\Rightarrow n - 1 = N - 1 \ \therefore n = N$

$\therefore No.of\ terms\ is\ n$

3. *The third term of a G.P is 10, and the sixth term is 80. Find the common ratio, the first term and the twelfth term.*

Solution

Given: $a_3 = 10, a_6 = 80$

Using $a_n = ar^{n-1}$

$\Rightarrow 10 = a.r^{3-1} = a.r^2$ ..... (1)

$\Rightarrow 80 = a.r^{6-1} = a.r^5$ ..... (2)

$eqn.(2) \div eqn.(1)$

$\Rightarrow \frac{80}{10} = \frac{a.r^5}{a.r^2} = r^3$

$\Rightarrow 8 = r^3 \ \therefore r = 2$

From eqn. (1)

$a = \frac{10}{r^2} = \frac{10}{4} = \frac{5}{2}$

$Now\ 12^{th} - term = a_{12} = a.r^{12-1} = a.r^{11}$

$\therefore 12^{th} - term = \frac{5}{2} \times 2^{11} = 5 \times 2^{10}$

$\therefore 1^{st} - term = \frac{5}{2}, 12^{th} - term = 5(2)^{10}$

4. *The $3^{rd} - term$ of a G.P is 2, and the $5^{th} - term$ is 18. Find two possible values of common ratio, and the $2^{nd} - term$ in each case.*

Solution

Given: $a_3 = 2, a_5 = 18$

Required: $r = ?, a_2 = ?$

Using $a_n = ar^{n-1}$

$\Rightarrow 2 = a.r^{3-1} = a.r^2$ ..... (1)

$\Rightarrow 18 = a.r^{5-1} = a.r^4$ ..... (2)

$eqn.(2) \div eqn.(1)$

$\Rightarrow \frac{18}{2} = \frac{a.r^4}{a.r^2} = r^2$

$\Rightarrow 9 = r^2 \ \therefore r = \pm 3$

Now from eqn. (1)

$a = \frac{2}{r^2} = \frac{2}{(\pm 3)^2} = \frac{2}{9}$

Now $a_2 = ar^{2-1} = ar = \frac{2}{9} \times \pm 3 = \pm\frac{2}{3}$

$\therefore$ For $1^{st}\ G.P$ with $r = 3, a = 2, the\ 2^{nd} - term\ is \frac{2}{3}$ and in the $2^{nd}\ G.P$ with $r = -3, a = 2, 2^{nd} - term\ is -\frac{2}{3}$

5. *The ratio of the 4th to the $12^{th} - term$ of a G.P with a positive common ratio is $\frac{1}{256}$. If the sum of the two terms is 61.68. find the $10^{th} - term$.*

Solution

Given: $\frac{a_4}{a_{12}} = \frac{1}{256} \Rightarrow \frac{a.r^3}{a.r^{11}} = \frac{1}{256}$ *or* $\frac{1}{r^8} = \frac{1}{256}$

$\Rightarrow r^8 = 256$. let $r^4 = x$

$\Rightarrow x^2 = 256 \ \therefore x = \pm 16$

Now $r^4 = \pm 16$ *or* $r^4 = 16\ (Taking\ the + ve\ value)$

$\therefore r = \pm 2$

Since r is required as $+ve$, then $r = 2$ is taken, neglecting the negative sign.

Also *given:* $a_4 + a_{12} = 61.68$

$\Rightarrow a.r^3 + a.r^{11} = 61.68$

$\Rightarrow a(r^3 + r^{11}) = 61.68$

$\therefore a = \frac{61.68}{(r^3 + r^{11})} = \frac{61.68}{(2^3 + 2^{11})} = 0.03$

Now $a_{10} = ar^9 = 0.03 \times 2^9 = 15.36$

$\therefore$ The $10^{th} - term\ is$ 15.36

The Sum and Product of convenient terms of a G.P

Examples:

1. *Find three numbers in a G.P such that*
   a) *Their sum is 19 and their product is 216*
   b) *Their sum is 21 and the sum of their squares is 651*
   c) *The sum of their squares is 819 and their product is 729.*

Solution

Let the numbers be $\frac{a}{r}, a, ar$

a) *Given:* $\frac{a}{r} + a + ar = 19$ *or* $a\left(\frac{1}{r} + r + 1\right) = 19$..... (1)

Also $\frac{a}{r} \cdot a \cdot ar = 216$ *or* $a^3 = 216 \therefore a = \sqrt[3]{216} = 6$ ..... (2)

Substituting eqn. (2) in to eqn. (1)

$\Rightarrow \left(\frac{1}{r} + r + 1\right) \cdot 6 = 19$

$\Rightarrow \left(\frac{r^2+r+1}{r}\right) \times 6 = 19$

$\Rightarrow 6r^2 + 6r + 6 = 19r$ *or* $6r^2 - 13r + 6 = 0$

$\Rightarrow 6r^2 - 9r - 4r + 6 = (3r - 2)(2r - 3) = 0$

$\therefore r = \frac{2}{3}, \frac{3}{2}$

$\therefore$ The numbers are: $\frac{6}{\frac{2}{3}}, 6, 6 \times \frac{2}{3}$ *for* $r = \frac{2}{3}$ *or* $9, 6, 4$

For $= \frac{3}{2}$, the numbers are in the reverse order of $9, 6, 4$

b) *Given:* $\frac{a}{r} + a + ar = 21$ *or* $a\left(\frac{1}{r} + r + 1\right) = 21$..... (1)

Also $\left(\frac{a}{r}\right)^2 + (a)^2 + (ar)^2 = 651$ or $a^2\left(\frac{1}{r^2} + r^2 + 1\right) = 651$ ..... (2)

Squaring equation (1)

$\Rightarrow a^2\left(\frac{1}{r} + r + 1\right)^2 = 441$

$\Rightarrow a^2\left[\left(\frac{1}{r} + r\right)^2 + 2\left(\frac{1}{r} + r\right) + 1\right] = 441$

$\Rightarrow a^2\left[\frac{1}{r^2} + r^2 + 2 + 2\left(\frac{1}{r} + r\right) + 1\right] = 441$

$\Rightarrow a^2\left[\left(\frac{1}{r^2} + r^2 + 1\right) + 2\left(\frac{1}{r} + r + 1\right)\right] = 441$ ..... (3)

Dividing eqn. (2) by eqn. (3)

$\Rightarrow \frac{a^2\left(\frac{1}{r^2}+r^2+1\right)}{a^2\left[\left(\frac{1}{r^2}+r^2+1\right)+2\left(\frac{1}{r}+r+1\right)\right]} = \frac{651}{441}$ *or* $441\left(\frac{1}{r^2} + r^2 + 1\right) = 651\left(\frac{1}{r^2} + r^2 + 1\right) + 1302\left(\frac{1}{r} + r + 1\right)$

$\Rightarrow (651 - 441)\left(\frac{1}{r^2} + r^2 + 1\right) + 1302\left(\frac{1}{r} + r + 1\right) = 0$

$\therefore 210\left(\frac{1}{r^2} + r^2 + 1\right) + 1302\left(\frac{1}{r} + r + 1\right)$

Let $y = \frac{1}{r} + r \Rightarrow y^2 - 2 = \frac{1}{r^2} + r^2$

$\Rightarrow 210(y^2 - 2 + 1) + 1302(y + 1) = 0$

$\Rightarrow 210y^2 + 1302y + 1092 = 0$

Solving the equation,

$y = -1, -\frac{26}{5}$

Now for $y = -1, y = \frac{1}{r} + r$

$\Rightarrow \frac{1}{r} + r + 1 = 0$ *or* $r^2 + r + 1 = 0$

Solving the equation, no real values of r

For $y = -\frac{26}{5}$

$\Rightarrow \frac{1}{r} + r + \frac{26}{5} = 0 \; or \; or \; 5r^2 + 26r + 5 = 0$

Solving the equation, $r = -5, -\frac{1}{5}$

For $r = -5$ in eqn. (1) $. i.e. a\left(\frac{1}{r} + r + 1\right) = 21$

$\Rightarrow a\left(-\frac{1}{5} - 5 + 1\right) = 21 \therefore a = -5$

$\therefore$ The numbers are: $\frac{-5}{-\frac{1}{5}}, -5, -5 \times -\frac{1}{5} \; or \; 25, -5, 1$

For $r = -\frac{1}{5}$, the numbers are in the reverse order

c) *Left as an exercise.*

ANSWER: $\boldsymbol{r = \pm 3, r = \pm\frac{1}{3}: No.s \; are: 3, 9, 27 \; or \; 27, 9, 3}$

2. *Find five numbers in a G.P such that:*
   a) *Their product is 32 and the product of the last two terms is 108*
   b) *Their sum is 7 ¾, their product is unity.*

Solution

Let the numbers be: $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

a) *Given:* $\frac{a}{r^2}.\frac{a}{r}.a.ar.ar^2 = 32 \Rightarrow a^5 = 32 \; \therefore a = 2$

Also $ar.ar^2 = 108 \Rightarrow r^3 = \frac{108}{a^2} = \frac{108}{4} = 27$

$\therefore r = 3$

$\Rightarrow$ The numbers are: $\frac{2}{9}, \frac{2}{3}, 2, 6, 18$

b) *Given:* $\frac{a}{r^2} + \frac{a}{r} + a + ar + ar^2 = \frac{31}{4}$

$\Rightarrow a\left(\frac{1}{r^2} + \frac{1}{r} + 1 + r + r^2\right) = \frac{34}{4}$ ..... (1)

Also $\frac{a}{r^2}.\frac{a}{r}.a.ar.ar^2 = 1 \Rightarrow a^5 = 1 \; \therefore a = 1$ ..... (2)

Eqn. (2) in to eqn. (2)

$\frac{1}{r^2} + \frac{1}{r} + 1 + r + r^2 = \frac{31}{4} \; or \; \frac{1}{r} + r + \frac{1}{r^2} + r^2 = \frac{27}{4}$

Let $y = \frac{1}{r} + r \Rightarrow y^2 - 2 = \frac{1}{r^2} + r^2$

$\Rightarrow y + y^2 - 2 = \frac{27}{4} \; or \; y + y^2 = \frac{35}{4}$

$\Rightarrow 4y^2 + 4y - 35 = 0 \; or \; (2y - 5)(2y + 7) = 0$

$\therefore y = \frac{5}{2}, y = -\frac{7}{2}$

$For \; r = \frac{5}{2}, \Rightarrow r + \frac{1}{r} = \frac{5}{2} \; or \; 2r^2 - 5r + 2 = 0$

$\Rightarrow 2r^2 - 4r - r + 2 = 0$

$\Rightarrow (2r - 1)(r - 2) = 0 \; \therefore r = 2, \frac{1}{2}$

For $r = 2$, the numbers are: $\frac{1}{4}, \frac{1}{2}, 1, 2, 4$

Also for $r = -\frac{7}{2} \Rightarrow r + \frac{1}{r} = -\frac{7}{2} \; or \; 2r^2 + 7r + 2 = 0$

$r = \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times 2}}{4} = \frac{-7 \pm \sqrt{33}}{4}$

The values of r can be neglected since they are irrational, the numbers are:

$\frac{1}{4}, \frac{1}{2}, 1, 2, 4$

Task

1. *The sum of three numbers in a G.P is 35 and their product is 1000. Find the numbers.* **ANSWER**: *For* $\boldsymbol{r = 2}$, $\boldsymbol{nos.are\ 5, 10, 20}$; *for* $\boldsymbol{r = \frac{1}{2}}$, $\boldsymbol{nos.are\ 20, 10, 5}$
2. *The sum of three numbers in a G.P is* $\frac{124}{5}$ *and their product is 64. Find the numbers.* **ANSWER**: *For* $\boldsymbol{r = 5\ or \frac{1}{5}}$, $\boldsymbol{nos.are \frac{4}{5}, 4, 20}$
3. *Find three numbers in a G.P such that;*
   i) *Their sum is* $\frac{2}{5}$ *and their product is* -8. **ANSWER**: $\boldsymbol{\frac{2}{5}, -2, 10}$
   ii) *Their sum is* 28 *and their product is* 512. **ANSWER**: $\boldsymbol{4, 8, 16}$ *or* $\boldsymbol{16, 8, 4}$

TO FIND THE SUM OF $n-terms$ OF A G.P

Let $a, ar, ar^2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ar^{n-1}$ be the $n-terms$ in a G.P.

Let $S_n = a + ar + ar^2 + \cdots\ldots\ldots\ldots\ldots\ldots + ar^{n-2} + ar^{n-1}$

…… (1)

S_n is the sum of $n-terms$

Multiplying eqn. (1) by r

$\Rightarrow rS_n = ar + ar^2 + ar^3 + \cdots\ldots\ldots\ldots\ldots\ldots + ar^{n-1} + ar^n$ …… (2)

$eqn.(1) - eqn.(2)$

$\Rightarrow S_n - rS_n = a - ar^n$

$\Rightarrow S_n(1-r) = a(1-r^n)$ *or* $S_n = \frac{a(1-r^n)}{1-r}$

$$\boxed{\boldsymbol{S_n = \frac{a(1-r^n)}{1-r}}, \text{ where } \boldsymbol{r \neq 1}}$$

NOTE:

1) In case $\boldsymbol{r = 1}$, $\boldsymbol{S_n} = a + a + a + \cdots\ldots\ldots\ldots\ldots\ldots + a(n-times) = \boldsymbol{na}$
2) In case $\boldsymbol{|r| < 1}, then\ \boldsymbol{S_n = \frac{a(1-r^n)}{1-r}}$
3) In case $\boldsymbol{|r| > 1}, then\ \boldsymbol{S_n = \frac{a(r^n-1)}{r-1}}$

Examples

1. *Find the sum of 11 terms of a G.P*
   a) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
   b) $3 + 33 + 333 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution

a) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Common ratio: $r = -\frac{\frac{1}{2}}{1} = \frac{\frac{1}{4}}{-\frac{1}{2}} = -\frac{1}{2}$

Given: $r = -\frac{1}{2}, a = 1, n = 11$

Using $S_n = \frac{a(1-r^n)}{1-r}$ since $|r| < 1$

$\Rightarrow S_n = \frac{1\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)} = \frac{1+\frac{1}{2^{11}}}{1+\frac{1}{2}}$

$= \frac{2^{11}+1}{2^{11}\times\frac{3}{2}} = \frac{2^{11}+2}{3\times 2^{10}} = \frac{2050}{3072} = \frac{1025}{1536}$

b) $3 + 33 + 333 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots$

Common ratio: $r = \frac{33}{3} = \frac{333}{33} = 11$

Given: $r = 11\,, a = 3\,, n = 11$

Using $S_n = \frac{a(r^n - 1)}{r-1}$ since $|r| > 1$

$\Rightarrow S_n = \frac{3(11^{11}-1)}{11-1} = \frac{3}{10}(11^{11} - 1)$

2. *Find how many terms of the G.P* $1 + 3 + 9 + \cdots \ldots \ldots \ldots$ *are required to make a total of more than 1 million.*

Solution

$1 + 3 + 9 + \cdots \ldots \ldots \ldots$, has $r = 3$

Using $S_n = \frac{a(r^n - 1)}{r-1}$, $|r| > 1$

$\Rightarrow S_n = \frac{1(3^n-1)}{3-1} = \frac{1}{2} \times (3^n - 1)$

$\Rightarrow 10^6 = \frac{1}{2} \times (3^n - 1)$

$\Rightarrow 2 \times 10^6 + 1 = 3^n$

$\Rightarrow 3^n = 2\,000\,001$

$\Rightarrow \log 3^n = \log 2\,000\,001$

$\therefore n = \frac{\log\ 2\,000\,001}{\log\ 3} = 13.206$

Thus the number of terms required to make a total of more than 1 million will be 14.

3. *What is the smallest number of terms of the G.P* $5\,, 10\,, 20\,, \ldots \ldots \ldots$ *can give a sum greater than* $500\,000$

Solution

$5\,, 10\,, 20\,, \ldots \ldots \ldots$, has $r = 2$

Using $S_n = \frac{a(r^n - 1)}{r-1}$, $|r| > 1$

$\Rightarrow S_n = \frac{5(2^n-1)}{2-1} = 5 \times (2^n - 1)$

$\Rightarrow 500\,000 = 5 \times (2^n - 1)$

$\Rightarrow 100\,000 + 1 = 2^n$

$\Rightarrow 2^n = 100\,001$

$\Rightarrow \log 2^n = \log 100\,001$

$\therefore n = \frac{\log\ 100\,001}{\log\ 2} = 16.6097$

Thus the number of terms required to make a total exceeding 500 000 is 17

4. *The sum of the first eight terms of a G.P (of real terms) is five times the sum of the first four terms. Find the common ratio.*

Solution

Let the G.P be $a\,, ar\,, ar^2\,, ar^3\,, \ldots \ldots \ldots \ldots$

$S_8 =$ Sum of first $8 -$ *terms* $= \frac{a(1-r^8)}{1-r}$

$S_4 =$ Sum of first $4 -$ *terms* $= \frac{a(1-r^4)}{1-r}$

Given: $S_8 = 5S_4$

$\Rightarrow \frac{a(1-r^8)}{1-r} = \frac{5a(1-r^4)}{1-r}$

$\Rightarrow 1 - r^8 = 5(1 - r^4)$

$\Rightarrow 1 - r^8 = 5 - 5r^4$

$\therefore r^8 - 5r^4 + 4 = 0$

Let $r^4 = x$

$\Rightarrow x^2 - 5x + 4 = 0 \ or \ (x-1)(x-4) = 0$
$\therefore x = 1, 4$
$\Rightarrow r^4 - 1 = 0 \ or \ (r^2-1)(r^2+1) = 0$
$\therefore r^2 - 1 = 0 \ or \ r = \pm 1 \ ; \ r^2 + 1 = 0, has \ no \ real \ roots$
Now for $r = 1$, the series is $a + a + a + a + \cdots\ldots\ldots\ldots$
But then, $S_8 = 8a$ and $S_4 = 4a$
So $S_8 \neq 4S_4$, hence $r \neq 1$
In case $r = -1$, we get $S_8 = 0$ and $S_4 = 0$
Thus $S_8 = 5S_4 \ for \ r = -1$
Again $r^4 - 4 = 0 \ or \ (r^2-2)(r^2+2) = 0$
$\therefore r^2 - 2 = 0 \ or \ r = \pm\sqrt{2}$
$\therefore r = -1 \ ; \ \pm\sqrt{2}$

5. *The sum of the last three terms of a G.P having $n-$ terms is 1024 times the sum of the first $3-$ terms of the progression. If the third term is 5, find the last term.*

Solution

Let the G.P be $a, ar, ar^2, ar^3, \ldots\ldots\ldots\ldots ar^{n-3}, ar^{n-2}, \ ar^{n-1}$
Given: $ar^{n-3} + \ ar^{n-2} + ar^{n-1} = 1024 \times S_3$
$a_3 = ar^2 = 5$
Also $S_3 = \frac{a(1-r^3)}{1-r}$
$\Rightarrow ar^{n-3} + \ ar^{n-2} + ar^{n-1} = 1024 \times \frac{a(1-r^3)}{1-r}$
$\Rightarrow r^{n-3} + r^{n-2} + r^{n-1} = 1024 \times \frac{(1+r+r^2)(1-r)}{1-r}$
$\Rightarrow r^{n-3}[1 + r + r^2] = 1024(1 + r + r^2)$
$\Rightarrow r^{n-3} = 1024$ ..... (1)
Also $ar^2 = 5$ ..... (2)
Now last term, $a_n = ar^{n-1}$ ..... (3)
$Eqn.(2) \div eqn.(3) to \ eliminate \ a$
$\Rightarrow \frac{ar^2}{ar^{n-1}} = \frac{5}{a_n}$
$\therefore a_n = 5 \times r^{n-3}$ ..... (4)
$Eqn.(1) \ in \ to \ eqn.(4)$
$\Rightarrow a_n = 5 \times r^{n-3} = 5 \times 1024 = 5120$
$\therefore Last \ term \ is \ 5120$

6. *The second term and the third term of a G.P are 24 and $12(b+1)$ respectively. Find b if the sum of the first $3-$ terms of progression is 76.*

Solution

Given: $a_2 = 24, a_3 = 12(b+1), S_3 = 76$
$a_2 = ar = 24$ ..... (1)
$a_3 = ar^2 = 12(b+1)$ ..... (2)
$S_3 = \frac{a(1-r^3)}{1-r} = 76$ ..... (3)
$Eqn.(1) \div eqn.(2) \ to \ eliminate \ a$
$\Rightarrow \frac{ar}{ar^2} = \frac{24}{12(b+1)} \ \ or \ \ \frac{1}{r} = \frac{2}{b+1}$
$\Rightarrow r = \frac{b+1}{2}$ ..... (4)
From $eqn.(3)$

$\frac{a(1-r^3)}{1-r} = 76$ *or* $\frac{a(1+r+r^2)(1-r)}{1-r} = 76$

$\Rightarrow a(1+r+r^2) = 76$ …… (5)

Eqn. (5) ÷ *eqn.* (1) *to eliminate* a

$\Rightarrow \frac{a(1+r+r^2)}{ar} = \frac{76}{24} = \frac{19}{6}$

$\Rightarrow 6 + 6r + 6r^2 = 19r$

$\Rightarrow 6r^2 - 13r + 6 = 0$ *or* $(2r-3)(3r-2) = 0 \ \therefore r = \frac{2}{3}, \frac{3}{2}$

From eqn. (4) $r = \frac{b+1}{2}$

For $r = \frac{b+1}{2}, r = \frac{2}{3}$

$\Rightarrow \frac{b+1}{2} = \frac{2}{3}$ *or* $4 = 3b + 3 \ \therefore b = \frac{1}{3}$

For $r = \frac{b+1}{2}, r = \frac{3}{2}$

$\Rightarrow \frac{b+1}{2} = \frac{3}{2}$ *or* $1 = b + 1 \ \therefore b = 2$

$\therefore b = \frac{1}{3}, 2$

7. *The* $n^{th} - term$ *of a G.P is* a_n*. Given that* $a_1 = 1, a_2 = 1 + x, \ a_3 = 5 + x^2, a_4 = a + x^4$*. Find a and x, hence determine the sum of the first* $10 - terms$ *of a G.P.*

Solution

Given: $a_1 = 1, a_2 = 1 + x, a_3 = 5 + x^2, a_4 = \ a + x^4$

$\Rightarrow r = \frac{1+x}{1} = \frac{5+x^2}{1+x} = \frac{a+x^4}{5+x^2}$

$\Rightarrow \frac{1+x}{1} = \frac{5+x^2}{1+x}$ *or* $(1+x)^2 = 5 + x^2$

$\Rightarrow 1 + 2x + x^2 = 5 + x^2$

$\Rightarrow 1 + 2x = 5 \ \therefore x = 2$

$\therefore r = 1 + x = 1 + 2 = 3$

From $\frac{a+x^4}{5+x^2} = r$

$\Rightarrow \frac{a+x^4}{5+x^2} = 3 \ \therefore \frac{a+2^4}{5+2^2} = 3$

$\Rightarrow a + 16 = 27 \ \therefore a = 11$

$\therefore a = 11, x = 2$

$S_{10} = \frac{1\times(3^{10}-1)}{3-1} = \frac{1}{2} \times (3^{10} - 1) = 29\,524$

Task

1. *The ratio of the* $9^{th} - term$ *of a G.P to the* $6^{th} - term$ *is* -8*, and the* $5^{th} - term$ *is 16. Ding the G.P.*

   ANSWER: $\mathbf{1}, \mathbf{-2}, \mathbf{4}, \mathbf{-8}, \mathbf{16}$

2. *The* $10^{th} - term$ *of a G.P is double the* $12^{th} - term$*. If the* $3^{rd} - term$ *is 6. Find the* $5^{th} - term.$

   ANSWER: **3**

3. *The sum of the first three terms of a G.P is 7 and the fourth term exceeds the first by 7. Find the seventh term.*

   ANSWER: **64**

PROOFS INVOLVING SUMMATION OF A G.P $.i.e.\ S_n = \frac{a(r^n-1)}{r-1}$ **FORMULAE AND** $n^{th} - TERM\ a_n = ar^{n-1}$

Examples

1. *The* 4th *,* 7th *and* 10th *terms of a G.P are* l *,* m *and* n *respectively. Prove that* $m^2 = ln$.

Solution

Given: $a_4 = l\ , a_7 = m\ , a_{10} = n$

$\Rightarrow l = ar^3$ ..... (1)

$\Rightarrow m = ar^6$ ..... (2)

$\Rightarrow n = ar^9$ ..... (3)

$Eqn.(1) \div eqn.(2)\ to\ eliminate\ a$

$\Rightarrow \frac{l}{m} = \frac{ar^3}{ar^6} = \frac{1}{r^3}\ \therefore r^3 = \frac{m}{l}$ ..... (4)

$Eqn.(2) \div eqn.(3)\ to\ eliminate\ a$

$\Rightarrow \frac{m}{n} = \frac{ar^6}{ar^9} = \frac{1}{r^3}\ \therefore r^3 = \frac{n}{m}$ ..... (5)

$Eqn.(4) = eqn.(5)$

$\Rightarrow \frac{m}{l} = \frac{n}{m}\ \therefore\ m^2 = ln$

2. *If* p^{th} *,* q^{th} *,* r^{th} *term of a G.P are respectively equal to* a *,* b *,* c *, then prove that* $a^{q-r}.b^{r-p}.c^{p-q} = 1$

Solution

Let the $1^{st} - term = A$ and $common\ ratio = R$

Using $A_n = AR^{n-1}$

Given: $A_p = a = AR^{p-1}$ ..... (1)

$A_q = b = AR^{q-1}$ ..... (2)

$A_r = c = AR^{r-1}$ ..... (3)

Now $a^{q-r}.b^{r-p}.c^{p-q} = [AR^{p-1}]^{q-r}.[AR^{q-1}]^{r-p}.[AR^{r-1}]^{p-q}$

$= A^{(q-r)}.R^{(p-1)(q-r)}.A^{(r-p)}.R^{(q-1)(r-p)}.A^{(p-q)}.R^{(r-1)(p-q)}$

$= A^{(q-r+r-p+p-q)}.R^{(p-1)(q-r)(q-1)(r-p)+(r-1)(p-q)}$

$= A^0.R^{pq-pr-q+r+qr-pq-r+p+pr-qr-p+q}$

$= A^0.R^0 = 1$

3. *If* x *,* y *and* z *are* n^{th} *,* $(2n)^{th}$ *and* $(3n)^{th}$ *terms of a G.P, show that* $y^2 = xz$.

Solution

Given: $a_n = x\ , a_{2n} = y\ and\ a_{3n} = z$

$\Rightarrow x = ar^{n-1}\ , y = ar^{2n-1}\ , z = ar^{3n-1}$

$\Rightarrow \frac{x}{y} = \frac{ar^{n-1}}{ar^{2n-1}} = r^{n-1-2n+1} = r^{-n}$

$\therefore \frac{y}{x} = r^n$ ..... (1)

$\frac{y}{z} = \frac{ar^{2n-1}}{ar^{3n-1}} = r^{2n-1-3n+1} = r^{-n}$

$\therefore \frac{z}{y} = r^n$ ..... (2)

$eqn.(1) = eqn.(2)$

$\Rightarrow \frac{z}{y} = \frac{y}{x}\ \therefore\ y^2 = xz$

4. *The pth term of a G.P is p and the qth term is q. show that the nth term is given as* $p.\left[\frac{p^{n-q}}{q^{n-p}}\right]^{\frac{1}{p-q}}$

Solution

Given: $a_p = p = ar^{p-1}$ ..... (1)

$a_q = q = ar^{q-1}$ ..... (2)

In these two equations, a and r are unknowns. Eliminating a from the equations,

$\Rightarrow \frac{p}{q} = \frac{ar^{p-1}}{ar^{q-1}} = r^{p-q}$

$\therefore r = \left[\frac{p}{q}\right]^{\frac{1}{p-q}}$ ..... (3)

eqn. (3) *in to eqn.* (1) to find a,

From $p = ar^{p-1} \Rightarrow a = \frac{p}{r^{p-1}} = \frac{p}{\left(\left[\frac{p}{q}\right]^{\frac{1}{p-q}}\right)^{p-1}} = \frac{p}{\left[\frac{p}{q}\right]^{\frac{p-1}{p-q}}}$

$\therefore a = \frac{p}{\left[\frac{p}{q}\right]^{\frac{p-1}{p-q}}}$ ..... (4)

Now $a_n = ar^{n-1}$

$\Rightarrow a_n = \frac{p}{\left[\frac{p}{q}\right]^{\frac{p-1}{p-q}}} \times \left[\left[\frac{p}{q}\right]^{\frac{1}{p-q}}\right]^{n-1}$

$= \frac{p}{\left[\frac{p}{q}\right]^{\frac{p-1}{p-q}}} \times \left[\frac{p}{q}\right]^{\frac{n-1}{p-q}} = p \times \left[\frac{p}{q}\right]^{\frac{n-1}{p-q} - \frac{p-1}{p-q}} = p.\left[\frac{p}{q}\right]^{\frac{n-p}{p-q}} = p \times \left[\frac{p^{n-p}}{q^{n-p}}\right]^{\frac{1}{p-q}}$

Task:

Prove that the $(n+1)^{th}$ term of a G.P of which the first term is a and the third term is b, is equal to $(2n+1)^{th}$ term of a G.P of which the first term is a and the fifth term is b.

5. *If S be the sum, P the product and R be sum of reciprocal of $n-$ terms in a G.P. prove that $P^2\,R^n = S^n$.*

Solution

Let $a, ar, ar^2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots ar^{n-1}$ be $n-terms$ in a G.P with $|r| < 1$

$\Rightarrow S = a + ar + ar^2 + \cdots\ldots\ldots\ldots\ldots\ldots + ar^{n-1}$

$\therefore S = \frac{a(1-r^n)}{1-r}$ ..... (1)

$\Rightarrow P = a.ar.ar^2.\ldots\ldots\ldots\ldots\ldots\ldots\ldots ar^{n-1}$

$P = a^{1+1+1+1+\cdots\ldots\ldots+n}.r^{1+2+3+\cdots\ldots\ldots\ldots\ldots+n-1}$

$= a^n.r^{\frac{n(n-1)}{2}}$ $\left[\because 1+2+3+\cdots\ldots\ldots+n-1 \text{ is an A.P with } a=1,\ d=1, n=n-1 \ \therefore S_n = \frac{n-1}{2}\left(2+(n-1+1)\right) = \frac{n(n-1)}{2}\right]$

$\therefore P = a^n.r^{\frac{n(n-1)}{2}}$ ..... (2)

$\Rightarrow R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \cdots\ldots\ldots\ldots\ldots\ldots + \frac{1}{ar^{n-1}}$

$= \frac{1}{a}\left[\frac{1}{1} + \frac{1}{r} + \frac{1}{r^2} + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{r^{n-1}}\right]$

$= \frac{1}{a}\left[\frac{1.\left[\left(\frac{1}{r}\right)^n - 1\right]}{\frac{1}{r}-1}\right]$ $\left[\because Since\ |r|>1, then\ using\ S_n = \frac{a(r^n-1)}{r-1}\right]$

$\therefore R = \frac{1}{a}\left[\frac{(1-r^n)r}{(1-r)r^n}\right]$ ..... (3)

$eqn.(1) in\ to\ eqn.(3)$

$\Rightarrow R = \frac{1}{a}.\left[\frac{S}{a}.\frac{r}{r^n}\right] = \frac{S.r}{a^2.r^n} = \frac{S}{a^2.r^{n-1}}$

Now $R^n = \left[\frac{S}{a^2.r^{n-1}}\right]^n = \frac{S^n}{a^{2n}.r^{n(n-1)}}$

$\Rightarrow a^{2n}.r^{n(n-1)}R^n = S^n$ ..... (4)

But from $eqn.(1)$

$\Rightarrow P = a^n.r^{\frac{n(n-1)}{2}}\ \therefore \frac{P}{a^n} = r^{\frac{n(n-1)}{2}}$

Squaring

$\Rightarrow \frac{P^2}{a^{2n}} = r^{n(n-1)}$ ..... (5)

$eqn.(5) in\ to\ eqn.(4)$

$\Rightarrow S^n = a^{2n}.r^{n(n-1)}R^n = a^{2n} \times \frac{P^2}{a^{2n}} \times R^n$

$\therefore\ P^2\ R^n = S^n$

6. *If S_1 , S_2 , S_3 be respectively the sum of n , $2n$, $3n$ terms of a G.P, prove that* $S_1(S_3 - S_2) = (S_2 - S_1)^2$

Solution

Given: $S_1 = S_n = \frac{a(1-r^n)}{1-r}$..... (1)

$S_2 = S_{2n} = \frac{a(1-r^{2n})}{1-r}$..... (2)

$S_3 = S_{3n} = \frac{a(1-r^{3n})}{1-r}$..... (3)

For all $|r| < 1$

Now $S_3 - S_2 = \frac{a(1-r^{3n})}{1-r} - \frac{a(1-r^{2n})}{1-r}$

$= \frac{a}{1-r}.(r^{2n} - r^{3n}) = \frac{ar^{2n}}{1-r}.(1-r^n)$ ..... (4)

Again $S_1(S_3 - S_2) = \frac{a(1-r^n)}{1-r} \times \frac{ar^{2n}}{1-r}.(1-r^n)$

$\Rightarrow S_1(S_3 - S_2) = \frac{ar^{2n}(1-r^n)^2}{(1-r)^2}$ ..... (5)

$\boldsymbol{R.H.S} = (S_2 - S_1)^2 = \left[\frac{a(1-r^{2n})}{1-r} - \frac{a(1-r^n)}{1-r}\right]^2$

$= a^2\left[\frac{r^n - r^{2n}}{1-r}\right]^2 = a^2\left[\frac{r^n(1-r^n)}{1-r}\right]^2 = \frac{ar^{2n}(1-r^n)^2}{(1-r)^2}$

$\therefore \mathbf{R.H.S = L.H.S}$

7. *If S_n represent sum of $n-$ terms of a G.P whose first term and common ratio are a and r respectively. Prove that* $S_1 + S_2 + S_3 + \cdots\ldots\ldots\ldots\ldots\ldots + S_n = \frac{an}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$

Solution

Given: $S_n = \frac{a(1-r^n)}{1-r}$

$S_1 = \frac{a(1-r^1)}{1-r} = a\ ;\ S_2 = \frac{a(1-r^2)}{1-r}\ ;\ S_3 = \frac{a(1-r^3)}{1-r}$

$S_1 + S_2 + S_3 + \cdots\ldots\ldots\ldots\ldots\ldots + S_n$

$= a + \frac{a(1-r^2)}{1-r} + \frac{a(1-r^3)}{1-r} + \cdots\ldots\ldots\ldots + \frac{a(1-r^n)}{1-r}$

$= \frac{a}{1-r}[(1-r) + (1-r^2) + (1-r^3) + \cdots\ldots\ldots\ldots\ldots + (1-r^n)]$

$= \frac{a}{1-r}[(1+1+1+1+\cdots\ldots n\ times) - (r + r^2 + r^3 + \cdots\ldots + r^n)]$

$= \frac{a}{1-r}\left[n - \frac{r(1-r^n)}{1-r}\right] = \frac{an}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$

$\therefore S_1 + S_2 + S_3 + \cdots \ldots\ldots\ldots\ldots\ldots + S_n = \frac{an}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$

8. *The sum of $2n-$ terms of a G.P whose first term is a and the common ratio r is equal to the sum of $n-$ terms of a G.P whose first term is b and common ratio r^2. Prove that b is equal to the sum of the first two terms of the series.*

Solution

Given: $S_{2n} = S_n$

1st *series:* $a, ar, ar^2, ar^3, \ldots\ldots\ldots\ldots\ldots, ar^{2n-1}$

2nd *series:* $b, br^2, br^4, br^6, \ldots\ldots\ldots\ldots\ldots, br^{n-1}$

Now $S_{2n} = \frac{a(1-r^{2n})}{1-r}, S_n = \frac{b[1-(r^2)^n]}{1-r^2}$

From $S_{2n} = S_n$

$\Rightarrow \frac{a(1-r^{2n})}{1-r} = \frac{b(1-r^{2n})}{1-r^2} \quad \therefore \frac{a(1-r^{2n})}{1-r} = \frac{b(1-r^{2n})}{(1+r)(1-r)}$

$\Rightarrow a = \frac{b}{1+r} \quad \therefore b = a + ar$

Thus b is the sum of the first two terms of the series.

Task

1) *If in a G.P, $(p+q)^{th}$ term is m and $(p-q)^{th}$ is n. show that p^{th} term is $\sqrt{mn}$ and the qth term is $m.\frac{2q-p}{2q}.n.\frac{p}{2q}$*

2) *There are $n-$ terms in a G.P, show that the nth root of their product is equal to the square root of the product of its first and last terms.*

Other forms of proofs in G.Ps

1. *if a, b, c are in a G.P and $a^x = b^y = c^z$, prove that* $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}$

Solution

a, b, c are in a G.P if

$\frac{b}{a} = \frac{c}{b}$ *or* $b^2 = ac$ ..... (1)

Expressing ac in terms of b

$a^x = b^y \Rightarrow a = b^{y/x}$ ..... (2)

$b^y = c^z \Rightarrow c = b^{y/z}$ ..... (3)

eqn. (2), (3) *in to* (1)

$\Rightarrow b^2 = b^{y/x}.b^{y/z} = b^{\frac{y}{z}+\frac{y}{x}}$

$\therefore 2 = \frac{y}{z} + \frac{y}{x}$ *or* $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}$

2. *if a, b, c, d are in a G.P, show that:*
   i) $(a+b), (b+c), (c+d)$ are in a G.P
   ii) $(a-b)^2, (b-c)^2, (c-d)^2$ are in a G.P
   iii) $a^2 - b^2, b^2 - c^2, c^2 - d^2$

Solution

i) a, b, c, d are in a G.P if:

$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$ *or* $b = ar, c = br = ar^2, d = cr = ar^3$

Now $(a+b), (b+c), (c+d)$ are in a G.P if:

$\frac{b+c}{a+b} = \frac{ar+ar^2}{a+ar} = \frac{r(1+r)}{1+r} = r$

$\frac{c+d}{b+c} = \frac{ar^2+ar^3}{ar+ar^2} = \frac{r^2(1+r)}{r(1+r)} = r$

Since $\frac{b+c}{a+b}=\frac{c+d}{b+c}=r$, then $(a+b),(b+c),(c+d)$ are in a G.P

ii) $(a-b)^2,(b-c)^2,(c-d)^2$ are in a G.P if:

$\frac{(b-c)^2}{(a-b)^2}=\frac{(c-d)^2}{(b-c)^2}$

Now $\frac{(b-c)^2}{(a-b)^2}=\frac{(ar-ar^2)^2}{(a-ar)^2}=\frac{a^2r^2(1-r)^2}{a^2(1-r)^2}=r^2$

Also $\frac{(c-d)^2}{(b-c)^2}=\frac{(ar^2-ar^3)^2}{(ar-ar^2)^2}=\frac{a^2r^4(1-r)^2}{a^2r^2(1-r)^2}=r^2$

Since $\frac{(b-c)^2}{(a-b)^2}=\frac{(c-d)^2}{(b-c)^2}=r^2$, then $(a-b)^2,(b-c)^2,(c-d)^2$ are in a G.P

iii) $a^2-b^2,\ b^2-c^2,c^2-d^2$ are in a G.P if:

$\frac{b^2-c^2}{a^2-b^2}=\frac{c^2-d^2}{b^2-c^2}$

Now $\frac{b^2-c^2}{a^2-b^2}=\frac{(ar)^2-(ar^2)^2}{a^2-(ar)^2}=\frac{a^2r^2(1-r^2)}{a^2(1-r^2)}=r^2$

Also $\frac{c^2-d^2}{b^2-c^2}=\frac{(ar^2)^2-(ar^3)^2}{(ar)^2-(ar^2)^2}=\frac{a^2r^4(1-r^2)}{a^2r^2(1-r^2)}=r^2$

Thus $a^2-b^2,\ b^2-c^2,c^2-d^2$ are in a G.P since $\frac{b^2-c^2}{a^2-b^2}=\frac{c^2-d^2}{b^2-c^2}$

3. *If a,b,c,d are in a G.P, show that:*

i) $(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2$

ii) $\frac{ab-cd}{b^2-c^2}=\frac{a+c}{b}$

Solution

i) a,b,c,d are in a G.P if:

$\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$ *or* $b=ar,c=br=ar^2,d=cr=ar^3$

Now $(b-c)^2+(c-a)^2+(d-b)^2=(ar-ar^2)^2+(ar^2-a)^2+(ar^3-ar)^2$

$=a^2r^2(1-r)^2+a^2(r^2-1)^2+a^2r^2(r^2-1)^2$

$=a^2[r^2(1-2r+r^2)+r^4-2r^2+1+r^2(r^4-2r^2+1)]$

$=a^2[r^2-2r^3+r^4+r^4-2r^2+1+r^6-2r^4+r^2]$

$=a^2[r^6-2r^3+1]=a^2(r^3-1)^2$

Now $\mathbf{R.H.S}=(a-d)^2=(a-ar^3)^2=a^2(1-r^3 1)^2=a^2(r^3-1)^2$

Thus $\mathbf{R.H.S}=\mathbf{L.H.S}$

ii) $\frac{ab-cd}{b^2-c^2}=\frac{a+c}{b}$

$\frac{ab-cd}{b^2-c^2}=\frac{a.ar-ar^2.ar^3}{a^2r^2-(ar^2)^2}=\frac{1-r^4}{r^2-r^4}=\frac{1-r^4}{r(1-r^2)}$

$=\frac{(1-r^2)(1+r^2)}{r(1-r^2)}=\frac{1+r^2}{r}$

Now $\frac{a+c}{b}=\frac{a+ar^2}{ar}=\frac{1+r^2}{r}$

Thus $\mathbf{R.H.S}=\mathbf{L.H.S}$

Task

If a,b,c,d are in a G.P, show that

i) $a^2b^2c^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=a^3+b^3+c^3$

ii) $\frac{(a+b+c)^2}{a^2+b^2+c^2}=\frac{a+b+c}{a-b+c}$

iii) $(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2)$

iv) $a^2+b^2,ab+bc,b^2+c^2$, are in a G.P

4. *If a,b,c form an A.P, b,c,a form a G.P, show that $\frac{1}{c},\frac{1}{a},\frac{1}{b}$ form an A.P.*

Solution

If a, b, c *form an A.P, then*

$b - a = c - b$ *or* $a + c = 2b$ ..... (1)

b, c, a *, form a G.P if;*

$\frac{c}{b} = \frac{a}{c}$ *or* $ab = c^2$ ..... (2)

Now for $\frac{1}{c}, \frac{1}{a}, \frac{1}{b}$ to be in an A.P, then

$\frac{1}{a} - \frac{1}{c} = \frac{1}{b} - \frac{1}{a} \Rightarrow \frac{2}{a} = \frac{1}{b} + \frac{1}{c}$

Generating $\frac{2}{a} = \frac{1}{b} + \frac{1}{c}$ from eqn. (1) and eqn. (2)

Eqn. (1) $\div$ eqn. (2)

$\Rightarrow \frac{a+c}{ab} = \frac{2b}{c^2}$

But $ab = c^2$,

$\Rightarrow \frac{1}{b} + \frac{c}{c^2} = \frac{2b}{ab} \therefore \frac{2}{a} = \frac{1}{b} + \frac{1}{c}$

5. *If* a, b, c *are in A.P,* a, x, b *and* b, y, c *are in a G.P, show that* x^2, b^2, y^2 *are in an A.P.*

Solution

a, b, c are in an A.P if;

$b - a = c - b$ *or* $2b = a + c$ ..... (1)

a, x, b are in a G.P if;

$\frac{x}{a} = \frac{b}{x}$ *or* $x^2 = ab$ ..... (2)

b, y, c are in a G.P if;

$\frac{y}{b} = \frac{c}{y}$ *or* $y^2 = bc$ ..... (3)

Now x^2, b^2, y^2 are in an A.P if;

$b^2 - x^2 = y^2 - b^2$ *or* $2b^2 = x^2 + y^2$ ..... (4)

Generating eqn. (4) from the other three equations

Now eqn. (2) + eqn. (3)

$\Rightarrow x^2 + y^2 = ab + bc = b(a + c)$

From eqn. (1), $2b = a + c$

$\therefore x^2 + y^2 = 2b^2$, which is eqn. (4). Hence x^2, b^2, y^2 are in an A.P.

Task

1. *If* a, b, c *are in G.P, prove that* $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ *are in an A.P.*
2. *If* b, a, c *are in an A.P and* c, b, a *are in a G.P, show that* $\frac{a+b}{b^2} = \frac{2}{c}$

Combination of A.P and G.P questions

1) Three numbers are in an A.P and their sum is 21, if $1, 5, 15$ *be added to them respectively, they form a G.P. find the numbers.*

Solution

Let the 3 numbers in an A.P be $a - d, a, a + d$

Given: $a - d + a + a + d = 21 \therefore a = 7$

Also: $(a - d + 1), (a + 5), (a + d + 15)$ *are in a G.P*

$\Rightarrow \frac{a+5}{a-d+1} = \frac{a+d+15}{a+5}$, but $a = 7$

$\Rightarrow \frac{7+5}{7-d+1} = \frac{7+d+15}{7+5}$

$\Rightarrow \frac{12}{8-d} = \frac{22+d}{12}$ *or* $144 = (8-d)(22+d) = 176 + 8d - 22d - d^2$

$\Rightarrow d^2 + 14d - 32 = 0$

$\Rightarrow d^2 + 16d - 2d - 32 = 0$ *or* $(d-2)(d+16) = 0$

$\therefore d = 2, -16$

Thus for $a = 7, d = 2$, the numbers are: $6, 12, 24$

For $a = 7, d = -16$, the numbers are: $6, 12, 24$

2) *The sequence of three numbers a, b, c is an A.P whose sum is 18. If a and b are each increased by 4 and c increased by 36, the new numbers form a G.P. find a, b, c.*

Solution

a, b, c are in an A.P if;

$b - a = c - b$ *or* $2b = a + c$ ..... (1)

$a + b + c = 18$ ..... (2)

Given: $a + 4, b + 4, c + 36$ *are in a G.P if;*

$\frac{b+4}{a+4} = \frac{c+36}{b+4} \Rightarrow (b+4)^2 = (a+4)(c+36)$ ..... (3)

Solving the equations simultaneously

Eqn. (1) in to eqn. (2)

$\Rightarrow b + 2b = 18, b = 6$

Put $b = 6$ in to eqn. (1) and (3)

$\Rightarrow a + c = 12$ ..... (4)

$\Rightarrow 100 = (a+4)(c+36)$ ..... (5)

From eqn. (4) $a = 12 - c$..... (6)

Eqn. (6) in to (5)

$\Rightarrow 100 = (12 - c + 4)(c + 36) = (16 - c)(c + 36)$

$\Rightarrow 100 = -c^2 - 20c + 576$

$\therefore c^2 + 20c + 476 = 0 \ \therefore c = 14, -34$

From eqn. (6), $a = 12 - c$

For $c = 14, a = 12 - 14 = -2$

For $c = -34, a = 12 + 34 = 46$

Hence numbers are: $-2, 6, 14$ *or* $46, 6, -34$

3) *The 1st, 10th and 28th terms of an A.P are three successive terms of a G.P. find the common ratio of a G.P. given that the sum of the first 28 terms of an A.P is 210, find its first term.*

Solution

Given: $a_1 = a, a_{10} = a + 9d, a_{28} = a + 27d$

Now $a, a + 9d, a + 27d$ are in a G.P

$\Rightarrow \frac{a+9d}{a} = \frac{a+27d}{a+9d}$ ..... (1)

Let the three consecutive terms of a G.P be: a, ar, ar^2

Note: the first term of an A.P is equal to the first term of a G.P

$\Rightarrow ar = a + 9d$ *or* $ar - a = 9d$ ..... (2)

$\Rightarrow ar^2 = a + 27d$ *or* $ar^2 - a = 27d$ ..... (3)

Given: $S_{28} = \frac{28}{2}(2a + 27d) = 210$

$\Rightarrow 2a + 27d = 15$ ..... (4)

$eqn.(2) \div eqn.(3)$

$\Rightarrow \frac{ar-a}{ar^2-a} = \frac{r-1}{r^2-1} = \frac{1}{3}$

$\Rightarrow \frac{r-1}{(r+1)(r-1)} = \frac{1}{3}$ *or* $3 = r+1 \ \ \therefore r = 2$

Substitute $r = 2$ in to eqn. (2)

$\Rightarrow 2a - a = 9d \therefore a = 9d$ ..... (5)

eqn. (5) in to eqn. (4)

$\Rightarrow 2(9d) + 27d = 15 \ \ \therefore d = \frac{15}{45} = \frac{1}{3}$

$\therefore 1^{st}\ term = 9d = 9 \times \frac{1}{3} = 3$

4) *The sum of the first three terms of two series: one an A.P and the other a G.P is the same. If the first term of each of these is* $\frac{2}{3}$ *and the common difference of an A.P is equal to the common ratio of the G.P, find the sum of each series to 2o terms.*

Solution

In an A.P, $S_3 = \frac{3}{2}(2a + 2d) = 3a + 3d$

In a G.P, $S_3 = \frac{a(1-r^3)}{1-r} = \frac{a(1-r)(1+r+r^2)}{1-r} = a(1 + r + r^2)$

Given: $a = \frac{2}{3}, r = d$

$\Rightarrow 3a + 3d = a(1 + r + r^2)$

$\Rightarrow 3 \times \frac{2}{3} + 3r = \frac{2}{3}(1 + r + r^2)$ *or* $6 + 9r = 2 + 2r + 2r^2$

$\Rightarrow 2r^2 - 7r - 4 = 0$ *or* $(2r + 1)(r - 4) = 0$

$\therefore r = 4, -\frac{1}{2}$

Now when $r = d = 4$, for an A.P,

$S_{20} = \frac{20}{2}\left(2 \times \frac{2}{3} + 19 \times 4\right) = 773\frac{1}{3}$

For a G.P, $S_{20} = \frac{\frac{2}{3}(4^{20}-1)}{4-1} = \frac{2}{9}(4^{20} - 1)$

Also when $r = d = -\frac{1}{2}$, for an A.P

$S_{20} = \frac{20}{2}\left(2 \times \frac{2}{3} + 19 \times -\frac{1}{2}\right) = -\frac{245}{3}$

For a G.P, $S_{20} = \frac{\frac{2}{3}\left(1-\left(-\frac{1}{2}\right)^{20}\right)}{1-\left(-\frac{1}{2}\right)} = \frac{1}{9}\frac{(2^{20}-1)}{2^{18}}$

5) *Find three numbers* a, b, c *between 2 and 18 such that their sum is 25, the numbers* $2, a, b$ *are consecutive terms of an A.P and the numbers* $b, c, 18$ *are the consecutive terms of a G.P.*

Solution

Given: $a + b + c = 18$ ..... (1)

$2, a, b$ form an A.P

$\Rightarrow a - 2 = b - a$ *or* $2a = b + 2$ ..... (2)

Also $b, c, 18$ form a G.P

$\Rightarrow \frac{c}{b} = \frac{18}{c}$ *or* $c^2 = 18b$ ..... (3)

Solving the equations simultaneously

Eqn. (3) in to eqn. (2)

$\Rightarrow 2a = \frac{c^2}{18} + 2$ *or* $36a = c^2 + 36$ ..... (4)

Eliminating b from eqn. (1) and eqn. (2)

From eqn. (2), $b = 2a - 2$

Substituting In equation (1)

$\Rightarrow a + 2a - 2 + c = 25$

$\Rightarrow 3a + c = 27$ ..... (5)
From eqn. (5), $c = 27 - 3a$
Substituting In equation (4)
$\Rightarrow 36a = (27 - 3a)^2 + 36$
$\Rightarrow 36a = 9(9 - a)^2 + 36$ .....$(\div 9)$
$\Rightarrow 4a = 81 - 18a + a^2 + 4 \ or\ a^2 - 22a + 85 = 0$
$\Rightarrow (a - 5)(a - 17) = 0 \ \therefore a = 5\,, 17$
From $c = 27 - 3a$
For $a = 5\,, c = 27 - 15 = 12$
For $a = 17\,, c = 27 - 51 = -24\ (Rejected)$
Thus $a = 5\,, c = 12$ are used to fond b.
From eqn. (3), $b = \frac{c^2}{18}$, for $c = 12$
$\Rightarrow b = \frac{144}{18} = 8$
$\therefore a = 5\,, b = 8\,. c = 12$
Hence the numbers are: $5\,, 8\,, 12$

6) *Three integers form an increasing G.P. if the third is decreased by 16 an A.P is formed. If the second number is decreased by 2, a G.P is formed. Find the numbers.*

Solution

Let the integers be: $a\,, ar\,, ar^2$
Decreasing the 3rd number by 16 gives $a\,, ar\,, ar^2 - 16$ are in an A.P
$\Rightarrow ar - a = ar^2 - 16 - ar$
$\Rightarrow 2ar - a = ar^2 - 16$ ..... (1)
Decreasing the 2nd number by 2 from the new set of numbers give $a\,, ar - 2\,, ar^2 - 16$ are in a G.P
$\Rightarrow \frac{ar-2}{a} = \frac{ar^2-16}{ar-2}$
$\Rightarrow (ar - 2)^2 = a(ar^2 - 16) \ or\ a^2r^2 - 4ar + 4 = a^2r^2 - 16a$
$\Rightarrow -4ar + 4 = -16a \ or\ 4 = a(4r - 16)$
$\therefore a = \frac{1}{r-4}$ ..... (2)
$eqn.(2) in\ to\ eqn.(1)$
$\Rightarrow 2\left(\frac{1}{r-4}\right)r - \frac{1}{r-4} = \frac{1}{r-4} \times r^2 - 16$
$\Rightarrow \frac{2r-1}{r-4} = \frac{r^2-16r+64}{r-4} \ \ or\ \ 2r - 1 = r^2 - 16r + 64$
$\Rightarrow r^2 - 18r + 64 = 0 \ or\ (r - 5)(r - 13) = 0$
$\therefore r = 5\,, 13$
From eqn. (2), $a = \frac{1}{r-4}$
For $r = 5\,, a = 1$
For $r = 13\,, a = \frac{1}{9}$
Thus the numbers are formed by $a = 1\,, r = 5\,. i.e. 1\,, 5\,, 25$

Task

1. *The* 2nd , 4th *and* 8th *terms of an A.P series are in a G.P series. The sum of the* 3rd *and* 5th *terms is 20. Find the sum of the first 20 terms of the progression.* **ANSWER**: 525

2. *An arithmetic series and geometric series have r as the common difference and the common ratio respectively. The first term of the arithmetic series is 1 and the first term of the geometric series is 2. If the* 4th *term of the arithmetic series is equal to the sum of the third and fourth terms of the geometric series, find three possible values of r when* $|r| < 1$, *find in the form* $p + q\sqrt{2}$ *the sum of the first* $10 -$ *terms of the arithmetic series.* **ANSWER**: $\frac{-70+45\sqrt{2}}{2}$

3. *A G.P and an A.P have the same first term. The sums of their first, second and third terms are* 6 , 10 , 10.5 *and 18 respectively. Calculate the sum of their fifth terms.* (***Uneb* 1995**)

   HINT:
   For an A.P: $a = 1^{st}\ term$, $a_2 = a + d$, $a_3 = a + 2d$, $a_5 = a + 4d$
   For a G.P: $a = 1^{st}\ term,$, $a_2 = ar$, $a_3 = ar^2$, $a_5 = ar^4$
   $Now\ a + a = 6\ \therefore a = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$
   $\Longrightarrow a + d + ar = 10.5\ \therefore 3 + d + 3r = 10.5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$
   $\Longrightarrow a + 2d + ar^2 = 18\ \therefore 3 + 2d + 3r^2 = \cdots\ldots\ldots\ldots\ldots\ldots\ldots (3)$
   $Solving\ eqn.(2) and\ eqn.(3) gives; 3 + 6r - 3r^2 = 3\ \therefore 2r - r^2 = 0$
   $\therefore r = 0 , r = 2$
   $From\ eqn.(2)\ d = 10.5 - 3 - 6 = 1.5\ \therefore d = 1.5$
   $Now\ a_5 = a + 4d + ar^4 = Sum\ of\ their\ fifth\ terms$
   $\therefore 3 + 4 \times 1.5 + 3 \times 2^4 = 57$

4. *The* $2^{nd}\ term$, $5^{th}\ term\ and\ 11^{th}\ term$ *of an A.P are in a G.P. if the* $11^{th}\ term$ *is 4, find the difference between the sums of the* $8 -$ *terms of each progressions.*

   Solution

   Given: $a_2 = a + d$, $a_5 = a + 4d$, $a_{11} = a + 10d = 4$ *are in a G.P;*
   $Let\ a = 1^{st}\ term\ d = common\ difference$
   $\Longrightarrow \frac{a+4d}{a+d} = \frac{a+10d}{a+4d} = r = common\ ratio$
   $\Longrightarrow (a + 4d)^2 = (a + 10d)(a + d)$ …… (1)
   Also $a + 10d = 4$ …… (2)
   Solving eqn. (2) and eqn. (1)
   $\Longrightarrow (4 - 10d + 4d)^2 = 4(4 - 10d)\ \ or\ \ 4(2 - 3d)^2 = 4(4 - 10d)$
   $\Longrightarrow 4 - 12d + 9d^2 = 4 - 9d\ \therefore 9d^2 - 3d = 0$
   $\therefore d = 0 , \frac{1}{3}$
   From eqn. (2) and for $d = \frac{1}{3}$, $a = 4 - \frac{10}{3} = \frac{2}{3}$
   For an A.P, $S_8 = \frac{8}{2}\left[2 \times \frac{2}{3} + 7 \times \frac{1}{3}\right] = 4 \times \frac{11}{3} = \frac{44}{3}$
   Now the common ratio, $r = \frac{a+4d}{a+d} = \frac{\frac{2}{3}+\frac{4}{3}}{\frac{2}{3}+\frac{1}{3}} = 2$
   Now the G.P is; $(a + d) , (a + 4d) , (a + 10d)$
   $1^{st}\ term = a + d = \frac{2}{3} + \frac{1}{3} = 1$

Using $S_n = \frac{a(r^n-1)}{r-1}$

$\Rightarrow S_8 = \frac{2^8-1}{2-1} = 255$

$\therefore S_{8(G.P)} - S_{8(A.P)} = 255 = \frac{44}{3} = \frac{721}{3}$

5. *If a, b, c are in a G.P, show that $\log a, \log b, \log c$ are in an A.P.*

TO FIND THE SUM TO INFINITY OF A G.P WHOSE COMMON RATIO IS LESS THAN ONE

Let $a, ar, ar^2, ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots ar^{n-1}$ be a G.P with $|r| < 1$

Sum of $n-terms$ is $S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$

Since $|r| < 1$, then $r^2 < r, r^3 < r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots r^n < r^{n-1}$

As the power of r goes on increasing, the corresponding term in a G.P decrease in value. So assuming that n becomes indefinitely large, r^n becomes indefinitely small. $i.e. r^n \longrightarrow 0$

Thus from $S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$

As $n \longrightarrow \infty, r^n \longrightarrow 0$

$$\therefore \boldsymbol{S_\infty} = \frac{\boldsymbol{a}}{\boldsymbol{1-r}} \text{ , where } |r| < 1 \text{ or } -1 < r < 1$$

This is the sum to infinity. $i.e.$ the limiting value to which the sum approaches/tends to.

$i.e. a + ar + ar^2 + ar^3 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{a}{1-r} = \boldsymbol{S_\infty}$

Examples

1. *Express the recurring decimals to fraction form in lowest term.*
   a) $0.1\dot{7}$
   b) $0.0\dot{7}$
   c) $1.\overline{45}$

Solution

a) $0.1\dot{7} = 0.17777\ldots\ldots\ldots\ldots$

$0.1\dot{7} = 0.1 + 0.07 + 0.007 + 0.0007 + \cdots\ldots\ldots\ldots$

$= \frac{1}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \cdots\ldots\ldots\ldots\ldots\ldots\ldots$

$= \frac{1}{10} + \left\{\frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \cdots\ldots\ldots\ldots\ldots\ldots\right\}$

The bracketed series is a G.P with first term as $\frac{7}{100}$ and common ratio as $\frac{1}{10}$

$\Rightarrow \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \cdots\ldots\ldots\ldots\ldots\ldots\ldots = \boldsymbol{S_\infty} = \frac{\frac{7}{100}}{1-\frac{1}{10}} = \frac{7}{90}$

$\therefore 0.1\dot{7} = \frac{1}{10} + \frac{7}{90} = \frac{8}{45}$

b) $0.0\dot{7} = 0.07777\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= 0.07 + 0.007 + 0.0007 + \cdots\ldots\ldots\ldots\ldots\ldots$

$= \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \cdots\ldots\ldots\ldots\ldots\ldots = \boldsymbol{S_\infty} = \frac{\frac{7}{100}}{1-\frac{1}{10}} = \frac{7}{90}$

$\therefore 0.0\dot{7} = \frac{7}{90}$

c) $1.\overline{45} = 1.45454545\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= 1 + 0.45 + 0.0045 + 0.00045 + \cdots\ldots\ldots\ldots$

$$= 1 + \left\{\frac{45}{100} + \frac{45}{10000} + \frac{45}{1000000} + \cdots \ldots\ldots\ldots\ldots\right\}$$

$$= 1 + \frac{\frac{45}{100}}{1-\frac{1}{100}} = \frac{45}{99} + 1 = \frac{16}{11}$$

$\therefore 1.\overline{45} = \frac{16}{11}$

2. a) *Find the first term of a G.P whose sum to infinite terms is 8 and the second term is 2*

b) The sum of infinite terms of a G.P is 15 and the sum of their squares is 45. Find the series.

Solution

a) *Given:* $S_\infty = 8 = \frac{a}{1-r}$, $ar = 2$

Dividing the equations

$\frac{8}{2} = \frac{\frac{a}{1-r}}{ar} = \frac{1}{r-r^2}$ or $4r - 4r^2 = 1 \ \therefore 4r^2 - 4r + 1 = 0$

$\therefore r = \frac{1}{2}$

$\therefore 1^{st}$ term $= a = \frac{2}{r} = \frac{2}{\frac{1}{2}} = 4$

b) Let $a, ar, ar^2, ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ be the G.P with $|r| < 1$

Given: $S_\infty = a + ar + ar^2 + ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{a}{1-r} = 15$ ..... (1)

Sum to infinity of squares $= 45$

$\Longrightarrow S_\infty = a^2 + a^2r^2 + a^2r^4 + a^2r^6 \ldots\ldots\ldots\ldots\ldots\ldots = \frac{a^2}{1-r^2} = 45$ ..... (2)

From eqn. (1), $a = 15(1-r)$, substituting in eqn. (2)

$\Longrightarrow \frac{[15(1-r)]^2}{1-r^2} = 45$ *or* $\frac{225(1-r)^2}{(1+r)(1-r)} = 45$

$\Longrightarrow \frac{225(1-r)}{1+r} = 45 \ \therefore 5 - 5r = 1 + r \ \therefore r = \frac{2}{3}$

But $a = 15(1-r) \Longrightarrow a = 15\left(1 - \frac{2}{3}\right) = 5$

Thus the series is $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots$

3. *The second term of a G.P is 24 and its sum to infinity is 100. Find two possible values of the common ratio and the corresponding first terms.*

Solution

Given: $ar = 24$..... (1)

$S_\infty = a + ar + ar^2 + ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{a}{1-r} = 100$

$\therefore a = 100 - 100r$ ..... (2)

Eqn. (2) in to eqn. (1) to eliminate a

$(100 - 100r)r = 24$ *or* $25r - 25r^2 - 6 = 0$

$\therefore 25r^2 - 25r + 6 = 0$

$\Longrightarrow r = \frac{25 \pm \sqrt{(25)^2 - 4 \times 25 \times 6}}{2 \times 25} = \frac{25 \pm 5}{50}$

$\therefore r = \frac{3}{5}, \frac{2}{5}$

From eqn. (1), for $r = \frac{3}{5}$, $a = \frac{24}{\frac{3}{5}} = 40$

For $r = \frac{2}{5}$, $a = \frac{24}{\frac{2}{5}} = 60$

$\therefore \frac{3}{5}, 40 ; \frac{2}{5}, 60$

4. *The sum of infinite terms of a G.P is 4 and the sum of their cubes is 192. Find the series.*

Solution

Let the series be: $a\,, ar\,, ar^2\,, ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Cubing gives $a^3\,, a^3r^3\,, a^3r^6\,, a^3r^9 \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Given: $S_\infty = a^3 + a^3r^3 + a^3r^6 + a^3r^9 \ldots\ldots\ldots\ldots\ldots = \frac{a^3}{1-r^3} = 192$

$\frac{a^3}{1-r^3} = \frac{a^3}{(1-r)(1+r+r^2)} = 192$ …… (1)

Also $S_\infty = a + ar + ar^2 + ar^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{a}{1-r} = 4$

$\frac{a}{1-r} = 4$ *or* $a = 4(1-r)$ …… (2)

$eqn.(2) in\ to\ eqn.(1)$

$\Longrightarrow \frac{[4(1-r)]^3}{(1-r)(1+r+r^2)} = 192$

$\Longrightarrow \frac{64(1-r)^3}{(1-r)(1+r+r^2)} = 192$ *or* $\frac{(1-r)^2}{(1+r+r^2)} = 3$

$\Longrightarrow 1 - 2r + r^2 = 3 + 3r + 3r^2$

$\therefore 2r^2 + 5r + 2 = 0$ *or* $(2r+1)(r+1) = 0$

$\therefore r = -\frac{1}{2}\,, -1$, rejecting $r = -1$ since $|r| < 1$

For $r = -\frac{1}{2}\,, a = 4 - 4 \times -\frac{1}{2} = 6$

$\therefore$ The series is $6\,, -3\,, \frac{3}{2}\,, -\frac{3}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Task

1. *Show that there are two possible G.Ps, in each of which the first term is 8, and the sum of three terms is 14. Find the sum to infinity in each.*
2. *The sum infinity of a G.P with positive common ratio is 9 and the sum of the first two terms is 5. Find the first four terms of the G.P.*

   ANSWER: $\mathbf{3}\,, \mathbf{2}\,, \frac{\mathbf{4}}{\mathbf{3}}\,, \frac{\mathbf{8}}{\mathbf{9}}$
3. *Show that there are two G.P's in which the second term is* $-\frac{4}{3}$ *and the sum of the first* $3-terms$ *is* $\frac{28}{9}$*. Show that one of these progressions converges to a sum of 3.*
4. *If* $x = 1 + a + a^2 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +\infty\,, y = 1 + b + b^2 + \cdots\ldots\ldots\ldots\ldots\ldots\ldots +\infty$*, prove that if* $|a| < 1$ *and* $|b| < 1$ *then* $1 + ab + a^2b^2 + \cdots\ldots\ldots\ldots\ldots\ldots = \frac{xy}{x+y-1}$
5. *Show that the sum of the series* $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \frac{1}{2^7} + \cdots\ldots to\ \infty = \frac{19}{24}$
6. a) If $S_1\,, S_2\,, S_3\,, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots S_n$ are sums of infinite G.P series whose first terms are $1\,, 2\,, 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, n$ and the common ratios are $\frac{1}{2}\,, \frac{1}{3}\,, \frac{1}{4}\,, \ldots\ldots\ldots, \frac{1}{n+1}$ respectively. Show that $S_1 + S_2 + S_3 + \cdots\ldots\ldots\ldots + S_n = \frac{n(n+3)}{2}$

   b) If $S_1\,, S_2$ *and* S are sums of $n - terms, 2n - terms$ and to infinity of a G.P respectively, show that $S_1(S_1 - S) = S(S_1 - S_2)$

   Solution

   a) $S_\infty = \frac{a}{1-r}$

   $S_1 = \frac{1}{1-\frac{1}{2}} = 2\,, S_2 = \frac{2}{1-\frac{1}{3}} = 3\,, S_3 = \frac{3}{1-\frac{1}{4}} = 4\,, S_n = \frac{n}{1-\frac{1}{n+1}} = n + 1$

L. H. S: $S_1 + S_2 + S_3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + S_n = 2 + 3 + 4 + \cdots \ldots\ldots\ldots + n + 1$
This is an A.P with $1^{st}\ term = 2\ , common\ diffreerence = 1$ and $last\ term = n + 1$
Using $S_N = \frac{N}{2}(1^{st}\ term + last\ term)$. Finding N, using
Using $a_N = A + (N-1)D\ \ or\ N = \frac{a_N - A}{D} + 1$
$\Longrightarrow N = \frac{n+1-2}{1} + 1 = n$
$\therefore S_n = \frac{n}{2}(2 + n + 1) = \frac{n}{2}(n + 3)$

b) *Given:* $S_n = S_1\ , S_{2n} = S_2\ , S_\infty = S_3$*; let* $1^{st}\ term = a, common\ ratio = r$
Using $S_n = \frac{a(1-r^n)}{1-r} = S_1$ ..... (1)
$S_{2n} = \frac{a(1-r^{2n})}{1-r} = S_2$ ..... (2)
$S_\infty = \frac{a}{1-r} = S$ ..... (3)
Now $S_1(S_1 - S) = \frac{a(1-r^n)}{1-r}\left[\frac{a(1-r^n)}{1-r} - \frac{a}{1-r}\right]$
$= \frac{a^2(1-r^n)}{(1-r)^2}[1 - r^n - 1] = \frac{-a^2r^n(1-r^n)}{(1-r)^2}$
$\therefore S_1(S_1 - S) = \frac{a^2r^n(r^n-1)}{(1-r)^2}$
Also $S(S_1 - S_2) = \frac{a}{1-r}\left[\frac{a(1-r^n)}{1-r} - \frac{a(1-r^{2n})}{1-r}\right]$
$= \frac{a^2}{(1-r)^2}[1 - r^n - 1 + r^{2n}]$
$\therefore S(S_1 - S_2) = \frac{a^2}{(1-r)^2}(r^{2n} - r^n) = \frac{a^2r^n(r^n-1)}{(1-r)^2}$
$\therefore S_1(S_1 - S) = S(S_1 - S_2) = \frac{a^2r^n(r^n-1)}{(1-r)^2}$

5. *Find the sum to infinity of the G.P* $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \ldots\ldots\ldots\ldots\ldots$ *and show that is S denotes* S_∞ *and* S_n *denotes sum of* $n -$ *terms then* $S - S_n < \frac{1}{1000}$ *if* $n \geq 11$.

Solution

$S = \frac{1}{1-\frac{1}{2}} = 2\ , S_n = \frac{a(1-r^n)}{1-r} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$
Now $S - S_n < \frac{1}{1000} \Longrightarrow 2 - 2\left(1 - \frac{1}{2^n} < \frac{1}{1000}\right)$
$\Longrightarrow \frac{2}{2^n} < \frac{1}{1000}\ \ or\ 2^{n-1} < 10^{-3}$
Introducing logarithms
$\Longrightarrow \log(2^{n-1}) < \log 10^{-3}$
$\Longrightarrow (1-n)\log 2 < -3\log 10$
$\Longrightarrow 1 - n < -\frac{3}{\log\ 2}$
$\therefore n > 1 + \frac{3}{\log\ 2}\ \ or\ n > 10.96578425$
$\therefore n \geq 11$

PROOF BY INDUCTION

Proof by induction is a mathematical technique that uses reasoning that if "proposition P_n" is true for a particular value of n, say $n = 1$ then it must be true for the next value of $n.i.e.\ n = 2$, and now if it is true for $n = 2$, then it must be also true for $n = 3$ and hence true for all positive integral values of n.

Characteristics of positive integral values (Natural numbers)

Axiom of Induction:

If T is a set of natural numbers with properties;

i) $1 \in T$ and

ii) $k \in T$ implies $k + 1 \in T$

, then T is a set of Natural numbers N.

Note:

Both conditions must be satisfied for the Axiom of induction to hold.

Illustration

Consider $T = \{1, 2, 3\}$

Clearly $1 \in T$, but is not true that $k + 1 \in T$ every time $k \in T\ e.g\ 3 \in T$ but $4 \notin T$. Thus $T \neq N$

Also let $T = \{10, 11, 12, \ldots\ldots\ldots\}$

Clearly if $k \in T$, then $k + 1 \in T$ but $1 \notin T$. Thus again $T \neq N$

Now let $T = \{1, 2, 3, \ldots\ldots\ldots\}$. Both $1, k, k + 1 \in T$, hence $T = N$.

Statement of Principle of Proof by Induction

For every $n \in N$, let P_n be a statement which is either true of false. If

i) P_1 is true and

ii) Whenever this statement is true for $n = k$, it is true for $n = k + 1$, then P_n is true for all $n \in N$

In general, to apply the statement of proof by induction, two things must be done.

i) First verify that P_1 is true and

ii) Verify that P_{k+1} is true whenever P_k is true.

Examples

Prove by induction that

a) $n^3 + 2n$ is divisible by 3

b) $n^3 - n$ is divisible by 6

c) $3^{2n} - 1$ is a multiple of 8

d) $9^n - 1$ is a multiple of 8

e) $8^n - 7n + 6$ is divisible by 7

Solution

a) Let $P_n = n^3 + 2n$

Verify that P_1 is true

$\Rightarrow P_1 = 1^3 + 2(1) = 3$, which is divisible by $3\ .i.e.P_1 = 3(1)$

Verify that whenever P_n is true for a natural number, say $n = k$, then it is true for the next natural number $n = k + 1$. To this you must assume that P_k is true for some$k \in N$. This assumption is the *induction hypothesis.*
Assume P_k is true $.i.e. k^3 + 2k$ is divisible by 3.
Next show thatP_k implies P_{k+1} $.i.e..i.e.(k+1)^3 + 2(k+1)$ is divisible by 3.
$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$
$= (k^3 + 2k) + 3k^2 + 3k + 3$
$= P(k) + 3(k^2 + k + 1)$
Since $P(k)$ is divisible by 3 and $3(k^2 + k + 1)$ is also divisible by 3, $P(k+1)$ is divisible by 3. Thus both conditions of principle of proof by induction are satisfied. You must conclude that$n^3 + 2n$ is divisible by 3 for$n \in N$.

NOTE:
The symbol $P(k)$ in the working stand for "$k^3 + 2k$". Don't confuse it with the symbol P_k which stands for the sentence "$k^3 + 2k$ is divisible by 3".
Similarly $P(k+1)$ stands for "$(k+1)^3 + 2(k+1)$"

b) Let $P_n = n^3 - n$
For $n = 1$, $P_1 = 1^3 - 1 = 0 = 6(0)$. Hence it is divisible by 6
Now for $n = k$, let P_k be true $.i.e. k^3 - k$ is divisible by 6.
Now for $n = k + 1$, P_{k+1} must also be true $.i.e.(k+1)^3 - (k+1)$ is divisible by 6
But $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$
$= (k^3 - k) + 3k^2 + 3k = P(k) + 3(k^2 + k)$
Since $P(k)$ is divisible by 6 and $3(k^2 + k)$ is divisible by 6 for any k≥ 1 and $k \in N$ then $P(k+1)$ is divisible by 6. Hence $n^3 - n$ is divisible by 6for$n \in N$.

c) Let $P_n = 3^{2n} - 1$
For $n = 1, P_1 = 3^{2(1)} - 1 = 8 = 8(1)$. Hence P_1 is true.
Now for $n = k$, let P_k be true $.i.e.\ 3^{2k} - 1$ be a multiple of 8.
Now for $n = k + 1$, P_{k+1} must also be true $.i.e. 3^{2(k+1)} - 1$a multiple of 8.
But $3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k}.3^2 - 1 = 9.3^{2k} - 1$
$$= 9\left[3^{2k} - \frac{1}{9}\right] = 9\left[3^{2k} - 1 + 1 - \frac{1}{9}\right]$$
$$= 9\left[(3^{2k} - 1) + \frac{8}{9}\right] = 9(3^{2k} - 1) + 8 = 9.P(k) + 8$$
Since $9.P(k)$ is a multiple of 8 and 8 is a multiple of 8, then$P(k+1)$ is a multiple of 8. Thus $3^{2n} - 1$ is a multiple of 8 for $n \in N$.

d) Let $P_n = 9^n - 1$ be a multiple of 8.
For $n = 1, P_1 = 9 - 1 = 8 = 8(1)$,hence P_1 is true.
Now for $n = k$, let P_k be true $.i.e.\ 9^k - 1$ be a multiple of 8.
Now for $n = k + 1$, P_{k+1} must also be true $.i.e. 9^{k+1} - 1$a multiple of 8
But $9^{k+1} - 1 = 9^k.9 - 1 = 9.[9^k - 1 + 1] - 1$
$= 9(9^k - 1) + 9 - 1 = 9.P(k) + 8$
Since $9.P(k)$ is a multiple of 8 and 8 is a multiple of 8, then$P(k+1)$ is a multiple of 8. Thus $9^n - 1$ is a multiple of 8 for$n \in N$.

e) Let $P_n = 8^n - 7n + 6$
For $n = 1, P_1 = 8^1 - 7.1 + 6 = 7 = 7(1)$, hence P_1 is true.
Now for $n = k$, let P_k be true $.i.e.\ 8^k - 7k + 6$ be divisible by 7.

Now for $n = k + 1$, P_{k+1} must also be true $.i.e. 8^{k+1} - 7(k+1) + 6$ be divisible by 7

But $8^{k+1} - 7(k+1) + 6 = 8^k.8 - 7k - 7 + 6 = 8^k.8 - 7k - 1$

$= 8[8^k - 7k + 6 + 7k - 6] - 7k - 1$

$= 8(8^k - 7k + 6) + 8(7k) - 48 - 1 = 8.P(k) + 7(k - 7)$

Since $8.P(k)$ is divisible by 7 and $7(k-7)$ is divisible by 7, then $P(k+1)$ is divisible by 7. Thus $8^n - 7n + 6$ is divisible by 7 for $n \in N$

Task

Prove by induction that for positive integer of n,

i) $n^3 + 3n^2 - 10n$ is divisible by 3
ii) $7^n - 4^n + 1^n$ is divisible by 6
iii) $10^n - 1$ is divisible by 9
iv) $7^n - 3^n$ is a multiple of 4
v) $10^{2n-1} + 1$ is divisible by 11
vi) $3^{2n} + 7$ is a multiple of 8
vii) $n(n+1)(n+2)$ is a multiple of 6
viii) $2^{4n-1} - 1$ is a multiple of 15
ix) $n^3 + 6n^2 + 8n$ is divisible by 3

Extension of Proof by Induction Principle

Suppose that for some proposition P_n where n is a positive integer and that P_1 is false/meaningless but true for $P_k \Rightarrow P_{k+1}$, clearly P_n cannot be proved for all positive integral values of n. however, provided that there is some positive integer m such that P_m is true, then the induction P_n holds for all integral values of n greater than or equal to m.

Examples

Prove by induction that $2^n + 3^{2n-3}$ *for* $n \geq 2$ *is always divisible by 7.*

Solution

Let $P_n = 2^n + 3^{2n-3}$

For $n = 2$, $P_2 = 2^2 + 3^{2\times2-3} = 7$, hence proof holds for $n = 2$

Now for $= k$, let P_k be true $i.e. 2^k + 3^{2k-3}$ be divisible by 7

Now for $n = k + 1, P_{k+1}$ must be true $i.e.$ $2^{k+1} + 3^{2(k+1)-3}$ is divisible by 7.

But $2^{k+1} + 3^{2(k+1)-3} = 2^k.2 + 3^{2k-1}$

$= 2^k.2 + 3^{2k}.3^{-1} = 2^k.2 + \frac{3^{2k}}{3}$

$= 2[2^k + 3^{2k-3} - 3^{2k-3}] + \frac{3^{2k}}{3} = 2(2^k + 3^{2k-3}) - 2.3^{2k-3} + \frac{3^{2k}}{3}$

$= 2(2^k + 3^{2k-3}) - 2.3^{2k}.3^{-3} + \frac{3^{2k}}{3} = 2(2^k + 3^{2k-3}) - 2.\frac{3^{2k}}{27} + \frac{3^{2k}}{3}$

$= 2(2^k + 3^{2k-3}) + 3^{2k}\left(-\frac{2}{27} + \frac{1}{3}\right)$

$= 2(2^k + 3^{2k-3}) + 3^{2k}.\frac{7}{27} = 2(2^k + 3^{2k-3}) + 7.3^{2k-3}$

$\therefore 2^{k+1} + 3^{2k-1} = 2P(k) + 7(3^{2k-3})$

Since $2P(k)$ is divisible by 7 and $7(3^{2k-3})$ is also divisible by 7, then $2^n + 3^{2n-3}$ is divisible by 7 for $n \geq 2$.

Proof by Induction of a finite series

Here a formula of $n-terms$ of a sum of finite series and a general term($n^{th}-term$) of the series must be known to prove by induction.

Examples

1. *Prove by induction that*
   a) $1+2+3+\cdots\ldots\ldots\ldots\ldots\ldots.+n=\frac{n(n+1)}{2}$
   b) $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\ldots\ldots\ldots\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$
   c) $1^2+2^2+3^2+\cdots\ldots\ldots\ldots\ldots\ldots.+n^2=\frac{n(n+1)(2n+1)}{6}$
   d) $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{1}{2^n}=\frac{2^n-1}{2^n}$

Solution

a) Let $P_n=1+2+3+\cdots\ldots\ldots\ldots\ldots\ldots.+n=\frac{n(n+1)}{2}$

For $n=1, P_1=1=\frac{1(1+1)}{2}=1$, thus $LHS=RHS$. Hence P_1 holds.

Now for $n=k$

Let P_k be true. $i.e. 1+2+3+\cdots\ldots\ldots\ldots\ldots\ldots.+k=\frac{k(k+1)}{2}$

Now for $n=k+1, P_{k+1}$ must be true. $i.e$

LHS $=+2+3+\cdots\ldots\ldots\ldots\ldots\ldots.+k+(k+1)$

RHS $=\frac{k(k+1)}{2}+k+1$

NOTE: On LHS of P_{k+1}, $(k+1)$ has been added, hence this proof is done by adding the general term of the series after replacing n by $k+1$ on either side of P_k.

$\therefore$ **RHS** $=\frac{k(k+1)}{2}+k+1=(k+1)\left[\frac{k}{2}+1\right]$

$=\frac{(k+1)(k+2)}{2}=\frac{(k+1)[(k+1)+1]}{2}$

Hence proof holds for $n=k+1$. Since the proof holds for $n=1, n=k$ and $n=k+1$, then it is true for all integral values of n.

b) $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\ldots\ldots\ldots\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Let $P_n=\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\ldots\ldots\ldots\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

For $n=1, P_1=\frac{1}{1\times2}=\frac{1}{1+1}=\frac{1}{2}$. Hence P_1 is true.

Now for $n=k$

Let P_k be true. $i.e.\ \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\ldots\ldots\ldots\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

Now for $n=k+1, P_{k+1}$ must be true. $i.e$

$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots\ldots\ldots\ldots+\frac{1}{k(k+1)}+\frac{1}{k(k+1)(k+1+1)}=\frac{k}{k+1}+\frac{1}{k(k+1)(k+1+1)}$

$=\frac{1}{k(k+1)(k+2)}$

$=\frac{1}{(k+1)}\left[k+\frac{1}{k+2}\right]$

$=\frac{1}{(k+1)}\left[\frac{k^2+2k+1}{k+2}\right]=\frac{1}{(k+1)}.\frac{(k+1)^2}{(k+2)}=\frac{k+1}{k+2}=\frac{k+1}{[(k+1)+1]}$

Hence proof holds for $n=k+1$

Note:

LHS and **RHS** for P_{k+1} has been increased by $\frac{1}{n(n+1)}$ for $n=k+1$.

$i.e.\frac{1}{k(k+1)(k+1+1)}=\frac{1}{k(k+1)(k+2)}$

Since proof holds for $n=1,k,k+1$, the proof holds for all integral values of n.

c) Let $P_n=1^2+2^2+3^2+\cdots\ldots\ldots\ldots\ldots\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

For $n=1,P_1=1^2=1=\frac{1(1+1)(2\times1+1)}{6}=1$. Hence P_1 is true.

Now for $n=k$

Let P_k be true. $i.e.\ 1^2+2^2+3^2+\cdots\ldots\ldots\ldots\ldots\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$

Now for $n=k+1,P_{k+1}$ must be true $.i.e$

$1^2+2^2+3^2+\cdots\ldots\ldots\ldots\ldots\ldots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$

$=\frac{(k+1)}{6}[k(2k+1)+6(k+1)]$

$=\frac{(k+1)}{6}[2k^2+k+6k+6]$

$=\frac{(k+1)(2k^2+7k+6)}{6}=\frac{(k+1)(2k^2+3k+4k+6)}{6}$

$=\frac{(k+1)(2k+3)(k+2)}{6}$

$=\frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$, hence proof holds.

Since the proof holds for $n=1,n=k$ and $n=k+1$, then it is true for all integral values of n

d) Let $P_n=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{1}{2^n}=\frac{2^n-1}{2^n}$

For $n=1,P_1=\frac{1}{2}=\frac{2^1-1}{2^1}=\frac{1}{2}$.Hence P_1 is true.

Now for $n=k$

Let P_k be true. $i.e.\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{1}{2^k}=\frac{2^k-1}{2^k}$

Now for $n=k+1,P_{k+1}$ must be true $.i.e$

$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=\frac{2^k-1}{2^k}+\frac{1}{2^{k+1}}$

$=\frac{1}{2^{k+1}}\left[\frac{2^k-1}{2^{-1}}+1\right]=\frac{1}{2^{k+1}}[2(2^k-1)+1]$

$=\frac{1}{2^{k+1}}[2^{k+1}-1]=\frac{2^{k+1}-1}{2^{k+1}}$, hence proof holds.

Since the proof holds for $n=1,n=k$ and $n=k+1$, then it is true for all integral values of n

2. *Prove by induction that*

   a) $\sum_{r=2}^{n}\frac{1}{r^2-1}=\frac{3}{4}-\frac{2n+1}{2n(n+1)}$

   b) $\sum_{r=1}^{n}r(r+1)=\frac{1}{3}n(n+1)(n+2)$

   c) $\sum_{r=1}^{n}\frac{r}{2^r}=2-\frac{n+2}{2^n}$

   d) $\sum_{r=1}^{n}3^{r-1}=\frac{3^n-1}{2}$

Solution

a) $\sum_{r=2}^{n}\frac{1}{r^2-1}=\frac{3}{4}-\frac{2n+1}{2n(n+1)}\ ;r=2,3,4,\ldots\ldots\ldots\ldots\ldots,n$

Generating the series;

$\Rightarrow\frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}+\cdots\ldots\ldots\ldots\ldots+\frac{1}{n^2-1}$

$\Rightarrow\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{1}{n^2-1}=\frac{3}{4}-\frac{2n+1}{2n(n+1)}$

Let $P_n=\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{1}{n^2-1}=\frac{3}{4}-\frac{2n+1}{2n(n+1)}$

For $n = 2, P_2 = \frac{1}{3} = \frac{3}{4} - \frac{2.2+1}{2.2(2+1)} = \frac{3}{4} - \frac{5}{12} = \frac{1}{3}$. Hence P_2 is true.

For $n = k$, let P_k be true $. i.e. \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{k^2-1} = \frac{3}{4} - \frac{2k+1}{2k(k+1)}$

Now for $n = k+1, P_{k+1}$ must be true $. i.e$

$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{k^2-1} + \frac{1}{(k+1)^2-1} = \frac{3}{4} - \frac{2k+1}{2k(k+1)} + \frac{1}{(k+1)^2-1}$

$= \frac{3}{4} + \frac{1}{(k+1)^2-1} - \frac{2k+1}{2k(k+1)} = \frac{3}{4} + \frac{1}{k(k+2)} - \frac{2k+1}{2k(k+1)}$

$= \frac{3}{4} + \frac{2(k+1)-(2k+1)(k+2)}{2k(k+1)(k+2)}$

$= \frac{3}{4} + \frac{2k+2-(2k^2+4k+k+2)}{2k(k+1)(k+2)} = \frac{3}{4} - \frac{2k^2+3k}{2k(k+1)(k+2)}$

$= \frac{3}{4} - \frac{k(2k+3)}{2k(k+1)(k+2)} = \frac{3}{4} - \frac{2(k+1)+1}{2k(k+1)[(k+1)+1]}$

Hence P_{k+1} is true for $n = k+1$. Thus since proof holds for $n = 2, k, k+1$, then it holds for all $n \geq 2$

b) $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)\,; r = 1, 2, 3, 4, \ldots\ldots\ldots\ldots\ldots\ldots, n$

Generating the series;

$\Rightarrow 1.(1+1) + 2.(2+1) + 3.(3+1) + \cdots\ldots\ldots\ldots + n(n+1)$

$= \frac{1}{3}n(n+1)(n+2)$

$\Rightarrow 2 + 6 + 12 + \cdots\ldots\ldots\ldots\ldots\ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

Let $P_n = 2 + 6 + 12 + \cdots\ldots\ldots\ldots\ldots\ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

For $= 1, P_1 = 2 = \frac{1}{3}.1.(1+1)(1+2) = 2$. Hence P_2 is true.

For $n = k$, let proof be true $. i.e. 2 + 6 + 12 + \cdots\ldots\ldots\ldots\ldots\ldots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$

Now for $n = k+1, P_{k+1}$ must be true $. i.e$

$2 + 6 + 12 + \cdots\ldots\ldots + k(k+1) + (k+1)(k+2)$

$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$

$= (k+1)(k+2)\left[\frac{k}{3} + 3\right] = \frac{(k+1)(k+2)(k+3)}{3} = \frac{(k+1)[(k+1)+1][(k+1)+2]}{3}$

Hence P_{k+1} is true for $n = k+1$. Thus since proof holds for $n = 2, k, k+1$, then it holds for all integral values of n.

c) $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}\,; r = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots\ldots, n$

Generating the series;

$\Rightarrow \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots\ldots\ldots\ldots\ldots\ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

Let $P_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots\ldots\ldots\ldots\ldots\ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

For $n = 1, P_1 = \frac{1}{2} = 2 - \frac{1+2}{2^1} = \frac{1}{2}$. Hence P_1 is true.

For $n = k$, let the proof be true $. i.e. \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots\ldots\ldots\ldots\ldots\ldots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$

Now for $n = k+1, P_{k+1}$ must be true $. i.e$

$\Rightarrow \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots\ldots\ldots\ldots\ldots\ldots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$

$= 2 + \frac{k+1}{2^{k+1}} - \frac{k+2}{2^k} = 2 + \frac{1}{2^k}\left[\frac{k+1}{2} - k + 2\right]$

$= 2 + \frac{1}{2^{k+1}}[-k-3] = 2 - \frac{k+3}{2^{k+1}}$

$= 2 - \frac{[(k+1)+2]}{2^{k+1}}$

Hence P_{k+1} is true for $n = k+1$. Thus since proof holds for $n = 2, k, k+1$, then it holds for all integral values of n.

d) $\sum_{r=1}^{n} 3^{r-1} = \frac{3^n-1}{2}$; $r = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots, n$

Generating the series;

$\Rightarrow 3^{1-1} + 3^{2-1} + 3^{3-1} + \cdots \ldots\ldots\ldots\ldots + 3^{n-1} = \frac{3^n-1}{2}$

Let $P_n = 1 + 3 + 3^2 + \cdots \ldots\ldots\ldots\ldots + 3^{n-1} = \frac{3^n-1}{2}$

For $n = 1, P_1 = 1 = \frac{3^1-1}{2} = 1$. Hence P_1 is true.

For $n = k$, let the proof be true $. i.e.\ 1 + 3 + 3^2 + \cdots \ldots\ldots\ldots\ldots + 3^{k-1} = \frac{3^k-1}{2}$

Now for $n = k+1, P_{k+1}$ must be true $. i.e$

$\Rightarrow 1 + 3 + 3^2 + \cdots \ldots\ldots\ldots\ldots + 3^{k-1} + 3^{k+1-1} = \frac{3^k-1}{2} + 3^{k+1-1}$

$= \frac{3^k-1}{2} + 3^k = \frac{3^k+2.3^k-1}{2}$

$= \frac{3.3^k-1}{2} = \frac{3^{k+1}-1}{2}$

Hence P_{k+1} is true for $n = k+1$. Thus since proof holds for $n = 2, k, k+1$, then it holds for all integral values of n.

Task

1. *Prove by induction that*
   a) $1^3 + 2^3 + 3^3 + \cdots \ldots\ldots\ldots\ldots\ldots + n^3 = \frac{1}{4}n^2(n+1)^2$
   b) $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$
   c) $\sum_{r=1}^{n} r(r+1)(r+4) = \frac{1}{4}n(n+1)(n+4)(n+5)$
   d) $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \cdots \ldots\ldots\ldots\ldots\ldots + \frac{1}{n(n+3)} = \frac{11}{18} - \frac{1}{3}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right)$
   e) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \ldots\ldots\ldots\ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$
   f) $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$
   g) $\sum_{r=1}^{n} 2r(2r-1) = \frac{1}{3}n(n+1)(4n-1)$
2. *Prove that* $\frac{d}{dx}(x^n) = nx^{n-1}$ *for all positive integral values of n.*
3. *Show that* $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ *and prove by induction that*
   $$(1+x)^n = 1 + nx + \cdots \ldots\ldots + \binom{n}{r}x^r + \cdots \ldots\ldots\ldots\ldots + x^n$$
4. $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

SUMMATION OF A FINITE SERIES

There are various standard results that can be used to find the sum of finite series.

The two formulae of sums of A.P's and G. P's are:

1) The sum of n-terms of the A.P with first term a and common difference d is:

$$\boxed{S_n = \frac{1}{2}n(2a + (n-1)d)}$$

2) The sum of $n-terms$ of a G.P with first term a and common ratio r is:

$$\boldsymbol{S_n = \frac{a(1-r^n)}{1-r}} \quad for\ |r| < 1$$

The standard series are usually given as:

i) $\boldsymbol{1 + 2 + 3 + \cdots \ldots \ldots \ldots + n = \sum_r^n r = \frac{1}{2}n(n+1)}$

ii) $\boldsymbol{1^2 + 2^2 + 3^2 + \cdots \ldots + n^2 = \sum_r^n r^2 = \frac{1}{6}n(n+1)(2n+1)}$

iii) $\boldsymbol{1^3 + 2^3 + 3^3 + \cdots \ldots + n^3 = \sum_r^n r^3 = \frac{1}{4}n^2(n+1)^2 = [\sum_r^n r]^2}$

The standard series above can be used in summation of finite number of terms.

Examples

1. *Find the sum of the series:*
   a) $1 + 2 + 3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n+1)$
   b) $1^2 + 2^2 + 3^2 + \cdots \ldots \ldots \ldots \ldots + (n-1)^2$
   c) $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots \ldots + (2n)^3$

Solution

NOTE: in all the above series, the last term is not a general term for $n \in N$. The series exercise the form of a standard series, hence the result of standard series will hold for the last term.

a) $1 + 2 + 3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n+1)$

Using $1 + 2 + 3 + \cdots \ldots + N = \frac{1}{2}N(N+1)$

$\Rightarrow N = 2n + 1$

$\therefore 1 + 2 + 3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n+1) = \frac{1}{2}(2n+1)[(2n+1)+1]$

$= (2n+1)(n+1)$

b) $1^2 + 2^2 + 3^2 + \cdots \ldots \ldots \ldots \ldots \ldots + (n-1)^2$

Using $1^2 + 2^2 + 3^2 + \cdots \ldots \ldots \ldots \ldots \ldots + N^2 = \frac{1}{6}N(N+1)(2N+1)$

Taking $N = (n-1)$

$\Rightarrow 1^2 + 2^2 + 3^2 + \cdots \ldots \ldots \ldots \ldots \ldots + (n-1)^2 = \frac{(n-1)(n-1+1)[2(n-1)+1]}{6} = \frac{n(nn-1)(2n-1)}{6}$

c) $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n)^3$

Using $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots \ldots \ldots + N^3 = \frac{N^2(N+1)^2}{4}$

Taking $N = 2n$

$1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n)^3 = \frac{(2n)^2((2n)+1)^2}{4} = n^2(2n-1)^2$

2. *Find the sum of the series;*
   a) $1^3 + 3^3 + 5^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n+1)^3$
   b) $2^3 + 4^3 + 6^3 + \cdots \ldots \ldots \ldots \ldots + (2n)^3$
   c) $1^3 + 3^3 + 5^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n-1)^3$

Solution

a) $1^3 + 3^3 + 5^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n+1)^3$

(Summation of the cubes of odd terms of natural numbers)

Using $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots \ldots \ldots + N^3 = \frac{N^2(N+1)^2}{4}$

Now $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots + (2n)^3 + (2n+1)^3 = \frac{(2n+1)^2((2n+1)+1)^2}{4}$

$= (2n+1)^2(n+1)^2$

Finding the sum of the series; $2^3 + 4^3 + 6^3 + \cdots \ldots \ldots \ldots \ldots + (2n)^3$

$2^3 + 4^3 + 6^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n)^3 = 2^3[\ 1^3 + 2^3 + 3^3 + \cdots \ldots \ldots + n^3]$

$= 8\left[\frac{n^2(n+1)^2}{4}\right] = 2n^2(n+1)^2$

$\therefore 1^3 + 3^3 + 5^3 + \cdots \ldots \ldots + (2n+1)^3 = (2n+1)^2(n+1)^2 - 2n^2(n+1)^2$

$= (n+1)^2[(2n+1)^2 - 2n^2]$

$= (n+1)^2[4n^2 + 4n + 1 - 2n^2] = (n+1)^2(2n^2 + 4n + 1)$

b) $2^3 + 4^3 + 6^3 + \cdots \ldots \ldots + (2n)^3 = 2^3[\ 1^3 + 2^3 + 3^3 + \cdots \ldots \ldots + n^3]$

$= 8\left[\frac{n^2(n+1)^2}{4}\right] = 2n^2(n+1)^2$

c) $1^3 + 3^3 + 5^3 + \cdots \ldots \ldots \ldots \ldots \ldots \ldots + (2n-1)^3$

First summing the natural numbers to the $(2n)^{th}$ term,

$\Longrightarrow 1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots + (2n-1)^3 + (2n)^3 = \frac{(2n)^2(2n+1)^2}{4} = n^2(2n+1)^2$

Now summing the series $2^3 + 4^3 + 6^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n)^3$

$2^3 + 4^3 + 6^3 + \cdots \ldots \ldots \ldots \ldots \ldots + (2n)^3 = 2n^2(n+1)^2 (See\ previous\ number\ above)$

Now subtracting the series,

$\Longrightarrow 1^3 + 3^3 + 5^3 + \cdots \ldots \ldots \ldots + (2n-1)^3 = n^2(2n+1)^2 - 2n^2(n+1)^2$

$= n^2[(2n+1)^2 - (n+1)^2]$

$= n^2[4n^2 + 4n + 1 - n^2 - 2n - 1]$

$= n^2(2n^2 - 1)$

NOTE:

In this last example the series $1^3 + 3^3 + 5^3 + \cdots \ldots \ldots + (2n-1)^3$ has the last term as $(2n-1)^3$ and is the general term of the series for all $n \in N$

Alternative summation of the series

$$1^3 + 3^3 + 5^3 + \cdots \ldots \ldots + (2n-1)^3 = \sum_{r=1}^{n} (2r-1)^3$$

$$= \sum_{r=1}^{n} (8r^3 - 12r^2 + 6r - 1) = 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$= 8\left[\frac{n^2(n+1)^2}{4}\right] - 12\left[\frac{n(n+1)(2n+1)}{6}\right] + 6\left[\frac{n(n+1)}{2}\right] - n$$

$$= 2n^2(n+1)^2 - n(n+1)(2n+1) + 3n(n+1) - n$$

$$= n(n+1)[2n(n+1) - 2(2n+1) + 3] - n$$

$$= n(n+1)[2n^2 + 2n - 4n - 2 + 3] - n$$

$$= n(n+1)(2n^2 - 2n + 1) - n$$

$$= n[2n^3 - 2n^2 + n + 2n^2 - 2n + 1 - 1]$$

$$= n[2n^3 - n] = n^2(2n^2 - 1)$$

3. a) *It can be proved by induction that for positive n,*

$1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

From this result deduce that

$(n+1)^3 + (n+2)^3 + (n+3)^3 + \cdots \ldots \ldots \ldots + (2n)^3 = \frac{1}{4}n^2(3n+1)(5n+3)$

b) *Show that the sum of the series*

$(n+1)^2 + (n+2)^2 + (n+3)^2 + \cdots \ldots \ldots \ldots + (2n)^2 = \frac{1}{6}n(2n+1)(7n+1)$

Solution

a) $1^3 + 2^3 + 3^3 + \cdots \ldots \ldots \ldots + n^3 + (n+1)^3 + (n+2)^3 + (n+3)^3 + \cdots \ldots \ldots + (2n)^3$

$$= \frac{(2n)^2(2n+1)^2}{4} = n^2(2n^2+1)$$

Also $1^3 + 2^3 + 3^3 + \cdots \ldots\ldots\ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$

Subtracting the series,

$(n+1)^3 + (n+2)^3 + (n+3)^3 + \cdots \ldots\ldots\ldots + (2n)^3 = n^2(2n^2+1) - \frac{n^2(n+1)^2}{4}$

$= \frac{1}{4}n^2[4(4n^2 - 4n + 1) - (n^2 + 2n + 1)]$

$= \frac{1}{4}n^2[16n^2 + 16n + 4 - n^2 - 2n - 1] = \frac{1}{4}n^2[15n^2 + 14n + 3]$

$= \frac{1}{4}n^2[15n^2 + 9n + 5n + 3] = \frac{1}{4}n^2[3n(5n+3) + (5n+3)]$

$= \frac{1}{4}n^2(3n+1)(5n+3)$

b) From $1^2 + 2^2 + 3^2 + \cdots \ldots\ldots\ldots\ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

$\Longrightarrow 1^2 + 2^2 + 3^2 + \cdots \ldots\ldots + n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 + \cdots\ldots + (2n)^2$

$= \frac{1}{6}(2n)(2n+1)(2(2n)+1) = \frac{2n(2n+1)(4n+1)}{6}$

Subtracting the series,

$(n+1)^2 + (n+2)^2 + (n+3)^2 + \cdots \ldots\ldots\ldots + (2n)^2$

$= \frac{2n(2n+1)(4n+1)}{6} - \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}n(2n+1)(7n+1)$

$= \frac{n(2n+1)}{6}[2(4n+1) - (n+1)]$

$= \frac{n(2n+1)}{6}(8n + 2 - n - 1) = \frac{1}{6}n(2n+1)(7n+1)$

4. *Show that the sum of the series*

$3 + 6 + 11 + \cdots \ldots\ldots\ldots\ldots\ldots + (n^2+2) = \frac{1}{6}n(2n^2 + 3n + 13)$

Solution

For this series, (n^2+2) is the general term of the series for all $n \in N$

$\therefore 3 + 6 + 11 + \cdots \ldots\ldots\ldots\ldots\ldots + (n^2+2) = \sum_{r=1}^{n}(r^2+2) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2$

$= \frac{1}{6}n(n+1)(2n+1) + 2n$

$= \frac{n}{6}[(n+1)(2n+1) + 12] = \frac{n}{6}[2n^2 + 3n + 1 + 12]$

$= \frac{1}{6}n(2n^2 + 3n + 13)$

Task

1. *Show that the sum of the series;*

a) $2 + 10 + 30 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + (n^3 + n) = \frac{n(n+1)(n^2+n+2)}{4}$

b) $2 + 12 + 36 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + (n^3 + n^2) = \frac{n(n+1)(n+2)(3n+1)}{12}$

c) $2 + 6 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + (n^2 - n) = \frac{n(n+1)(n-1)}{3}$

d) $1.4 + 2.5 + 3.6 + \cdots \ldots\ldots\ldots\ldots + n(n+3) = \frac{n(n+1)(n+5)}{3}$

e) $2^2 + 5^2 + 8^2 + \cdots \ldots\ldots\ldots\ldots + (3n-1)^2 = \frac{n(6n^2+3n-1)}{6}$

f) $1.2 + 3.4 + 5.6 + \cdots \ldots\ldots\ldots\ldots + 2n(2n-1) = \frac{n(n+1)(4n-1)}{3}$

g) $1.2.5 + 2.3.6 + 3.4.7 + \cdots \ldots\ldots + n(n+1)(n+4) = \frac{n(n+1)(n+2)(3n+12)}{12}$

2. *Find the sum of the series*

a) $21^3 + 22^3 + 23^3 \ldots\ldots\ldots\ldots\ldots\ldots + 35^3$

b) $1^3 + 3^3 + 5^3 + \cdots \ldots\ldots\ldots\ldots\ldots + 25^3$

c) $1^3 - 2^3 + 3^3 - 3^3 + \cdots \ldots\ldots\ldots + (2n-1)^3 - (2n)^3$

ANSWER: $\boldsymbol{a)\ 352\,800}$ $\boldsymbol{b)\ 56\,953}$ $\boldsymbol{c) -n(4n+3)}$

INTEREST AND ANNUITIES

Simple and Compound Interest

When person A lends money to person B he makes a charge for the use of his money. The amount of money lent is called *Principal (P)* and the charge made for lending is called *Interest (I).* The sum of interest and principal is the *Amount* or *Total amount (A)*, which is the amount B will pay back to A after any interval of time. $i.e.\ (A = P + I)$

The *Rate per cent per annum (R %)* is the interest payable on principal for one year.

Types of Interest

- Simple interest
- Compound interest

Simple Interest (S.I)

Simple Interest is the interest that is paid on the principal only.

S.I is paid on a given principal, and is the same after equal intervals of time. The principal is always constant.

To find the Interest and Amount of a given sum in a given time at Simple Interest

When money is put out at S.I, the interest is payable for each year, but is not added to the principal.

Let P be the given principal and n the number of years for which the principal is lent, R the rate of interest per annum.

The interest of P for$1 - year = P.\frac{R}{100}$ and therefore for $n - years$ is $P.\frac{R}{100}.n$

$\therefore$ Amount at the end o**f** $1 - year = A_1 = P + P.\frac{R}{100} = P\left(1 + \frac{R}{100}\right)$

Since the principal is lent at S.I, the interest $P.\frac{R}{100}$ will be added to the amount at the end of each year.

Thus the amount at the end of $n - years$ will be;

$$A_n = P\left(1 + \frac{R}{100}\right) + \frac{PR}{100} + \frac{PR}{100} + \frac{PR}{100} + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{PR}{100}$$

$$= P + \underbrace{\frac{PR}{100} + \frac{PR}{100} + \frac{PR}{100} + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{PR}{100}}_{n - times}$$

$A_n = P + \frac{PRn}{100} = P\left(1 + \frac{nR}{100}\right)$, which is the amount at the end of n$-years$.

$$\therefore Amount\ for\ n - years, \boldsymbol{A_n = P\left(1 + \frac{nR}{100}\right)}$$

, which is the formula for amount at the end of $n - years$, where;

$P = Principal$

$n = Number\ of\ years\ principal\ is\ lent$

$R = Rate\ per\ cent\ per\ annum$

Note:

1. The amount at the end of each year constitute a series in an A.P with a common difference $\frac{PR}{100}$
2. Simple interest can be paid at different intervals $.i.e.$ monthly, quarterly, half yearly or yearly.

3. In $A = P\left(1 + \frac{nR}{100}\right)$, formulae contain four quantities $.i.e. P, A, R, n$ and if three are given, the fourth can be evaluated.
4. When working out the interest due on a sum of money for a period between two specific dates, always take a year as containing 365 days.
5. For purposes of interest, the day a deposit is first made does not count. However the day on which money is withdrawn does count. Thus if someone deposits money on an account on 15th March and withdraws it on17^{th} April, he is not paid interest for 15th March but he is paid interest for 17th April.

Examples

1. *Find the simple interest on and the amount at simple interest of;*
   a) $Sh.8000$ for $2\frac{1}{4}$ years at $8\frac{1}{5}\%\ p.a.$
   b) $Sh.480$ for 2 months at $3\frac{3}{4}\%\ p.a.$

Solution

a) Simple Interest $= \frac{P.R.n}{100}$; $P = 8000, R = 8.2\%, n = 2.25\ years$

$= \frac{8000 \times 8.2 \times 2.25}{100} = 1\,476$

$Amount = P + I = 8000 + 1476 = 9\,476\ /=$

b) $P = 480, R = 3.75\%, n = \frac{20}{12}\ years$

$\therefore Simple\ Interest = \frac{480 \times 3.75 \times \frac{20}{12}}{100} = 30$

$Amount = 30 + 480 = 510\ /=$

2. *Calculate to the nearest shilling, the simple interest on;*
   a) $Sh.550$ from $March$ 6 to May 6 at $11\%\ p.a.$
   b) $Sh.990$ from $August$ 20 to $December$ 13 at $9\frac{1}{2}\%\ p.a.$

Solution

a) Simple Interest $= \frac{P.R.n}{100}$, $P = 550, R = 11\%$,

$n = (31 - 6)days + (30 + 6)\ days = 61\ days = \frac{61}{365}\ years$

$[April = 30\ days, May\ 6 = 6\ days, March\ 6 = (31 - 6)days]$

$\therefore Simple\ Interest = \frac{550 \times 11 \times \frac{61}{365}}{100} = 10\ /=$

b) $P = 990, R = 9\frac{1}{2}\%$,

$n = (31 - 20)days + (30 + 31 + 30 + 13)days = 115days = \frac{115}{365}\ years$

$\therefore Simple\ Interest = \frac{990 \times 9\frac{1}{2} \times \frac{115}{365}}{100} = 30\ /=$

3. a) In what time will $Sh.3\,200$ amount to $Sh.4\,188$ at $9\frac{1}{2}\%\ p.a.$ simple interest?
   b) A sum of money invested at $9\%\ p.a.$ simple interest amounts after 3 years to $sh.7\,620$. Find the sum invested.

Solution

a) *Given:* $P = 3\,200, A = 4\,188, R = 9\frac{1}{2}\%,\ n =?$

From Simple Interest $= \frac{P.R.n}{100} \Longrightarrow n = \frac{100I}{P.R}$

But $Interest = Amount - Principal$

$\therefore I = 4188 - 3200 = 988$

$\therefore n = \frac{100 \times 988}{3200 \times 9\frac{1}{2}} = 3\frac{1}{4}\text{years}$

b) *Given: Amount* $= 7\,620 = Principal + Interest$

$Interest = \frac{P.R.n}{100} = \frac{P \times 9 \times 3}{100}$

Now $7\,620 = P + \frac{27P}{100} = \frac{127P}{100}$ $\therefore P = 6\,000/=$

Task

1. *Find the un known quantities in the table below*

Principal	**Interest**	**Amount**	**Time**	**Rate** $\%\ p.a.$
$sh.650$	$sh.156$	……	$3\ years$	……
……	$sh.1\,942.25$	……	$1\frac{1}{4}\ years$	$9\frac{1}{4}$
$sh.8\,450$	……		$9\ months$	……
……	$sh.170.40$		$16\ months$	$11\frac{1}{4}$
$sh.1\,818$	……	$sh.1\,908.90$	……	$8\frac{1}{3}$

ANSWER:

a) $sh.806$; 8% b) $sh.16\,800$; $sh.18\,742.50$ c) $sh.549.25$; $8\frac{2}{3}\%$ d) $sh.1\,136$; $sh.1\,306.40$

e) $sh.90.90$; $219\ days$

2. *Two men both invest the same sum, one at* $9\frac{3}{4}\%\ p.a.$ *if the difference in the interest they receive is* $sh.100$ *, what sum each man invested is.*
ANSWER: $\boldsymbol{sh.20\,000}$

3. *Two men both invest* $sh.800$ *at* 10% *and* $8\frac{1}{2}\%\ p.a$ *respectively. What is the difference in the interest they earn after* $9\ months$*?*
ANSWER: $\boldsymbol{sh.9}$

4. *A club of 320 members has to meet a bill of* $sh.9\,500$ *in 5 months time. Each member subscribes* $sh.30$ *and the money accumulates at* $8\frac{1}{4}\%\ p.a$ *simple interest. What balance remains when the bill has been paid?*
ANSWER: $\boldsymbol{sh.430}$

$$\left[\begin{array}{l} \textbf{\textit{Hint}}: Total\ principal = 30 \times 320 = 9\overset{600}{=}, n = \frac{5}{12}\ , I = \frac{9600 \times 8\frac{1}{4} \times \frac{5}{12}}{100} = 330 \\ Now\ New\ amount = 330 + 9600 = Sh.\,9\,930\ \therefore Balance = 9930 - 9500 = 430/= \end{array}\right]$$

5. *A sum of money amounts at simple interest to* $sh.819$ *in 2 years at* $8\frac{1}{2}\%\ p.a.$ *to what would it amount in 1 year 9 months at* $8\%\ p.a$*?* **ANSWER**: $\boldsymbol{sh.798}$

$$\left[\begin{array}{l} \textbf{\textit{Hint}}: Let\ sum = P = Principal, A = P + I\ , I = \frac{P \times 8\frac{1}{2} \times 2}{100} = 0.17P \\ Now\ 819 = P + 0.17P\ , P = 700 \\ Now\ at\ 8\%\ p.a, amount, A = 700 + \frac{700 \times 8\frac{1}{2} \times 1\frac{9}{12}}{100} = 798/= \end{array}\right]$$

6. *Diana invests* $\frac{2}{5}$ *of her capital at* $15\%\ p.a$ *and the remainder at* $10\%\ p.a$ *simple interest. After 25 cents in the shillings has been deducted for tax, she receives* $sh.720$ *interest each year. What is the size of her capital?*
ANSWER: $\boldsymbol{sh.8000}$

7. *A man borrows sh.*5000 *at* 10 ½% *p.a payable yearly. After one year he pays back sh.*2500*. How much will he need to clear his date completely at the end of another one year?* **ANSWER**: ***sh.*3 343**

$$\left[\begin{array}{l}\boldsymbol{Hint}: P = 5000\,, R = 10.5\%, Amount\ after\ 1\ year = 5000 + \frac{10.5}{100} \times 5000 = 5\ 525 \\ \quad Balance\ after\ paying\ off\ 2500 = 5525 - 2500 = 3025 \\ Now\ amount\ after\ another\ 1\ year = 3025 + 3025 \times \frac{10.5}{100} = 3\ 343\ /=\end{array}\right]$$

8. *Godfrey borrows sh.*20 000 *from a finance corporation and pays* 10% *interest p.a on an amount still owing at the beginning of that year. After one year he pays sh.*5 000 *back, sh.*6 000 *a year later and sh.*7 000 *at the end of the third year. How much does he still need to clear the debt completely?* **ANSWER**: ***sh.*6 970**

$$\left[\begin{array}{l}\boldsymbol{Hint}: \quad P = 20\ 000\,, R = 10\%\ p.a. Amount\ \ after\ 1\ year = 20000 + \frac{10}{100} \times 20000 = 22\ 000; \\ \quad Amount\ remaining\ after\ paying\ 5000 = 22\ 000 - 5\ 000 = 17\ 000 \\ \quad Expected\ amount\ after\ another\ year = 17000 + \frac{10}{100} \times 17000 = 18\ 700; \\ \quad Balance\ after\ paying\ sh.6000 = 18\ 700 - 6000 = 12\ 700/= \\ \quad Expected\ amount\ after\ another\ 3^{rd}\ year = 12\ 700 + 12\ 700 \times \frac{10}{100} = 13\ 970/= \\ \quad Now\ balannce\ after\ clearing\ sh.7000 = 13\ 970 - 7000 = 6\ 970/= \\ \boldsymbol{NOTE}: Principal\ is\ not\ constant\ every\ year\ but\ interest\ is\ the\ same.\end{array}\right]$$

COMPOUND INTEREST (C.I)

Compound interest is the interest calculated on the principal and the accrued interest.

The interest unpaid is added to the principal and for the next period the interest is computed. *i.e.* Compound interest is the interest on the growing principal.

NOTE

1. Both principal and compound interest change from period to period since the principal changes.
2. Interest earned on the principal is not withdrawn.
3. Compound interest can be computed annually, half yearly, quarterly or monthly.
4. When a given amount is due after a specified period of time, its present value (principal for the next period) is that sum together with the interest for that period produces the given amount.

To find Interest and Amount of a given sum at Compound Interest

If the money is lent at compound interest the interest is added each year to the principal and for the following year the interest is calculated on their sum.

Let P be the principal and n the number or years for which the principal is lent at r% per annum compound interest.

The amount at the end of one year will be $= P + \frac{Pr}{100} = P\left(1 + \frac{r}{100}\right) = PR$

The amount at the end of two years will be $= PR + PR.\frac{r}{100} = PR\left(1 + \frac{r}{100}\right) = PR^2$

The amount at the end of three years will be $= PR^2 + PR^2.\frac{r}{100} = PR^2\left(1 + \frac{r}{100}\right) = PR^3$

Then arguing in this way, the amount at the end of n years will be$= PR^n$

Thus the amount, A at the end of n years for principal P lent at r% per annum is

$$\boxed{\boldsymbol{A = P\left(1 + \frac{r}{100}\right)^n \text{ or } A = PR^n}}$$

, where $A = Amount$

$P = Principal$

$r = rate\ per\ cent\ per\ annum$

$R = \left(1 + \frac{r}{100}\right)$

Deduction

From $A = PR^n$, $\boxed{P = \frac{A}{R^n}}$ this enables us to find the sum which will produce amount A in n years.

Also $\boxed{R = \sqrt[n]{\frac{A}{P}}}$, from which interest can be calculated.

Again from $R^n = \frac{A}{P}$, $\log(R^n) = \log\left(\frac{A}{P}\right)$

$\Rightarrow n\log R = \log A - \log P$ $\boxed{\therefore n = \frac{\log A - \log P}{\log R}}$, from which time taken for P to amount to A can be evaluated.

Note

The amount in each year constitute terms in a G.P with first term PR and common ratio $\left(1 + \frac{r}{100}\right)$

Examples

1. *Find the compound interest on;*
   a) $Sh.10\,000\ for\ 2\ years\ at\ 10\%\ p.a\ paid\ annually$
   b) $Sh.6\,000\ for\ 3\ years\ at\ 10\%\ p.a\ paid\ annually$

Solution

a) $P = 10\,000, n = 2, r = 10\%$

$Interest, I = Amount - Principal$

$I = P\left(1 + \frac{r}{100}\right)^n - P = 10\,000\left(1 + \frac{10}{100}\right)^2 - 10\,000 = Sh.2\,100$

b) $P = 6\,000, n = 3, r = 10\%$

$A = 6000\left(1 + \frac{10}{100}\right)^3 = 7\,986$

$\therefore Interest = 7\,986 - 6\,000 = sh.1\,986$

2. *Prove that in order that a sum of money may double itself in 10 years by investment at compound interest payable annually, the interest should be approximately 7.2 per cent per annum.*

Solution

Let $P = Amount\ invested\ for$

$n = 10\ years, which\ amounts\ to$

$A = 2P\ at\ a\ rate\ of$

$r = r\%\ per\ annum$

Using $A = P\left(1 + \frac{r}{100}\right)^n$,

$\Rightarrow 2P = P(1 + 0.01r)^{10}\ \ or\ \ 2 = (1 + 0.01r)^{10}$

Introducing logarithms

$\Rightarrow \log 2 = \log(1 + 0.01r)^{10} = 10\log(1 + 0.01r)$

$\Rightarrow \frac{\log 2}{10} = \log(1 + 0.01r)$

$\Rightarrow 2^{0.1} = 1 + 0.01r\ \ \therefore r = \frac{2^{0.1} - 1}{0.01} \approx 7.2\%$

3. *How many years (correct to the nearest whole year) does it take a sum of money to double in value if compound interest is paid at a rate of* $9\%\ p.a$*?*

Solution

Let the sum be P be invested for $n - years$ to amount to $2P$ at a rate of $9\%\ p.a$

Using $A = P\left(1 + \frac{r}{100}\right)^n$,

$\Rightarrow 2P = P\left(1 + \frac{9}{100}\right)^n\ \ or\ \ 2 = 1.09^n$

Introducing logarithms

$\Rightarrow \log 2 = \log 1.09^n = n\log 1.09$

$\Rightarrow \therefore n = \frac{\log 2}{\log 1.09} \approx 8\ years$

To find Amount of a given sum for $\left(n + \frac{1}{m}\right)$ $years$ at Compound Interest

In business transactions when the time contains a fraction of a year, allow Simple Interest for the fraction of a year, and Compound Interest for the whole years.

Let P be invested for $\left(n + \frac{1}{m}\right)\ years$ at a rate per cent per annum ($r\%$) Compound Interest.

$Amount,\ \ A \quad = Amoun\_after\ n - years + Interest\ for \frac{1}{m}\ years$

$after\ \left(n + \frac{1}{m}\right)\ years \qquad (principal)\ due\ to\ C.I$

$on\ principal\ (Amount\ due\ to\ C.I)$

Amount after $n - years = P(1 + r)^n = PR^n$

Now principal for $\frac{1}{m}\ years = PR^n$

Interest for $\frac{1}{m}\ years = PR^n \times r \times \frac{1}{m} = PR^n \times \frac{r}{m}$

Now Total Amount $= PR^n + PR^n \times \frac{r}{m} = PR^n\left(1 + \frac{r}{m}\right)$

$$\boxed{A = PR^n\left(1 + \frac{r}{m}\right)}$$

Example

1. *Find Compound Interest on;*
   a) $Sh.6\,000\ for\ 2\frac{1}{2}\ years\ at\ 10\%\ p.a\ paid\ annually$
   b) $Sh.8\,000\ for\ 1\frac{3}{4}\ years\ at\ 7.5\%\ p.a\ paid\ annually$

Solution

a) $P = 6\,000, n = 2, \frac{1}{m} = \frac{1}{2}, r = 10\%\ p.a.$

Using $A = PR^n\left(1 + \frac{r}{m}\right), R = 1 + r = 1 + \frac{10}{100} = 1.1$

Amount after $2\frac{1}{2}\ years = A = 6\,000 \times (1.1)^2 \times \left(1 + \frac{10}{100} \times \frac{1}{2}\right) = 7\,623$

Now $Compound\ Interest = Amount - Principal$

$= 7\,623 - 6\,000 = sh.\,1\,623$

b) $P = 8\,000\,, n = 1\,, \frac{1}{m} = \frac{3}{4}\,, r = 7.5\%\ p.a.$

Amount after $1\frac{3}{4}\ years = A = 8\,000 \times \left(1 + \frac{7.5}{100}\right)^1 \times \left(1 + \frac{7.5}{100} \times \frac{3}{4}\right) = Sh.\,9\,083.75$

$\therefore Compound\ Interest = 9\,083.75 - 8\,000 = Sh.\,1\,083.75$

2. *A person owes a merchant Sh. 5 702.27 to be paid at the end of $2\frac{1}{4}$ years from now. How much the merchant ought to accept now in order to clear off the accounts taking the money to be worth 6% p.a compound interest?*

Solution

Let $P = Sum\ to\ be\ invested\,, A = Sh.\,5\,702.27\,, r = 6\%\ p.a, n = 2\,, \frac{1}{m} = \frac{1}{4}$

Using $A = PR^n\left(1 + \frac{r}{m}\right)\,, R = 1 + r = 1 + \frac{6}{100}$

$\Longrightarrow 5\,702.27 = P \times \left(1 + \frac{6}{100}\right)^2 \times \left(1 + \frac{6}{100} \times \frac{1}{4}\right)$

$\Longrightarrow 5\,702.27 = P(1.06)^2 \times (1.015)$

$\therefore P = \frac{5\,702.27}{(1.06)^2 \times (1.015)} = Sh.\,5\,000$

$\therefore$ The merchant has to accept now $Sh.\,5\,000$ in order to get $Sh.\,5\,702.27$ after $2\frac{1}{4}\ years$ at a rate of $6\%\ p.a\ C.I$

Note

- The rate r% must be in per annum
- If the investment is for only n years, then $\frac{1}{m} = 0$. Hence the formula $A = P(1 + r)^n$

3. *A sum of money is put out at compound interest amounts in 2 years to R.s 2 809 and in 3 years to R.s 2 977.54. Find the rate of interest and the original sum.*

Solution

Amount for 2 years = $R.s\ 2\,809$

Amount for 3 years = $R.s\ 2\,977.54$

$\Longrightarrow A = P(1 + r)^2 = 2\,809$ ..... (1)

$\Longrightarrow A = P(1 + r)^3 = 2\,977.54$ ..... (2)

Dividing the equations

$\Longrightarrow \frac{P(1+r)^2}{P(1+r)^3} = \frac{2\,809}{2\,977.54}$

$\therefore 1 + r = 1.06\,0\ \ or\ \ r = 0.06$

From eqn. (1)

$P(1 + r)^2 = 2\,809 \Longrightarrow P(1 + 0.06)^2 = 2\,809$

$\therefore P = \frac{2\,809}{(1.06)^2} = R.s\ 2\,500$

4. *A person wishes to divide Rs. 65 050 between his two sons who are 13 and 15 years old respectively in such a way that their shares if invested at 4% compound interest should produce the same amount when they become 18 years of age. Find the share of each.*

Solution

Let the share of the young son = $Rs.x$

$\therefore$ The share of the older son= $65\,050 - x$

The young son has to go for 5 years and the elder son for 3 years to become 18 years old.

Case I: younger son

$P = x\,, n = 5\,, r = 4\%\,, A =?$

$A = x(1 + 0.04)^5 = x.(1.04)^5$

Case II: Elder son

$P = 65\,050 - x\,, n = 3\,, r = 4\%\,, A =?$

$A = (65\,050 - x)(1 + 0.04)^3 = (65\,050 - x).(1.04)^3$

Now the amounts must be the same

$\Longrightarrow x.(1.04)^5 = (65\,050 - x).(1.04)^3$

$\Longrightarrow x.(1.04)^2 = 65\,050 - x$

$\therefore 1.0816x = 65\,050 \Longrightarrow = Rs.31\,250$

∴ The share of the younger son is Rs. 31 250 and the share of the elder son is $Rs.65\,050 - 31\,250 = Rs.33\,800$

Find the sum if the difference between the simple interest and compound interest on the sum is £30 at 9% p.a for 3 years.

Solution

Let $P = sum\,, r = 9\%\ p.a\,, n = 3\ years$

At Simple Interest, simple interest $= \frac{P \times 9 \times 3}{100} = 0{,}27P$

At Compound Interest, compound interest $= Amount - Principal$

$compound\ interest = P\left(1 + \frac{9}{100}\right)^3 - P = P(1.09)^3 - P$

$= 0.295029P$

Now $0.295029P - 0.27P = 30\ \ or\ \ 0.025029P = 30$

$\therefore P = £1198.6$

Task

1. *Find the compound interest on;*
   a) $Sh.12\,000\ for\ 2\frac{1}{2}\ years\ at\ 10\%\ p.a.$
   b) $Sh.10\,000\ for\ 1\frac{1}{4}\ years\ at\ 5\%\ p.a.$

   ANSWER: *a)* $\boldsymbol{Sh.3\,246}$ *b)* $\boldsymbol{Sh.631.25}$
2. *Find the compound interest on the sums, giving your answers to 3 significant figures*
   a) $Sh.500\ for\ 4\ years\ at\ 10\%\ p.a.$
   b) $Sh.18\,765\ for\ 4\ years\ at\ 9\%\ p.a.$
   c) $Sh.15\,000\ for\ 3\ years\ at\ 20\%\ p.a.$

   ANSWER: *a)* $\boldsymbol{Sh.232}$ *b)* $\boldsymbol{Sh.7\,690}$ c) $\boldsymbol{Sh.10\,920}$
3. *On a certain sum, the compound interest for 2 years is Sh.91.35 and the simple interest is Sh. 90. Fond the sum and the rate of interest per annum.*

   ANSWER: $\boldsymbol{Sh.1\,500}\,, \boldsymbol{3\%}$
4. A sum of money invested at compound interest, payable yearly, amounts to Sh. 2 704 at the end of the second year and to Sh. 2 812.16 at the end of the third year. Find the rate of interest and the sum.

   ANSWER: $\boldsymbol{Sh.25\,000}\,, \boldsymbol{4\%}$
5. *A person deposited in a bank £1000 at 4% p.a interest being added annually. After 5 years the rate of interest was raised to 5% and after four more years,*

the rate was further increased to 6%. The money was withdrawn at the end of 12 years. Find the amount.

ANSWER: $\boldsymbol{Sh.1\,761.40}$

6. *A person has two sons aged 12 and 15 years respectively. He wants to divide Sh.* 51 783 *between them and deposit the share of each separately in the bank so that each gets the same amount when he is 18 years old. If the bank pays compound interest at the rate of* $5\%\ p.a$*, what should be the share of each?*

ANSWER: $\boldsymbol{Sh.24\,000\ for\ young\ son;\ Sh.27\,783\ for\ elder\ son}$

To find Amount on a sum (P) in given n-years, interest payable q-times a year at a rate r% per annum

Suppose you deposit \$ 1 000 in a savings and a loan that pays 8% compounded semi annually, how much will the savings and the loan owe you at the end of 2 years?

Solution

Compounded semi annually means "Interest is paid to your account at the end of each 6 month period, and the interest will in turn earn interest"

Annual rate $r = 8\%$

Semi annual rate$= \frac{8}{2} = 4\%$

Now amount after 6 months $= 1000 + \frac{4}{100} \times 1000 = 1000(1 + 0.04)$

Principal for the next 6 months $= 1000(1 + 0.04)$

Now amount for 1 year $= 1000(1 + 0.04) + 1000(1 + 0.04) \times \frac{4}{100}$

$$= 1000(1 + 0.04)^2$$

Again amount for another 6 months

$$= 1000(1 + 0.04)^2 + 1000(1 + 0.04)^2 \times \frac{4}{100} = 1000(1 + 0.04)^3$$

This amount for 2 years $= 1000(1 + 0.04)^3 + 1000(1 + 0.04)^3 \times \frac{4}{100}$

$$= 1000(1 + 0.04)^4$$

Now the amount that would be expected at the end of 6 years will be

$$= 1000(1 + 0.04)^{12}$$

In general,

"If the interest is payable $q - times$ a year and r is the nominal rate, the amount of P in $n - years$ or $q.n$ intervals is;

$$\boldsymbol{A = P\left(1 + \frac{r}{q}\right)^{n.q}}$$

NOTE

1. Since the principal P represents the initial amount in the account and A represents the amount in the account n-years later, P is called *present value* of the account and A is called the *future value* of the account.
2. $r\%\ p.a$ is the annual rate/nominal rate and $\frac{r}{q}\%\ per \frac{1}{q^{th}}\ annum$ is the rate for compounding period/interval.

Example

1. *Find how much Peter will have on the account at the end of 2 years if he invests a sum of* \$ 100 *in an account paying* $8\%\ p.a$ *interest compounded;*
   a) *Annually*

b) *Semi annually*
c) *Quarterly*
d) *Weekly*
e) *Daily*
f) *Hourly*

Solution

Using $\boldsymbol{A = P\left(1+\frac{r}{q}\right)^{n.q}}$		
Compounding frequency	$\boldsymbol{q}$	$\boldsymbol{A = 100\left(1+\frac{0.08}{q}\right)^{2q}}$
Annually	1	\$116.64
Semi annually	2	\$116.9859
Quarterly	4	\$117.1659
Weekly	52	\$117.3367
Daily	365	\$117.3490
Hourly	8 760	\$117.3501

NOTE:

- $1\ year = 12\ months = 365\ days = 52\ weeks = 8760\ hours$
- Semi annually $= \frac{1}{2} \times 12 = 6\ months$; Now in $1\ year, q = 2$
- Quarterly $= \frac{1}{4} \times 12 = 3\ months$; now in $1\ year, q = 4$
- Weekly $= \frac{1}{7} \times 365 \approx 52\ weeks\ \therefore q = 52$

2. *Find compound interest on;*
   a) $Sh.5\,000\ for\ 1\frac{1}{2}\ years\ at\ 5\%\ p.a$ *compound interest is paid half yearly*
   b) $Sh.1\,000\ for\ 1\ year\ at\ 10\%\ p.a$ *compound interest is paid quarterly*
   c) $Sh.10\,000\ for\ 2\frac{1}{2}\ years\ at\ 15\%\ p.a$ *compound interest is paid weekly*

Solution

Using $A = P\left(1+\frac{r}{q}\right)^{n.q}$, $A = P + I$

a) $P = 5000$, $q = 2$, $n = 1\frac{1}{2}\ years = \frac{3}{2} years$, $r = 5\% = 0.05$

$A = 5000\left(1+\frac{0.05}{2}\right)^{\frac{3}{2}\times 2} = 5000(1.025)^3 = Sh.5\,384.45$

$\therefore Compound\ Interest = A - P = 5\,384.45 - 5\,000 = Sh.384.45$

b) $P = 1000$, $q = 4$, $n = 1$, $r = 10\% = 0.1$

$A = 1000\left(1+\frac{0.1}{4}\right)^{4\times 1} = Sh.1\,103.83$

$\therefore Compound\ Interest = 1\,103.83 - 1\,000 = Sh.103.83$

c) $P = 10\,000$, $q = 52$, $n = 2$, $r = 15\% = 0.15$

$A = 10000\left(1+\frac{0.15}{52}\right)^{52\times 2} = Sh.13\,492.76$

$\therefore Compound\ Interest = 13\,492.76 - 10\,000 = Sh.3\,492.76$

Task

1. *Find the amount the principal* $Sh.10\,000$ *will accumulate at a rate of* $5\%\ p.a$ *compound interest for* $2\ years$ *if compound interest is paid;*
   i) Annually **ANSWER:** $\boldsymbol{Sh.1\,025}$
   ii) Half yearly **ANSWER:** $\boldsymbol{Sh.1\,038.13}$
   iii) Quarterly **ANSWER:** $\boldsymbol{Sh.1\,044.86}$

2. *How many years to the nearest year will it take a sum of money to double if it is invested at 15% compounded annually.*
   ANSWER: $\mathbf{5\ years}$. $\left[Use\ A = P\left(1+\frac{r}{q}\right)^{n.q}\right]$
3. *How many years to the nearest year will it take money to quadruple if it is invested at 20% compounded annually.* **ANSWER**: $\mathbf{8\ years}$. $\left[Use\ A = P\left(1+\frac{r}{q}\right)^{n.q}\right]$
4. *If you deposit* \$5 000 *in an account paying 9% compounded daily, how much will you have in an account in 5 years. Compute your answer to the nearest cent.* **ANSWER**: $\mathbf{\$7\ 841.12}$

Other problems related to compound interest

Problems relating

- Depreciation
- Change of population growth
- Annuity calculations

, are majorly on the principle of compound interest.

DEPRECIATION

This happens when the value of a new item decreases with passage of time. Depreciation can be expressed in terms of an *annual percentage rate*, so as to calculate the decrease in value after a given period of time.

Let P = price of a new commodity, which depreciates at $r\%\ p.a$ for $n - years$.

The price after n-years will be calculated as below.

Value after 1 year $= P \times \left(\frac{100-r}{100}\right) = P\left(1-\frac{r}{100}\right)$

Value after 2 years $= P\left(1-\frac{r}{100}\right) \times \left(\frac{100-r}{100}\right) = P\left(1-\frac{r}{100}\right)^2$

Value after 3 years $= P\left(1-\frac{r}{100}\right)^2 \times \left(\frac{100-r}{100}\right) = P\left(1-\frac{r}{100}\right)^3$

Value after n years , $\boldsymbol{A = P\left(1-\frac{r}{100}\right)^n}$

Example

1. *A machine costing* £15 000 *is depreciated at 10% p.a. how long will it take to reduce the value to* £1 400*?*

Solution

The cost of machine, $P = £15\ 000$

Rate of depreciation, $r = 10\% = 0.1$

Value of the machine after n years, $A = £1\ 400$

But value of the machine after n years is $A = P\left(1-\frac{r}{100}\right)^n$

$\Longrightarrow 1400 = 15000(1-0.1)^n$

$\Longrightarrow \frac{1400}{15000} = (0.9)^n$

$\Longrightarrow \log\left(\frac{14}{150}\right) = \log(0.9)^n = n\log 0.9$

$\therefore n = \frac{\log\left(\frac{14}{150}\right)}{\log\ 0.9} = \frac{0.8451-1.8751}{-1.9542} = 22.5\ years$

2. *A machine costing £30 000 has an estimated life time of 15 years. If the depreciation is at a rate of 20% p.a, find the scrap value of the machine.*

Solution

The cost of the machine = $P = £30\,000$

Life of the machine = $n = 15\ years$

Rate of depreciation = $r = 20\% = 0.2$

$\therefore$ The value of the machine at the end of 15th year = $P\left(1 - \frac{r}{100}\right)^n$

$= 30\,000(1 - 0.2)^{15} = 30\,000 \times (0.8)^{15} = £1\,055$

3. *A machine was bought at Rs. 25 000 on Jan. 1, 1981. It was depreciated at 10% p.a. It was sold on 31.12.1983 at Rs. 20 047.50. Find out the percentage of profit on the reduced cost.*

Solution

The cost of the machine = $P = Rs.\,25\,000$

Life of the machine = $n = 3\ years$ $(i.e.\ Period\ from\ Jan.\ 1, 1981\ to\ 31.12.1983)$

Rate of depreciation = $r = 10\%$

Expected value after $n = 3\ years = P\left(1 - \frac{r}{100}\right)^3$

$= 25\,000(1 - 0.1)^3 = 25\,000(0.9)^3$

$= Rs.\,18\,225$

Selling price = $Rs.\,20\,047.50$

Profit realized= $20\,047.50 - 18\,225 = Rs.\,1\,822.5$

$\%\ profit = \frac{1822.5}{18225} \times 100 = 10\%$

Task

1. A machine depreciates in value each year at a rate of 10% of its value at the beginning of a year. The machinery was purchased for Rs. 10 000. Obtain to the nearest rupee its value at the end of the tenth year. ***Answer*: *Rs*. 3 483**
2. *A machine depreciates at the rate of 7% of its value at the beginning of the year. If the machine was purchased for Rs. 8 500, what is the minimum number of complete years at the end of which the worth of the machine will be less that or equal to half of its original price?*

   ANSWER: 10 *years*
3. *John bought a new car for Sh. 650 000. The car depreciated in value by 25%, 15%, 10% and 5% of its value at the beginning of each year for four successive years. Find its value at the end of the fourth year.*
   If after the fourth year it depreciated at a uniform annual rate of 4%, find its value at the end of 10 years (Give your answers to 3 significant figures).

   ANSWER: *Sh*. 354 000 , *Sh*. 277 000
4. *A woman opened a hair dressing saloon but went out of business three years later. She sold off various pieces of equipment for Sh. 12 000. This represented a uniform rate of depreciation of 15% p.a. estimate the cost price of the equipment giving your answers to 3 s.f.* **ANSWER: *Sh*. 19 500**
5. *A man bought a sewing machine for Sh. 10 800. After 12 years he sold it for Sh. 4 000. At what annual rate did the machine depreciate? (Answer to the nearest integer)*
   What was the value of the machine when it was 6 years old? (Answer to the 3 s.f)

   ANSWER: 8 % , *Sh*. 65 500

POPULATION GROWTH

When the population P increases at a uniform rate $r\%\ p.a$ for $n-\ years$, at the end of the $n^{th}\ year$, the population can be calculated as below.

Initial (present) population $= P$

Rate on increase $= r\%$

Population after 1 year $= P + P \times \frac{r}{100} = P\left(1 + \frac{r}{100}\right)$

Population after 2 years $= P\left(1 + \frac{r}{100}\right) + P\left(1 + \frac{r}{100}\right) \times \frac{r}{100}$

$$= P\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right) = P\left(1 + \frac{r}{100}\right)^2$$

Population after 3 years $= P\left(1 + \frac{r}{100}\right)^2 + P\left(1 + \frac{r}{100}\right)^2 \times \frac{r}{100}$

$$= P\left(1 + \frac{r}{100}\right)^2\left(1 + \frac{r}{100}\right) = P\left(1 + \frac{r}{100}\right)^3$$

$\therefore$ Population after n years $= \boldsymbol{P\left(1 + \frac{r}{100}\right)^n}$

Note

If the population is decreasing at a rate r% with passage of time, then the population after n years can be evaluated as $\boldsymbol{P\left(1 - \frac{r}{100}\right)^n}$

Example

1. *The population of a town increases yearly at a rate of 25 per 1000. It is now 378 530, what it be 3 years hence?*

   Solution

   Present population, $P = 378\,530$

   Time, $n = 3\ years$

   Rate of growth, $r = \frac{25}{1000} = 2.5\%$

   $\therefore$ Population after 3 years $= P\left(1 + \frac{r}{100}\right)^n$

   $$= 378\,530(1 + 0.025)^3$$

   $$= 378\,530(1.025)^3 = 407\,635$$

2. *In a certain town, population raised at 20 per 1000. Find the number of years in which the population will be doubled assuming there is neither immigration nor migration.*

   Solution s

   Let present population $= P$

   Rate of growth $= \frac{20}{1000} = 2\% = 0.02$

   Time of growth $= n - years$

   Population after n-years $= 2P$

   But population after n years $= P(1 + 0.02)^n = P(1.02)^n$

   $\Longrightarrow 2P = P(1.02)^n\ \ or\ 2 = (1.02)^n$

   $\log 2 = \log(1.02)^n = n \log 1.02$

   $\therefore n = \frac{\log 2}{\log 1.02} = 35\ years$

3. *In a Military Academy, 150 trainees were recruited in 2000. The recruiting team planned to increase the number of trainees each subsequent year at a rate of 20% of the number for the previous year. Determine the number expected to be in 2010.*

Solution

In 2000 there were 150

In 2001 there were $150 + 0.2(150) = 150(1 + 0.2) = 150(1.2)$

In 2002 there were $150(1.2) + (150)(1.2)(0.2) = 150(1.2)(1 + 0.2) = 150(1.2)^2$

In 2003 the number will be $150(1.2)^2 + 150(1.2)^2(0.2)$

$= 150(1.2)^2(1 + 0.2) = 150(1.2)^2$

∴ In 2010, after 10 years, the number is $150(1.2)^{10} \approx 929$ trainees.

Task

1. *The population of a country increases every year by 2.4% of the population at the beginning of that year. In what time will the population double itself?*
   ANSWER: **29 *years***
2. *In Jinja district, 6500 pupils registered for PLE 1996.* In 1997 the number of students increased by 1250 and in each subsequent year the number of pupils increased by 1/5 more than the previous year. Find the number of pupils that will be registered in 2006. **ANSWER**: **40 000** *to* 3 *s. f*
3. *If Kenya has a population of 23 million people and a doubling time of 19 years, and growth continues at the same rate, find the population in;*
   a) 10 years. **ANSWER**: **33 *million***
   b) 30 years. **ANSWER**: **69 *million***
4. *If the population in a particular third world country is growing at 4% compounded annually how long will it take the population to double?*
   ANSWER: **18 *years***

ANNUITY CALCULATIONS

FUTURE VALUE OF ANNUITY

Bankers call a series of equal annual payment made at equal intervals of time an *Annuity.*

Terms associated with Annuities

i) **Immediate/Ordinary Annuity**

If each of the annuity payment is made at the end of each period (time interval), is called *immediate annuity.*

ii) **Future value of an annuity**

Is the sum of all the annuity payments plus any accumulated interest.

To find the Amount of Annuity left unpaid for a given number of years, allowing Compound Interest.

Let R be the periodic deposit, i the rate per period and n the number of periods in the given years.

Case I: Payment at the end of each period.

At the end of the 1st year, R is due and the amount of this sum in the remaining $(n-1)$ year is $R(1+i)^{n-1}$

At the end of 2nd year, another R is due and the amount of this sum in the remaining $(n-2)$ years is $R(1+i)^{n-2}$

$\therefore$ Future value $= R(1+i)^{n-1} + R(1+i)^{n-2} + \cdots \ldots \ldots + R(1+i)^2 + R(1+i)^1 + R$

$= R[1 + (1+i) + (1+i)^2 + \cdots \ldots \ldots + (1+i)^{n-2} + (1+i)^{n-1}]$

$= R[1 + (1+i) + (1+i)^2 + \cdots \ldots \ldots \ldots \ldots \ldots to\ n - terms]$

$= \frac{R[(1+i)^n - 1]}{(1+i)-1} = \frac{R[(1+i)^n - 1]}{i}$

In formula form,

Future value of Immediate Annuity is:

$\boldsymbol{FV = PMT.\frac{(1+i)^n - 1}{i}}$

$PMT =$ Periodic payment

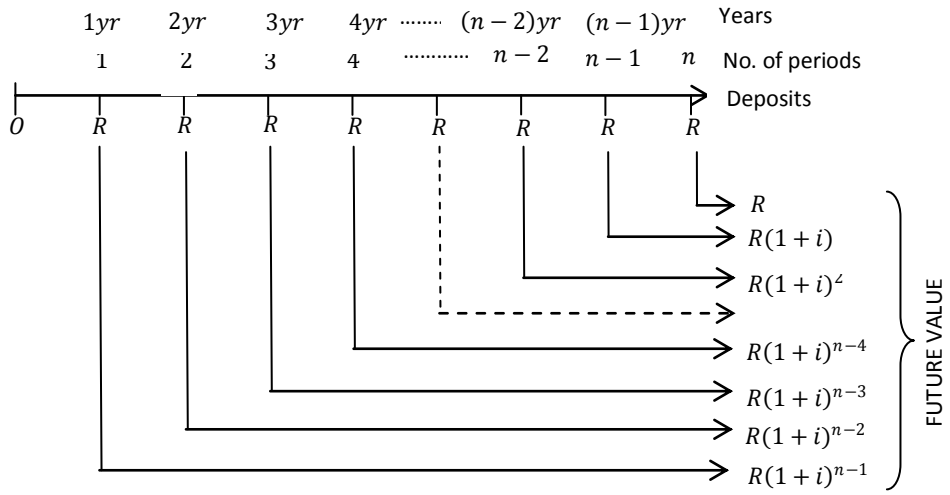
$i =$ Rate per period

$n =$ Number of payments (Periods)

$FV =$ Future Value (Amount)

[Payments are made at the end of each period]

Generalization of the above series using time line



Hence the sum of all these annuities plus any interest is;

$= R + R(1+i) + R(1+i)^2 + \cdots\ldots\ldots\ldots + R(1+i)^{n-4} + R(1+i)^{n-3} + R(1+i)^{n-2} + R(1+i)^{n-1}$

NOTE:

1. The future value of annuity is considered when payments are made at the end of each period (Immediate Annuity)
2. Banks usually state interest rates as *Annual Percentage Rates*(APRS)

Case II: Payment at the beginning of every period (payment for every period)

At the beginning of 1st year, R is due and amount of this sum for n years is $R(1+i)^n$

At the beginning of 2nd year, another R is due and the amount of the remaining $(n-1)$ year is $R(1+i)^{n-1}$

$\therefore$ Future value $= R(1+i)^n + R(1+i)^{n-1} + \cdots\ldots\ldots\ldots\ldots + R(1+i)^2 + R(1+i)$

$= R[(1+i) + (1+i)^2 + \cdots\ldots\ldots + (1+i)^{n-1} + (1+i)^n]$

$= R(1+i)[1 + (1+i) + (1+i)^2 + \cdots\ldots\ldots + (1+i)^{n-2} + (1+i)^{n-1}]$

$= R(1+i)[1 + (1+i) + (1+i)^2 + \cdots\ldots\ldots to\ n - terms]$

$= \frac{R(1+i)[(1+i)^n - 1]}{i}$

In formula form,

Future value of an Annuity due is:

$$\boldsymbol{FV = PMT.\frac{(1+i)}{i}.[(1+i)^n - 1]}$$

$PMT =$ Periodic payment

$i =$ Rate per period

$n =$ Number of payments (Periods)

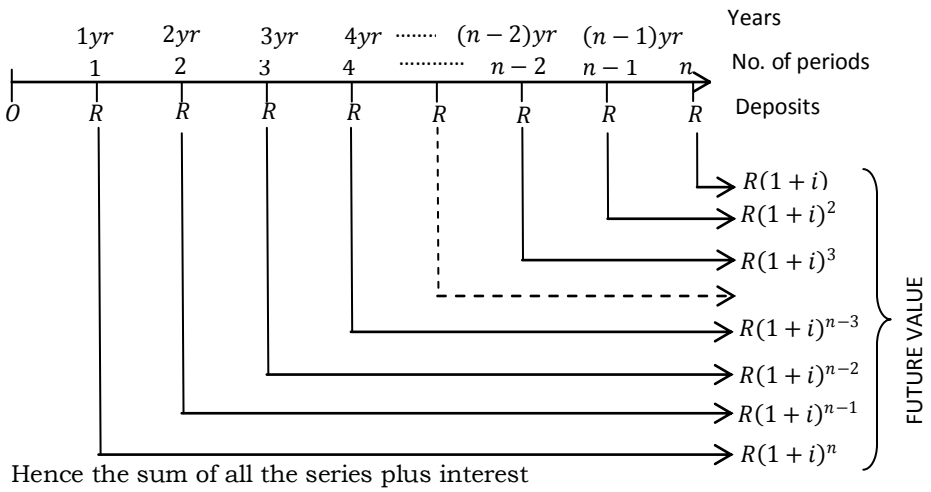
$FV =$ Future Value (Amount)

[Payments are made at the beginning of each period/payments are made for every period]

Case III: Annuity due

Is an annuity where the first payment falls due in the beginning of the first period or interval, hence Case II.

Generation of Case II series above using time line.



Hence the sum of all the series plus interest

$\therefore$ Future value $= R(1+i)^n + R(1+i)^{n-1} + \cdots\ldots\ldots\ldots\ldots + R(1+i)^2 + R(1+i)$

$= R(1+i)[1 + (1+i) + (1+i)^2 + \cdots\ldots\ldots + (1+i)^{n-2} + (1+i)^{n-1}]$

General examples:

(Immediate Annuity)

1. *A person wishes to deposit* $\$\,1000$ *at the end of each year in a bank which pays* 10% *p.a Compound Interest. If installments are allowed to accumulate, what will be the total accumulation at the end of 10 years?*

Solution

At the end of $1yr, 1^{st}$ installment of $\$\,1000$ amounts in 9 years to: $1000(1+0.1)^9 = 1000(1.1)^9$

The 2nd installment of $\$\,1000$ amounts in 8 years to: $1000(1+0.1)^8 = 1000(1.1)^8$

The last installment of $\$\,1000$ amounts to: $1000(1+0.1)^0 = 1000$

So the total accumulated at the end of 10 years is:

$Total = 1000 + 1000(1.1) + 1000(1.1)^2 + \cdots\ldots\ldots\ldots + 1000(1.1)^8 + 1000(1.1)^9$, which is a G.P of 10 terms.

$$= \frac{1000[(1.1)^{10}-1]}{1.1-1} = \frac{1000[(1.1)^{10}-1]}{0.1} = \$\,15\,937$$

Alternatively, using the formula,

$P = \$\,1000, i = 0.1, n = 10, A = ?$

$A = \frac{P}{i}[(1+i)^n - 1]$

$= \frac{1000}{0.1}[(1+0.1)^{10} - 1] = \$\,15\,937$

(Immediate Annuity)

2. *Jessica deposits* $\$\,100$ *every 6 months in an account that pays 6% Compound Interest compounded semi annually. If she made 6 deposits one at the end of each interest payment period over 3 years, how much money will be on the account after the last deposit is made?*

Solution

$\%\ rate\ p.a = 6, \%\ rate\ per\ 6-months = \frac{6}{2} = 3\%\ .i.e.\ i = 0.03, no.of\ periods = 6$

At the end of the 1st $6\ months$, \$100 is due, and this amounts to:

$\$\ 100(1+0.03)^5$

The 2nd installment of \$ 100 amounts to: $\$\ 100(1+0.03)^4 = \$\ 100(1.03)^4$

The 3rd installment of \$ 100 amounts to: $\$\ 100(1+0.03)^3 = \$\ 100(1.03)^3$

The $last$ installment of \$ 100 amounts to: $\$\ 100(1+0.03)^0 = \$\ 100(1.03)^0 = \$\ 100$

Thus the total accumulated at the end of the 6th deposit (at the end of 3 years) is:

$= 100 + 100(1.03) + 100(1.03)^2 + 100(1.03)^3 + \cdots \ldots . + 100(1.03)^5$

$= 100[1 + 1.03 + (1.03)^2 + \cdots \ldots . + (1.03)^5]$

$= \frac{100[(1.03)^6 - 1]}{1.03 - 1} = \$\ 646.84$

Alternatively, using the formula

$P = \$\ 100\ , i = \frac{6}{2} = 3\% = 0.03\ , n = 3\ yrs = 6\ periods, A = ?$

$A = 100\left[\frac{(1.03)^6 - 1}{0.03}\right] = \$\ 646.84$

(Immediate Annuity)

3. *When Jim started his first job after he finished college, he opened an Individual Retirement Account (IRA). He plans to contribute* \$ 2 500 *per year for 38 years until he reaches age 62. He hopes to earn an average annual percentage rate (APR) of 8% over the 38 year period.*
   a) *If Mr. Jim contributes to his IRA at the rate he plans, how much will his account be worth when he is 62 years old?*
   b) *How much interest will be earned on the account?*

   Solution

   Given: $P = \$\ 2\ 500\ , i = 8\% = 0.08\ , n = 38\ yrs$

a) At the end of 1st year \$ 2 500 is due, and this amounts to:
   $\$\ 2\ 500(1+0.08)^{37} = \$\ 2\ 500(1.08)^{37}$ for the remaining years.
   The 2nd installment of \$ 2 500 amounts to:
   $\$\ 2\ 500(1+0.08)^{36} = \$\ 2\ 500(1.08)^{36}$
   The 2nd installment of \$ 2 500 amounts to:
   $\$\ 2\ 500(1+0.08)^{35} = \$\ 2\ 500(1.08)^{35}$
   $\therefore$ the $last$ deposit amounts to:
   $\$\ 2\ 500(1+0.08)^0 = \$\ 2\ 500(1.08)^0 = \$\ 2\ 500$
   $\therefore$ the accumulated amount after 38 years is:
   $= 2500 + 2500(1.08) + 2500(1.08)^2 + \cdots \ldots \ldots + 2500(1.08)^{37}$
   $= 2500[1 + 1.08 + (1.08)^2 + \cdots \ldots \ldots + (1.08)^{37}]$
   $= \frac{2500[(1.08)^{38} - 1]}{1.08 - 1} = \$\ 550\ 789.86$

b) Mr. Jim will pay$\$\ 2500 \times 38 = \$95\ 000$ in to his IRA, so the account will earn:
   $\$\ (550\ 789.86 - 95\ 000) = \$\ 455\ 789.86$

4. *John, procrastinating, does not make his first \$ 2000 deposit in to an IRA account until he is 32, but then he continues to deposit \$ 2000 on every birthday until he is 65. If his account also earns 9% compounded annually, how much (to the nearest dollar) will he have on his account when he makes the last deposit on his* 65th *birth day?*

Solution

The 1st deposit is at when John is 32 years.

The *last* deposit is at when John is 65 years.

$\therefore$ number of deposits in all= 34, at the end of every year $i = 9\%\ p.a, P = \$\ 2\ 000$

The 1st deposit (at 32 years) grows to $2000(1 + 0.09)^{33}$

The 2nd deposit (at 33 years) grows to $2000(1 + 0.09)^{32}$

$\therefore$ the last deposit (at 65 years) grows to $2000(1 + 0.09)^{0}$

$\therefore$ the total amount accumulated at the end of the 65th birthday is;

$$= 2000 + 2000(1 + 0.09) + 2000(1 + 0.09)^2 + \cdots \ldots \ldots \ldots + 2000(1 + 0.09)^{33}$$

$$= 2000[1 + 1.09 + (1.09)^2 + \cdots \ldots \ldots \ldots \ldots \ldots + (1.09)^{33}]$$

$$= \frac{2000\,[(1.09)^{34} - 1]}{1.09 - 1} = \$\ 393\ 965$$

5. *A man puts $Rs.10$ at the end of every year in the savings Bank at 2 ½ % compound interest. How much will his savings amount to in 15 years?*

Solution

At the end of 1st year, $Rs.10$ is due, and amounts to $10(1 + 0.025)^{14}$

At the end of the 2nd year, $Rs.10$ is again due, and amounts to $10(1 + 0.025)^{13}$

The last deposit grows to $10(1 + 0.025)^{0}$

$\therefore$ total amount $= 10 + 10(1.025) + 10(1.025)^2 + \cdots \ldots \ldots \ldots \ldots + 10(1.025)^{14}$

$$= 10[1 + 1.025 + (1.025)^2 + \cdots \ldots \ldots \ldots \ldots . . + (1.025)^{14}]$$

$$= \frac{10[(1.025)^{15} - 1]}{1.025 - 1} = Rs.179.32$$

6. *A man deposits* $\$\ 100$ *per month in to an account earning 9% compounded monthly for 5 years. How much interest does he receive?*

Solution

At the end of the 1st month, 1st deposit grows to $\$\ 100\left(1 + \frac{0.09}{12}\right)^{59}$ in the remaining 59 months.

At the end of the 2nd month, 2nd deposit grows to $\$\ 100\left(1 + \frac{0.09}{12}\right)^{58}$ in the remaining 58 months.

$\therefore$ at the end of the last month of deposit, deposit grows to $\$\ 100\left(1 + \frac{0.09}{12}\right)^{0} = \$\ 100$

$\therefore$ total accumulation at the end of 5 years(60 *months*) is:

$$= 100 + 100\left(1 + \frac{0.09}{12}\right)^{1} + 100\left(1 + \frac{0.09}{12}\right)^{2} + \cdots \ldots \ldots \ldots \ldots + 100\left(1 + \frac{0.09}{12}\right)^{59}$$

$$= 100\left[1 + \left(1 + \frac{0.09}{12}\right) + \left(1 + \frac{0.09}{12}\right)^{2} + \cdots \ldots \ldots . . + \left(1 + \frac{0.09}{12}\right)^{59}\right]$$

$$= \frac{100\left[\left(1 + \frac{0.09}{12}\right)^{60} - 1\right]}{\left(1 + \frac{0.09}{12}\right) - 1} = \frac{100\left[\left(1 + \frac{0.09}{12}\right)^{60} - 1\right]}{0.0075} = \$\ 7\ 542.41$$

Now total amount deposited= $60 \times 100 = 6\ 000$

$Interest = Value - Deposits$

interest= $7\ 542.41 - 6\ 000 = \$\ 1\ 542.41$

Note:

Since 9% is compounded monthly, then $i = \frac{0.09}{12} = 0.0075$

7. *Jane deposits* $\$2000$ *a year in to an IRA account earning 9% compounded annually. She makes her first deposit on her* 24th *birthday and her last deposit on her* 31st *birthday (8 deposits in all). Making no additional deposits, she leaves the accumulated amount from the 8 deposits in an account, earning interest at 9% compounded annually until her* 65th *birthday. How much to the nearest dollar will be in her account on her* 65th *birthday?*

Solution

Given: $P = \$2000, i = 9\% = 0.09, n = 8$

At the end of the 1st year, $\$2000$ is due and amounts to $2000(1 + 0.09)^7$ in the remaining 7 years.

At the end of the 2nd year, $\$2000$ is again due and amounts to $2000(1 + 0.09)^8$ in the remaining 6 years.

$\therefore$ the last deposit of $\$2000$ grows to $2000(1 + 0.09)^0 = 2000$

$\therefore$ the total amount for the 8 years of depositing is:

$$= 2000 + 2000(1.09) + 20002000(1.09)^2 + \cdots \ldots \ldots \ldots \ldots + 2000(1.09)^7$$
$$= 2000[1 + 1.09 + (1.09)^2 + \cdots \ldots \ldots \ldots \ldots + (1.09)^7]$$
$$= \frac{2000[(1.09)^8 - 1]}{1.09 - 1} = \$22\,056.94759$$

Now since no other deposit after the 8 −deposits, the money grows at compound interest at a rate of 9% $p.a$ for 34 years after the 31st *birthday.*

Using $A = P(1 + i)^n$

$$A = 22\,056.94759(1 + 0.09)^{34} = \$413\,092$$

(Annuity due)

8. *A person invests* £2000 *every year with a company which pays interest 10% p.a. he allows his deposit to accumulate at compound interest. Find the amount to the credit of the person at the end of the* 5th *year.*

Solution

At the beginning of the 1st year, £2000 is due and his sum amounts in 5 years to

$£2000(1 + 0.1)^5$

At the beginning of the 2nd year, another £2000 is due and his sum amounts in 4 years to

$£2000(1 + 0.1)^4$

The last deposit (5th year) grows to $£2000(1 + 0.1)^1$

$\therefore$ total amount accumulated

$$= 2000(1 + 0.1) + 2000(1 + 0.1)^2 + \cdots \ldots \ldots . + 2000(1 + 0.1)^5$$
$$= 2000(1.1)[1 + 1.1 + (1.1)^2 + \cdots \ldots \ldots \ldots + (1.1)^4]$$
$$= \frac{2000\,(1.1)[(1.1)^5 - 1]}{1.1 - 1} = \frac{2000\,(1.1)[(1.1)^5 - 1]}{0.1} = £13\,431.22$$

Alternatively by formula method

$P = £2000, n = 5, i = 0.1, A = \frac{P(1+i)}{i}[(1 + i)^n - 1]$

$A = £2000 \times \frac{1.1}{0.1} \times [(1.1)^5 - 1] = £13\,431.22$

(Annuity due)

9. *Mr. John deposits* £2000 *per year in the bank for 10 years at 5% p.a. find how much will he receive at the end of 10 years if the money accumulates at compound interest.*

Solution

At the end of the 10th deposit, the last deposit grows to $£2000(1+0.05)$
At the end of the 9th deposit, the amount grows to $£2000(1+0.05)^2$
$\therefore$ the 1st deposit grows to $£2000(1+0.05)^{10}$
$\therefore$ total amount accumulated $= 2000(1.05) + 2000(1.05)^2 + \cdots \ldots \ldots \ldots + 2000(1.05)^{10}$

$$= 2000(1.05)[1 + 1.05 + (1.05)^2 + \cdots \ldots \ldots \ldots + (1.05)^9]$$
$$= \frac{2000\,(1.05)[(1.05)^{10}-1]}{1.05-1} = \frac{2000\,(1.05)[(1.05)^{10}-1]}{0.05}$$
$$= £\,26\,413.57$$

10. *Jack operates an account with a certain bank which pays a compound interest rate of 13.5% p.a. he opened the account at the beginning of the year with Sh. 500 000 and deposits the same amount of money at the beginning of every year. Calculate how much will he receive at the end of 9 years. After how long will the money have accumulated to Sh. 3.32 million.*

Solution

At the beginning of the 1st year, Sh. 500 000 is due, and the amount in 9 years is
$500\,000(1+0.135)^9 = 500000(1.135)^9$
At the beginning of the 2nd year, another Sh. 500 000 is due and amounts in 8years to
$500\,000(1+0.135)^8 = 500000(1.135)^8$
$\therefore$ the last deposit in the 9th year, amounts to $500\,000(1+0.135)^1 = 500\,000(1.135)$
$\therefore$ the total amount $= 500000(1.135) + 500000(1.135)^2 + \cdots \ldots \ldots . + 500000(1.135)^9$

$$= 500000(1.135)[1 + 1.135 + (1.135)^2 + \cdots \ldots \ldots + (1.135)^8]$$
$$= \frac{500000\,(1.135)[(1.135)^9-1]}{1.135-1} = \frac{500000\,(1.135)[(1.135)^9-1]}{0.135}$$
$$= Sh.\,8\,936\,281$$

Let the money accumulate in $n-years$ to 3.32 million
The 1st deposit grows to $500\,000(1+0.135)^n = 500\,000(1.135)^n$ in $n-years.$
The 2nd deposit grows to $500\,000(1+0.135)^{n-1} = 500\,000(1.135)^n$ in $(n-1)-years.$
$\therefore$ the nth deposit grows to $500\,000(1.135)$
$\therefore$ the total amount after $n-years$

$$= 500\,000(1.135) + 500\,000(1.135)^2 + \cdots \ldots \ldots + 500\,000(1.135)^n$$
$$= 500\,000(1.135)[1 + 1.135 + (1.135)^2 + \cdots \ldots \ldots . + (1.135)^{n-1}]$$
$$= \frac{500\,000(1.135)[(1.135)^n-1]}{1.135-1} = \frac{500\,000(1.135)[(1.135)^n-1]}{0.135}$$

$\therefore 3.32 \times 10^6 = \frac{500\,000(1.135)[(1.135)^n-1]}{0.135}$
$0.789779735 = 1.135^n - 1 \ \ or \ \ 1.135^n = 1.789779735$
$\ln(1.135)^n = \ln 1.789779735$
$\therefore n = \frac{\ln 1.789779735}{\ln 1.135} = 4.6\ years$

11. *Five million shillings are invested each year at a rate of 20% interest. In how many years will it accumulate to have more than 50 million shillings?*

Solution

Since deposits are made each/every year, this is a case of *annuity due.*

For the 1st year, $Sh.5\,000\,000$ is due, and amounts in n years to $Sh.5\,000\,000(1+0.2)^n$

For the 2nd year, another $Sh.5\,000\,000$ is due, and amounts in $n-1$ years to $Sh.5\,000\,000(1+0.2)^{n-1}$

$\therefore$ the last deposit in the nth year amounts to $Sh.5\,000\,000(1+0.2)$

$\therefore$ total accumulation after n years is:

$$= 5\,000\,000(1.2) + 5\,000\,000(1.2)^2 + \cdots\ldots\ldots\ldots + 5\,000\,000(1.2)^n$$
$$= 5\,000\,000(1.2)[1 + 1.2 + (1.2)^2 + \cdots\ldots\ldots + (1.2)^{n-1}]$$
$$= \frac{5\,000\,000(1.2)[(1.2)^n-1]}{1.2-1} = \frac{5\,000\,000(1.2)[(1.2)^n-1]}{0.2}$$

Given the total accumulation is $Sh.50\,000\,000$

$$\Rightarrow 50\,000\,000 = \frac{5\,000\,000(1.2)[(1.2)^n-1]}{0.2}$$

$$\Rightarrow \frac{5}{3} = 1.2^n - 1 \ or\ 1.2^n = \frac{8}{3}$$

$$\Rightarrow \ln 1.2^n = \ln\left(\frac{8}{3}\right) \ \therefore n = \frac{\ln\left(\frac{8}{3}\right)}{\ln 1.2} = 5.379666948$$

$\therefore$ in 6 years the money would have accumulated more than 50 million.

Task

1. *Find the amount of annuity due of £2000 per year payable in the beginning of every year for 10 years at 5% p.a* **ANSWER: £26 413.60**
2. *Five million shillings are invested each year at a rate of 15% interest. In how many years will it accumulate to more than Sh. 50 million?*
   ANSWER: 6 *years*
3. *A man puts Rs. 10 at the end of every year in the savings Bank at 2 ½ % compound interest. How much will his savings amount to in 15 years?*
   ANSWER: *Rs.* 179
4. *A company sets aside a sum of £20 000 annually to enable it pay off a debenture issue of £230 000 at the end of 10 years. Assuming that the sum accumulates at 4% p.a compound interest, find the surplus after paying off a debenture.*

   HINT:
   £20 000 will amount to (at the end of 10 years) $20\,000(1.04)^{10}$
   The 2nd *sum amounts to* $20\,000(1.04)^9$
   $\therefore$ total money at the end of the 10th *year wil be*:
   $20\,000(1.04)[1 + 1.04 + (1.04)^2 + \cdots\ldots\ldots\ldots\ldots\ldots + (1.04)^9]$
   $= \frac{20\,000(1.04)[(1.04)^{10}-1]}{0.04} = £249\,080$
   Thus the surplus $= 249\,080 - 230\,000 = £19\,080$

5. *Phionah deposited* $1000 *every 6 months in to an account earning 8% compounded semi annually. How much will the* $1000 *amount to at the end of 10 years? How much of this value is interest?*

HINT:

$1^{st}\ deposit\ grows\ to\ 1000\left(1+\frac{0.08}{2}\right)^{19}\ in\ 19\ months$

$2^{nd}\ deposit\ grows\ to\ 1000\left(1+\frac{0.08}{2}\right)^{18}\ in\ 19\ months$

$\therefore last\ deposit\ grows\ to\ 1000\left(1+\frac{0.08}{2}\right)^{0}$

$\therefore Total\ value = 1000[1+1.04+(1.04)^2+\cdots\ldots\ldots+(1.04)^{19}]$

$= \$\ 29\ 778.4$

$\therefore Interest = 29\ 778.4 - 20\times 1000 = \$\ 9\ 778.4$

6. *Beginning in January, a person plans to deposit* $\$\ 100$ *at the end of each month in to an account earning 9% compounded monthly. Each year taxes must be paid on the interest earned during that year. Find the interest earned during each year for the first* $3-$*years.*

HINT:

$i=\frac{0.09}{12}=0.0075\ per\ month, P=\$\ 100\ , n=1\ year=12\ months.$

$Value\ after\ 1^{st}\ year=\frac{100[(1+0.0075)^{12}-1]}{0.0075}=\$\ 1\ 250.76$

$Interest\ for\ 1^{st}\ year=1\ 250.76-12\times 100=\$\ 50.76$

$Value\ after\ 2^{nd}\ year=\frac{100[(1+0.0075)^{24}-1]}{0.0075}=\$\ 2\ 618.85$

$Interest\ after\ 2\ years=2\ 618.85-24\times 100=\$\ 218.85$

$\therefore Interest\ for\ 2^{nd}\ year=218.85-50.76=\$\ 168.09$

$Value\ after\ 3^{rd}\ year=\frac{100[(1+0.0075)^{36}-1]}{0.0075}=\$\ 4\ 115.27$

$\therefore Interest\ after\ 3\ years=4\ 115.27-36\times 100=\$\ 512.27$

$Now\ interest\ for\ 3^{rd}\ year$

$= Interest\ for\ 3\ years\ minus\ interest\ for\ the\ 1^{st}\ two\ years$

$= 512.27-218.85=\$\ 296.42$

7. *What is the annuity at the end of 5 years if* $\$\ 100$ *per month is deposited in to an account earning 9% compounded monthly? How much of this value is interest?* ***Answer***: $\$\ \mathbf{1\ 542.41}$

8. *A man pays a premium of* $£\ 100$ *at the beginning of every year to an insurance company on the understanding that at the end of fifteen years he can receive back the premiums which he has paid with 5% compound interest. What should he receive? (Give your answer correct to* $3\ s.f$*)*

HINT:

$At\ the\ beginning\ of\ 1^{st}\ year, £\ 100\ is\ due\ and\ amounts\ to\ £\ 100(1+0.05)^{15}$ $for\ remaining\ 15\ years.$

$At\ the\ beginning\ of\ 2^{nd}\ year, another\ £\ 100\ is\ due\ and\ amounts\ to$ $£\ 100(1+0.05)^{14}\ for\ the\ remaining\ 14\ years.$

$The\ last\ deposit\ grows\ to\ £\ 100(1+0.05)$

$\therefore Total = 100(1.05)+100(1.05)^2+\cdots\ldots\ldots\ldots+100(1.05)^{15}$

$=\frac{100(1.05)[(1.05)^{15}-1]}{0.05}=2\ 265.749177$

≈ 22700

9. *A man earned in a certain year* $£\ 2000$ *from a certain source and his annual earnings from this time continued to increase at a rate of 5%. Find to the nearest £ the whole amount he received from this source in this year and the next seven years. Give your answer correct to* $3\ s.f$.

HINT:
Amount in the 1st *year* $= 2000(1 + 0.05) = £2100$
Now amount in 2nd *year grows to* $2000(1 + 0.05)^7$ *in the remaining* 7 *years or grows to* $2000(1 + 0.05)^6$ *in the remaining* 6 *years, taking* $£2100$ *as the beginning amount.*
Amount in 3rd *year grows to* $2000(1 + 0.05)^6$ *in the remaining* 6 *years or grows to* $2000(1 + 0.05)^5$ *in the remaining* 5 *years.*
∴ *Amount in the last year grows to* $2000(1 + 0.05)^0$
∴ *Total amount accumulated in the next* 7 *years is*:

$$= 2000 + 2000(1.05) + 2000(1.05)^2 + \cdots \ldots \ldots + 2000(1.05)^7$$
$$= 2000[1 + 1.05 + (1.05)^2 + \cdots \ldots . + (1.05)^7]$$
$$= \frac{2000[(1.05)^8 - 1]}{0.05} = £19\,098.21775$$
$$\approx £19\,100$$

10. *A credit society gives out a compound interest of 4.5% p.a. Mugaga deposits Sh. 300 000 at the beginning of each year. How much money will he have at the end of 4 years if there are no withdraws during this period.* ***(Uneb 2006) Answer***: ***Sh.*** **1 341 213**
11. *A man invests* $£100$ *at the end of 1971,* $£200$ *at the end of1972,* $£300$ *at the end of 1973,* $£400$ *at the end of 1974,* $£500$ *at the end of 1975. If all interest accumulated at* $5\%\ p.a$*, what do his investments amount to at the end of 1976?*

HINT:
$£100$ *is invested for* 5 *years,* $£200$ *for* 4 *years,* $£300$ *for* 3 *years,* $£400$ *for* 2 *years* $£500$ *for* 1 *year*
Total value at the end of 1976 *is*;
$A = 100(1.05)^5 + 100(1.05)^4 + 100(1.05)^3 + 100(1.05)^2 + 100(1.05) = £1\,684.2$

PERMUTATIONS AND COMBINATIONS

FACTORIALS:

Are used in the computations of permutations and combinations

Definition:

The continued product of the first $n-$ *natural numbers* $i.e.$ the continued product of n-consecutive integers beginning with 1 and ending with n, is *ctorial* n , denoted by the symbol $n!$

$$\therefore \boldsymbol{n!} = \boldsymbol{1 \times 2 \times 3 \times \ldots\ldots\ldots\ldots \times n}$$
$$= \boldsymbol{n(n-1)(n-2)(n-3)\ldots\ldots\ldots \times 2 \times 1}$$

Examples

1. *How many different 3-digit number can be constructed from numerals 5,7 and 8 taking each once?*

   Solution

   3-choices arise for the 1st numeral and for each choice; there are a further of 2-choices for the 2nd numeral. The third numeral is the one that is left
   $\therefore 3 \times 2 \times 1 = 6$ choices for the 1st *and* 2nd numerals combined exist.
   $i.e. 3! = 3 \times 2 \times 1 = 6$

2. *How many 4-digit numbers can be constructed using numerals 1,2,3 and 4 once each?*

   Solution

   4-choices arise for the 1st numeral; each choice leaves 3-ways to select2^{nd} numeral
   $\Rightarrow 4 \times 3 = 12$ *ways* of selecting the first 2-numerals exist.
   Each combination of the first two numerals leaves 2-ways to select the third numeral.
   $\Rightarrow$ *Ther are* $4 \times 3 \times 2 \times 1 = 24$ *ways* of selecting first 3-numerals. The last nmeral s the one left.
   $\therefore 4 \times 3 \times 2 \times 1 = 4! = 24$ are the number of ways.

3. *Evaluate*

   a) $6!$ d) $\frac{3!}{0!}$

   b) $\frac{8!}{3!}$ e) $\frac{5!}{(5-2)!2!}$

   c) $(7-2)!$

   Solution

   a) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

   b) $\frac{8!}{3!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{3\times2\times1} = 6720$

   c) $(7-2)! = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

   d) $\frac{3!}{0!} = \frac{3\times2\times1}{1} = 6 \quad [\because 0! = 1]$

   e) $\frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5\times4\times3\times2\times1}{(3\times2\times1).(2\times1)} = 10$

   Note

   1) From the definition of n!
      $n! = n(n-1)(n-2)(n-3)\ldots\ldots\ldots \times 2 \times 1$

$= n \times (n-1)!$

$\therefore n = \frac{n!}{(n-1)!}$

2) $2n! = 2 \times n!\,;\ (2n)! = (2n)(2n-1)(2n-2)\ldots\ldots \times 2 \times 1$

Example

1. *Simplify*
   a) $\frac{(2n+1)!}{(2n-1)!}$
   b) $\frac{n!}{(n+1)!}$
   c) $\frac{(n+1)!}{(n-1)!}$
   d) $\frac{(2n)!}{(2n+2)!}$

Solution

a) $\frac{(2n+1)!}{(2n-1)!} = \frac{(2n+1)(2n+1-1)(2n+1-2)\times\ldots\ldots\ldots\ldots\times 2\times 1}{(2n-1)!}$

$= \frac{(2n+1)(2n)(2n-1)!}{(2n-1)!} = 2n(2n+1)$

b) $\frac{n!}{(n+1)!} = \frac{n!}{(n+1)(n+1-1)!} = \frac{n!}{(n+1)\times n!} = \frac{1}{n+1}$

c) $\frac{(n+1)!}{(n-1)!} = \frac{(n+1)(n+1-1)(n+1-2)\ldots\ldots\times 2\times 1}{(n-1)!}$

$= \frac{(n+1)\times n\times(n-1)!}{(n-1)!} = n(n+1)$

d) $\frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+2-1)(2n+2-2)\ldots\ldots\times 2\times 1}$

e) $= \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{2(n+1)(2n+1)}$

2. *Write each of the following in factorial notation*
   a) $4 \times 3 \times 2 \times 1$
   b) $6 \times 5 \times 4$
   c) $\frac{(7\times 6)\times(3\times 2\times 1)}{2}$

Solution

a) $4 \times 3 \times 2 \times 1 = 4!$

b) $6 \times 5 \times 4 = \frac{(6\times 5\times 4)\times 3\times 2\times 1}{3\times 2\times 1} = \frac{6!}{3!}$

c) $\frac{(7\times 6)\times(3\times 2\times 1)}{2} = \frac{[(7\times 6)\times 5\times 4\times 3\times 2\times 1]\times(3\times 2\times 1)}{(2\times 1)\times(5\times 4\times 3\times 2\times 1)} = \frac{7!3!}{2!5!}$

Note

Some other terms have been introduced but without changing the meaning of the question so as to apply $n!$ definition.

PERMUTATIONS

The knowledge of permutations helps in finding *the number of ways a set of objects may be grouped/arranged under certain conditions.*

Definition

Each arrangement/*ordered* set of objects which can be obtained from a given set of objects by taking some or all at a time is called *permutation of the given set of objects.*

Notation

If the number of objects in the given set is n and the number of objects in each permutation is r $(i.e. n \geq r)$, the total number of permutation is denoted as $^{n}\boldsymbol{p}_{r}$

Consider the example below

Qn. *Assuming in the department of math, there are four teachers A, B, C and D; a committee consisting of two members is to be formed for drafting the math syllabus, list the possible groups.*

ANSWER: AB, AC, AD, BC, BD, CD

Hence there is 6-groups altogether $i.e. 6$ *combinations*

Qn. *Suppose one is to be selected as a president of the syllabus committee and the other as a secretary, list the possible groups.*

ANSWER:

Let the 1st one denote the president and the 2nd as the secretary.

The possibilities are;

$A_{president}\ B_{secretary}\ , BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC$

Hence there is now 12-possibilities altogether.

⇒ This problem is not only one of *grouping* but also *ordering* them.

Thus each of the 12-*arrangements* made out of 4-letters A, B, C, D by taking 2-letters at a time viz: $AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC$ is a *permutation of the 4-letters taken 2-at a time. i.e.* $^{4}\boldsymbol{p}_{2} = \mathbf{12}$

NOTE

1. Permutations involve *ordering* and *arranging*. $i.e. AB$ *and* BA are different; AB means A is the president , B is the secretary, while BA means B is the president and A is the secretary.
   However for purpose of *grouping, AB* and *BA* mean one and same group.
2. Grouping is more in *combinations*

IMPORTANT RESULTS OF PERMUTATIONS

a) *Permutations of n-different objects*
Permutations of n-different objects taken r-at a time $r < n$ is

$$^{n}\boldsymbol{p}_{r} = \frac{\boldsymbol{n!}}{\boldsymbol{(n-r)!}} = \boldsymbol{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}$$

b) *Permutations in which objects are repeated*
The number of permutations of n-different objects taken r-at a time in which each object is repeated r-times in any arrangement is $\boldsymbol{n^{r}}$

c) *Permutations of objects not all different*
The number of permutations of n- objects of which p-objects are alike of one kind, q-objects are alike of second kind, r-objects are alike of a third kind, and all the rest are all different is

d) *Circular permutations*

$$\frac{\boldsymbol{n!}}{\boldsymbol{p!q!r!}}$$

In circular permutations, fix the position of one object and then arrange the remaining $(n-1)$ out of n-objects in all the possible ways. This can be done in only

$$\boxed{(n-1)!\,ways}$$

e) *Restricted permutations*

i) The number of permutations of n-different objects taken r-at a time in which p-particular objects do not occur is

$$\boxed{{}^{n-p}p_r}$$

ii) The number of permutations of n-different objects taken r-at a time in which p-particular objects are present is

$$\boxed{{}^{n-p}p_{r-p} \times {}^{r}p_{p}}$$

Examples

1. *Find the value of n if:*
   a) ${}^{n}p_1 = 3$
   b) ${}^{n}p_2 = 2$
   c) ${}^{2n}p_2 = 6$
   d) ${}^{3n}p_2 = 30$

   Solution

   a) ${}^{n}p_1 = 3 \Rightarrow \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = 3 \ \therefore n = 3$

   b) ${}^{n}p_2 = 2 \Rightarrow \frac{n!}{(n-2)!} = 2 \ or \ \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) = 2$

   $\Rightarrow n^2 - n - 2 = 0 \ \therefore n = 2, -1 \ \ Required\ n = 2$

   c) ${}^{2n}p_2 = 6 \Rightarrow {}^{2n}p_2 = \frac{(2n)!}{(2n-2)!} = \frac{(2n)(2n-1)(2n-2)!}{(2n-2)!} \ \therefore 2n(2n-1) = 6 \ or \ 4n^2 - 2n - 6 = 0$

   solving$n = 1.5, -1$. Since n is a counting number, then $n = 2$

   d) ${}^{3n}p_2 = 30 \Rightarrow {}^{3n}p_2 = \frac{(3n)!}{(3n-2)!} = \frac{(3n)(3n-1)(3n-2)!}{(3n-2)!}$

   $\therefore 3n(3n-1) = 30 \ or \ 9n^2 - 3n - 30 = 0$. Solving $n = 2, -\frac{5}{3} \ \therefore$

   $Required\ n = 2$

2. *Find n if*
   i) ${}^{n}p_5 : {}^{n}p_3 = 2 : 1$
   ii) ${}^{2n+1}p_{n-1} : {}^{2n-1}p_n = 3 : 5$
   iii) ${}^{56}p_{n+6} : {}^{54}p_{n+3} = 30800 : 1$

   Solution

   i) ${}^{n}p_5 : {}^{n}p_3 = 2 : 1 \Rightarrow {}^{n}p_5 \,/\, {}^{n}p_3 = \frac{2}{1} \ or \ \frac{\frac{n!}{(n-5)!}}{\frac{n!}{(n-3)!}} = \frac{n!}{(n-5)!} \times \frac{(n-3)!}{n!} = 2$

   $\Rightarrow \frac{(n-3)!}{(n-5)!} = 2 \ or \ \frac{(n-3)(n-4)(n-5)!}{(n-5)!} \ \therefore \ (n-3)(n-4) = 2$

   $\Rightarrow n^2 - 7n + 10 = 0 \ \therefore n = 2 \ or \ 5$

   $Since\ n > r, then\ required\ n = 5\ (Reject\ n = 2)$

   ii) ${}^{2n+1}p_{n-1} : {}^{2n-1}p_n = 3 : 5 \ \Rightarrow {}^{2n+1}p_{n-1} \,/\, {}^{2n-1}p_n = 3 \,/\, 5$

   $\therefore \frac{(2n+1)!}{[(2n+1)-(n-1)]!} \div \frac{(2n-1)!}{[(2n-1)-n]!} = \frac{3}{5}$

$\frac{(2n+1)!}{(n+2)!}\times\frac{(n-1)!}{(2n-1)!}=\frac{3}{5}\Rightarrow\frac{(2n+1)!}{(2n-1)!}\times\frac{(n-1)!}{(n+2)!}$

$\frac{(2n+1)(2n+1-1)(2n+1-2)!}{(2n-1)!}\times\frac{(n-1)!}{(n+2)(n+2-1)(n+2-2)(n+2-3)!}=\frac{3}{5}$

$\frac{(2n+1)(2n)(2n-1)}{(2n-1)!}\times\frac{(n-1)!}{(n+2)(n+1).n.(n-1)!}=\frac{3}{5}$

$\Rightarrow\frac{2n(2n+1)}{n(n+1)(n+2)}=\frac{3}{5}\ \ or\ \ \frac{2(2n+1)}{(n+1)(n+2)}=\frac{3}{5}$

$\therefore 10(2n+1)=3(n^2+3n+2)\ or\ 3n^2-11n-4=0$

$\Rightarrow(n-4)(3n+1)=0\ \ \therefore n=1,-\frac{1}{3}\ Required\ n=1$

$and\ non\ integer\ n=-\frac{1}{3}\ rejected$

iii) ${}^{56}p_{n+6}:{}^{54}p_{n+3}=30800:1$

${}^{56}p_{n+6}\ /\ {}^{54}p_{n+3}=30800/1\Rightarrow\frac{56!}{[56-(n+6)]!}\div\frac{54!}{[54-(n+3)]!}=\frac{30800}{1}$

$\Rightarrow\frac{56!}{[50-n]!}\div\frac{54!}{[51-n]!}=\frac{30800}{1}\ \Rightarrow\frac{56!}{54!}\times\frac{[51-n]!}{[50-n]!}=30800$

$\Rightarrow\frac{56\times55\times54!}{54!}\times\frac{(51-n)(51-n-1)!}{(50-n)!}=30800$

$\therefore 3080(51-n)=30800\ \ or\ \ 51-n=10\ \ \therefore n=41$

Task

1) *If* ${}^{10}p_{n-1}:{}^{11}p_{n-2}=30:11$, *find n*

   ANSWER: $\boldsymbol{n=7}$

2) *Find n if*

   a) $5\times{}^{56}p_{n+6}=12\times{}^{n-1}p_3$

   b) ${}^{n+1}p_5:{}^{n-1}p_5=9:2$

   c) ${}^{2n-1}p_n:{}^{2n+1}p_{n-1}=5:3$

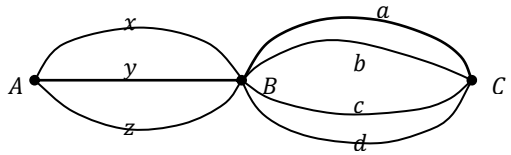
THE FUNDAMENTAL PRINCIPLE OF COUNTING

Consider the figure below

$A,B\ and\ C$ are 3-stations.

$x,y\ and\ z$ are 3-routes from A to B

$a,b,c\ and\ d$ are 4-routes from B to C



There are 3-independent ways of going from A to B, and corresponding to each way of going from A to B, there are 4-ways of going from B to C.

Hence there $3\times4=12\ ways$ of going from A to C.

$i.e.\ xa,xb,xc,xd,ya,yb,yc,yd,za,zb,zc,zd$

With this background the fundamental rules are;

1. **Multiplication principle**

   If one operation can be performed independently in **m** different ways, and another operation can be performed independently in **n** different ways, the total number of ways in which both operations can be performed simultaneously is $(\boldsymbol{m\times n})\boldsymbol{ways}$

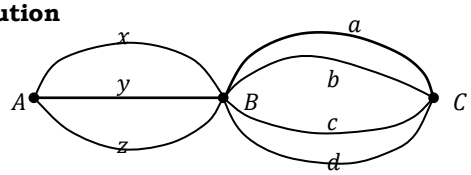
2. **Addition principle**

If one operation can be performed in **m** ways, and another operation can be performed in **n** ways, and only one operation can be done at a time, then either of the two operations can be performed in $(\boldsymbol{m}+\boldsymbol{n})\ \boldsymbol{ways}$

Examples

1. *There are four bus lines between A and B and three bus lines between B and C.*
   a) *In how many ways can a man travel by bus from A to C by way of B?*
   b) *In how many ways can a man travel round trip by bus from A to C by way of B, if he doe not want to use a bus line more than once?*

Solution



a) $4 \times 3 = 12\ ways$

b) Travelling from A to C, $no.of\ ways = 4 \times 3 = 12\ ways$. Since the man doe not want to use the bus line more than once, the number of bus lines from C to B are 2 and from B to A are 3.

$\therefore$ $no.$ of ways in which he can travel from C to A by way of B is $2 \times 3 = 6\ ways$

$\therefore$ the total umber of ways in which a man can travel round trip by bus from A to C by way of B not using bus line more than once is $12 \times 6 = 72\ way$

General examples on permutations

1) *Find the number of words that can be formed by considering all possible permutations of the letters of the word "FATHER". How many of these words begin with A and end with R?*

Solution

Since all the letters are different,

$n = 6\ , r = 6\ (Taking\ all\ the\ letters\ for\ a\ word)$

$\Rightarrow {}^{n}p_r = {}^{6}p_6 = 6! = \frac{6!}{0!} = 720\ ways$

Since A and R are fixed, permutations are to be done on FTHE

$\therefore n = 4\ , r = 4\ \Rightarrow {}^{4}p_4 = 24\ ways$

2) *How many different words containing all the letters of the word TRIANGLE can be formed?*

How many of them

a) Begin with T?
b) Begin with E?
c) Begin with T and end with E?

Solution

All the letters are different

$n = 8, r = 8\ \ \therefore {}^{8}p_8 = 8!$

a) Since T is fixed for all words, permutations are to be made on R ,I ,A,N,G,L,E

$\therefore n = 7, r = 7 \Rightarrow {}^{7}p_7 = 7!$

b) Since E is now fixed, permutations are to be made on the remaining letters

$\therefore n = 7, r = 7 \Rightarrow {}^{7}p_7 = 7!$

c) Now T and E are fixed, hence permutations are to be made on the remaining 6-letters

$\therefore n = 6, r = 6 \Rightarrow {}^{6}p_6 = 6!$

3) i) *In how many ways can the letters of the word "MOBILE" be arranged so that the consonants always occupy the odd places?*

ii) *How many words can be formed out of the letters "ARTICLE" so that the vowels occupy the even places?*

Solution

i) There are 6-letters of which 3 (M, B, L) are consonants and 3 (I, O, E) are vowels.

The positions to be filled up with consonants are as below

1	2	3	4	5	6
(M)	–	(B)	–	(L)	–

The total no. of ways consonants can be filled $= {}^{3}p_3 = 3! = 6\ ways$

The total no. of ways 3-vowels can be arranged $= {}^{3}p_3 = 3! = 6\ ways$

By fundamental principle,

Total arrangements $= 6 \times 6 = 36\ ways$

ii) There are 7-letters of which 3 (I, O, E) are vowels

1	2	3	4	5	6	7
–	(A)	–	(I)	–	(E)	–

Arrangements of vowels$= {}^{3}p_3 = 3! = 6\ ways$

Remaining 4-letters can be arranged in 4 places in ${}^{4}p_4 = 4!\ ways$

By fundamental theorem, total no. of arrangements$= 3! \times 4! = 144\ ways$

4) *In how many ways can the letters of the word "STRANGE" be arranged so that the vowels may appear in the odd places?*

Solution

There are 7 letters with 2 (A, E) vowels.

1	2	3	4	5	6	7
(A)	–	(E)		–		–

The 2 vowels occupy only in the 2 out of 4 positions, in which this can be done in

$${}^{4}p_2 = \frac{4!}{2!} = 12\ ways$$

The rest of the letters occupy empty spaces. This can be done in

$${}^{5}p_5 = 5! = 120\ ways$$

$$\therefore total\ no.\ of\ arrangements\ are\ 120 \times 12 = 1440\ ways$$

5) *How many words can be formed from the letters of the word "FATEPUR" when*

i. The 3-letters PUR occur together?

ii. Vowels occur at even places?

Solution

i. There are 8 letters. Since the 3 (PUR) letters occur together, we may regard them as forming one letter, so there are now 6 letters for permutation.
⇒ The 6 letters can be arranged in ${}^{6}p_{6} = 6!\ ways$
⇒ The 3 (PUR) letters can be arranged in ${}^{3}p_{3} = 3!\ ways$
By fundamental principle, total arrangement $= 6! \times 3!\, ways$

ii.

1	2	3	4	5	6	7	8
–	(A)	–	(E)	–	(U)	–	$(-)$

3-vowels occupy 3 out of 4 positions in
${}^{4}p_{3} = \frac{4!}{1!} = 24\ ways$
Remaining consonants occupy 5-empty spaces $(1, 3, 5, 7, 8)$ in
${}^{5}p_{5} = 5! = 120\ ways$
$\therefore total\ no.of\ arrangements = 120 \times 24 = 2880\ ways$

6) *How many different words containing all the letters of the word "TRIANGLE" can be formed? How many of them*
   i) *When consonants are never together?*
   ii) *When no two vowels are never together?*
   iii) *When consonants and vowels are both always together?*
   iv) *Vowels occupy odd places?*
   v) *Relative positions of vowels and consonants remain unaltered?*

Solution

TRIANGLE contain 8- different letters, which can be arranged in ${}^{8}p_{8} = 8!\ ways$
Consonants are (T, R, N, G, L)
Vowels are (I, A, E)

i) When consonants are never together

$$= \begin{pmatrix} Total\ arrangements \\ of\ 8-letters \end{pmatrix} - \begin{pmatrix} consonants\ when \\ altogether \end{pmatrix}$$

Permutations for consonants when together $= {}^{5}p_{5} = 5!$
Permutations for the word with consonants together $.i.e.(TRNGL), I, A, E\ .i.e.$ 4-letters can be arranged in ${}^{4}p_{4} = 4!\ ways$
⇒ The total arrangements when consonants are together $= 5! \times 4!$
∴ When consonants are never together $= 8! - 4! \times 5!\, ways$ can arise.

ii) When no two vowels are together $= 8!$ − permutation for vowels when together. Different arrangement for 2 vowels out of 3

$$= {}^{3}p_{2} = \frac{3!}{1!} = 3!\, ways$$

Arrangement for vowels (two) occurring together $= {}^{2}p_{2} = 2!\, ways\ arise$
Arrangement for the remaining consonants $= {}^{5}p_{5} = 5!\, ways\ arise$
$\therefore Total\ arrangement\ for\ vowels = 2! \times 3! \times 5!$
∴When no two vowels occur together $= (8! - 2! \times 3! \times 5!)\ ways\ arise$

iii) When consonants and vowels are both always together $.i.e.(TRNGL), (IAE)$ give two letters that can be arranged in
${}^{2}p_{2} = 2!\, ways$
Consonants are arranged in ${}^{5}p_{5} = 5!$
Vowels are arranged in ${}^{3}p_{3} = 3!$

∴ Total arrangements= $2! \times 3! \times 5!$

iv) Vowels occupy odd places

1	2	3	4	5	6	7	8
(A)	–	(I)	–	(E)	–	( )	–

Vowels occupy 3 out of 4 positions in ${}^4p_3 = 4!\,ways$

Remaining consonants occupy 5 places in ${}^5p_5 = 5!\,ways$

∴ Total arrangements $4! \times 5!$

v) T R I A N G L E

⇒ For consonants, ${}^5p_5 = 5!\,ways$ occur

⇒ For vowels, ${}^3p_3 = 3!\,ways$ occur

∴ Relative positions of vowels and consonants remain unaltered in the word is $5! \times 3!\,ways$

7) How many permutations can be made out of the letters of the word "MISSISSIPPI" taken altogether?

Solution

MISSISSIPPI contains repeated letters $.i.e.$ "S" occurs 4-times, "I" occurs 4-times and "P" occurs 2-times.

Using permutation of objects not all different $.i.e.\ \frac{n!}{p!q!r!}$, ⇒ number of permutations for 11-letters will be= $\frac{11!}{4!4!2!}$

8) *In how many ways can the letters of the word "CONSTITUTION" be arranged? How many of these will have the letter "N" both at the beginning and at the end?*

Solution

Word "CONSTITUTION" consists of 12-letters having O, N, I two times each, T- 3 times and

U, C, S once each.

Required number of different ways is $= \frac{12!}{2!2!2!!3!} = 9979200\,ways$

For having both N at the beginning and at the end, then N is fixed at the beginning and at the end, there by leaving 10-letters having the above characteristics.

∴ Required number of ways is $= \frac{10!}{2!2!3!} = 151200$

9) *In how many ways can the letters of the word "ARRANGE" be arranged? How many of these arrangements are there in which*

a) The two R's come together?
b) The two R's do not come together?
c) The two R's and the two A's come together?

Solution

ARRANGE has 7-letters out of which 2 are A, 2 are R, 1 is N, 1 is G and 1 is E

∴ Required no. of ways $= \frac{7!}{2!2!} = 1260\,ways$

a) Two R's come together give the word $A\ (RR)\ A\ N\ \ G\ E$ consisting of now 6-letters.

Now word has 2 A's, N, G, E are all one.

$\Rightarrow Required\ arrangements = \frac{6!}{2!} = 360\ ways$

$\left[For\ 2R's, arrangement = \frac{2!}{2!} = 1\right]$

b) Required no. of arrangements $= 1260 - 360 = 900\ ways$

c) Treating both R and A as one individual letter the word becomes $(AA)(RR)NGE$ consisting of 5-letters that are different

$\therefore No.of\ ways = 5! - 120\ ways$

10) a) H*ow many different arrangements can be made by using all the letters in the word "MATHEMATICS"? How many of them begin with*

i) *C?* ii) *T?*

b) How many permutations can be made out of the letters of the word "INDEPENDENCE"? How many of them have vowels occurring together?

Solution

a) MATHEMATICS has 11-letters of which 'M' occurs twice, 'A' occurs twice and 'T' also occurs twice. Remaining letters $.i.e. 11 - 6 = 5\ letters\ H, E, I, C, S$ are all different from one another.

$\therefore$ No. of permutations of these 11- letters taken all at a time is

$\frac{11!}{2!2!2!} = 4989600\ ways$

i) Fixing C in first position, then the permutation shall be done on the remaining letters. $i.e. 10\ letters$ with the above stated characteristics

$$\text{No. of ways} = \frac{10!}{2!2!2!} = 453\,600\ ways$$

ii) Fixing T in first position, permutation is to be done on the remaining letters. $i.e. T, M, A, T, H, E\ M, A, I, C, S\ .i.e.$ in these 10-letters, M and A both occur twice, T-no longer occurs twice for one of them occupied the first position.

$$\therefore \text{No. of ways} = \frac{10!}{2!2!} = 907\,200\ ways$$

b) INDEPENDENCE has 12-letters in which 'N' occurs thrice, 'D' occurs twice, 'E' occurs 4-times and the remaining 3-letters I, P, C are all different from each other.

$\therefore$ No. of permutations of 12-letters taking all at a time= $\frac{8!}{3!2!4!}$

INDEPENDENCE, consists of 5-vowels $i.e. I, E, E, E, E\ (4 - letters\ are\ similar)$

Taking the 5-vowels as one letter, then $(IEEEE)NDPNDNC$ is the word of 8-letters consisting of 3 N's and 2 D's

$\therefore Arrangement = \frac{8!}{3!2!}$

Permutation of the tied/bracketed vowels= $\frac{5!}{4!}$

$\therefore$ Total no. of arrangements $= \frac{8!}{3!2!} \times \frac{5!}{4!}$

11) a) How *many numbers greater than 1 million can be formed with digits 4, 5, 5, 0, 5, 4, 3?*

Solution

There are 7 digits in all, of which there are 2-fours, 3-fives and the rest are different.

$\therefore$ Total no. of arrangements $= \frac{7!}{2!3!} = 420$

Of these numbers, some begin with 0 and are less than 1-million, so they must be rejected.

Fixing 0 at the beginning, possible no. of permutations are= $\frac{6!}{2!3!} = 60$

$\therefore$ Required number $= 420 - 60 = 360$

b) *(i) Find all the six digit numbers formed from the digits 0, 1, 2, 3, 4, 5.*

(ii) How many numbers of 4-digits can be formed from out of the digits 3, 4, 5, 6, 7, 8, 0 if no digit is to be repeated?

Solution

i) The digits can be arranged in ${}^{6}p_{6} = 6!\, ways$

Out of these 6! Ways, there are numbers that begin with "0"

$\therefore$ Number of such numbers=, ${}^{5}p_{5} = 5!$

6-digit number $= 6! - 5! = 600$

ii) From the 7-digits, 4-digit number is to be formed.

Number of ways for this, ${}^{7}p_{4} = \frac{7!}{3!}$

Out of these numbers, there are numbers starting with 0-digit hence reducing to 3- digit number.

Number of ways for this $= {}^{6}p_{3} = \frac{6!}{3!}$

Required numbers $= \frac{7!}{3!} - \frac{6!}{3!} = 720$

c) *How many odd numbers of 5-digits can be formed with digits 3, 2, 7, 4, 0 when no digit is repeated?*

Solution

Odd numbers have 3 or 7 digit in the last place.

Odd numbers starting with 0 and have 3 or 7 in the last place exist.

Let us find all 5-digit odd numbers with 3 in last place $= {}^{4}p_{4} = 4! = 24\, ways$

Now all 5-digit odd numbers with 7 in last place $= {}^{4}p_{4} = 4! = 24\, ways$

Now finding all odd numbers with 0 in first place but ending with 3

$.i.e. 4 - digit\ number$

$= {}^{3}p_{3} = 3! = 6\, ways$

Also finding numbers with 0 in first place and 7 in the last place. $i.e. 4 - digit\ number$

$= {}^{3}p_{3} = 3! = 6\, ways$

Now total number of odd numbers with 0 in first place $= 6 + 6 = 12\, numbers$

Total number of odd numbers $= 24 + 24 = 48$

$\therefore$ Number of odd numbers with 5-digits $= 48 - 12 = 36$

d) (i) *How many numbers of 4-digits can be formed with digits 3, 5, 7, 8, 9, o digit being repeated? How many of these will be greater 8000?*

ii) *A number of 4-different digits is to formed using digits 1, 2, 3, 4, 5, 6, 7in all possible ways. Find how many of such numbers can be formed. Hence how many such numbers are greater than 3400?*

Solution

i) 4-digit numbers $= {}^{5}p_{4} = 5!$

For 4-digit number to be greater than 8000, 8 or 9 must occupy the first place.

Permutations of the remaining digits $.i.e$ 3out of 4 after the first being occupied by 8 or 9

$= {}^4p_3 = 4!\ ways$

Since numbers are of 2 types $.i.e\ 8\ or\ 9$ starting digits, then

Total number of numbers= $4! \times 2 = 48$

Alternatively

Number greater than 8000= $5! - (Numbers\ less\ than\ 8000)$

Finding $4 - digit\ numbers < 8000$

Numbers < 8000 begin with 3, 5 or 7

Since first place is being occupied in 3-ways, then

$4 - digit\ numbers < 8000 = 3 \times$ Permutation of the 3-digits after the first one being occupied by 3 or 5 or 7

$\therefore Numbers < 8000 = 3 \times {}^4p_3 = 3 \times 4!$

Required $numbers > 8000 = 5! - 3 \times 4! = 48$

ii) 4-digit numbers formed out of 7-digits in ${}^7p_4 = \frac{7!}{3!}\ ways$

For a number to be greater than 3400,

Case I

The first digit on LHS must be 4, 5, 6, 7 and the 3-digis occupied in ${}^6p_3 = \frac{6!}{3!}$

The first place in 4-ways

Case II

The first digit on LHS is 3, the second digit must be 5, 6, 7 and the other two places are occupied by any of the two remaining 5-digits

Case III

The first digit on LHS is occupied by 3, second digit by 4 and the other two places are occupied by two of the remaining digits

For case I: Total arrangement= $4 \times {}^6p_3$

For case II: Total arrangement = $3 \times {}^5p_2$

For case III: Total arrangement $= {}^5p_2$

Total $= 4 \times {}^6p_3 + 3 \times {}^5p_2 + {}^5p_2 = 560\ ways$

e) *How many numbers greater than 7000 can be formed with digits 3, 5, 7, 8, 9 no digit being repeated?*

Solution

Cases in the question

Case I: All numbers with 5-digits are > 7000

This is done in ${}^5p_5 = 5!\ ways$

Case II: There will be 4-digit numbers which are> 7000. These start with 8, 9 or 7 but never with 3 or 5

A. 4-digit number > 7000, starting with 8 or 9 are:

$2 \times {}^4p_3 \left[\begin{matrix}\because\ the\ remaining\ 3 - digits \\ are\ permutated\ out\ of\ 4\end{matrix}\right]$

B. If the number is to start with 7, the 3-digits must be filled up from the 4-remaninig digits in ${}^4p_3 = 4!\ ways$

$\therefore Total\ arrangement = {}^4p_3 + 2 \times {}^4p_3 + 5! = 192$

Task

1. **Uneb 2004**

How many arrangements can be made from the letters of the name "MISSISSIPPI"

a. *When all letters are taken at a time?*

b. *If the two letters P begin every word?*

ANSWER: ***a.*** $\frac{11!}{4!4!2!}$ ***b.*** $\frac{9!}{4!4!}$ $\left[\begin{matrix}HINT: MISSISSI(PP)I\ has\ 10\ letters\ of\ which\ 9\\ are\ to\ be\ permutated\end{matrix}\right]$

2. *How many different words can be formed with the letter of the word "CAPTAIN"? In how many of these C and T are never together?*

ANSWER: $\frac{7!}{2!}$, *taking C and T as one unit*$\Rightarrow \frac{6!}{2!}, (CT)\ occur\ in\ {}^2p_2 = 2!\ \therefore total = \frac{6!}{3!} \times 2!$

3. *In how many ways the letters of the word "PETROL" can be arranged? How many of these do not begin with P? How many begin with P but do not end with L? Also find the number of words which can be formed if O and L are never together?*

ANSWER: ***i)*** *720* ***ii)*** $720 - 120$ ***iii)*** $120 - 24$ ***iv)*** $720 - 2 \times 120$

4. *How many permutations can be made out of letters all taken together*

a) *EXAMINATION*

b) *PERMUTATION*

c) *ACCOUNTANT*

d) *MATAAMA*

ANSWER: ***a)*** *4 989 600* ***b)*** $\frac{11!}{2!}$ ***c)*** $\frac{10!}{2!2!2!}$ ***d)*** $\frac{7!}{2!4!}$

Circular permutation

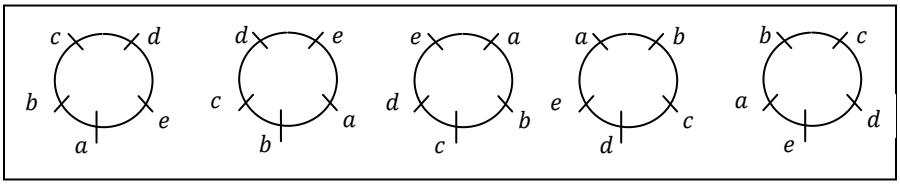
1. *In how many ways can 5 persons be seated on a round table conference?*

Solution

Let the 5 persons be denoted by letters a, b, c, d and e

NOTE

If these persons are to be seated in a *row* the five linear arrangements are; $abcde, bcdea, cdeab, deabc, eabcd$, are all different from one another. But if are to be seated at a round table in the same sense as in the figure below,



All the arrangements are the same, for starting with different letters and reading them in clockwise direction.$i.e.\ abcde, bcdea, cdeab, deabc, eabcd$, which are the same as the linear arrangements.

∴ A single circular arrangement of the n letters is equivalent to n different linear arrangements.

i.e. **n × a circular arrangement = n! × (linear arrangement of n − letters)**

⇒ **No. of circular arrangements = (n − 1)!**

Let the circular arrangement = x

$\Rightarrow 5 \times x = 5!$ $(linear\ arrangements\ of\ 5 - letters)$

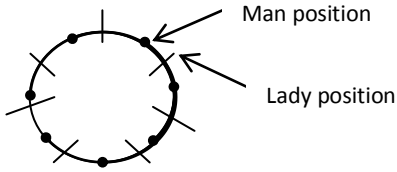
$\therefore x = \frac{1}{5} \times 5! = 4!$

$\therefore No.\ of\ circular\ arrangements\ of\ 5 - persons = (5-1)! = 4!$

2. a) *In how many ways can 7 gentlemen and 7 ladies sit at a round table so that no two ladies are together?*
b) In how many ways can 5 gentlemen and 3 ladies sit at a round table so that no two ladies are together?

Solution

a)

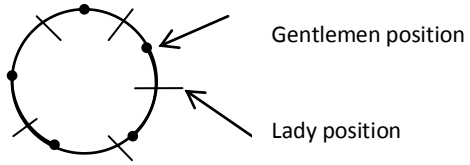


No. of ways in which 7 gentlemen can sit on round table= $(7-1)! = 6!$

Since no two ladies can be sitting together, ladies can be put in between the men $.i.e.$ there are exactly 7 places, 7 ladies can use. This can be done in 7! Ways.

$\therefore Total\ no.\ of\ ways = 6! \times 7! = 3\ 628\ 800$

b)



5 gentlemen sit on a round table in $(5-1)! = 4!\ ways$

There are 5 positions 3 ladies can occupy in ${}^{5}p_3 = \frac{5!}{2!}\ ways$

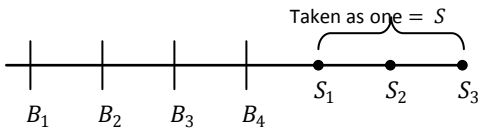
$\therefore Total\ nu.\ of\ ways\ required = 4! \times \frac{5!}{2!} = 1440\ ways$

Task

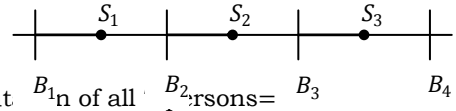
1. *In how many ways can 7 people be arranged at a round table so that two particular persons be together?*
2. *A family of 4 brothers and 3 sisters is to be arranged for a photograph in one row. In how many ways can they be seated if*
   a) *All sisters sit together?*
   b) *No two sisters sit together?*

HINT:

a) $Total\ no.\ of\ arrangements\ in\ which\ all\ sisters\ sit\ together = 5! \times 3! = 720$



∴ Permutation of $B_1B_2B_3B_4S = 5!$
Now permutation of $S_1S_2S_3 = 3!$
Total permutation= $5! \times 3!$

ii) 

Permutation of all persons=
Permutations of sisters seated together
$= 5! \times 3!$
∴ Total= $7! - 5! \times 3! = 4320$

3. *There are 6 books of Economics, 3 of Math and 2 of Accountancy. In how many ways can these be placed on a shelf if the books on the same subject are to be together?*
HINT: 6E's, 3M's and 2A's are different⇒ $6! \times 3! \times 2!$

4. *A library has 5 copies of one book, 4 copies of each of two books, 6 copies of each of 3 books and a single copy of 8 books. In how many ways can all the books be arranged on the shelf?*
HINT: total number of books of same kind= $1 \times 5 + 4 \times 2 + 6 \times 3 + 8 =$ 39 *books* of which same types are 5, 4, 4, 6, 6, and 6
∴ Total number of arrangements= $\frac{39!}{5!4!4!6!6!6!} = \frac{39!}{5!(4!)^2(6!)^3}$

COMBINATIONS
Definition:
The different groups/collections/selections that can be formed out of a given set of objects by taking some or all of them at a time (without regard to the order of arrangements) are called
Combinations
NOTATION
The number of combinations of n-different objects taken r-at a time is denoted as

$$^nC_r = \frac{^np_r}{r!} = \frac{n!}{r!(n-r)}$$, called *combinatorial coefficient*

IMPORTANT DEFINATIONS IN COMBINATIONS

a) **Complementary combinations**

i) $^nC_r = {}^nC_{n-r}$.*i.e.*The number of combinations of n different objects taken r at a time is equal to the number of combinations of n different objects taken $(n-r)$ at a time.

Proof

$$^nC_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)![n-n+r]} = \frac{n!}{(n-r)![n-(n-r)]} = {}^nC_{n-r}$$

ii) $^nC_n = {}^nC_0 = 1$

Proof

${}^{n}C_{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1$

Also ${}^{n}C_{0} = \frac{n!}{n!(n-0)!} = \frac{n!}{n!0!} = 1$

iii) $\boxed{{}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1} \quad \text{OR} \quad {}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}}$

i.e. The number of combination of n objects taken r at a time plus the number of combinations of n taken (r+1) or (r-1) in the above formulae respectively at a time is equal to the number of combinations (n+1) objects taken (r+1) or respectively at a time.

Proof

${}^{n}C_{n} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

${}^{n}C_{n} + {}^{n}C_{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)![n-(r+1)]!} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)![n-r-1]!}$

$= \frac{n!}{(n-r-1)!r!}\left[\frac{1}{n-r} + \frac{1}{r+1}\right]$ *taking out common factor*

$= \frac{n!}{(n-r-1)!r!}\left[\frac{r+1+n-r}{(n-r)(r+1)}\right]$ *adding two fractions* $= \frac{n!}{(n-r-1)!r!} \cdot \frac{(n+1)}{(n-r)(r+1)}$

$= \frac{(n+1)!}{(n-r)!(r+1)!}\begin{bmatrix} \because (n-r-1)!\,(n-r) = (n-r)! \\ r!\,(r+1) = (r+1)! \\ n!\,(n+1) = (n+1)! \end{bmatrix}$

$= \frac{(n+1)!}{[(n+1)-(r+1)]!(r+1)!}$ $[\because (n+1)-(r+1) = n-r]$

$= {}^{n+1}C_{r+1}$

b) **Restricted combinations**

i. The number of combinations on n objects taken r at a time in which p particular objects are always included is

$\boxed{{}^{n-p}C_{r-p}}$

ii. The number of combinations of n objects taken r at a time in which p particular objects never occur is $\boxed{{}^{n-p}C_{r}}$

c) **Total number of combinations of n-objects all different**

The total number of combinations of n-different objects taken some or all at a time is

$\boxed{{}^{n}C_{1} + {}^{n}C_{2} + {}^{n}C_{3} + \ldots\ldots\ldots\ldots + {}^{n}C_{n} = 2^{n} - 1}$

d) **Group division**

The number of ways in which $(m+n)$objects m and n being un equal, can be divided in to two groups containing m and n objects respectively is

$\boxed{\frac{(m+n)!}{m!n!}}$

Example

1. *Evaluate*

   i) ${}^{6}c_3$ ii) ${}^{7}c_2$ iii) ${}^{4}c_4$ iv) ${}^{5}c_1$

   Solution

   i) ${}^{6}c_3 = \frac{6!}{3!3!} = 20$ iii) ${}^{4}c_4 = \frac{4!}{4!0!} = 1$

   ii) ${}^{7}c_2 = \frac{7!}{5!2!} = 21$ iv) ${}^{5}c_1 = \frac{5!}{4!1!} = 5$

2. *Find n if:*

   a) ${}^{n}c_2 = 3$ b) ${}^{2n}c_3 : {}^{n}c_2 = 44:3$ c) ${}^{28}c_{2n} : {}^{24}c_{2n-4} = 225:11$

   Solution

   a) ${}^{n}c_2 = \frac{n!}{(n-2)!2!} = 3 \Rightarrow \frac{n(n-1)(n-2)!}{(n-2)!2!} = 3$

   $\Rightarrow \frac{n(n-1)}{2!} = 3 \; or \; n^2 - n - 6 = 0 \; \therefore n = 3, -2$

   Taking $n = 3$; $n = -2$ *is rejected*

   b) ${}^{2n}c_3 : {}^{n}c_2 = \frac{(2n)!}{(2n-3)!3!} \div \frac{n!}{(n-2)!2!} = \frac{44}{3} \Rightarrow \frac{(2n)!}{(2n-3)!3!} \times \frac{(n-2)!2!}{n!} = \frac{44}{3}$

   $\Rightarrow \frac{2n(2n-1)(2n-2)}{3!} \times \frac{(n-2)!2!}{n(n-1)(n-2)!} = \frac{44}{3}$

   $\Rightarrow \frac{2n(2n-1)(2n-2)(2n-3)}{6} \times \frac{2}{n(n-1)} = \frac{44}{3} \; or \; \frac{4(2n-1)}{3} = \frac{44}{3}$

   $\therefore 2n - 1 = 11 \; or \; n = 6$

3. *Find n if* ${}^{n}p_4 = 30.\,{}^{n}c_5$

   Solution

   $\frac{n!}{(n-4)!} = 30 \times \frac{n!}{(n-5)!5!} \Rightarrow \frac{1}{(n-4)(n-5)!} = \frac{30}{(n-5)!5!}$

   $\Rightarrow \frac{1}{n-4} = \frac{1}{5!} = \frac{1}{4} \; \therefore n - 4 = 4 \; or \; n = 8$

4. *There are 12 men and 9 women. In how man ways can they stand in a row so that no two women are together?*

   Solution

   Let the 12 men (denoted by M's) be put in a row as shown below.

   × M × M × M × M × M × M × M × M × M × M × M × M ×

   In order that no two women may be put together the women can only be put in places marked as ×, and the men are 12, the number of such places is 13.

   ∴ In order to place the 9 women we must choose any 9 out of 13 places. *i.e.* ${}^{13}c_9$ or ${}^{13}c_4 = \frac{13!}{9!4!} = 715 \; ways$

5. *In how many ways can a committee of 3 ladies and 4 gentlemen be appointed from a meeting consisting of 8 ladies and 7 gentlemen?*

   Solution

   3 ladies can be selected from 8 in ${}^{8}c_3$ ways.

   4 gentlemen can be selected from 7 in ${}^{7}c_4$ ways

   ∴ The number of ways in which a committee can be selected is ${}^{8}c_3 \times {}^{7}c_4 = 1\,960 \; ways$

6. *A committee of 4 men and 3 women is to be formed from 10 men and 8 women. In how many ways can the committee be formed?* (***uneb* 2011**)

Solution

4 men can be selected from 10 men in ${}^{10}c_4$

3 women are selected from 8 women in ${}^{8}c_3$

Total number of ways of selecting committee $={}^{10}c_4 \times {}^{8}c_3 = 11\,760\ ways$

7. *In how many ways can a committee of 6 men and 2 women be formed out of 10 men and 5 women?*

Solution

6 men selected from 10 men in ${}^{10}c_6$ ways

2 women selected from 5 women in ${}^{5}c_2$

Total number of ways of selecting a committee $={}^{10}c_6 \times {}^{5}c_2 = 2\,100$

8. *A group of nine has to be selected from 10 boys and 8 girls. It can consist of either 5 boys and 4 girls or 4 boys and 5 girls. How many different groups can be chosen?*(***Uneb* 2006**)

Solution

Case I

5 boys selected from 10 boys in ${}^{10}c_5$

4 girls selected from 8 girls in ${}^{8}c_4$

Total number of selection $= {}^{10}c_5 \times {}^{8}c_4$

Case II

4 boys selected from 10 boys in ${}^{10}c_4$

5 girls selected from 8 girls in ${}^{8}c_5$

Total number of selection $= {}^{10}c_4 \times {}^{8}c_5$

$\therefore$ Overall total number of selection $={}^{10}c_5 \times {}^{8}c_4 + {}^{10}c_4 \times {}^{8}c_5 = 29\,400$

9. *In a multinational company, 3 branches in a particular country are managed by 4, 5 and 8 persons. In how many ways can 20 persons be allotted to the different branches?*

Solution

Let the branches be A, B and C

For branch A

Let the 1st 4 persons be selected out 20 persons.

Number of ways $={}^{20}c_4$

Having selected 4 persons for branch A, 5 persons can be selected out of the remaining 16 for **branch B** in ${}^{16}c_5 ways$

After this 8 persons for **branch C** out of the remaining 11 in ${}^{11}c_8\ ways$

By fundamental theorem, selection of persons can be made in ${}^{20}c_4 \times {}^{16}c_5 \times {}^{11}c_8\ ways$

10. *At an election 3 wards of a town are canvassed by 4, 5 and 8 respectively. If 20 men volunteer, in how many ways can they be allotted to the different wards?*

Solution

Let 3 wards be A, B and C

Let A be canvassed by the 1st 4 out of 20 in ${}^{20}c_4\ ways$

B will be canvassed by 5 out of remaining 16 in ${}^{16}c_5\ ways$

C will be canvassed by 8 out of the remaining 11 in ${}^{11}c_8$ *ways*

∴ Total number of ways $= {}^{20}c_4 \times {}^{16}c_5 \times {}^{11}c_8$

11. *From 6 boys and 4 girls, 5 are to be selected for admission for a particular course. In how many ways can this be done if there must be exactly 2 girls?*

Solution

Type of selection	**Boys (6)**	**Girls (4)**
A	3	2

⇒ A can be done in ${}^{6}c_3 \times {}^{4}c_2 = 120$ *ways*

12. *From 5 gentlemen and 6 ladies, a committee of 5 is to be formed. In how many ways can this be done if*
    a) The committee is to include at least a lady
    b) There is no restriction about its formation
    c) Not more than 3 gentlemen?

Solution

a)

Lady (6)	**Gentlemen (5)**	**No. ways of formation**
1	4	${}^{6}c_1 \times {}^{5}c_4$
2	3	${}^{6}c_2 \times {}^{5}c_3$
3	2	${}^{6}c_3 \times {}^{5}c_2$
4	1	${}^{6}c_4 \times {}^{5}c_1$
5	0	${}^{6}c_5 \times {}^{5}c_0$

∴ Number of ways in which a committee includes at least a lady is

$= {}^{6}c_1 \times {}^{5}c_4 + {}^{6}c_2 \times {}^{5}c_3 + {}^{6}c_3 \times {}^{5}c_2 + {}^{6}c_4 \times {}^{5}c_1 + {}^{6}c_5 \times {}^{5}c_0$

$= 30 + 50 + 200 + 75 + 6 = 461$ *ways*

b) When there is no restriction about formation 5 are selected out of 11 in

${}^{11}c_5 = 231$ *ways*

c)

Gentlemen (5)	**Lady (6)**	**Type of committee**	**No. of ways**
0	5	A	${}^{5}c_0 \times {}^{6}c_5$
1	4	B	${}^{5}c_1 \times {}^{6}c_4$
2	3	C	${}^{5}c_2 \times {}^{6}c_3$
3	2	D	${}^{5}c_3 \times {}^{6}c_2$

∴ Number of ways in which a committee consisting of not more than 3 gentlemen is formed in

${}^{5}c_0 \times {}^{6}c_5 + {}^{5}c_1 \times {}^{6}c_4 + {}^{5}c_2 \times {}^{6}c_3 + {}^{5}c_3 \times {}^{6}c_2 = 6 + 75 + 200 + 150 = 431$ *ways*

13. *The question paper on math and statistics contains 10 questions divided in to two groups of 5 questions each. In how many ways can an examinee select 6 questions taking at least 2 questions from each group?*

Solution

Group (A) [5 questions]	**Group (B) [5 questions]**	**No. of ways**
2	4	${}^{5}c_2 \times {}^{5}c_4$
3	3	${}^{5}c_3 \times {}^{5}c_3$
4	2	${}^{5}c_4 \times {}^{5}c_2$
5	1	
6	0	

Not considered since at least two questions from each group are to be considered.

$\therefore$ Total number of selection $= {}^5c_2 \times {}^5c_4 + {}^5c_3 \times {}^5c_3 + {}^5c_4 \times {}^5c_2$

$= 50 + 100 + 50 = 200\ ways$

14. *A council consists of 11 members of which 5 are permanent and the remaining are temporary. A solution is declared to be passed by the council if it receives at least 7 votes including all the permanent members. Find the number of ways in which the resolution can be passed. What will be the number of ways if;*
    a) *For passing a resolution, only a simple majority was selected*
    b) *Any 7 or more votes could secure the passing of the resolution?*

Solution

Permanent vote (5)	**Temporary vote (6)**	**No. of selection**
5	2	${}^5c_5 \times {}^6c_2 = 15$
5	3	${}^5c_5 \times {}^6c_3 = 20$
5	4	${}^5c_5 \times {}^6c_4 = 15$
5	5	${}^5c_5 \times {}^6c_5 = 6$
5	6	${}^5c_5 \times {}^6c_6 = 1$

$\therefore$ Total number of ways $= 15 + 20 + 15 + 6 + 1 = 57$

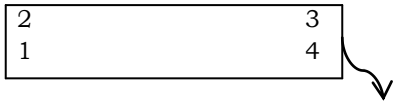
a) For simple majority either 6 or 7 or 8 or 9 or 10 or 11 votes out of 11 are required. This is done in
${}^{11}c_6 + {}^{11}c_7 + {}^{11}c_8 + {}^{11}c_9 + {}^{11}c_{10} + {}^{11}c_{11} = 1024\ ways$

b) Number of ways $= {}^{11}c_7 + {}^{11}c_8 + {}^{11}c_9 + {}^{11}c_{10} + {}^{11}c_{11} = 562\ ways$

15. *A council consists of 10 members, 6 belonging to the party A and 4 to the party B. in how many ways can a committee of 5 be selected so that the members of the party A are in the majority?*

Solution

A (6)	**B (4)**	**No. of ways**
5	0	${}^6c_5 \times {}^4c_0 = 6$
4	1	${}^6c_4 \times {}^4c_1 = 60$
3	2	${}^6c_3 \times {}^4c_2 = 120$
2	3	
1	4	



, cannot be taken since party (A) must be the majority.

$\therefore$ Total number of selection $= 6 + 60 + 120 = 186$

16. *Among 20 members of a cricket club there are 2 wicket keepers and 5 bowlers. In how many ways can 11 be chosen so as to include only one of the wicket keepers and at least 3 bowlers?*

Solution

Type	**Wicket players** [2 *max*]	**Bowler [5max]**	**No. of players other than Wicket of Bowlers [13 max]**	**Number of ways**
A	1	3	7	${}^2c_1 \times {}^5c_3 \times {}^{13}c_7$
B	1	4	6	${}^2c_1 \times {}^5c_4 \times {}^{13}c_6$
C	1	5	5 (Exhausted)	${}^2c_1 \times {}^5c_5 \times {}^{13}c_5$

$\therefore$ Total number of ways $= {}^{2}c_1 \times {}^{5}c_3 \times {}^{13}c_7 + {}^{2}c_1 \times {}^{5}c_4 \times {}^{13}c_6 + {}^{2}c_1 \times {}^{5}c_5 \times {}^{13}c_5 = 54\,054\ ways$

16. *A cricket team of 11 players is to be formed from 16 players including 4 bowlers and 2 wicket keepers. In how many ways can a team be formed so that the team consists of at least 3 bowlers and at least one wicket keeper?*

Solution

Wicket keeper (2 max)	**bowlers (4 max)**	**Neither wicket nor keeper (10 max)**
1	3	7
2	4	5
1	4	6
2	3	6

Further two possibilities

$\therefore$ Total selection $= {}^{2}c_1 \times {}^{4}c_3 \times {}^{10}c_7 + {}^{2}c_1 \times {}^{4}c_4 \times {}^{10}c_6 + {}^{2}c_2 \times {}^{4}c_3 \times {}^{10}c_6 + {}^{2}c_2 \times {}^{4}c_4 \times {}^{10}c_5 = 2\,472\ ways$

3. a) *A bag contains a mixture of 8 Rupee, 6 Fifty paise and 4 Twenty paise coins. In how many ways selection of 3 coins can be made so that*

   i) *all the three are Rupee coins*
   ii) *one is of each denomination*
   iii) *none is a Rupee coin*

Solution

i) For all to be Rupee coins, the 3 coins are selected from 8 Rupee in
$${}^{8}c_3 \times {}^{6}c_0 \times {}^{4}c_0 = 56\ ways$$

ii) One of each denomination is to be chosen; 1 Rupee can be chosen out of 8 in ${}^{8}c_1$, 1 Fifty paise is taken out of 6 in ${}^{6}c_1$ and 1 Twenty paise in ${}^{4}c_1$ ways.
$$\therefore \text{Total number of choices} = {}^{8}c_1 \times {}^{6}c_1 \times {}^{4}c_1 = 192\ ways$$

iii) None is a Rupee coin if the 3 coins are to be selected from Fifty paise and Twenty paise *. i. e.* a total of $6 + 4 = 10$
$$\therefore \text{Total number of ways} = {}^{10}c_3 = 120\ ways$$

b) *A box contains 7 red, 6 white and 4 blue balls. How many selection of 3 balls can be made so that*

   i) *All three are red balls*
   ii) *None is a red ball*
   iii) *There is one ball of each color?*

Solution

i) All 3 balls are red and are selected from only 7 red balls in ${}^{7}c_3 = 35\ ways$

ii) No red ball, the 3 balls are selected from the other colors *. i. e.* out of $6 + 4 = 10$ balls
$$\therefore {}^{10}c_3 = 120\ ways\ of\ selection$$

iii) One ball of each color is selected in ${}^{7}c_1 \times {}^{6}c_1 \times {}^{4}c_2 = 168\ ways$

c) *A committee of 4 persons is to be appointed from 3 officers of the production department, 3 officers of the sales department, 2 officers of the purchase department and 1 cost accountant. In how many ways should a committee be formed so that*

i) *There must be one from each category*
ii) *It should have at least one from purchase department?*

Solution

i) Number of ways of forming a committee from each category $= {}^{3}c_1 \times {}^{3}c_1 \times {}^{2}c_1 \times {}^{1}c_1 = 18\ ways$

ii) A committee with at least 1 from purchase department
= No. of ways of (1 − from purchase department +3 from other depts.)
+(2 − from purchase dept.+ 2 from other depts.)
$= {}^{2}c_1 \times {}^{7}c_3 + {}^{2}c_2 \times {}^{7}c_2 = 92\ ways$

d) *An urn contains 8 white and 3 red balls. In how many ways should 2 balls be drawn so that*

i) *Both are white*
ii) *Both are red*
iii) *One is of each color?*

Solution

i) 2 white balls are drawn from 8 in ${}^{8}c_2 = 28\ ways$
ii) 2 red balls are drawn from 3 in ${}^{3}c_2 = 3\ ways$
iii) No. of ways in which 1 white and 1 red can be drawn out of 8 white and 3red $= {}^{8}c_1 \times {}^{3}c_1 = 24\ ways$

e) *A subcommittee of 16 members is to be formed out of a group of 7 men and 4 ladies. In how many ways can the subcommittee be formed in order to consist of*

i) *Exactly two ladies*
ii) *At least 3 ladies*

Solution

i) 2 ladies out of 4 and 4 men out of 7 can be selected in ${}^{4}c_2 \times {}^{7}c_4 = 210\ ways$
ii) The subcommittee will consist of at least 2 ladies if there are 2, 3 or 4 ladies in it in each case the number of men would be 4, 3 and 2 respectively.
No. of ways $= {}^{4}c_2 \times {}^{7}c_4 + {}^{4}c_3 \times {}^{7}c_3 + {}^{4}c_4 \times {}^{7}c_2 = 371\ ways$

f) *A bag contains 6 white and 9 black balls. Two drawings of 4 balls are made such that*

i) *The balls are replaced before the second trial*
ii) *The balls are not replaced before the second trial*
In how many ways should the first drawing give 4 white and the second 4 black balls in each case? What if no restriction of which is the first or second drawn, find the number of ways this can be done in each case.

Solution

i) No. of ways in which 4 white balls are drawn $= {}^{6}c_4$ ways
No. of ways in which 4 black balls are drawn $= {}^{9}c_4$ ways

ii) At the first trial number of ways of drawing 4 white $= {}^{6}c_4$ ways. When 4 white balls have been removed the bag contains 2 white and 9 black balls
∴ No. of ways $= {}^{9}c_4$ ways
If no restriction; (i) ${}^{15}c_4 \times {}^{15}c_4$ (ii) ${}^{15}c_4 \times {}^{11}c_4$

Task

1. A student has to answer 8 out of 10 questions in an examination.
   a) How many choices have he?
   b) How many if he must answer the first three questions
   c) How many if he must answer at least 4 of the first five questions?

Solution

a) 8 out of 10 can be selected in ${}^{10}c_8 = 45\ ways$

b) Since the first three questions are compulsory only 5 out of the remaining 7 questions must be selected. This can be done in ${}^{7}c_5 = 21\ ways$

c) **Case I:** 4 from the first 5 questions and 4 from the remaining 5 questions

Case II: 5 from the first 5 questions and 3 from the 3 remaining questions

$\therefore$ No. of ways $= {}^{5}c_4 \times {}^{5}c_4 + {}^{5}c_5 \times {}^{5}c_3 = 35$

2. How many combinations can be formed of 8 counters marked 1. 2, 3, 4, 5, 6, 7 , 8 taking them 4 at a time, there being at least one odd and one even counter in each combinations?

   Solution

   ${}^{4}c_3 \times {}^{4}c_1 + {}^{4}c_2 \times {}^{4}c_2 + {}^{4}c_1 \times {}^{4}c_3 = 68$

3. Find the number of combinations of the words "ACCOMMODATION", "COLLEGE" taken 4 at a time. **ANSWER**: For ACCOMMODATION: **167** , COLLEGE, $\boldsymbol{{}^{5}c_4 + {}^{2}c_1 \times {}^{4}c_2 + {}^{2}c_2 = 18}$

4. i) ${}^{n}c_{10} = {}^{n}c_{12}$ ii) ${}^{n}c_7 = {}^{n}c_3$ iii) ${}^{n}c_5 + {}^{n}c_3 = 6$ **ANSWER**: (i) 22 (ii) 10

5. Find the value of r if
   a) ${}^{n}p_r = 30240$ and ${}^{n}c_r = 252$ **ANSWER**: r = 5
   b) ${}^{n}c_r = {}^{n}c_{2r-5}$

   A committee of 4 men and 3 women is to be formed from 1o men and 8 women. In how many ways can the committee be formed? (***Uneb* 2011**)
   ANSWER: $\boldsymbol{{}^{10}c_4 \times {}^{8}c_3 = 11\,760}$

POLYNOMIAL FUNCTIONS AND EQUATIONS

Polynomial functions of one variable

A Polynomial in the symbol x over the real numbers $\mathbb{R}$, is an algebraic expression that can be written:
$\boldsymbol{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \ldots \ldots \ldots \ldots + a_1 x + a_0}$, where n is a non-negative integer, the *coefficient* $\boldsymbol{a_i}$in$\mathbb{R}$, and $\mathbb{R}\boldsymbol{a_n \neq 0}$
The number **n** is said to be the *degree* of a polynomial and $\boldsymbol{a_n}$ is called the *leading coefficient.*
A *polynomial function* $\boldsymbol{f(x)}$ is an algebraic function that can be written as:
$\boldsymbol{f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \ldots \ldots \ldots \ldots + a_1 x + a_0}$, where x is a variable whose domain is the set of real number $\mathbb{R}$

Examples of polynomial functions

i) $f(x) = x^5 + 2x^3 + 1$, degree $= 5$
ii) $f(x) = 2x^3 + 5x + 1$, degree $= 3$

Examples of non-polynomial functions

a) $f(x) = \sqrt{x} + 2x^3 + 1\,;\ x^{\frac{1}{2}}$ non permissible
b) $r(x) = \frac{3}{2}ix^2 + 2\,;\frac{3}{2}i \notin \mathbb{R}$

Division of a polynomial by another polynomial

Two methods may be applied

- Long division
- Synthetic division

The polynomial $\boldsymbol{f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \ldots + a_1 x + a_0}$ can be divided by a DIVISOR to either get a QUOTIENT only or *quotient* + REMAINDER. The polynomial being divided is called a DIVIDEND.

LONG DIVISION

Steps followed are:

1) Arrange the dividend in descending powers of the variable
2) Insert with 0 coefficients any missing terms of less than degree n
3) Divide the 1st term of the divisor in to the 1st term of the dividend
4) Multiply the divisor by the result from step (3) above, line up like terms
5) Subtract as arithmetic and bring down now the remaining terms
6) Repeat the process until the degree of the remainder is less than that of the divisor.

NOTE:

Division of a polynomial by a divisor will be possible if the degree of polynomial is greater than that of divisor.

Examples

Divide

a) $f(x) = 5 + 4x^3 - 3x$ by $2x - 3$
b) $f(x) = x^6 - 2x^4 + 2$ by $x^2 - x - 2$
c) $f(x) = x^5 + 2x^3 + 2$ by $x^3 + x + 1$

Solution

a)

$$
\begin{array}{r|l}
 & 2x^2+3x+3 \quad \text{Quotient} \\
\hline
\text{Divisor}\ 2x-3 & 4x^3+0x^2-3x+5 \quad \text{Dividend} \\
 & 4x^3-6x^2 \\
\hline
 & 6x^2-3x+5 \\
 & 6x^2-9x \\
\hline
 & 6x+5 \\
 & 6x-9 \\
\hline
 & 14 \quad \text{Remainder}
\end{array}
$$

$\therefore \frac{f(x)}{2x-3} = 2x^2+3x+3+\frac{14}{2x-3}$ *or* $f(x) = (2x^2+3x+3)(2x-3)+14$

In general, $\boxed{Dividend = (Divisor)(Quotient) + Remainder}$

NOTE:

i) Divisor $2x-3$ is *linear*

ii) Dividend $4x^3-3x+5$ is of degree $n = 3$

iii) Quotient $2x^2+3x+3$ is of degree $n-1 = 2$

iv) Remainder 14 is a *constant*

b)

$$
\begin{array}{r|l}
 & x^4+x^3+x^2+3x+6 \\
\hline
x^2-x-2 & x^6+0x^5-2x^4+0x^3+x^2+0x-2 \\
 & x^6-x^5-2x^4 \\
\hline
 & x^5+0x^4+0x^3+x^2+0x \\
 & x^5-x^4-2x^3 \\
\hline
 & x^4+2x^3+x^2+0x-2 \\
 & x^4-x^3-2x^2 \\
\hline
 & 3x^3+3x^2+0x-2 \\
 & 3x^3-3x^2-6x \\
\hline
 & 6x^2+6x-2 \\
 & 6x^2+6x-12 \\
\hline
 & 12x+10
\end{array}
$$

$\therefore \frac{f(x)}{x^2-x-2} = x^4+x^3+x^2+3x+\frac{12x+10}{x^2-x-2}$

NOTE:

i) Divisor x^2-x-2 is *a Quadratic*

ii) Dividend x^6-2x^4+2 by x^2-x-2 is of degree $n = 6$

iii) Quotient $x^4+x^3+x^2+3x+6$ is of degree $n-2 = 4$

iv) Remainder $12x+10$ is *linear*

c)
$$x^3 + 0x^2 - x - 2 \;\overline{)\; x^5 + 0x^4 + 2x^3 + 0x^2 + 0x + 2}$$

$$x^4 + x^3 + x^2 + 3x + 6$$

$$x^5 + 0x^4 + x^3 + x^2$$

$$x^3 - x^2 + 0x + 2$$

$$x^3 + 0x^2 + x + 1$$

$$-x^2 - x + 1$$

$\therefore \frac{f(x)}{x^3+x+1} = x^2 + 1 + \frac{-x^2-x+1}{x^3+x+1}$

NOTE:

i) Divisor $x^3 + x + 1$ is *a Cubic*

ii) Dividend $x^5 + 2x^3 + 2$is of degree n = 5

iii) Quotient $x^2 + 1$ is of degree n − 3 = 2

iv) Remainder $-x^2 - x + 1$ is *Quadratic*

Summary of the results:

Degree of Polynomial	Divisor	Degree of Quotient	Remainder
$\boldsymbol{n}$	$(\boldsymbol{ax+b})$ *Linear*	$n-1$	$\boldsymbol{R}$ *Constant*
	$(\boldsymbol{ax+b})^2$ *or* $(\boldsymbol{ax+b})(\boldsymbol{px+q})$ *or* $\boldsymbol{ax^2+bx+c}$ *Quadratic*	$n-2$	$\boldsymbol{Ax+B}$ *Linear*
	$(\boldsymbol{ax+b})^3$ *or* $(\boldsymbol{ax+b})(\boldsymbol{px+q})(\boldsymbol{mx+n})$ *or* $\boldsymbol{ax^3+bx^2+cx+d}$ *Cubic*	$n-3$	$\boldsymbol{Ax^2+Bx+C}$

The above table is important in finding the remainder basing on the type of the divisor. It is so much applied in remand theorem.

SYNTHETIC DIVISION

Key steps in synthetic division process

To divide $f(x) = Polynomial$ by$\boldsymbol{x-a}$

1. Arrange the coefficients of $f(x)$ in descending powers of x. write zero as the coefficient for each missing power
2. After writing the divisor in the form$(x-a)$, use $\boldsymbol{a}$ to generate the 2nd and 3rd row s as follows: bring down the 1st coefficient of the dividend and multiplying it by$\boldsymbol{a}$**;** then add the product to the 2nd coefficient of the dividend. Multiplying this sum by $\boldsymbol{a}$ and add the product to the 3rd coefficient of the dividend.
   Repeat the procedure until a product is added to the constant term of the$f(x)$.
3. The last nu number to the right in the 3rd row of numbers is the remainder. The other numbers in the 3rd row are the coefficients of the quotient which is of degree< $f(x)$.

Examples

Use synthetic division to find the quotient and remainder resulting from dividing;

a) $f(x) = 4x^5 - 30x^3 - 50x - 2$ by $x + 3$

b) $f(x) = 2x^4 + 3x^3 - x - 5$ by $x + 2$

Solution

a) $Divisor = x + 3 = x - (-3) \ \therefore a = -3$

$Dividend = 4x^5 + 0x^4 - 30x^3 + 0x^2 - 50x - 2$

$1^{st}\ row$:	4	0	-30	0	-50	-2
$2^{nd}\ row$:		-12	36	-18	54	-12
-3	4	-12	6	-18	4	-14

$\therefore$ The quotient is: $4x^4 - 12x^3 + 6x^2 - 18x + 4$

The remainder is: -14

$\therefore \frac{f(x)}{x+3} = 4x^4 - 12x^3 + 6x^2 - 18x + 4 - \frac{14}{x+3}$

b) $Divisor = x + 2 = x - (-2) \ \therefore a = -2$

$Dividend = 2x^4 + 3x^3 + 0x^2 - x - 5$

	2	3	0	-1	$-$
		-4	2	-4	10
-2	2	-1	2	$-$	

Dividend Coefficient

Remainder

$Quotient\ Coefficient$

$\therefore$ The quotient is: $2x^3 - x^2 + 2x - 5$

The remainder is: 5

$\therefore \frac{f(x)}{x+2} = 2x^3 - x^2 + 2x - 5 - \frac{5}{x+2}$

THE REMAINDER AND FACTOR THEOREM

Remainder Theorem

If $\boldsymbol{f(x)}$ **is** a polynomial of degree n , $n > 0$ and if $\boldsymbol{c}$ is a number, then the remainder in division of $\boldsymbol{f(x)}$ by $\boldsymbol{(x - c)}$ is $\boldsymbol{f(c)}$**.** In symbols,

$\boldsymbol{f(x) = (x - c).Q(x) + f(c)}$ **,** where $\boldsymbol{Q(x)}$ is a quotient of degree $\boldsymbol{n - 1}$

Proof:

Dividing $\boldsymbol{f(x)}$by $\boldsymbol{(x - c)}$, you obtain a quotient and a remainder **r** (a real number)

$\Rightarrow \frac{f(x)}{(x-c)} = Q(x) + \frac{r}{(x-c)}$

$\therefore f(x) = Q(x).(x - c) + r$

This is an identity, true for all values of x.

If $x = c$, then $f(xc) = Q(c).(c - c) + r$

$\therefore \boldsymbol{r = f(c)}$

NOTE:

The theorem is stated in terms of $\boldsymbol{(x - c)}$**,** the *difference* between x and the number c.

The factor Theorem

A number c is a zero of polynomial function f of degree $n, n > 0$, if and only if $x - c$ is a factor of $f(x)$.

Proof:

From the remainder theorem $f(x) = Q(x).(x - c) + r$, if $f(c) = 0$ (Remainder$f(c) = 0$), then $f(x) = (x - c).Q(x)$.

Consequently $(x - c)$ is a factor of $f(x)$.

Now from $f(x) = (x - c).Q(x)$, for $x = c$ then $f(c) = Q(c).(c - c) = 0$. Thus for a polynomial function $f(x)$, if $x - c$ is its factor, then $f(c) = 0$. $i.e.$ remainder is zero.

To test whether or not a linear polynomial $mx + b$, $m \neq 0$ is a factor of $f(x)$ you write;

$mx + b = m\left(x + \frac{b}{m}\right) = m\left[x - \left(-\frac{b}{m}\right)\right]$

, and see whether $f\left(-\frac{b}{m}\right) = 0$

By factor theorem,$mx + b$ is a factor of $f(x)$ if and only if$f\left(-\frac{b}{m}\right) = 0$

Examples

1. *Find the remainder when*$x^5 + x - 9$ *is divided by;*
   a) $x + 2$
   b) $x - 3$

Solution

Let $f(x) = x^5 + x - 9$

a) $x + 2 = x - (-2)$

$\therefore f(-2) = (-2)^5 + (-2) - 9 = -43$

b) $x - 3$

$\therefore f(3) = (3)^5 + (3) - 9 = 239$

2. *Find the remainder when* $4x^3 - 5x + 4$ *when divided by;*
   a) $2x - 1$
   b) $2x + 3$

Solution

Let $f(x) = 4x^3 - 5x + 4$

a) $2x - 1 = 2\left(x - \frac{1}{2}\right)$

$\therefore f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 5\left(\frac{1}{2}\right) + 4 = 2$

$\therefore f\left(-\frac{3}{2}\right) = 4\left(-\frac{3}{2}\right)^3 - 5\left(-\frac{3}{2}\right) + 4 = -2$

3. *Find the value of constant a if;*
   i) $x^3 - 3x^2 + ax + 5$ has a remainder 17 when divided by $x - 3$
   ii) $x^5 + 4x^4 - 6x^2 + ax + 2$ has a remainder 6 when divided by $x + 2$
   iii) $4x^3 - ax + 5$ has a remainder 2 ½ when divided by $2x - 1$
   iv) $3x^3 + ax^2 - 6x + 3$ has a remainder 5 when divided by $3x + 1$

Solution

i) Let $f(x) = x^3 - 3x^2 + ax + 5$, Remainder= 17 when divided by $x - 3$

$\therefore$ Remainder= $f(3) = 17$

$\Rightarrow (3)^3 - 3(3)^2 + a(3) + 5 = 17$

$\therefore 3a + 5 = 17 \ \therefore a = 4$

ii) Let $f(x) = x^5 + 4x^4 - 6x^2 + ax + 2$, Remainder= 6 when divided by $x + 2$;
Divisor $= x + 2 = x - (-2)$
Remainder= $f(-2) = 6$
$\Rightarrow (-2)^5 + 4(-2)^4 - 6(-2)^2 + a(-2) + 2 = 6$
$\therefore -2a + 10 = 6 \ \therefore a = 2$

iii) Let $f(x) = 4x^3 - ax + 5$, Remainder= $2\frac{1}{2}$ when divided by $2x - 1$
Divisor $= 2x - 1 = 2\left(x - \frac{1}{2}\right)$
Remainder= $f\left(\frac{1}{2}\right) = 2\frac{1}{2}$
$\Rightarrow 4\left(\frac{1}{2}\right)^3 - a\left(\frac{1}{2}\right) + 5 = 2\frac{1}{2}$
$\therefore \frac{a}{2} + \frac{11}{2} = \frac{5}{2} \ \therefore a = -6$

iv) Let $f(x) = 3x^3 + ax^2 - 6x + 3$, Remainder= 5 when divided by $3x + 1$
Divisor $= 3x + 1 = 3\left(x + \frac{1}{3}\right) = 3\left[x - \left(\frac{1}{3}\right)\right]$
Remainder= $f\left(-\frac{1}{3}\right) = 5$
$\Rightarrow 3\left(-\frac{1}{3}\right)^3 + a\left(-\frac{1}{3}\right)^2 - 6\left(-\frac{1}{3}\right) + 3 = 5$
$\therefore \frac{a}{9} + \frac{46}{9} = 5 \ \therefore a = -1$

4. *Show that;*
   i) $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$
   ii) $2x^3 + x^2 - 8x - 4$is divisible by $2x + 1$
   iii) $x + 2$ and $x + 1$ are factors of $2x^3 + 5x^2 + x - 2$

Solution

i) Let $f(x) = 12x^3 + 16x^2 - 5x - 3$
$f(x)$is divisible by $2x - 1$ if Remainder $= 0$
Remainder= $f\left(\frac{1}{2}\right) = 0 \ \left[\because 2x - 1 = 2\left(x - \frac{1}{2}\right)\right]$
$\Rightarrow f\left(\frac{1}{2}\right) = 12\left(\frac{1}{2}\right)^3 + 16\left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) - 3 = 0$
$\therefore 2x - 1$is a factor of $12x^3 + 16x^2 - 5x - 3$ or $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$

ii) Let $f(x) = 2x^3 + x^2 - 8x - 4$
$2x + 1 = 2\left[x - \left(-\frac{1}{2}\right)\right]$is a factor of $f(x)$ if Remainder $= 0$
Remainder $= f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 - 8\left(-\frac{1}{2}\right) - 4 = 0$
$\therefore 2x + 1$is a factor of$2x^3 + x^2 - 8x - 4$

iii) Let $f(x) = 2x^3 + 5x^2 + x - 2$
$x + 2 = x - (-2)$is a factor of $f(x)$ if Remainder $= f(-2) = 0$
$\Rightarrow f(-2) = 2(-2)^3 + 5(-2)^2 + (-2) - 2 = 0$. Hence $x + 2$ is a factor.
Also $x + 1 = x - (-1)$ is a factor of $f(x)$ if Remainder $= f(-1) = 0$
$\Rightarrow f(-1) = 2(-1)^3 + 5(-1)^2 + (-1) - 2 = 0$. Hence $x + 1$ is a factor.

NOTE:
For all the above examples the factors and divisors used are all *linear*. Now let us look at examples of non-linear divisors.

Examples

1. *Find the remainder when;*
   a) $x^4 - 5x^3 + 6x^2 - 7$ is divided by $(x-1)(x-3)$
   b) $x^4 + x^2 - 7$ is divided by $x^2 - 4$
   c) $x^5 - 7x^3 + 4x - 2$ is divided by $(x+1)(x-1)(x-3)$
   d) $x^5 + x^2 - 4x - 3$ is divided by $(x^2-4)(x+1)$

Solution

a) Let $f(x) = x^4 - 5x^3 + 6x^2 - 7$
Divisor $= (x-1)(x-3)$ *. i. e. Quadratic divisor.*
Let Remainder= $Ax + B$ *. i. e. Linear remainder.*
$\Rightarrow f(x) = (x-1)(x-3).Q(x) + Ax + B$
For $x = 1$
$\Rightarrow f(1) = (1-1)(1-3).Q(1) + A(1) + B = A + B \ \therefore f(1) = A + B$
Also from $f(x) = x^4 - 5x^3 + 6x^2 - 7 \Rightarrow f(1) = (1)^4 - 5(1)^3 + 6(1)^2 - 7 = -5$
$\therefore f(1) = -5$
$\therefore A + B = -5$ ..... (1)
Also for $x = 3$
$\Rightarrow f(3) = (3-1)(3-3).Q(3) + A(3) + B = 3A + B \ \therefore f(3) = 3A + B$
Now from $f(x) = x^4 - 5x^3 + 6x^2 - 7 \Rightarrow f(3) = (3)^4 - 5(3)^3 + 6(3)^2 - 7 = -7$
$\therefore f(3) = -7$
$\therefore 3A + B = -7$ ..... (2)
Solving (1) and (2) simultaneously, eqn. (1)− eqn. (2)
$\Rightarrow -2A = 2 \ \therefore A = -1$
From eqn. (1), $B = -5 - A = -5 + 1 = -4$
$\therefore$ Remainder $= -x - 4$

b) Let $f(x) = x^4 + x^2 - 7$
Divisor $= (x^2 - 4) = (x+2)(x-2)$ *. i. e. Quadratic divisor.*
Let Remainder= $Ax + B$ *. i. e. Linear remainder.*
$\Rightarrow f(x) = (x+2)(x-2).Q(x) + Ax + B$ *or* $x^4 + x^2 - 7 = (x+2)(x-2).Q(x) + Ax + B$
For $x = 2$
$f(2) = 2A + B = (2)^4 + (2)^2 - 7 = 13$
$\therefore 2A + B = 13$ ..... (1)
For $x = -2$
$f(-2) = -2A + B = (-2)^4 + (-2)^2 - 7 = 13$
$\therefore -2A + B = 13$ ..... (2)
Solving (1) and (2) simultaneously, eqn. (1)+ eqn. (2)
$\Rightarrow 2B = 26 \ \therefore B = 13$
From eqn. (1), $B = \frac{1}{2}(13 - B) = \frac{1}{2}(13 - 13) = 0$
$\therefore$ Remainder $= 13$

c) Let $f(x) = x^5 - 7x^3 + 4x - 2$
Divisor $= (x+1)(x-1)(x-3)$ *. i. e. Cubic divisor.*
Let Remainder= $Ax^2 + Bx + C$ *. i. e. Quadratic remainder.*
$\Rightarrow f(x) = (x+1)(x-1)(x-3).Q(x) + Ax^2 + Bx + C = x^5 - 7x^3 + 4x - 2$
For $x = 1$
$f(1) = A + B + C = (1)^5 - 7(1)^3 + 4(1) - 2 = -4$
$\therefore A + B + C = -4$ ..... (1)

For $x = -1$

$f(-1) = A - B + C = (-1)^5 - 7(-1)^3 + 4(-1) - 2 = 0$

$\therefore A - B + C = 0$ ..... (2)

For $x = 3$

$f(3) = 9A + 3B + C = (3)^5 - 7(3)^3 + 4(3) - 2 = 64$

$\therefore 9A + 3B + C = 64$ ..... (3)

Solving the equations simultaneously

$Eqn.(1) - eqn.(2)$ to eliminate C

$$\begin{array}{rl} & A + B + C = -4 \\ - & A - B + C = 0 \\ \hline & 2B \quad = -4 \quad \therefore B = -2 \end{array}$$

$Eqn.(1) - eqn.(3)$ to eliminate C

$$\begin{array}{l} A - B + C \quad = 0 \\ 9A + 3B + C = 64 \\ \hline -8A - 4B \quad = -64 \end{array}$$

From $B = -2$, $-8A = -64 + 4B = -64 - 8 = -72$

$\therefore A = 9$

From $eqn.(1)$, $A + B + C = -4$

$\Rightarrow 9 - 2 + C = -4 \therefore C = -11$

$\therefore Remainder = 9x^2 - 2x - 11$

d) Let $f(x) = x^5 + x^2 - 4x - 3$

Divisor $= (x+1)(x^2 - 4) = (x+1)(x-2)(x+2)$. $i.e. Cubic\ divisor.$

Let Remainder= $Ax^2 + Bx + C$. $i.e. Quadratic\ remainder.$

$\Rightarrow f(x) = (x+2)(x-2)(x+1).Q(x) + Ax^2 + Bx + C = x^5 + x^2 - 4x - 3$

For $x = 2$

$f(2) = 4A + 2B + C = (2)^5 + (2)^2 - 4(2) - 3 = 25$

$\therefore 4A + 2B + C = 25$ ..... (1)

For $x = -2$

$f(-2) = 4A - 2B + C = (-2)^5 + (-2)^2 - 4(-2) - 3 = -23$

$\therefore 4A - 2B + C = -23$ ..... (2)

For $x = -2$

$f(-1) = 4A - 2B + C = (-1)^5 + (-1)^2 - 4(-1) - 3 = 1$

$\therefore A - B + C = 1$ ..... (3)

Solving the equations simultaneously

$Eqn.(1) - eqn.(2)$ to eliminate C

$$\begin{array}{rl} & 4A + 2B + C = 25 \\ - & 4A - 2B + C \ = -23 \\ \hline & 4B \quad = 48 \end{array}$$

$\therefore B = 12$

$Eqn.(2) - eqn.(3)$ to eliminate C

$$\begin{array}{rl} & 4A + 2B + C = 25 \\ - & A - B + C \ = 1 \\ \hline & 3A - B \quad = -24 \end{array}$$

From $B = 12$, $3A = B - 24 = 12 - 24 = -12$
$\therefore A = -4$
From $eqn.(3)$, $A - B + C = 1$
$\Rightarrow -4 - 12 + C = 1 \therefore C = 17$
$\therefore Remainder = -4x^2 + 12x + 17$

NOTE:
If the divisor is not factorisable to linear factors, you can use long division to find the remainder.

2. *Find the quotient when;*
   a) $2x^3 - 3x^2 + 2x - 4$ is divided by $x - 2$
   b) $x^4 - x^3 + x - 5$ is divided by $x + 1$

Solution

a) Let $f(x) = 2x^3 - 3x^2 + 2x - 4$
$\Rightarrow f(x) = (x-2).Q(x) + R\,; R = f(2) = constant$
Now $f(2) = 2(2)^3 - 3(2)^2 + 2(2) - 4 = 4$
$\therefore f(x) = (x-2).Q(x) + 4$
$\Rightarrow \frac{f(x)-4}{x-2} = Q(x)$. Here $Q(x)$ has no remainder.
$\therefore Q(x) = \frac{2x^3-3x^2+2x-4-4}{x-2} = \frac{2x^3-3x^2+2x-8}{x-2}$
Using long division,

$$\begin{array}{r|l} x-2 & 2x^2 + x + 4 \\ \hline & 2x^3 - 3x^2 + 2x - 8 \\ & \underline{2x^3 - 4x^2} \\ & \quad x^2 + 2x - 8 \\ & \quad \underline{x^2 - 2x} \\ & \qquad 4x - 8 \\ & \qquad \underline{4x - 8} \end{array} \qquad \therefore Q(x) = 2x^2 + x + 4$$

Alternatively
Let $f(x) = 2x^3 - 3x^2 + 2x - 4, Degree: n = 3$
Divisor $= x - 2$
$\therefore f(x) = (x-2).Q(x) + Remainder$
But $Remainder = f(2) = 2(2)^3 - 3(2)^2 + 2(2) - 4 = 4$
$\therefore f(x) = (x-2).Q(x) + 4$
Since the quotient has degree $n - 1 = 3 - 1 = 2$, then the quotient must be a quadratic. $.i.e.\, Ax^2 + Bx + C$
$\Rightarrow f(x) = (x-2).(Ax^2 + Bx + C) + 4$
$\therefore 2x^3 - 3x^2 + 2x - 4 \equiv (x-2).(Ax^2 + Bx + C) + 4$
$\equiv Ax^3 + Bx^2 + Cx - 2Ax^2 - 2Bx - C + 4$
$\equiv Ax^3 + x^2(B - 2A) + x(C - 2B) + (4 - 2C)$
Comparing coefficients;
For x^3 : $2 = A$
For x^2 : $-3 = B - 2A; B = -3 + 2A = -3 + 4 = 1$
For x : $2 = C - 2B \Rightarrow C = 2 + 2B = 2 + 2 = 4$
For $constants$: $; -4 = 4 - 2C \Rightarrow 2C = 8 \therefore C = 4$
$\therefore Q(x) = 2x^2 + x + 4$

b) Let $f(x) = x^4 - x^3 + x - 5; Divisor = x + 1$

$\Rightarrow f(x) = (x+1).Q(x) + R\,; R = f(-1) = constant$

Now $f(-1) = (-1)^4 - (-1)^3 + (-1) - 5 = -4$

$\therefore f(x) = (x+1).Q(x) - 4$

$\Rightarrow \frac{f(x)+4}{x+1} = Q(x)$. Here $Q(x)$ has no remainder.

$\therefore Q(x) = \frac{x^4-x^3+x-5+4}{x+1} = \frac{x^4-x^3+x-1}{x+1} = \frac{x^4-x^3+0x^2+x-1}{x+1}$

By long division.

$$
\begin{array}{r|l}
x+1 & x^3 - 2x^2 + 2x - 1 \\
\hline
 & x^4 - x^3 + 0x^2 + x - 1 \\
 & \underline{x^4 + x^3} \\
 & -2x^3 + 0x^2 + x - 1 \\
 & \underline{-2x^3 - 2x^2} \\
 & 2x^2 + x - 1 \\
 & \underline{2x^2 + 2x} \\
 & -x - 1 \\
 & \underline{-x - 1} \\
 & 0
\end{array}
$$

$\therefore Q(x) = x^3 - 2x^2 + 2x - 1$

Alternatively

Let $f(x) = x^4 - x^3 + x - 5, Degree: n = 4$

Divisor $= x + 1$

$\therefore f(x) = (x+1).Q(x) + Remainder$

But $Remainder = f(-1) = (-1)^4 - (-1)^3 + (-1) - 5 = -4$

$\therefore f(x) = (x+1).Q(x) - 4$

Since the quotient has degree $n - 1 = 4 - 1 = 3$, then the quotient must be a cubic. $.i.e. Ax^3 + Bx^2 + Cx + D$

$\Rightarrow f(x) = (x+1).(Ax^3 + Bx^2 + Cx + D) - 4$

$\therefore x^4 - x^3 + x - 5 \equiv (x+1).(Ax^2 + Bx + C) - 4$

$\equiv Ax^4 + Ax^3 + Bx^3 + Bx^2 + Cx^2 + Cx + Dx + D - 4$

$\equiv Ax^4 + x^3(A+B) + x^2(B+C) + x(C+D) + D - 4$

Comparing coefficients;

For $x^4 : 1 = A$

For $x^3 : -1 = A + B \Rightarrow B = -1 - A = -1 - 1 = -2$

For $x^2 : 0 = B + C \Rightarrow C = -B = -(-2) = 2$

For $x : 1 = C + D \Rightarrow D = 1 - C = 1 - 2 = -1$

For $constants: ; -5 = D - 4 \Rightarrow D = -1$

$\therefore Q(x) = x^3 - 2x^2 + 2x - 1$

NOTE:

To determine the quotient using method II, given the polynomial function and divisor, first establish the degree of the quotient basing on that of the divisor, then determine the coefficients as presented as above.

3. *Determine the quotient when;*
   a) $x^4 + x^2 - 7$ is divided by $x^2 - 4$
   b) $x^5 - 7x^3 + 4x - 2$ is divided by $(x^2 - 1)(x - 3)$

Solution

a) Let $f(x) = x^4 + x^2 - 7, Degree: n = 4$

Divisor $= x^2 - 4$
$\therefore f(x) = (x^2 - 4).Q(x) + Remainder$
Let the remainder$= Ax + B$
$\Rightarrow f(x) = (x^2 - 4).Q(x) + Ax + B$
$\therefore f(x) = (x + 2)(x - 2).Q(x) + Ax + B$
$Or\ x^4 + x^2 - 7 \equiv (x + 2)(x - 2).Q(x) + Ax + B$
First obtaining the remainder,
For $x = 2$
$\Rightarrow (2)^4 + (2)^2 - 7 \equiv (2 + 2)(2 - 2).Q(2) + A(2) + B$
$\therefore 2A + B = 13$ ..... (1)
For $x = -2$
$\Rightarrow (-2)^4 + (-2)^2 - 7 \equiv (-2 + 2)(2 - 2).Q(-2) + A(-2) + B$
$\therefore -2A + B = 13$ ..... (1)
Solving eqn. (1) and eqn. (2) simultaneously,
$eqn.(1) + eqn.(2)\ to\ eliminate\ A$
$2B = 26\ \therefore B = 13$
From eqn. (1), $A = \frac{1}{2}(13 - B) = \frac{1}{2}(0) = 0$
Remainder = 13
Now $f(x) = (x + 2)(x - 2).Q(x) + 13$
Since the degree of $f(x)$ is $n = 4$ and the divisor is of degree $n - 2 = 2$, then the quotient is a quadratic $.i.e.mx^2 + nx + p$
Let the quotient $Q(x) = mx^2 + nx + p$
$\Rightarrow f(x) = (x + 2)(x - 2).(mx^2 + nx + p) + 13$
$\therefore x^4 + 0x^3 + x^2 - 7 \equiv (x + 2)(x - 2).(mx^2 + nx + p) + 13$
$\equiv (x^2 - 4)(mx^2 + nx + p) + 13$
$\equiv mx^4 + nx^3 + px^2 - 4mx^2 - 4nx - 4p + 13$
$\equiv mx^4 + nx^3 + x^2(p - 4m) - 4nx + (-4p + 13)$
Comparing coefficients;
For $x^4 : 1 = m$
For $x^3 : 0 = n$
For $x^2 : 1 = p - 4m \Rightarrow p = 1 + 4m = 1 + 4 = 5$
For $x : 0 = -4n\ \therefore n = 0$
For $constants: ; -7 = 13 - 4p\ \therefore p = 5$
$\therefore Q(x) = x^2 + 5$

b) Let $f(x) = x^5 - 7x^3 + 4x - 2, Degree: n = 5$
Divisor $= (x^2 - 1)(x - 3)$
$\therefore f(x) = (x^2 - 1)(x - 3).Q(x) + Remainder$
Let the remainder$= px^2 + qx + r$
$\therefore x^5 - 7x^3 + 4x - 2 \equiv (x^2 - 1)(x - 3).Q(x) + px^2 + qx + r$
$Or\ x^5 - 7x^3 + 4x - 2 \equiv (x + 1)(x - 1)(x - 3).Q(x) + px^2 + qx + r$
First obtaining the remainder,
For $x = 1$
$(1)^5 - 7(1)^3 + 4(1) - 2 \equiv (1 + 1)(1 - 1)(1 - 3).Q(1) + p(1)^2 + q(1) + r$
$\therefore p + q + r = -4$ ..... (1)
For $x = -1$
$(-1)^5 - 7(-1)^3 + 4(-1) - 2 \equiv (-1 + 1)(-1 - 1)(-1 - 3).Q(-1) + p(-1)^2 + q(-1) + r$

$\therefore p - q + r = 0$ ..... (2)
For $x = 3$
$(3)^5 - 7(3)^3 + 4(3) - 2 \equiv (3+1)(3-1)(3-3).Q(3) + p(3)^2 + q(3) + r$
$\therefore 9p + 3q + r = 64$ ..... (3)
Solving eqn. (1), eqn. (2) and eqn. (3) simultaneously,
$eqn.(1) - eqn.(2)\ to\ eliminate\ r$
$\Rightarrow 2q = -4 \therefore q = -2$
$eqn.(2) - eqn.(3)\ to\ eliminate\ r$
$\Rightarrow -8p - 4q = -64\ , q = -2$
$\therefore -8p + 8 = -64\ , p = 9$
From eqn. (2), $p - q + r = 0 \Rightarrow r = q - p = -2 - 9 = -11$
$\therefore Remainder = 9x^2 - 2x - 11$
Now $\therefore f(x) = (x^2 - 1)(x - 3).Q(x) + 9x^2 - 2x - 11$
Since the degree of $f(x)$ is $n = 5$ and the divisor is of degree $n - 3 = 2$, then the quotient is a quadratic $.i.e.Ax^2 + Bx + C$
Let the quotient $Q(x) = Ax^2 + Bx + C$
$\Rightarrow f(x) = (x^2 - 1)(x - 3).(Ax^2 + Bx + C) + 9x^2 - 2x - 11$
$\therefore f(x) \equiv (x^2 - 1)(x - 3).(Ax^2 + Bx + C) + 9x^2 - 2x - 11$
$\equiv (x^3 - 3x^2 - x + 3)(Ax^2 + Bx + C) + 9x^2 - 2x - 11$
$\equiv Ax^5 + Bx^4 + Cx^3 - 3x^4 - 3Bx^3 - 3Cx^2 - Ax^3 - Bx^2 - Cx + 3Ax^2 + 3Bx + 3C + 9x^2 - 2x - 11$
$\therefore f(x) \equiv Ax^5 + x^4(B - 3) + x^3(C - 3B - A) + x^2(-3C - B + 3A + 9) + x(-C + 3B - 2) + 3C - 11$
Given $f(x) = x^5 + 0x^4 - 7x^3 + 0x^2 + 4x - 2$
Comparing coefficients,
For $x^5 : 1 = A$
For $x^4 : 0 = B - 3 \Rightarrow B = 3$
For $x^3 : -7 = C - 3B - A \Rightarrow C = -7 + 9 + 1 = 3$
For $x^2 : 0 = -3C - B + 3A + 9 \Rightarrow C = \frac{-3+3+9}{3} = 3\ As\ before$
For $x : 4 = -C + 3B - 2 \Rightarrow C = 9 - 2 - 4 = 3\ As\ before$
For $constants: ; -2 = 3C - 11 \Rightarrow C = 3\ As\ before$
$\therefore Q(x) = x^2 + 3x + 3$

NOTE:
The above examples on determination of remainder and quotient when a polynomial is divided by a divisor using the above methods can still be done using long division.

4. *Find the constants* $p\ , q\ , r$ *if;*
   a) $x^3 - px + q$ is divided by $x^2 - 3x + 2$ leaves a remainder $4x - 1$
   b) $px^4 + qx^3 + 3x^2 - 2x + 3$ has a remainder $x + 1$ when divided by $x^2 - 3x + 2$
   c) $(x - 1)$ and $(x + 1)$ are factors of the expression $x^3 + px^2 + qx + r$ a nd it leaves a remainder of 12 when divided by $x - 2$

Solution

a) Let $f(x) = x^3 - px + q$
Divisor $= x^2 - 3x + 2 = (x - 1)(x - 2)$

Remainder $= 4x - 1$

$\Rightarrow f(x) \equiv Q(x).(x-1)(x-2) + 4x - 1$

$\therefore x^3 - px + q \equiv Q(x).(x-1)(x-2) + 4x - 1$

For $x = 1$, $(1)^3 - p(1) + q \equiv Q(1).(1-1)(1-2) + 4(1) - 1$

$\Rightarrow 1 - p + q = 3 \ \therefore q - p = 2$ ..... (1)

For $x = 2$, $(2)^3 - p(2) + q \equiv Q(2).(2-1)(2-2) + 4(2) - 1$

$\Rightarrow 8 - 2p + q = 7 \ \therefore q - 2p = -1$ ..... (2)

Solving eqn. (1) and eqn. (2) simultaneously

$eqn.(1) - eqn.(2)$, eliminate q

$\Rightarrow p = 3$

From $eqn.(1), q - p = 2 \Rightarrow q = 2 + 3 = 5$

$\therefore p = 2, q = 5$

Alternatively

By long division

$$
\begin{array}{r|l}
x^2 - 3x + 2 & x + 3 \\
\hline
 & x^3 + 0x^2 - px + q \\
 & \underline{x^3 - 3x^2 + 2x} \\
 & \quad 3x^2 + (-p-2)x + q \\
 & \quad \underline{3x^2 - 9x + 6} \\
 & \quad (-p+7)x + q - 6
\end{array}
$$

Comparing the remainder,

$(-p + 7)x + q - 6 \equiv 4x - 1$

$\Rightarrow -p + 7 = 4 \ \therefore p = 3$

$\Rightarrow q - 6 = -1 \ \therefore q = 5$

b) Let $f(x) = px^4 + qx^3 + 3x^2 - 2x + 3$

Divisor $= x^2 - 3x + 2 = (x-1)(x-2)$

Remainder $= x + 1$

$\Rightarrow f(x) \equiv Q(x).(x-1)(x-2) + x + 1$

$\therefore px^4 + qx^3 + 3x^2 - 2x + 3 \equiv Q(x).(x-1)(x-2) + x + 1$

For $x = 1$, $p(1)^4 + q(1)^3 + 3(1)^2 - 2(1) + 3 \equiv Q(1).(1-1)(1-2) + 1 + 1$

$\Rightarrow p + q + 3 - 2 + 3 = 2 \ \therefore p + q = -2$ ..... (1)

For $x = 2$, $p(2)^4 + q(2)^3 + 3(2)^2 - 2(2) + 3 \equiv Q(2).(2-1)(2-2) + 2 + 1$

$\Rightarrow 16p + 8q + 12 - 4 + 3 = 3$

$\therefore 16p + 8q = -8 \ or \ 2p + q = -1$ ..... (2)

Solving eqn. (1) and eqn. (2) simultaneously

$eqn.(1) - eqn.(2)$, eliminate q

$\Rightarrow -p = -1 \ \therefore p = 1$

From $eqn.(1), \ p + q = -2 \Rightarrow q = -2 - 1 = -3$

$\therefore p = 1, q = -3$

c) Let $f(x) = x^3 + px^2 + qx + r$

Divisor $= (x-1)$, Remainder $= 0$

Divisor $= (x+1)$, Remainder $= 0$

Divisor $= (x-2)$, Remainder $= 12$

$\Rightarrow f(x) \equiv Q(x).(x-1)(x+1)$

$\therefore x^3 + px^2 + qx + r \equiv Q(x).(x-1)(x+1)$

For $x = 1, (1)^3 + p(1)^2 + q(1) + r \equiv Q(1).(1-1)(1+1)$

$\Rightarrow p + q + r = -1$ ..... (1)

For $x = -1, (-1)^3 + p(-1)^2 + q(-1) + r \equiv Q(-1).(-1-1)(-1+1)$

$\Rightarrow p - q + r = 1$ ..... (2)

$\Rightarrow f(x) \equiv H(x).(x-2) + 12$

$\therefore x^3 + px^2 + qx + r \equiv H(x).(x-2) + 12$
For $x = 2\,, (2)^3 + p(2)^2 + q(2) + r \equiv H(2).(2-2) + 12$
$\Rightarrow 4p + 2q + r = 4$ ..... (3)
Eliminating r from eqn. (1) and eqn. (2), $eqn.(1) - eqn.(2)$
$2q = -2 \therefore q = -1$
Eliminating r from eqn. (2) and eqn. (3), $eqn.(2) - eqn.(3)$
$-3p - 3q = -3$ *or* $p + q = 1$
But $q = -1 \ \therefore p = 1 - q = 1 + 1 = 2$
From (1) $r = -1 - p - q = -1 - 2 + 1 = -2$
$\therefore p = 2\,, q = -1\,, r = -2$

5. *Given the polynomial* $f(x) = Q(x).g(x) + R(x)$ *where* $Q(x)$ *is the quotient,* $g(x) = (x-\alpha)(x-\beta)$ *and* $R(x)$ *is the remainder, show that* $R(x) = \frac{(x-\beta)f(\alpha)+(\alpha-x)f(\beta)}{\alpha-\beta}$ *when* $f(x)$ *is divided by* $g(x)$.
Hence find the remainder when $f(x)$ *is divided by* $x^2 - 9$ *given that* $f(x)$ *divided by* $x - 3$ *and when divided by* $x + 3$*, the remainders are respectively* 2 *and* -3.

Solution

$f(x) = Q(x).g(x) + R(x)$
Divisor $= g(x) = (x-\alpha)(x-\beta)$
Remainder $= R(x) = Ax + B$
$\Rightarrow f(x) = Q(x).(x-\alpha)(x-\beta) + Ax + B$
Now for $x = \alpha$
$f(\alpha) = A\alpha + B$ ..... (1)
Now for $x = \beta$
$f(\beta) = A\beta + B$ ..... (2)
Since $f(x)$ is known, $f(\alpha)$ and $f(\beta)$ will be known. Now finding the unknowns A and B,
Eliminating B from eqn. (1) and eqn. (2), $eqn.(1) - eqn.(2)$
$\Rightarrow f(\alpha) - f(\beta) = A(\alpha - \beta)$
$\therefore A = \frac{f(\alpha)-f(\beta)}{\alpha-\beta}$ ..... (3)
From eqn. (1), $f(\alpha) = A\alpha + B \Rightarrow B = f(\alpha) - A\alpha$
$\Rightarrow B = f(\alpha) - \alpha\left[\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right] = \frac{\alpha f(\alpha)-\beta f(\alpha)-\alpha f(\alpha)+\alpha f(\beta)}{\alpha-\beta}$
$\therefore B = \frac{\alpha f(\beta)-\beta f(\alpha)}{\alpha-\beta}$ ..... (4)
$\therefore R(x) = Ax + B$ becomes
$R(x) = \frac{f(\alpha)-f(\beta)}{\alpha-\beta}x + \frac{\alpha f(\beta)-\beta f(\alpha)}{\alpha-\beta} = \frac{xf(\alpha)-xf(\beta)+\alpha f(\beta)-\beta f(\alpha)}{\alpha-\beta}$
$\therefore R(x) = \frac{(x-\beta)f(\alpha)+(\alpha-x)f(\beta)}{\alpha-\beta}$, as required.
From $f(x) = Q(x).(x-\alpha)(x-\beta) + Ax + B$
Divisor $= g(x) = (x-\alpha)(x-\beta) \equiv x^2 - 9 = (x-3)(x+3)$
By comparison, $\alpha = 3\,, \beta = -3$
Given: $f(x) = G(x).(x-3) + 2 \Rightarrow f(3) = 2 \equiv f(\alpha)$
$f(x) = H(x).(x+3) - 3 \Rightarrow f(-3) = -3 \equiv f(\beta)$
, where $G(x)$ and $H(x)$ are the new quotients due to different divisors.
Remainder= $Ax + B$ is given by $\frac{(x-\beta)f(\alpha)+(\alpha-x)f(\beta)}{\alpha-\beta}$

$\Rightarrow R(x) = \frac{(x-3)(2)+(3-x)(-3)}{3-(-3)} = \frac{5x-3}{6}$

$\therefore$ Remainder when $f(x)$ is divided by $x^2 - 9$ is $\frac{5x-3}{6}$

6. *A polynomial $P(x)$ leaves a remainder 6 when divided by $x^2 - 4$ and a remainder $x + a$ when divided by $x^2 - x - 2$. Find;*
   a) *The value of the constant a*
   b) *The remainder when $P(x)$ is divided by;*
      i) $x^2 + 3x + 2$
      ii) $x^3 + x^2 - 4x - 4$

Solution

Case I:

$P(x) = Q(x).(x^2 - 4) + 6$

$= Q(x).(x + 2)(x - 2) + 6$

For $x = 2$, $P(2) = 6$

For $x = -2$, $P(-2) = 6$

Case II:

$P(x) = H(x).(x^2 - x - 2) + x + a$

$= H(x).(x - 2)(x + 1) + x + a$

For $x = 2$, $P(2) = 2 + a$

For $x = -1$, $P(-1) = -1 + a$

a) But $P(2) = 6 = 2 + a \therefore a = 4$

b) (i) $P(x) = G(x).(x^2 + 3x + 2) + Ax + B$

[$\because$ *Remainder is $Ax + B$(Linear)since Divisor is Quadratic*]

$P(x) = G(x).(x + 1)(x + 2) + Ax + B$

For $x = -1$, $P(-1) = -A + B$

For $x = -2$, $P(-2) = -2A + B$

But $P(-1) = -1 + a = -A + B$; $-1 + a = -1 + 4 = 3$

$\Rightarrow -A + B = 3$ ..... (1)

Also $P(-2) = -2A + B = 6$..... (2)

Eliminating B from eqn. (1) and eqn. (2)

$eqn.(1) - eqn.(2)$

$\therefore A = -3$

From eqn. (1) $-A + B = 3$

$\Rightarrow B = A + 3 = -3 + 3 = 0 \therefore B = 0$

Remainder when $P(x)$ is divided by $x^2 + 3x + 2$ is $-3x$

(ii) $P(x) = F(x).(x^3 + x^2 - 4x - 4) + mx^2 + nx + l$

$= F(x).(x + 1)(x + 2)(x - 2) + mx^2 + nx + l$

For $x = 2$, $P(2) = 4m + 2n + l$

For $x = -2$, $P(-2) = 4m - 2n + l$

For $x = -1$, $P(-1) = m - n + l$

But $P(2) = 6, P(-2) = 6, P(-1) = 3$

$\Rightarrow 4m + 2n + l = 6$ ..... (i)

$\Rightarrow 4m - 2n + l = 6$ ..... (ii)

$\Rightarrow m - n + l = 3$ ..... (iii)

Eliminating l from eqn. (i) and eqn. (ii)

$eqn.(i) - eqn.(ii)$

$4n = 0 \therefore n = 0$

Eliminating l from eqn. (ii) and eqn. (iii)

$eqn.(ii) - eqn.(iii)$
$3m - n = 3, n = 0$
$\therefore 3m = 3 \Rightarrow m = 1$
From eqn. (iii) $m - n + l = 3 \Rightarrow l = 3 - m + n$
$\Rightarrow l = 3 - 1 + 0 = 2$
$\therefore$ Remainder is $x^2 + 2$

Task

1. *Find the value of the constant k so that* $f(x) = x^4 + kx^3 - 2kx^2 + 3x - 5$ *has* $(x - 1)$ *as a factor.* $\boldsymbol{Answer: k = -1}$
2. *Find the value of k so that* $f(x) = x^3 - x^2 + kx - 12$ *has a factor* $(x - 3)$
   $\boldsymbol{Answer: k = -2}$
3. *Find the value of a if;*
   i) $x^3 - 3x^2 + ax + 5$ *has a remainder 17 when divided* $x - 3$.
   $\boldsymbol{Answer: 4}$
   ii) $x^3 + x^2 - 2ax + a^2$ *has remainder 8 when divided by* $x - 2$
   $\boldsymbol{Answer: 2}$
   iii) $x - a$ *will divide* $f(x) = x^2 + 4x + 2$ *with a remainder of* -1
   $\boldsymbol{Answer: a = -3, -1}$
4. *Find the values of a and b if;*
   i) When $x^5 + 4x^2 + ax + b$ *is divided by* $x^2 - 1$ *the remainder is* $2x + 3$
   $\boldsymbol{Answer: a = 1, b = -1}$
   ii) When $ax^4 + bx^3 - x^2 + 2x + 3$ *has a remainder* $3x + 5$ *when divided by* $x^2 - x - 2$
   $\boldsymbol{Answer: a = 1, b = -1}$
   iii) $x - 3$ *and* $x + 7$ *are factors of the quadratic* $ax^2 + 12x + b$
   $\boldsymbol{Answer: a = 3, b = -63}$
5. a) *If* $f(x) = x^2 + px + q$ *is exactly divisible by* $x - a$ *and* $x - b$, *show that* $p = -a - b$ *and*
   $q = ab$
   b) *Show that;*
   i) $x - a$ *is a factor of* $x^n - a^n$ *when* $n = even$
   ii) $x + a$ *is a factor of* $x^n + a^n$ *when* $n = odd$

$$\left[\begin{array}{l}\boldsymbol{HINT}: \\ f(-a) = (-a)^n - a^n. If\ n\ is\ odd, (-a)^n = -a^n\ and\ f(-a) = 0. Thus\ x + a\ is\ a\ facor\ of \\ x^n + a^n. Q(x) = x^{n-1} - ax^{n-2} + ax^{n-3} - \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a^{n-1}\end{array}\right]$$

   c) *Show that the remainder when the polynomial* $f(x)$ *is divided by* $x - a$ *is* $f(a)$. *Show further that if* $f(x)$ *is divided by* $(x - a)(x - b)$ *where* $a \neq b$, *then the remainder is;* $\left[\frac{f(a) - f(b)}{a - b}\right]x + \left[\frac{af(b) - bf(a)}{a - b}\right]$

   d) *A polynomial* $f(x)$ *is divided by* $x^2 - a^2$ *where* $a \neq 0$ *and the remainder is* $px + q$ *prove that* $p = \frac{1}{2a}[f(a) - f(-a)], q = \frac{1}{2a}[f(a) + f(-a)]$
6. *When a polynomial* $p(x)$ *is divided by* $2x + 1$, *the quotient is* $x^2 - x + 4$ *and the remainder is 3. Find* $p(x)$ $[\boldsymbol{HINT}: p(x) = (2x + 1).(x^2 - x + 4) + 3]$
7. *When a polynomial* $p(x)$ *is divided by* $3x - 4$, *the quotient is* $x^3 + 2x + 2$ *and the remainder is* -1. *Find* $p(x)$. $[\boldsymbol{HINT}: p(x) = (3x - 4).(x^3 + 2x + 2) - 1]$

8. a) *When the quadratic expression* $ap^2 + bp + c$ *is divided by* $p - 1$, $p - 2$ *and* $p + 1$, *the remainders are* 1 , 2 *and* 25 *respectively. Determine the factors of the expression.*
   [***HINT***: *Solve eqns.* $a + b + c = 1, 4a - 12b + c = 2$ *and* $a - b + c = 25$. *Then* $f(p) = 4p^2 - 12p + 9$. *Now factoring* $f(p) = (2p - 3)(2p - 3)$]

   b) Express $2x^3 + 5x^2 - 4x - 3$ in the form $(x^2 + x - 2).Q(x) + Ax + B$ where $Q(x)$a polynomial in is x and A and B are constants. Determine the values of constants A and B and the expression $Q(x)$. ***Answer***: $\boldsymbol{Q(x) = 2x + 3}$, $\boldsymbol{A = -3}$, $\boldsymbol{B = 3}$ (***Uneb* 1993**)

9. *if* $4x^3 + kx^2 + px + 2$ *is divisible by* $x^2 + \lambda^2$, *prove that* $kp = 8$.
   [***HINT***: *Let* $f(x) = 4x^3 + kx^2 + px + 2 \equiv (x^2 + \lambda^2)(Ax + B)$,
   since the quotient must be of degree 1 *when the divisor is of degree* 2 *for* $f(x)$ *is of degree* 3]
10. *A polynomial expression* $P(x)$ *when divided by* $x - 1$ *leaves a remainder of* 3 *and when divided by* $x - 2$ *leaves a remainder of* 1. *Show that when divided by* $(x - 1)(x - 2)$ *it leaves a remainder of* $-2x + 5$.
    [***HINT***: $P(x) = (x - 1).Q(x) + 3$ *or* $P(1) = 3$
    $P(x) = (x - 2).H(x) + 1$ *or* $P(2) = 1$
    Now $P(x) = (x - 1)(x - 2).G(x) + Ax + B$
    $P(1) = A + B = 3$; $P(2) = 2A + B = 1$
    Solving the eqns. $A = -2, B = 5$
    $\therefore$ *Remainder* $= -2x + 5$]
11. *Find the quotient and the remainder when* $f(x)$ *is divided by a linear factor*
    i) $f(x) = 2x^3 - 3x^2 + 2x - 8$, $x - 2$
    ii) $f(x) = x^7 - 31x^2$, $x - 2$
    ANSWER: (i) $\boldsymbol{2x^2 + x + 4}$; $\boldsymbol{0}$ (ii) $\boldsymbol{x^6 + 2x^5 + 4x^4 + 8x^3 + 16x^2 + x + 2}$; $\boldsymbol{2}$
12. *Find the constant p such that* $x^2 + 2$ *is a factor of* $x^4 - 6x^2 + p$. *Hence factorize it.* **ANSWER**: $\boldsymbol{p = -16}$, $\boldsymbol{(x^2 + 2)(x - 2\sqrt{2})(x + 2\sqrt{2})}$

REPEATED ROOTS

(Equal Roots)

From factor theorem which states that "*If for a polynomial function* $f(x)$ *if* $f(a) = 0$ *then* $(x - a)$ *is a factor of* $f(x)$"

If $f(x)$ has a repeated factor $(x - a)$. *i.e.* $f(x) = (x - a)^2.g(x)$, then;

$f'(x) = \frac{d}{dx}[(x - a)^2.g(x)] = g(x).2(x - a) + (x - a)^2.g'(x)$

$= (x - a)\left[(x - a).g'(x) + 2g(x)\right]$

i.e. If $f(x)$ has a repeated factor $(x - a)$, then $f'(x)$ has a repeated factor $(x - a)$. This is true if and only if $f(a) = 0$, $f'(a) = 0$.

Examples

1. *Without using long division find the remainder when* $x^3 - 5x^2 + 6x - 2$ *is divided by* $(x - 2)^2$.

Solution

Let $f(x) = x^3 - 5x^2 + 6x - 2$

$\Rightarrow f(x) = (x - 2)^2.Q(x) + Ax + B$ ..... (*)

For $x = 2$

$f(2) = 2A + B = (2)^3 - 5(2)^2 + 6(2) - 2$

$\therefore 2A + B = -2$ ..... (1)

No other condition linking A and B can be obtained from eqn. (*) without involving $Q(x)$.

But from $f(x) = x^3 - 5x^2 + 6x - 2$, $f'(x) = 3x^2 - 10x + 6$

$f'(x) = \frac{d}{dx}[(x-2)^2.Q(x) + Ax + B] = Q(x).2(x-2) + (x-2)^2.Q'(x) + A$

$\Rightarrow 3x^2 - 10x + 6 = Q(x).2(x-2) + (x-2)^2.Q'(x) + A$

$= (x-2)[2Q(x) + (x-2).Q'(x)] + A$

For $x = 2$

$f'(2) = 3(2)^2 - 10(2) + 6 = A \ \therefore A = -2$

From eqn. (1) $2A + B = -2 \Rightarrow B = -2 + 4 = 2$

$\therefore Remainder = -2x + 2$

2. *Find the constants m and n such that when $x^4 - mx^2 + n$ is divided by $(x+1)^2$ the remainder is $5x - 2$.*

Solution

Let $P(x) = x^4 - mx^2 + n = (x+1)^2.Q(x) + 5x - 2$

For $x = -1$

$P(-1) = (-1)^4 - m(-1)^2 + n = 5(-1) - 2$

$\therefore n - m = -8$ ..... (1)

Since the divisor is a repeated factor, then

$P'(x) = 4x^3 - 2mx = Q(x).2(x+1) + (x+1)^2.Q'(x) + 5$

$= (x+1)[2Q(x) + (x+1).Q'(x)] + 5$

For $x = -1$

$P'(-1) = 4(-1)^3 - 2m(-1) = 5$

$\Rightarrow 2m - 4 = 5 \ \therefore m = \frac{9}{2}$

From eqn. (1), $n = m - 8 = \frac{9}{2} - 8 = -\frac{7}{2}$

$m = \frac{9}{2}, n = -\frac{7}{2}$

3. *If the equation $3x^4 + 2x^3 - 6x^2 - 6x + p = 0$ has two equal roots, find the possible values of p.*

Solution

Let $f(x) = 3x^4 + 2x^3 - 6x^2 - 6x + p$

The equation $f(x) = 0$ has two equal roots if $f(x)$ has two equal factors.$i.e.$ *Repeated factors*.

Any linear factor of $f'(x)$ is a possible repeated factor of $f(x)$.

$f'(x) \equiv 12x^3 + 6x^2 - 12x - 6$

$\equiv 6(x-1)(x+1)(2x+1)$

For $(x-1)$ to be a repeated factor of $f(x)$, then $f(1) = 0$

$\Rightarrow f(1) = 3(1)^4 + 2(1)^3 - 6(1)^2 - 6(1) + p = 0 \ \therefore p = 7$

Similarly $(x+1)$ is a repeated factor of $f(x)$, if $f(-1) = 0$

$\Rightarrow f(-1) = 3(-1)^4 + 2(-1)^3 - 6(-1)^2 - 6(-1) + p = 0 \ \therefore p = -1$

For $(2x+1)$ to be a repeated factor of $f(x)$, then $f\left(-\frac{1}{2}\right) = 0$

$\Rightarrow f\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^4 + 2\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right)^2 - 6\left(-\frac{1}{2}\right) + p = 0$

$\therefore p = -\frac{23}{16}$

4. *Determine whether $f(x) \equiv 3x^4 - 8x^3 - 6x^2 + 24x - 13$ has any repeated factors, and if so find them.*

Solution

$f'(x) \equiv 12x^3 - 24x^2 - 12x + 24$

$\equiv 12(x^3 - 2x^2 - x + 2)$

$\equiv 12\big((x-1)(x^2 - x - 2)\big) \equiv 12(x-1)(x+1)(x-2)$

Now $f'(x) = 0$ if $x = 1, -1, 2$

Checking the value of $f(x)$ for these values of x,

$\Rightarrow f(1) = 0, f(-1) \neq 0, f(2) \neq 0$

So $(x+1)$ and $(x-2)$ are not factors of $f(x)$. Hence $(x-1)$ is the only repeated factor of $f(x)$

5. a) *Find the values of p and q which make* $x^4 + 6x^3 + 13x^2 + px + q$ *a perfect square.*

b) *The polynomial* $p(x) \equiv x^4 + 4x^3 + bx^2 + cx + d$ *is a perfect square of degree two. Show that* $c + 8 = 2b, 16d = c^2$. *Given that* $P(x)$ *leaves a remainder of* 4 *when divided by* $x + 1$, *determine the possible values of the constants* b, c *and* d.

Solution

a) Since $p(x) \equiv x^4 + 4x^3 + bx^2 + cx + d$ is of degree four(4^0), then its perfect square must be a quadratic.

Let $x^4 + 6x^3 + 13x^2 + px + q \equiv (Ax^2 + Bx + C)^2 \equiv [(Ax^2 + Bx) + C]^2$

$\equiv (Ax^2 + Bx)^2 + 2C(Ax^2 + Bx) + C^2$

$\equiv A^2x^4 + 2ABx^3 + B^2x^2 + 2ACx^2 + 2BCx + C^2$

$\equiv A^2x^4 + (2AB)x^3 + (B^2 + 2AC)x^2 + (2BC)x + C^2$

Comparing coefficients

For x^4: $1 = A^2 \ \therefore A = \pm 1$

For x^3: $6 = 2AB \ \therefore B = \frac{3}{A} = \pm 3$

For x^2: $13 = B^2 + 2AC \ \therefore C = \frac{13 - B^2}{2A} = \frac{13-9}{2(\pm 1)} = \pm 2$

For x: $p = 2BC \ \therefore p = 2(\pm 3)(\pm 2) = \pm 12$

For $Constants$: $q = C^2 \ \therefore q = (\pm 2)^2 = 4$

$\therefore p = 12, q = 4$

b) Let $P(x) = x^4 + 4x^3 + bx^2 + cx + d \equiv (Ax^2 + Bx + D)^2$

$\equiv [(Ax^2 + Bx) + D]^2$

$\equiv (Ax^2 + Bx)^2 + 2D(Ax^2 + Bx) + D^2$

$\equiv A^2x^4 + 2ABx^3 + B^2x^2 + 2ADx^2 + 2BDx + D^2$

$\equiv A^2x^4 + (2AB)x^3 + (B^2 + 2AD)x^2 + (2BD)x + D^2$

Comparing coefficients

For x^4: $1 = A^2 \ \therefore A = \pm 1$

For x^3: $4 = 2AB \ \therefore B = \frac{2}{A} = \pm 2$

For x^2: $b = B^2 + 2AD \ \therefore b = 4 \pm 2D$

For x: $c = 2BD \ \therefore c = \pm 4D$

For Constants: $d = D^2 \ \therefore q = (\pm 2)^2 = 4$

From $b = 4 + 2D$ and $c = 4D$ (Taking the + sign for both), eliminating D from the equations,

$2b = 8 + 4D$, but $c = 4D$

$\Rightarrow 2b = 8 + c \ \therefore c + 8 = 2b$

From $d = D^2$ and $c = 4D$ *or* $D = \frac{c}{4}$, eliminating D from the equations,

$\Rightarrow d = \left(\frac{c}{4}\right)^2 \ \therefore 16d = c^2$

Given that $P(x) = Q(x).(x+1) + 4$

For $x = -1$, $P(-1) = 4$

From $P(x) = x^4 + 4x^3 + bx^2 + cx + d$

$\Rightarrow P(-1) = (-1)^4 + 4(-1)^3 + b(-1)^2 + c(-1) + d$

$\Rightarrow 4 = 1 - 4 + b - c + d$

$\therefore b - c + d = 7$ ..... (1)

From $16d = c^2$ *or* $d = \frac{c^2}{16}$

$\therefore b - c + \frac{c^2}{16} = 7$ ..... (2)

From $c + 8 = 2b$ *or* $\frac{c+8}{2} = b$

Eqn. (2) becomes,

$\frac{c+8}{2} - c + \frac{c^2}{16} = 7$ .....$(\times 16)$

$\Rightarrow 64 + 8c - 16c + c^2 = 112$ *or* $c^2 - 8c - 48 = 0$

Solving the equation, $c = 12, -4$

$\therefore b = \frac{c+8}{2} = \frac{12+8}{2}$ *or* $\frac{-4+8}{2}$

$\therefore b = 10, c = 12; b = 2, c = -4$

Now from $d = \frac{c^2}{16}$

For $c = -4$, $d = \frac{16}{16} = 1$

For $c = 12$, $d = \frac{144}{16} = 9$

$\therefore b = 10, c = 12, d = 9; b = 2, c = -4, d = 9$

Task

1) *Find the remainder when* $x^3 - 5x^2 + 7$ *is divided by* $(x-1)^2$. **ANSWER**: $\mathbf{10 - 7x}$
2) *If the equation* $2x^3 - 9x^2 + 12x + p = 0$ *has two equal roots, find the possible values of p.* **ANSWER**: $\mathbf{-5, -4}$
3) *Find the value of a for which the function* $2x^3 - ax^2 - 12x - 7$ *has a repeated factor.* **ANSWER**: $\mathbf{3}$
4) *If* $(x+1)^2$ *is a factor of* $2x^4 + 7x^3 + 6x^2 + Ax + B$. *Find the values of A and B.* **ANSWER**: $\boldsymbol{A = 7, B = 0}$
5) *Given that* $x^2 + c^2$ *is a factor of* $5x^4 + px^3 + qx^2 + rx + 4$, *show that* $5r^2 + 4p^2 - 4pr = 0$
6) *Show that if* $x^3 + 3px^2 + qx + r$ *is a perfect cube for all real values of* x *then* $q^3 = 27r^2$
7) *Show that* $x^2 + 3$ *is a factor of* $x^3 - x^2 + 3x - 3$
8) *Find the constant m for which* $x^2 + 1$ *is a factor of* $mx^4 + x^2 - 1$. **ANSWER**: $\mathbf{2}$
9) *Determine whether the given functions have any repeated factors, and if any, find them.*
   a) $x^4 - 16$ **ANSWER**: $\boldsymbol{No}$
   b) $x^4 - 18x^2 + 81$ **ANSWER**: $\boldsymbol{(x-3)(x+3)}$
   c) $2x^3 - 3x^2 + 1$ **ANSWER**: $\boldsymbol{x - 1}$
   d) $x(x^4 - 4)$ **ANSWER**: $\boldsymbol{No}$
10) *Find the constants p and q such that* $(x-1)$ *is a common factor of* $x^4 - 2px^2 + 2$ *and* $x^4 + x^2 - q$. **ANSWER**: $\boldsymbol{p = \frac{3}{2}, q = 2}$

SOLVING POLYNOMIAL EQUATIONS BY FACTORING

Important Theorems used:

a) **The Factor Theorem**

"For a polynomial $P(x)$, $x-a$ is a factor if and only if $P(a)=0$"

b) **The Rational Theorem**

"Let $P(x)$ be a polynomial of degree n with integral coefficients a_i and non zero constant term, $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+a_1x+a_0$ where $a_i \in I$.

If one of the roots of the equation $P(x)=0$ is $x=\frac{p}{q}$, $\frac{p}{q}\neq 0$, $q>0$ and $\frac{p}{q}$ is expressed in lowest terms, then p must be a factor of a_0 and q must be a factor of a_n

c) **The Fundamental Theorem of Algebra**

If the complex system consisting of all real and imaginary numbers, if $P(x)$ is a polynomial of degree n, $(n>0)$ with complex coefficients, then the equation $P(x)=0$ has exactly $n-$roots (provided a double root is counted as two roots, a triple is counted as $3-$ roots

Example

- $2ix^3+\sqrt{5}x^2+(3+2i)x+7-0$ has $3-$ roots
- $3x^4-11x^3+19x^2+25x-36=0$ has $4-$ roots

Other Theorems

- " If $P(x)$ is a polynomial of odd degree with real coefficients, then the equations $P(x)=0$ has at least on real root"
- The relationship between the roots of a polynomial equation and coefficients of the polynomial is;

  For the equation $\boldsymbol{a_nx^n+a_{n-1}x^{n-1}+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+a_1x+a_0=0}$ with $a_n\neq 0$, then

  Sum of roots $=-\frac{a_{n-1}}{a_n}$

  Product of roots $=\begin{cases}\frac{a_0}{a_n} & if\ n\ is\ even\\ -\frac{a_0}{a_n} & if\ n\ is\ odd\end{cases}$

Example

1. a) If one of the roots of the equation $P(x)=3x^4+13x^3+15x^2-4=0$ is in the form $x=\frac{p}{q}$, state the possible rational roots of the equation $P(x)=0$.

   b) Determine whether any of the possible rational roots really are roots. Then find all the other roots, real/imaginary.

Solution

a) $x=\frac{p}{q}$ is a possible rational root if p divides -4 and q divides 3

⇒ p could be equal to: $\pm1,\pm2\ or\ \pm4$ and

q could be equal to: $\pm1,\pm3$

$\therefore \frac{p}{q}=\pm1,\pm2,\pm4,\pm\frac{1}{3},\pm\frac{2}{3},\pm\frac{4}{3}$

b) Using Factor Theorem

From $P(x)=3x^4+13x^3+15x^2-4$

Let $x=1$, $P(1)\neq 0$ $\therefore x-1$ is not a factor

$x=-1\,,P(-1)\neq 0\ \therefore x+1$ is not a factor
$x=2\,,P(2)\neq 0\ \therefore x-2$ is not a factor
$x=-2\,,P(-2)=0\ \therefore x+2$ is a factor
Now by long division.

$$\begin{array}{r|l} & 3x^3+7x^2+x-2 \\ \hline x+2 & 3x^4+13x^3+15x^2+0x-4 \\ & \underline{3x^4+6x^3} \\ & \quad 7x^3+15x^2+0x-4 \\ & -\ \underline{7x^3+14x^2} \\ & \quad\quad x^2+0x-4 \\ & -\ \underline{x^2+2x} \\ & \quad\quad -2x-1 \\ & -\ \underline{-2x-1} \end{array}$$

$\therefore P(x)=(x+2)(3x^3+7x^2+x-2)=0$

Now finding roots of the depressed equation $3x^3+7x^2+x-2=0$
If $x=\frac{p}{q}$ is the root of the equation, p divides -2 and q divides 3
p could be equal to: $\pm1\,,\pm2$ and
q could be equal to: $\pm1\,,\pm3$
$\therefore \frac{p}{q}=\pm1\,,\pm2\ ,\pm\frac{1}{3}\,,\pm\frac{2}{3}$
Let $Q(x)=3x^3+7x^2+x-2=0$
For $x=1\,,Q(1)\neq 0\ \therefore x-1$ is not a factor
$x=-1\,,Q(-1)\neq 0\ \therefore x+1$ is not a factor
$x=2\,,Q(2)\neq 0\ \therefore x-2$ is not a factor
$x=-2\,,Q(-2)=0\ \therefore x+2$ is a factor
By long division,

$$\begin{array}{r|l} & 3x^2+x-1 \\ \hline x+2 & 3x^3+7x^2+x-2 \\ & \underline{3x^3+6x^2} \\ & \quad x^2+x-2 \\ & -\ \underline{x^2+2x} \\ & \quad\quad -x-2 \\ & \quad\quad \underline{-x-2} \end{array}$$

$\therefore Q(x)=(x+2)(3x^2+x-1)=0$

$\therefore P(x)=(x+2)^2(3x^2+x-1)=0$

Now solving $3x^2+x-1=0$
$x=\frac{-1\pm\sqrt{(1)^2+4(3)(1)}}{2(3)}=\frac{-1\pm\sqrt{13}}{6}$
∴Roots are: $x=\frac{-1\pm\sqrt{13}}{6}\,,x=-2\,,x=-2$

2. *Find all the zeros and if possible, all the other real zeros of;*
   a. $f(x)=x^3-x^2-14x+24$
   b. $f(x)=x^3-8x^2+5x+14$
   c. $f(x)=3x^3-x^2-6x+2$

Solution
For rational zeros, $f(x)=0$ is an equation to be solved.

a. $f(x) = x^3 - x^2 - 14x + 24 = 0$

Let $x = \frac{p}{q}$, p divides 24, q divides 1

p could be equal to: $\pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm8, \pm12, \pm24$ and

q could be equal to: ±1

∴Possible values of $x = \frac{p}{q} = \pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm8, \pm12, \pm24$

By factor theorem,

For $x = 1, f(1) \neq 0$ ∴ $x - 1$ is not a factor

$x = -1, f(-1) \neq 0$ ∴ $x + 1$ is not a factor

$x = -2, f(-2) \neq 0$ ∴ $x + 2$ is not a factor

$x = 2, f(2) = 0$ ∴ $x - 2$ is a factor

By long division,

$$
\begin{array}{r|l}
x - 2 & x^2 + x - 12 \\
\hline
 & x^3 - x^2 - 14x + 24 \\
 & \underline{x^3 - 2x^2} \\
 & \quad x^2 - 14x + 24 \\
 & \quad \underline{x^2 - 2x} \\
 & \quad\quad -12x + 24 \\
 & \quad\quad \underline{-12x - 24}
\end{array}
$$

∴ $f(x) = (x - 2)(x^2 + x - 12) = 0$

Solving $x^2 + x - 12 = 0$

$\Rightarrow (x + 4)(x - 3) = 0$

∴ $x = -4, 3$

∴The rational zeros are:

∴ $x = 2, -4, 3$

b. $f(x) = x^3 - 8x^2 + 5x + 14 = 0$

Let $x = \frac{p}{q}$, p divides 14, q divides 1

Possible values of $p = \pm1, \pm2, \pm7, \pm14$

Possible values of $q = \pm1$

Possible values of $x = \pm1, \pm2, \pm7, \pm14$

By factor theorem,

For $x = 1, f(1) \neq 0$ ∴ $x - 1$ is not a factor

$x = -1, f(-1) \neq 0$ ∴ $x + 1$ is a factor

$x = 2, f(2) = 0$ ∴ $x - 2$ is a factor

$x = 7, f(2) = 0$ ∴ $x - 7$ is a factor

∴ $f(x) = (x + 1)(x - 2)(x - 7) = 0$

The zeros of $f(x)$ are: $-1, 2, 7$

c. $f(x) = 3x^3 - x^2 - 6x + 2 = 0$

Let $x = \frac{p}{q}$, p divides 2, q divides 3

Possible values of $p = \pm1, \pm2$

Possible values of $q = \pm1, \pm3$

Possible values of $x = \pm1, \pm2, \pm\frac{1}{3}, \pm\frac{2}{3}$

By factor theorem,

For $x = 1, f(1) \neq 0$ ∴ $x - 1$ is not a factor

$x = -1, f(-1) \neq 0$ ∴ $x + 1$ is not a factor

$x = 2, f(2) \neq 0$ ∴ $x - 2$ is not a factor

$x = -\frac{1}{3}, f\left(-\frac{1}{3}\right) \neq 0$ ∴ $x + \frac{1}{3}$ is not a factor

$x = \frac{1}{3}, f\left(\frac{1}{3}\right) \neq 0$ ∴ $x - \frac{1}{3}$ is a factor

By long division

$$\begin{array}{r|l} & 3x^2 - 6 \\ \hline x - \frac{1}{3} & 3x^3 - x^2 - 6x + 2 \\ & \underline{3x^3 - x^2} \\ & \quad -6x + 2 \\ & \quad \underline{-6x + 2} \end{array} \qquad \therefore f(x) = (3x^2 - 6)\left(x - \tfrac{1}{3}\right) = 0$$

$\therefore f(x) = (3x^2 - 6)\left(x - \frac{1}{3}\right) = 0 \Rightarrow 3x^2 - 6 = 0 \ \therefore x = \pm\sqrt{2}$

The zeros of $f(x)$ are: $\pm\sqrt{2}, \frac{1}{3}$

3. *Factorize the polynomial $P(x)$, hence find the zeros of $P(x)$.*
   a) $P(x) = x^3 + 2x^2 - 9x - 18$
   b) $P(x) = x^4 - 11x^2 + 28$
   c) $P(x) = 2x^4 - 5x^3 - 11x^2 + 20x + 12$

Solution

a) $P(x) = x^3 + 2x^2 - 9x - 18$

Let $x = \frac{p}{q}$, p divides -18, q divides 1

Possible values of $p = \pm1, \pm2, \pm3, \pm6, \pm9, \pm18$

Possible values of $q = \pm1$

Possible values of $x = \pm1, \pm2, \pm3, \pm6, \pm9, \pm18$

By factor theorem,

For $x = 1, P(1) \neq 0 \ \therefore x - 1$ is not a factor

$x = -1, P(-1) \neq 0 \ \therefore x + 1$ is not a factor

$x = -2, P(-2) = 0 \ \therefore x + 2$ is a factor

$x = 3, P(3) = 0 \ \therefore x - 3$ is a factor

$x = -3, P(-3) \neq 0 \ \therefore x + 3$ is a factor

$\therefore P(x) = (x + 3)(x - 3)(x + 2)$

The zeros of $P(x)$ are: $\pm3, -2$

b) $P(x) = x^4 - 11x^2 + 28$

Let $x = \frac{p}{q}$, p divides 28, q divides 1

Possible values of $p = \pm1, \pm2, \pm3, \pm4, \pm7, \pm14, \pm28$

Possible values of $q = \pm1$

Possible values of $x = \pm1, \pm2, \pm3, \pm4, \pm7, \pm14, \pm28$

By factor theorem,

For $x = 1, P(1) \neq 0 \ \therefore x - 1$ is not a factor

$x = -1, P(-1) \neq 0 \ \therefore x + 1$ is not a factor

$x = 2, P(2) = 0 \ \therefore x - 2$ is a factor

$x = -2, P(-2) = 0 \ \therefore x + 2$ is a factor

$\therefore P(x) = (x + 2)(x - 2).M(x) = (x^2 - 4).M(x)$

Since $x + 2, x - 2$ are factors of $P(x)$, then $(x + 2)(x - 2) = x^2 - 4$ must also be a factor.

By long division to obtain $M(x)$

Divisor $= x^2 - 4$

$$\begin{array}{r|l} & x^2 - 7 \\ \hline x^2 - 4 & x^4 + 0x^3 - 11x^2 + 0x + 28 \\ & \underline{x^4 + 0x^3 - 4x^2} \\ & \quad -7x^2 + 0x + 28 \\ & \quad \underline{7x^2 + 0x + 28} \end{array} \qquad \therefore M(x) = x^2 - 7$$

$\therefore P(x) = (x+2)(x-2)(x^2-7) = (x+2)(x-2)\left(x \pm \sqrt{7}\right)$

The zeros of $P(x)$ are: $\pm 2, \pm\sqrt{7}$

c) $P(x) = 2x^4 - 5x^3 - 11x^2 + 20x + 12$

Let $x = \frac{p}{q}$, p divides 12, q divides 2

Possible values of $p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

Possible values of $q = \pm 1, \pm 2$

Possible values of $x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}$

By factor theorem,

For $x = 1, P(1) \neq 0 \ \therefore x - 1$ is not a factor

$x = -1, P(-1) \neq 0 \ \therefore x + 1$ is not a factor

$x = 2, P(2) = 0 \ \therefore x - 2$ is a factor

$x = -2, P(-2) = 0 \ \therefore x + 2$ is a factor

$x = 3, P(3) = 0 \ \therefore x - 3$ is a factor

$\therefore P(x) = (x+2)(x-2)(x-3).M(x) = (x^2-4)(x-3).M(x)$

Since $x+2, x-2, x-3$ are factors of $P(x)$, then $(x+2)(x-2)(x-3)$ must also be a factor. By long division to obtain $M(x)$

Divisor $= (x^2-4)(x-3) = x^3 - 3x^2 - 4x + 12$

$$\begin{array}{r|l} & 2x + 1 \\ \hline x^3 - 3x^2 - 4x + 12 & 2x^4 - 5x^3 - 11x^2 + 20x + 12 \\ & \underline{2x^4 - 6x^3 - 8x^2 + 24x} \\ & \quad x^3 - 7x^2 + 0x + 28 \\ & \quad \underline{x^3 - 7x^2 + 0x + 28} \end{array}$$

$\therefore P(x) = (x+2)(x-2)(x-3)(2x+1)$

The zeros of $P(x)$ are: $\pm 2, -\frac{1}{2}, -3$

Task

1) *Factorize $f(x)$, hence state the zeros of $f(x)$*

a) $f(x) = x^3 + 4x^2 + x - 6$

ANSWER: $(x+2)(x+3)(x-1)$; $Zeros$: $-2, -3, 1$

b) $f(x) = 6x^3 - 5x^2 - 2x + 1$

ANSWER: $(3x-1)(2x+1)(x-1)$; $Zeros$: $-\frac{1}{2}, \frac{1}{3}, 1$

c) $f(x) = 4x^3 - 13x - 6$

ANSWER: $(x-2)(2x+1)(2x+3)$; $Zeros$: $-\frac{1}{2}, -\frac{3}{2}, 2$

2) *Solve the equation giving real and imaginary roots if any;*

a) $2x^3 - 7x + 2 = 0$ **ANSWER**: $\frac{2\pm\sqrt{2}}{2}, -2$

b) $3x^3 - 4x^2 - 5x + 2 = 0$ **ANSWER**: $-1, 2, \frac{1}{3}$

c) $x^4 - 10x^2 + 9 = 0$ **ANSWER**: $\pm 1, \pm 3$

THE BINOMIAL THEOREM

Pascal's triangle

This is used to give the coefficients of the emerging terms during the expansion.

The coefficients are obtained as below.

Consider an array of triangular combinatorial coefficients.

		COLUMN (r)					
ROW (n)	0	0	1	2	3	4	5
	0	${}^0C_0$					
	1	${}^1C_0$	${}^1C_1$				
	2	${}^2C_0$	${}^2C_1$	${}^2C_2$			
	3	${}^3C_0$	${}^3C_1$	${}^3C_2$	${}^3C_3$		
	4	${}^4C_0$	${}^4C_1$	${}^4C_2$	${}^4C_3$	${}^4C_4$	

Superscript on the left of each of the coefficients denotes *row number* and the *subscript* on the right denote *column number*.

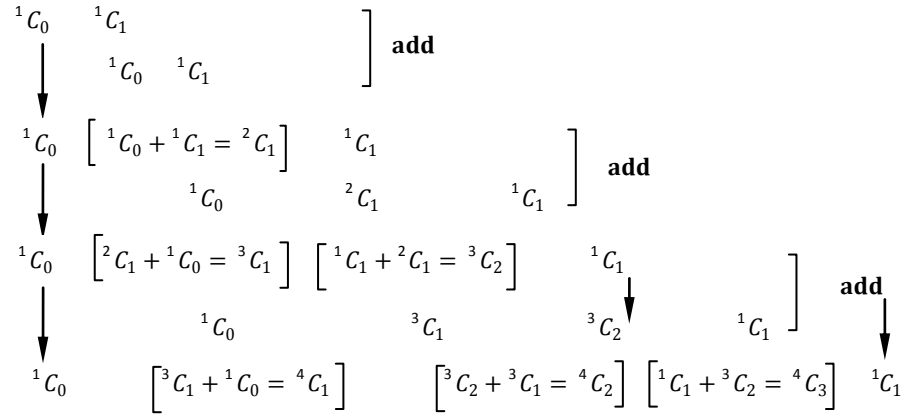
Note:

1. Superscript *must* be greater than or equal to subscript. *i.e.* for ${}^nC_r$, $n \geq r$
2. The above array is generated using ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

Simplifying the table

		COLUMN (r)					
ROW (n)	0	0	1	2	3	4	5
	0	1					
	1	1	${}^1C_1$				
	2	1	2	1			
	3	1	3	3	1		
	4	1	4	6	4	1	

From the above table, from the coefficients of each expansion, the coefficient of the next expansion can be obtained using ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$



Observation

	1	1				
		1	1			**add**
$(a+b)^2$ coeff:	**1**	**2**	**1**			
		1	2	1		**add**
$(a+b)^3$ coeff:	**1**	**3**	**3**	**1**		
		1	3	3	1	**add**
$(a+b)^4$ coeff:	**1**	**4**	**6**	**4**	**1**	
		1	4	6	4	1 **add**
$(a+b)^5$ coeff:	**1**	**5**	**10**	**10**	**5**	**1**

Pascal's triangle is

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Binomials

Definition

An algebraic expression containing two numbers/terms is a *binomial.*

Examples

$(x+y), (a+b), (3x-2y), (4a-3b)\ etc$

Pascal's triangle shall be applied to binomials raised to a real number. $i.e.\ (a+b)^n$, where n is a particular number.

$(a+b)^n$ when being expanded using Pascal's triangle has the following features:

i) There are $(n+1)$ *terms* in the expansion.
ii) The sum of the powers of 'a' and 'b' in each term equal to 'n' in the expansion.
iii) The first term of expansion is a^n and the last term is b^n.
iv) The power of 'a' decrease by 1 and subsequent terms and that of b start with 1 in the second term, and increasing by 1 in every succeeding term.

Examples

1) *Expand in descending powers of x in each of the following.*

i. $(x+6)^6$ iii. $(2x+1)^3$

ii. $\left(2x-\frac{1}{4}\right)^4$ iv. $\left(x-\frac{1}{x}\right)^5$

Solution

i. $(x+6)^6 = ?$

There are 7 *terms. i. e.*

$x^6 \quad x^5(6)^1 \quad x^4(6)^2 \quad x^3(6)^3 \quad x^2(6)^4 \quad x^1(6)^5 \quad (6)^6$

Coefficients from Pascal's triangle

1 6 15 20 15 6 1

$\Rightarrow (x+6)^6 = \mathbf{1}.x^6 + 6.x^5(6)^1 + 15.x^4(6)^2 + 20.x^3(6)^3 + \mathbf{15}.x^2(6)^4 + \mathbf{6}.x^1(6)^5 + \mathbf{1}.(6)^6$

$= x^6 + 36x^5 + 540x^4 + 4320x^3 + 19440x^2 + 46656x + 46656$

ii. $\left(2x - \frac{1}{4}\right)^4 = ?$

There are 5 *terms. i. e.*

$(2x)^4 \quad (2x)^3\left(-\frac{1}{4}\right)^1 \quad (2x)^2\left(-\frac{1}{4}\right)^2 \quad (2x)^1\left(-\frac{1}{4}\right)^3 \quad \left(-\frac{1}{4}\right)^4$

Coefficients from Pascal's triangle

1 4 6 4 1

$\Rightarrow \left(2x - \frac{1}{4}\right)^4 = \mathbf{1}.(2x)^4 + \mathbf{4}.(2x)^3\left(-\frac{1}{4}\right)^1 + \mathbf{6}.(2x)^2\left(-\frac{1}{4}\right)^2 + \mathbf{4}.(2x)^1\left(-\frac{1}{4}\right)^3 + \mathbf{1}.\left(-\frac{1}{4}\right)^4$

$= 16x^4 - 8x^3 + \frac{3}{2}x^2 - \frac{1}{8}x^3 + \frac{1}{256}$

iii. $(2x+1)^3 = ?$

There are 4 *terms. i. e.*

$(2x)^3 \quad (2x)^2(1)^1 \quad (2x)^1(1)^2 \quad (1)^3$

Coefficients from Pascal's triangle

1 3 3 1

$\Rightarrow (2x+1)^3 = \mathbf{1}.(2x)^3 + \mathbf{3}.(2x)^2(1)^1 + \mathbf{3}.(2x)^1(1)^2 + \mathbf{1}.(1)^3$

$= 8x^3 + 12x^2 + 6x + 1$

iv. $\left(x - \frac{1}{x}\right)^5 = ?$

There are 6 *terms. i. e.*

$x^5 \quad x^4\left(-\frac{1}{x}\right)^1 \quad x^3\left(-\frac{1}{x}\right)^2 \quad x^2\left(-\frac{1}{x}\right)^3 \quad x\left(-\frac{1}{x}\right)^4 \quad \left(-\frac{1}{x}\right)^5$

Coefficients from Pascal's triangle

1 5 10 10 5 1

$\Rightarrow \left(x - \frac{1}{x}\right)^5 = \mathbf{1}.x^5 + \mathbf{5}.x^4\left(-\frac{1}{x}\right)^1 + \mathbf{10}.x^3\left(-\frac{1}{x}\right)^2 + \mathbf{10}.x^2\left(-\frac{1}{x}\right)^3 + \mathbf{5}.x\left(-\frac{1}{x}\right)^4 + \mathbf{1}.\left(-\frac{1}{x}\right)^5$

$= x^5 - 5x^2 + \frac{10}{x} - \frac{10}{x^4} + \frac{\mathbf{5}}{x^7} - \frac{1}{x^{10}}$

Finding approximations by Pascal's triangle expansions

2) *Expand* $\left(3 + \frac{1}{8}x\right)^4$ *in ascending powers of x. by putting* $x = 0.2$ *in the first three terms of the expansion, find the approximate value of* $(3.025)^4$ *to* $3d.p.$

Solution

$\left(3 + \frac{1}{8}x\right)^4$ has $5 - terms. i. e.$

$(3)^4 \quad (3)^3\left(\frac{x}{8}\right) \quad (3)^2\left(\frac{x}{8}\right)^2 \quad (3)\left(\frac{x}{8}\right)^3 \quad \left(\frac{x}{8}\right)^4$

Coefficients: **1 4 6 4 1**

$\therefore \left(3 + \frac{1}{8}x\right)^4 = \mathbf{1}.(3)^4 + \mathbf{4}.(3)^3\left(\frac{x}{8}\right) + \mathbf{6}.(3)^2\left(\frac{x}{8}\right)^2 + \mathbf{4}.(3)\left(\frac{x}{8}\right)^3 + \mathbf{1}.\left(\frac{x}{8}\right)^4$

$= 81 + \frac{108}{8}x + \frac{54}{64}x^2 + \frac{12}{512}x^3 + \frac{x^4}{4096}$

$= 81 + \frac{27}{2}x + \frac{27}{32}x^2 + \frac{3}{128}x^3 + \frac{x^4}{4096}$

Taking the first 3 –terms

$\left(3 + \frac{1}{8}x\right)^4 = 81 + \frac{27}{2}x + \frac{27}{32}x^2$

For $x = 0.2$

$\Rightarrow \left(3 + \frac{1}{8} \times 0.2\right)^4 = 81 + \frac{27}{2} \times 0.2 + \frac{27}{32} \times (0.2)^2$

$\therefore (3.025)^4 \approx 83.734$

3) *Expand* $\left(1+\frac{1}{x}\right)^3$ *as far as* $\frac{1}{x^3}$. *Hence find the value of* $(1.025)^3$ *giving your answer to* $4\ s.f.$

Solution

$\left(1+\frac{1}{x}\right)^3$ has $4-terms.i.e.$

$(1)^3 \quad (1)^2\left(\frac{1}{x}\right)^1 \quad (1)^1\left(\frac{1}{x}\right)^2 \quad \left(\frac{1}{x}\right)^3$

Coefficients from Pascal's triangle: **1 3 3 1**

$\therefore \left(1+\frac{1}{x}\right)^3 = \mathbf{1}.(1)^3 + \mathbf{3}.(1)^2\left(\frac{1}{x}\right)^1 + \mathbf{3}.(1)^1\left(\frac{1}{x}\right)^2 + \mathbf{1}.\left(\frac{1}{x}\right)^3$

$= 1+\frac{3}{x}+\frac{3}{x^2}+\frac{1}{x^3}$

Now finding a suitable value of x for substitution

$\Rightarrow \left(1+\frac{1}{x}\right)^3 = (1.025)^3 \quad \therefore 1+\frac{1}{x} = 1.025 \ \ or\ x = 40$

$\therefore \left(1+\frac{1}{40}\right)^3 = (1.025)^3 = 1+\frac{3}{40}+\frac{3}{40^2}+\frac{1}{40^3} = 1.0779$

$\therefore (1.025)^3 \approx 1.077$

4) *Write down the expansion of* $\left(1+\frac{x}{4}\right)^4$ *af far as* x^3. *Hence find the value of* $(1.025)^4$ *correct to* $3d.ps.$

Solution

$\left(1+\frac{x}{4}\right)^4$ has $5-terms.i.e.$

$(1)^4 \quad (1)^3\left(\frac{x}{4}\right)^1 \quad (1)^2\left(\frac{x}{4}\right)^2 \quad (1)^1\left(\frac{x}{4}\right)^3 \quad \left(\frac{x}{4}\right)^4$

Coefficients from Pascal's triangle: **1 4 6 4 1**

$\therefore \left(1+\frac{1}{x}\right)^3 = \mathbf{1}.(1)^4 + \mathbf{4}.(1)^3\left(\frac{x}{4}\right)^1 + \mathbf{6}.(1)^2\left(\frac{x}{4}\right)^2 + \mathbf{4}.(1)^1\left(\frac{x}{4}\right)^3 + \mathbf{1}.\left(\frac{x}{4}\right)^4$

$\approx 1+x+\frac{3}{8}x^2+\frac{1}{16}x^3\ ;\ \left(\frac{x}{4}\right)^4 \ is\ ignored$

Now finding a suitable value of x for substitution

$\Rightarrow \left(1+\frac{x}{4}\right)^4 = (1.025)^4 \quad \therefore 1+\frac{x}{4} = 1.025 \ \ or\ x = 0.1$

$\therefore \left(1+\frac{0.1}{4}\right)^3 = (1.025)^4 = 1+0.1+\frac{3}{8}(0.1)^2+\frac{1}{16}(0.1)^3 = 1.104$

$\therefore (1.025)^4 \approx 1.104$

5) *Using Pascal's triangle, find the exact value of*

a) $(2.95)^3$

b) $(2.998)^4$

Solution

a) $(2.95)^3 = (3-0.05)^3 = (2+0.95)^3$ $[Use\ any\ of\ binomials\ of\ your\ choice]$

Using $(2.95)^3 = (3-0.05)^3$, has $4-terms.i.e.$

$(3)^3 \quad (3)^2(-0.05)^1 \quad (3)^1(-0.05)^2 \quad (-0.05)^3$

Coefficients from Pascal's triangle: **1 3 3 1**

$\Rightarrow (2.95)^3 = (3-0.05)^3 = \mathbf{1}.(3)^3 + \mathbf{3}.(3)^2(-0.05)^1 + \mathbf{3}.\ \ (3)^1(-0.05)^2 + \mathbf{1}.(-0.05)^3$

$= 27-1.35+0.0225-0.000125$

$= 25.672375$

b) $(2.998)^4 = (3-0.002)^4 = (2+0.998)^4$

Using $(2.998)^4 = (3-0.002)^4$, has $5-terms.i.e.$

$(3)^4 \quad (3)^3(-0.002)^1 \quad (3)^2(-0.002)^2 \quad (3)^1(-0.002)^3 \quad (-0.002)^4$

Coefficients from Pascal's triangle: **1 4 6 4 1**

$\Rightarrow (2.998)^4 = (3-0.05)^3$

$= \mathbf{1}.(3)^4 + \mathbf{4}.(3)^3(-0.002)^1 + \mathbf{6}.(3)^2(-0.002)^2 + \mathbf{4}.(3)^1(-0.002)^3 + \mathbf{1}.(-0.002)^4$

$= 81 - 0.216 + 0.000216$

$= 80.784$

Task

1) *Expand* $(2-x)^6$ *in ascending powers of* x *.Taking the first 3-terms, find the value of* $(1.998)^6$ *as accurately as possible.* ***Answer*****:** $\mathbf{63.617}$
2) *Write down the first three terms of* $(3-x)^6$*. Hence find the value of* $(2.998)^6$ *as accurately as possible.* ***Answer*****:** $\mathbf{80.784}$

Note:

1) Pascal's triangle is applicable to $(a+b)^n$ where n is relatively small. $i.e.\, n \leq 6$ and is a natural number.
2) If n is large or negative of fractional, Binomial theorem can be used.

THE BINOMIAL THEOREM

This states that if n is a particular number then

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1}b^1 + {}^nC_2 a^{n-2}b^2 + \ldots\ldots\ldots\ldots + {}^nC_n a^{n-n}b^n$$
$$= a^n + {}^nC_1 a^{n-1}b^1 + {}^nC_2 a^{n-2}b^2 + \ldots\ldots\ldots\ldots\ldots + b^n$$

However if few terms of an expansion are required the theorem is used in the form

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots + b^n$$

Note:

${}^nC_1 = \frac{n!}{(n-1)!} = n$, ${}^nC_2 = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2!}$, *and* ${}^nC_3 = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{3!}$ $e.t.c$

Putting all these results in the first stated Binomial theorem yields the second stated theorem above.

Example

Expand by Binomial theorem.

a) $\left(x+\frac{1}{x}\right)^5$ *as far as term in* $\frac{1}{x^5}$

b) $\left(3+\frac{x}{5}\right)^4$ *as far as term in* x^3

c) $\left(4+\frac{x}{8}\right)^8$ *as far as term in* x^4

Solution

a) $\left(x+\frac{1}{x}\right)^5 = ?$

Using $(a+b)^n = a^n + {}^nC_1 a^{n-1}b^1 + {}^nC_2 a^{n-2}b^2 + \ldots\ldots\ldots\ldots\ldots + b^n$

$\Rightarrow \left(x+\frac{1}{x}\right)^5 = x^5 + {}^5C_1 x^{5-1}\left(\frac{1}{x}\right)^1 + {}^5C_2 x^{5-2}\left(\frac{1}{x}\right)^2 + {}^5C_3 x^{5-3}\left(\frac{1}{x}\right)^3 + {}^5C_4 x^{5-4}\left(\frac{1}{x}\right)^4 + {}^5C_5 x^{5-5}\left(\frac{1}{x}\right)^5$

$= x^5 + \frac{5!}{4!1!}.\frac{x^4}{x} + \frac{5!}{3!2!}.\frac{x^3}{x^2} + \frac{5!}{2!3!}.\frac{x^2}{x^3} + \frac{5!}{1!4!}.\frac{x}{x^4}$

$\therefore \left(x+\frac{1}{x}\right)^5 = x^5+5x^3+10x+\frac{10}{x}+\frac{5}{x^3}+\frac{1}{x^5}$

b) $\left(3+\frac{x}{5}\right)^4 = ?$

$\left(3+\frac{x}{5}\right)^4 = (3)^4 + {}^4C_1(3)^{4-1}\left(\frac{x}{5}\right)^1 + {}^4C_2(3)^{4-2}\left(\frac{x}{5}\right)^2 + {}^4C_3(3)^{4-3}\left(\frac{x}{5}\right)^3 + {}^4C_4\left(\frac{x}{5}\right)^4$

$= 81 + \frac{4!}{3!1!}(3)^3\left(\frac{x}{5}\right) + \frac{4!}{2!2!}(3)^2\left(\frac{x}{5}\right)^2 + \frac{4!}{1!3!}(3)\left(\frac{x}{5}\right)^3 + \frac{x^4}{625}$

$\therefore \left(3+\frac{x}{5}\right)^4 = 81+\frac{108}{5}x+\frac{54}{25}x^2+\frac{12}{125}x^3$

c) $\left(4+\frac{x}{8}\right)^8 = (4)^8 + {}^8C_1(4)^7\left(\frac{x}{8}\right)^1 + {}^8C_2(4)^6\left(\frac{x}{8}\right)^2 + {}^8C_3(4)^5\left(\frac{x}{8}\right)^3 + {}^8C_4(4)^4\left(\frac{x}{8}\right)^4$

$\therefore \left(4+\frac{x}{8}\right)^8 = 65536+16384x+1792x^2+112x^3+\frac{35}{8}x^4$

The general term of Binomial expansion

Recall: $(a+b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots\ldots + {}^nC_na^{n-n}b^n$

Each term looks like ${}^nC_ra^{n-r}b^r$ where $0 \le r \le n$.

Thus the expression $\boldsymbol{{}^nC_ra^{n-r}b^r}$ is typical of each term in the expansion of $(a+b)^n$. Hence it is called **general term.**

Any specific term can be derived from the general term.

Example:

If $r = 2$, *the general term becomes;*

${}^nC_2a^{n-2}b^2 = \frac{n!}{(n-2)!2!}a^{n-2}b^2$, which is the 3rd *term* of the expansion of $(a+b)^n$.

$\therefore$ The 3rd *term* is obtained by letting $r = 2$.

General deduction

$\boldsymbol{{}^nC_ra^{n-r}b^r}$ represents $(r+1)^{th}$ *term* in the expansion of $(a+b)^n$ *for* $0 \le r \le n$

Example

1. *State the required term in the expansions written in ascending powers of x.*
   a) 10th *term in* $(1+x)^{15}$
   b) 8th *term in* $\left(2-\frac{x}{3}\right)^{12}$
   c) 4th *term in* $\left(2-\frac{3}{x}\right)^8$

Solution

Using $\boldsymbol{(a+b)^n}$ expansion with general term $\boldsymbol{{}^nC_ra^{n-r}b^r}$ for $\boldsymbol{(r+1)^{th}\ term}$

a) $(1+x)^{15}$ gives: $a = 1, b = x, n = 15, r+1 = 10 \Rightarrow r = 9$

$\therefore 10^{th}\ term = {}^{15}C_9(1)^{15-9}x^9 = \frac{15!}{(15-9)!9!}x^9$

$= \frac{151}{6!9!}x^9 = 5005x^9$ $[obtained\ from\ calculator]$

b) $\left(2-\frac{x}{3}\right)^{12}$ gives: $a = 2, b = -\frac{x}{3}, n = 12, r+1 = 8 \Rightarrow r = 7$

$\therefore 8^{th}\ term = {}^{12}C_7(2)^{12-7}\left(-\frac{x}{3}\right)^7 = \frac{12!}{5!7!}(2)^5.\frac{x^7}{(-3)^7}$

$= \frac{792\times32}{-2187}x^7 = -\frac{2816}{243}x^7$

c) $\left(2-\frac{3}{x}\right)^8$ gives: $a = 2, b = -\frac{3}{x}, n = 8, r+1 = 4 \Rightarrow r = 3$

$\therefore 4^{th}\ term = {}^{8}C_3(2)^{8-3}\left(-\frac{3}{x}\right)^3 = \frac{8!}{5!3!}(2)^5.\frac{(-3)^3}{x^3} = -\frac{48384}{x^3}$

Note:

Sometimes a coefficient of term can be required. The general term can be used to state the required coefficient.

The required coefficient of term is derived by finding a suitable *r.*

2. *Find the coefficient of* x^{10} *in the expansion of* $(2x-3)^{14}$.

Solution

From the general term $\boldsymbol{U_{r+1} = {}^{n}C_r a^{n-r}b^r}$ of expansion of $\boldsymbol{(a+b)^n}$

For $(2x-3)^{14}$, $a = 2x, b = -3, n = 14, r+1 = ? \Rightarrow r = ?$

$\Rightarrow U_{r+1} = {}^{14}C_r(2x)^{14-r}(-3)^r = {}^{14}C_r.(2)^{14-r}.(x)^{14-r}.(-3)^r$

Since term in x^{10} is required, then equating $x^{10} = (x)^{14-r}$ gives the value of r which is the key determinant of required coefficient.

$\Rightarrow 10 = 14 - r \ \therefore r = 4$

Thus the required coefficient is in the $5^{th}\ term$ of the expansion of $(2x-3)^{14}$

$\therefore U_5 = {}^{14}C_4.(2)^{14-4}.(x)^{14-4}.(-3)^4 = \frac{14!}{10!4!}.2^{10}.(-3)^4.x^{10}$

Required coefficient $= 1001 \times 2^{10} \times (-3)^4 = 83026944$

3. *Find the coefficient of* x^{-4} *in the expansion of* $\left(2-\frac{3}{x}\right)^8$.

Solution

For $\left(2-\frac{3}{x}\right)^8$, $a = 2, b = -\frac{3}{x}, n = 8, r+1 = ? \Rightarrow r = ?$

$\Rightarrow U_{r+1} = {}^{8}C_r(2)^{8-r}\left(-\frac{3}{x}\right)^r = {}^{8}C_r.(2)^{8-r}.(x)^{-r}.(-3)^r$

Finding a suitable value of *r;*

Since term in x^{-4} is required, then equating $x^{-4} = x^{-r}$ gives the value of r $.i.e.r = 4$

Thus the required coefficient is in the $5^{th}\ term$ of the expansion of $\left(2-\frac{3}{x}\right)^8$

$\therefore U_5 = {}^{8}C_4.(2)^{8-4}.(x)^{-4}.(-3)^4 = \frac{8!}{4!4!}.2^4.(-3)^4.x^{-4}$

Required coefficient $= \frac{8!}{4!4!}.2^4.(-3)^4 = 90\,720$

4. *Find the coefficient of* a^6x^{14} *in* $(a+x)^{20}$.

Solution

For $(a+x)^{20}$, $A = a, B = x, N = 20, R+1 = ? \Rightarrow R = ?$

$\Rightarrow U_{R+1} = {}^{20}C_R a^{20-R}x^R$

Finding a suitable value of *R;*

Since term in a^6x^{14} is required, then equating either $x^{14} = x^R$ giving the value of $R = 14$ or $a^6 = a^{20-R} \Rightarrow R = 14\ as\ before$

Thus the required coefficient is in the $15^{th}\ term$ of the expansion of $(a+x)^{20}$

$\therefore U_{15} = {}^{20}C_{14}a^{20-14}x^{14} = {}^{20}C_{14}.a^6x^{14}$

$\therefore coefficient = {}^{20}C_{14} = 38\,760$

5. *Write down the coefficient of the term in* x^4y^5 *in the expansion of* $(2x+3y)^9$

Solution

For $(2x+3y)^9$, $a = 2x, b = 3y, n = 9, r+1 = ? \Rightarrow r = ?$

$\Rightarrow U_{r+1} = {}^{9}C_r(2x)^{9-r}(3y)^r = {}^{9}C_r.(2)^{9-r}.(x)^{9-r}.(3)^r(y)^r$

Finding a suitable value of *r;*

Since term in x^4y^5 is required, then equating either $x^4 = x^{9-r}$ giving the value of $r = 5$ or $y^5 = y^r \Rightarrow r = 5$ *as before*

Thus the required coefficient is in the 6th *term* of the expansion of $(2x + 3y)^9$

$\therefore U_6 = {}^9C_5.(2)^{9-5}.(x)^{9-5}.(3)^5(y)^5 = {}^9C_5.(2)^{9-5}.(3)^5.x^4y^5$

$\therefore coefficient = {}^9C_5.(2)^{9-5}.(3)^5 = 489\,988$

6. *Expand* $\left(1+\frac{10}{x}\right)^{10}$ *in terms of descending powers of x as far as term in* $\frac{1}{x^3}$.

Find: *a) The* 8th *term*

b) The coefficient of x^{-8}

Solution

The expansion is left as an exercise.

a) Using $U_{r+1} = {}^nC_r a^{n-r}b^r$ for $(r+1)^{th}$ *term*

For $\left(1+\frac{10}{x}\right)^{10}$, $a = 1$, $b = \frac{10}{x}$, $n = 10$

Given: $r + 1 = 8 \ \therefore r = 7$

$\therefore \ U_8 = {}^{10}C_7(1)^{10-7}\left(\frac{10}{x}\right)^7 = {}^{10}C_7.\frac{10^7}{x^7} = \frac{10!}{7!3!}.\frac{10^7}{x^7}$

$\therefore 8^{th}\ term = \frac{1.2\times10^9}{x^7}$

b) From $U_{r+1} = {}^nC_r a^{n-r}b^r$, $a = 1$, $b = \frac{10}{x}$, $n = 10$, $r + 1 = ? \Rightarrow r = ?$

$U_{r+1} = {}^{10}C_r(1)^{10-r}\left(\frac{10}{x}\right)^r = {}^{10}C_r.10^r.x^{-r}$ $[\because (1)^{10-r} = 1]$

Finding a suitable value of *r;*

Equating $x^{-r} = x^{-8} \ \therefore r = 8$

$\Rightarrow U_9 = {}^{10}C_8.10^8.x^{-8}$

$\therefore coefficient = {}^{10}C_8.10^8 = \frac{10!}{2!8!}\times10^8 = 4.5\times10^8$

7. *Write down the middle terms of the expansions*

a) $(a+b)^{14}$

b) $(a-b)^{16}$

c) $(1+x)^9$

Solution

a) $(a+b)^{14}$, has $15-$ *terms*. Required term is the $\left(\frac{15+1}{2}\right)^{th}$ *value* $= 8^{th}$ *term*

$\left[\because For\ odd\ terms, the\ n^{th}\ term\ is\ located\ using\ \left(\frac{n+1}{2}\right)^{th}\ value\right]$

Using $U_{r+1} = {}^nC_r a^{n-r}b^r$ for $(r+1)^{th}$ *term*

$\Rightarrow 8 = r + 1 \ \therefore r = 7$

$\therefore Middle\ term = 8^{th}\ term = {}^{14}C_7a^{14-7}b^7$

$= {}^{14}C_7a^7b^7 = 3432a^7b^7$

b) $(a-b)^{16}$, has $17-$ *terms*(odd). Required term is the $\left(\frac{17+1}{2}\right)^{th}$ *value* $= 9^{th}$ *term*

The 9th *term* $= r + 1 \ \therefore r = 8$

$\Rightarrow U_9 = \ {}^{16}C_8a^{16-8}(-b)^8 = {}^{16}C_8a^8b^8 = 12\,870a^8b^8$

c) $(1+x)^9$, has $10-$ *terms*$(even)$. Hence there are $2-$ *middle terms*.

$\left[\because For\ even\ terms, the\ \frac{n^{th}}{2}\ and\ \left(\frac{n^{th}}{2}+1\right)\ are\ the\ middle\ terms\right]$

Required terms are $\left(\frac{10}{2}\right)^{th}$ *and* $\left(\frac{10}{2}+1\right)^{th}$ *terms*

For 5th *term*, $r + 1 = 5 \ \therefore r = 4$

$\Rightarrow 5^{th}\ term = {}^9C_4(1)^{9-4}(x)^4 = {}^9C_4.x^4 = 126x^4$

For $6^{th}\ term, r+1=6\ \therefore r=5$

$\Rightarrow 6^{th}\ term = {}^{9}C_5(1)^{9-5}(x)^5 = {}^{9}C_5.x^5 = 126x^5$

8. *Find the last three terms of* $(2x+y)^7$

Solution

$(2x+y)^7$ has $8-terms$

Required terms are: $8^{th}, 7^{th}\ and\ 6^{th}\ term$ each corresponding to $r-values$: $7, 6\ and\ 5$ respectively.

$\therefore 8^{th}-term = {}^{7}C_7(2x)^{7-7}y^7 = y^7$

$\therefore 7^{th}-term = {}^{7}C_6(2x)^{7-6}y^6 = 7\times 2xy^6 = 14xy^6$

$6^{th}-term = {}^{7}C_5(2x)^{7-5}y^5 = 21\times(2x)^2y^5 = 84x^2y^5$

9. *Write down the terms involving:*

a) $x^4\left(\frac{1}{x}\right)^2$

b) $x^3\left(\frac{1}{x}\right)^3$, in the expansion of $\left(x+\frac{1}{x}\right)^6$

Solution

For $\left(x+\frac{1}{x}\right)^6, a=x, b=\frac{1}{x}, n=6$

$\Rightarrow U_{r+1} = {}^{6}C_r.x^{6-r}\left(\frac{1}{x}\right)^r = {}^{6}C_r.x^{6-r}x^{-r}.(1)^r = {}^{6}C_r.x^{6-2r}.(1)^r$

a) Now term involving $x^4\left(\frac{1}{x}\right)^2 = x^4.x^{-2}.1^2 = x^2$ is equated to x^{6-2r}

$\Rightarrow x^2 = x^{6-2r}\ \therefore 2=6-2r\ From\ which\ r=2$

$\therefore 3^{rd}-term = {}^{6}C_2.x^{6-2\times2}.(1)^2 = 15x^2$

b) For term involving $x^3\left(\frac{1}{x}\right)^3 = x^3.x^{-3}(1)^3 = x^0$

Equating $x^0 = x^{6-2r} \Rightarrow 0=6-2r\ or\ r=3$

$\therefore 4^{th}-term = {}^{6}C_3.x^{6-2\times3}.(1)^3 = 20$

10. *Write down the constant terms in the expansion of:*

a. $\left(x-\frac{1}{x}\right)^8$

b. $\left(2x^2-\frac{1}{x}\right)^6$

Solution

A constant term does not contain x. finding the suitable value for r;

a. $\left(x-\frac{1}{x}\right)^8 = {}^{8}C_r.x^{8-r}.\left(\frac{1}{x}\right)^r = {}^{8}C_r.x^{8-r}.x^{-r}.(1)^r = {}^{8}C_r.x^{8-2r}.(1)^r$

For a constant term, $.x^{8-2r} = x^0 \Rightarrow 8-2r=0\ \therefore r=4$

The $constant\ term = 5^{th}\ term = {}^{8}C_4.x^{8-2\times4}.(1)^4 = {}^{8}C_4 = 70$

b. $\left(2x^2-\frac{1}{x}\right)^6 = {}^{6}C_r.(2x^2)^{6-r}.\left(-\frac{1}{x}\right)^r = {}^{8}C_r.2^{6-r}.x^{12-2r}.x^{-r}.(-1)^r$

$= {}^{8}C_r.2^{6-r}.x^{12-3r}.(-1)^r$

For a constant term, $.x^{12-3r} = x^0 \Rightarrow 12-3r=0\ \therefore r=4$

The $constant\ term = 5^{th}\ term = {}^{6}C_4.2^{6-4}.x^{12-3\times4}.(-1)^4$

$= {}^{6}C_4\times4 = 60$

11. *Find the coefficients of the terms indicated in the expansions below.*

a) $\left(x+\frac{1}{x}\right)^6, term\ in\ x^4$

b) $\left(2x+\frac{1}{x}\right)^7, term\ in\ \frac{1}{x^5}$

c) $\left(x-\frac{2}{x}\right)^8, term\ in\ x^6$

d) $\left(2+\frac{x}{2}\right)^{10}$, $term\ in\ x^7$

Solution

Using $U_{r+1} = {}^{n}C_r a^{n-r}b^r$ for $(r+1)^{th}\ term\ for\ (a+b)^n$

a) $\left(x+\frac{1}{x}\right)^6 = {}^{6}C_r.x^{6-r}.\left(\frac{1}{x}\right)^r = {}^{6}C_r.x^{6-r}.x^{-r}.(1)^r$

$= {}^{6}C_r.x^{6-2r}.(1)^r$

For term in x^4 a suitable value of r is;

$x^4 = x^{6-2r} \Rightarrow 4 = 6-2r \ \therefore r = 1$

$\therefore term\ in\ x^4 = {}^{6}C_1.x^{6-2\times1}.(1)^1 = 6x^4$

$\therefore coefficient\ in\ x^4 = 6$

b) $\left(2x+\frac{1}{x}\right)^7 = {}^{7}C_r.(2x)^{7-r}.\left(\frac{1}{x}\right)^r = {}^{7}C_r.2^{7-r}.x^{7-r}.x^{-r}.(1)^r$

$= {}^{7}C_r.2^{7-r}.x^{7-2r}.(1)^r$

For term in $\frac{1}{x^5}$ a suitable value of r is;

$x^{-5} = x^{7-2r} \Rightarrow -5 = 7-2r \ \therefore r = 6$

$\therefore term\ in\ \frac{1}{x^5} = {}^{7}C_6.x^{7-2\times6}.(1)^6 = 14x^{-5}$

$\therefore coefficient\ in\ \frac{1}{x^5} = 14$

c) $\left(x-\frac{2}{x}\right)^8 = {}^{8}C_r.x^{8-r}.\left(-\frac{2}{x}\right)^r = {}^{8}C_r.x^{8-r}.(-2)^r x^{-r}$

$= {}^{8}C_r.(-2)^r.x^{8-2r}$

For term in x^6 a suitable value of r is;

$x^6 = x^{8-2r} \Rightarrow 6 = 8-2r \ \therefore r = 1$

$\therefore term\ in\ x^6 = {}^{8}C_1.x^{8-2\times1}.(-2)^1 = -16x^6$

$\therefore coefficient\ in\ x^6 = -16$

d) $\left(2+\frac{x}{2}\right)^{10} = {}^{10}C_r.2^{10-r}.\left(\frac{x}{2}\right)^r = {}^{10}C_r.x^r.2^{10-r}.\left(\frac{1}{2}\right)^r$

$= {}^{10}C_r.2^{10-2r}.(1)^r.x^r$

For term in x^7 a suitable value of r is;

$x^7 = x^r \ \therefore r = 7$

$\therefore term\ in\ x^7 = {}^{10}C_7.x^7.2^{10-2\times7} = {}^{10}C_7.2^{-4}.x^7 = \frac{15}{2}x^7$

$\therefore coefficient\ in\ x^7 = \frac{15}{2}$

The binomial expansion of $(1+x)^n$

From $(a+b)^n = a^n + {}^{n}C_1 a^{n-1}b^1 + {}^{n}C_2 a^{n-2}b^2 + \ldots\ldots\ldots\ldots\ldots\ldots + b^n$, $if\ a = 1, b = x$, then;

$(1+x)^n = 1^n + {}^{n}C_1(1)^{n-1}x^1 + {}^{n}C_2(1)^{n-2}x^2 + \ldots\ldots\ldots\ldots\ldots\ldots + x^n$

$= 1 + {}^{n}C_1 x + {}^{n}C_2 x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + x^n$

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + x^n \quad (1)$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + x^n \quad (2)$$

NOTE:

1) The expansion of $(1+x)^n$ has $(n+1) - terms.$
2) The term in x^2 is the $3^{rd} - term$, the term in x^3 is the $4^{th} - term$ and the term in x^r is the $(r+1)^{th} - term.$

$$\therefore r^{th} - term = \binom{n}{r-1}.x^{r-1} = {}^{n}C_{r}.x^{r}$$

3) The coefficient of $x^r = {}^{n}C_r = \binom{n}{r}$

4) The form of the expansion given in (1) is useful when the coefficient of a larger power of x is required, or when general term is required.
The form of the expansion given in (2) is useful when the first few terms of an expansion are required.

5) The expansion of $(a+x)^n$ where n is a positive integer can be done as;

$(a+b)^n = \left[a\left(1+\frac{x}{a}\right)\right]^n = a^n\left(1+\frac{x}{a}\right)^n$

Replacing x *by* $\frac{x}{a}$ in the binomial expansion series gives;

$(a+x)^n = a^n\left[1+\binom{n}{1}\left(\frac{x}{a}\right)+\binom{n}{2}\left(\frac{x}{a}\right)^2+\cdots\ldots\ldots+\binom{n}{r}\left(\frac{x}{a}\right)^r+\cdots\ldots\ldots\ldots\ldots+\binom{n}{n}\left(\frac{x}{a}\right)^n x^n\right]$

$= a^n+\binom{n}{1}a^{n-1}x+\binom{n}{2}a^{n-2}x^2+\cdots\ldots\ldots\ldots+\binom{n}{r}a^{n-r}x^r+\cdots\ldots\ldots+\binom{n}{n}a^n x^n$

$$(a+x)^n = a^n+\binom{n}{1}a^{n-1}x+\binom{n}{2}a^{n-2}x^2+\ldots\ldots\ldots\ldots+\binom{n}{r}a^{n-r}x^r+\ldots\ldots\ldots+\binom{n}{n}a^n x^n$$

$$(a+x)^n = a^n+na^{n-1}x+\frac{n(n-1)}{2!}a^{n-2}x^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+x^n$$

Example:

1) *Write down the first* $3-$*terms in the expansion in ascending powers of* x

a) $\left(1-\frac{x}{2}\right)^{10}$

b) $(3-2x)^8$

Solution

Using $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+x^n$

Replacing x *by* $-\frac{x}{2}$

a) $\left(1-\frac{x}{2}\right)^{10} = 1+10\left(-\frac{x}{2}\right)+\frac{10\times9}{2!}\left(-\frac{x}{2}\right)^2+\ldots\ldots\ldots\ldots\ldots$

$= 1-5x+\frac{45}{4}x^2+\ldots\ldots\ldots\ldots\ldots$

b) Using $(a+x)^n = a^n+na^{n-1}x+\frac{n(n-1)}{2!}a^{n-2}x^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots+x^n$

Replacing x *by* $-2x$

$\Rightarrow (3-2x)^8 = 3^8+8(3)^7(-2x)+\frac{8(7)}{2!}(3)^6(-2x)^2+\ldots\ldots\ldots\ldots\ldots\ldots$

$= 3^8-16(3)^7x+112(3)^6x^2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

2) *Write down the binomial expansion in ascending powers of* x *as far as and including the term in* x^2 *for each of the following;*

a) $(1+x)(1-x)^9$

b) $(1-x)(1+2x)^{10}$

c) $(2+x)\left(1-\frac{x}{20}\right)^{20}$

d) $(1+x)^2(1-5x)^{14}$

e) $(1+x-2x^2)^8$

f) $(1-x+2x^2)^5$

Solution

a) $(1+x)(1-x)^9 = ?$

Expanding $(1-x)^9$ as far as x^2

$(1-x)^9 = [1+(-x)]^9 = 1 + 9(-x) + \frac{9(8)}{2!}(-x)^2 + \ldots\ldots\ldots\ldots\ldots$

$= 1 - 9x + 36x^2 + \ldots\ldots\ldots$

$\therefore (1+x)(1-x)^9 = (1+x)(1-9x+36x^2)$

$= 1(1-9x+36x^2) + x(1-9x+36x^2)$

$= 1 - 9x + 36x^2 + x - 9x^2 + \ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore (1+x)(1-x)^9 = 1 - 8x + 27x^2 \quad [terms\ in\ x^3\ are\ ignored]$

b) $(1-x)(1+2x)^{10}$

Expanding $(1+2x)^{10}$ as far as x^2

$(1+2x)^{10} = 1 + 10(2x) + \frac{10(9)}{2!}(2x)^2 + \ldots\ldots\ldots\ldots\ldots$

$= 1 + 20x + 180x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore (1-x)(1+2x)^{10} = (1-x)(1+20x+180x^2)$

$= 1(1+20x+180x^2) - x(1+20x+180x^2)$

$= 1 + 20x + 180x^2 - x - 20x^2 + \ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore (1+x)(1-x)^9 = 1 + 19x + 160x^2 \quad [terms\ in\ x^3\ are\ ignored]$

c) $(2+x)\left(1-\frac{x}{20}\right)^{20}$

Expanding $\left(1-\frac{x}{20}\right)^{20}$ as far as x^2

$\left(1-\frac{x}{20}\right)^{20} = \left[1+\left(-\frac{x}{20}\right)\right]^{20} = 1 + 20\left(-\frac{x}{20}\right) + \frac{20(19)}{2!}\left(-\frac{x}{20}\right)^2 + \ldots\ldots\ldots\ldots$

$= 1 - x + \frac{19}{40}x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore (2+x)\left(1-\frac{x}{20}\right)^{20} = (2+x)\left(1-x+\frac{19}{40}x^2\right)$

$= 2\left(1-x+\frac{19}{40}x^2\right) - x\left(1-x+\frac{19}{40}x^2\right)$

$= 2 - 2x + \frac{38}{40}x^2 + x - x^2 + \ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore (2+x)\left(1-\frac{x}{20}\right)^{20} = 2 - x - \frac{1}{20}x^2$

d) $(1+x)^2(1-5x)^{14}$

Expanding $(1+x)^2$ as far as x^2

$(1+x)^2 = 1 + 2x + \frac{2(1)}{2!}(x)^2 = 1 + 2x + x^2$

Expanding $(1-5x)^{14}$ as far as x^2

$(1-5x)^{14} = [1+(-5x)]^{14} = 1 + 14(-5x) + \frac{14(13)}{2!}(-5x)^2$

$= 1 - 70x + 2275x^2$

$\therefore (1+x)^2(1-5x)^{14} = (1+2x+x^2)(1-70x+2275x^2)$

$= 1(1-70x+2275x^2) + 2x(1-70x+2275x^2) + x^2(1-70x+2275x^2)$

$= 1 - 70x + 2275x^2 + 2x - 140x^2 + x^2[terms\ in\ x^3\ and\ higher\ power\ terms\ ignored]$

$\therefore (1+x)^2(1-5x)^{14} = 1 - 68x + 2136x^2$

e) $(1+x-2x^2)^8$

$(1+x-2x^2)^8 = [(1+x)-2x^2]^8 = [(1+x)+(-2x^2)]^8$

Using $(a+x)^n = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n}a^n x^n$

Replacing $a\ by\ (1+x)$ and $x\ by\ (-2x^2)$

$[(1+x)+(-2x^2)]^8 = (1+x)^8 + \binom{8}{1}(1+x)^7(-2x^2) + \binom{8}{2}(1+x)^6(-2x^2)^2 + \ldots\ldots$

Now expanding $(1+x)^8$ and $(1+x)^7$ as far as term in x^2

$(1+x)^8 = 1 + 8x + \frac{8(7)}{2!}x^2 = 1 + 8x + 28x^2$

$(1+x)^7 = 1 + 7x + \frac{7(5)}{2!}x^2 = 1 + 7x + 21x^2$

The $3^{rd} - term$ in the expansion of $(1 + x - 2x^2)^8$ yield higher power terms greater than 2

$\therefore (1 + x - 2x^2)^8 = (1 + 8x + 28x^2\ ) + \binom{8}{1}(1 + 7x + 21x^2)(-2x^2)$

$= 1 + 8x + 28x^2 + \binom{8}{1}(-2x^2)$ $[x^3\ and\ higher\ power\ terms\ neglected]$

$= 1 + 8x + 28x^2 - 16x^2$

$\therefore (1 + x - 2x^2)^8 = 1 + 8x + 12x^2$

f) $(1 - x + 2x^2)^5$

$(1 - x + 2x^2)^5 = [(1 - x) + 2x^2]^8 = (1 - x)^5 + \binom{5}{1}(1 - x)^4(2x^2) + \ldots\ldots\ldots$

$[other\ terms\ neglected\ because\ yiel\ unwanted\ tems]$

Now expanding $(1 - x)^5$ and $(1 - x)^4$ as far as term in x^2

$(1 - x)^5 = 1 - 5x + \frac{5(4)}{2!}x^2 = 1 - 5x + 10x^2$

$(1 - x)^4 = 1 - 4x + \frac{4(3)}{2!}x^2 = 1 - 4x + 6x^2$

$\therefore (1 - x + 2x^2)^5 = (1 - 5x + 10x^2\ ) + \binom{5}{1}(1 - 4x + 6x^2)(2x^2)$

$= 1 - 5x + 10x^2 + 10x^2(1 - 4x + 6x^2)$

$= 1 - 5x + 10x^2 + 10x^2$

$[x^3\ and\ higher\ power\ terms\ neglected]$

$\therefore (1 - x + 2x^2)^8 = 1 - 5x + 20x^2$

Task

Expand the following as far as term in x^3

a) $(1 - x - x^2)^4$
b) $(3 + x + x^3)^4$
c) $(3 - 2x + 2x^2)^4$
d) $(1 - x + x^2)^8$

ANSWER:

a) $1 - 4x + 2x^2 + 8x^3$
b) $81 + 108x + 54x^2 + 120x^3$
c) $81 - 216x + 324x^2 - 312x^3$
d) $1 - 8x + 36x^2 - 112x^3$

USING BINOMIAL EXPANSION TO FIND APPROXIMATIONS

1) *If x is so small that x^2 and higher power terms can be neglected, show that*

a) $(1 - x)^5\left(2 + \frac{x}{2}\right)^{10} \cong 2^9(2 - 5x)$
b) $(2x + 3)(1 - 2x)^{10} \cong 3 - 58x + 500x^2$
c) $\frac{x+3}{(1+x)^2} \cong 3 - 5x + 7x^2 - 9x^3$
d) $\frac{1+x}{1-x} \cong 1 + 2x + 2x^2 + 2x^3$

Solution

a) $(1 - x)^5\left(2 + \frac{x}{2}\right)^{10} \cong 2^9(2 - 5x)$

Since the **R.H.S** contains terms with highest power as *one $i.e.\ x^1$, then, ignoring terms in x^2 and higher power terms*

$(1 - x)^5 = 1 - 5x$

$\left(2+\frac{x}{2}\right)^{10} = 2^{10} + \binom{10}{1}2^9.\left(\frac{x}{2}\right) = 2^{10} + 5.2^9x$

$\therefore (1-x)^5\left(2+\frac{x}{2}\right)^{10} \cong (1-5x)(2^{10}+5.2^9x)$

$\cong 2^9[(1-5x)(2+5x)] \quad \because [2^9\ is\ factorised\ out]$

$\cong 2^9[2+5x-10x] \quad [x^2 - term\ ignored]$

$\therefore (1-x)^5\left(2+\frac{x}{2}\right)^{10} \cong 2^9(2-5x)$

b) $(2x+3)(1-2x)^{10} \cong 3-58x+500x^2$

Since the **R.H.S** contains terms with highest power as *two*. $i.e.\ x^2$, *then, ignoring terms in* x^3 *and higher power terms*

$(1-2x)^{10} = 1+10(-2x)+\frac{10(9)}{2!}(-2x)^2$

$= 1-20x+180x^2$

$\therefore (2x+3)(1-2x)^{10} \cong (2x+3)(1-20x+180x^2)$

$\cong 2x-40x^2+3-60x+540x^2$

$\therefore (2x+3)(1-2x)^{10} \cong 3-58x+500x^2$

c) $\frac{x+3}{(1+x)^2} \cong 3-5x+7x^2-9x^3$

$\frac{x+3}{(1+x)^2} = (x+3)(1+x)^{-2}$

Expanding $(1+x)^{-2}$ as far as term in x^3 neglecting higher power terms

$\Rightarrow (1+x)^{-2} = 1-2x+\frac{(-2)(-3)}{2!}x^2+\frac{(-2)(-3)(-4)}{3!}x^3$

$= 1-2x+3x^2-4x^3$

$\therefore \frac{x+3}{(1+x)^2} \cong (x+3)(1-2x+3x^2-4x^3) \cong x-2x^2+3x^3+3-6x+9x^2-12x^3$

$\therefore \frac{x+3}{(1+x)^2} \cong 3-5x+7x^2-9x^3$

d. $\frac{1+x}{1-x} \cong 1+2x+2x^2+2x^3$

$\frac{1+x}{1-x} = (1+x)(1-x)^{-1} \approx 1+2x+2x^2+2x^3$

Expanding $(1-x)^{-1}$ as far as term in x^3 neglecting higher power terms

$\Rightarrow (1-x)^{-1} = 1+x+\frac{(-1)(-2)}{2!}x^2+\frac{(-1)(-2)(-3)}{3!}x^3 = 1+x+x^2+x^3$

$\therefore \frac{1+x}{1-x} \cong (1+x)(1+x+x^2+x^3)$

$\cong 1+x+x^2+x^3+x+x^2+x^3 \quad [x^4 - term\ neglected]$

$\therefore \frac{1+x}{1-x} \cong 1+2x+2x^2+2x^3$

NOTE:

In *all* the above examples on binomial theorem, the power on the brackets is an integer not a rational number. *i.e.* of the form $\frac{p}{q}$.

EXPANSION OF $(1+x)^n$ *if* $|x|<1$ **AND n IS RATIONAL**

(***i.e. of the form*** p/q)

The binomial theorem used is;

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+ \ldots\ldots\ldots\ldots$$

Note:

$|x|<1\ means -1<x<+1\ (i.e.\ validity\ range\ of\ x\ for\ expansion)$

Application

- Expansion of expressions
- Finding the nth *root* of a number
- Finding errors in numbers

Examples

1) *Expand* $(1-3x)^{\frac{1}{4}}$ *in ascending powers of* x *as far as the term in* x^3. *Hence evaluate* $\sqrt[4]{13}$ *correct to* $3\ s.f$.

Solution

Using $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots$ for $|x| < 1$

$\Rightarrow (1-3x)^{\frac{1}{4}} = 1 + \frac{1}{4}(-3x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-3x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{3}\right)}{3!}(-3x)^3 + \ldots.$

$= 1 - \frac{3}{4}x - \frac{27}{32}x^2 - \frac{189}{128}x^3$

$(1-3x)^{\frac{1}{4}}$ is valid for expansion if $|-3x| < 1$ *or* $|3x| < 1$ *or* $-1 < 3x < 1$

$-\frac{1}{3} < x < \frac{1}{3}$ is the validity interval for expansion.

Since the value of x to be used to estimate $\sqrt[4]{13}$ in the expansion in the range $-\frac{1}{3} < x < \frac{1}{3}$,then

$\sqrt[4]{13} = \sqrt[4]{(16-3)}$ $[\because 13\ is\ expressed\ as\ a\ differencce\ of\ a\ no.\ with\ exact\ 4^{th}\ root]$

$= \sqrt[4]{(16-3)} = \sqrt[4]{16\left(1-\frac{3}{16}\right)} = 2\left(1-\frac{3}{16}\right)^{\frac{1}{4}}$

Now expressing x as a rational for the interval $-\frac{1}{3} < x < \frac{1}{3}$

Comparing $\sqrt[4]{\left(1-\frac{3}{16}\right)}$ *with* $\sqrt[4]{(1-3x)}$

$\Rightarrow \frac{3}{16} \equiv 3x \ \therefore x = \frac{1}{16}$, which is in the range $-\frac{1}{3} < x < \frac{1}{3}$

From $\sqrt[4]{(1-3x)} = 1 - \frac{3}{4}x - \frac{27}{32}x^2 - \frac{189}{128}x^3$, for $x = \frac{1}{16}$, then

$\sqrt[4]{\left(1-\frac{3}{16}\right)} \cong 1 - \frac{3}{4}\left(\frac{1}{16}\right) - \frac{27}{32}\left(\frac{1}{16}\right)^2 - \frac{189}{128}\left(\frac{1}{16}\right)^3 \cong 0.94946812$

$\therefore \sqrt[4]{13} = 2 \times 0.94946812 \approx 1.90$

2) *Expand* $(1-16x)^{\frac{1}{4}}$ *as far as* x^3. *Hence evaluate* $\sqrt[4]{39}$ *correct to* $5.s.f$. *Take* $x = \frac{1}{10000}$

Solution

$(1-16x)^{\frac{1}{4}} = 1 + \frac{1}{4}(-16x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-16x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{3}\right)}{3!}(-16x)^3 + \ldots.$

$= 1 - 4x - 24x^2 - 224x^3$

$\Rightarrow \left(1-\frac{16}{10000}\right)^{\frac{1}{4}} = 1 - \frac{4}{10000} - \frac{24}{10000^2} - \frac{224}{10000^3}$

$\left(\frac{9984}{10000}\right)^{\frac{1}{4}} \cong 0.99996$

$\frac{\sqrt[4]{256\times39}}{10} \cong 0.99996 \ \therefore \sqrt[4]{39} \cong \frac{0.99996\times10}{\sqrt[4]{254}} \cong \frac{0.99996\times10}{4}$

$\therefore \sqrt[4]{39} \cong 2.4999$

NOTE:

1) In general, if x is small so that successive powers of x quickly become negligible in value, then the sum of the first few terms in the expansion of $(1+x)^n$ gives the approximate value for $(1+x)^n$. The number of terms required to obtain a good approximation depends on;
   a) The value of x (the smaller x is, the fewer are the terms needed to obtain a good approximation)
   b) The accuracy required (an answer correct to 3 s.f needs fewer terms than the answer to 6 s.f)
2) When finding an approximation, the binomial expansion of $(1+x)^n$ and not $(a+x)^n$ should be used if n is rational.*e.g.* Find an approximate value of $(3.006)^{\frac{1}{5}}$. here use $3^{\frac{1}{5}}(1+0.002)^{\frac{1}{5}}$
3) *Expand* $(1-x)^{\frac{1}{3}}$ *in ascending powers of x as far as the fourth term. Taking* $x=\frac{1}{1000}$ *, find* $\sqrt[3]{37}$ *to* $6.sf$.

   Solution

   $$(1-x)^{\frac{1}{3}}=[1+(-x)]^{\frac{1}{3}}=1+\frac{1}{3}(-x)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(-x)^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(-x)^3$$
   $$=1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3$$

   Putting $x=\frac{1}{1000}$

   $$\Rightarrow\left(1-\frac{1}{1000}\right)^{\frac{1}{3}}=1-\frac{1}{9\times10^6}-\frac{5}{81\times10^9}=\frac{\sqrt[3]{999}}{10}=\frac{2998}{2999}$$

   $$\therefore\sqrt[3]{999}=\sqrt[3]{27\times37}=\frac{2998}{2999}$$

   $$\therefore\sqrt[3]{37}=\frac{29980}{2999\times\sqrt[3]{27}}\cong3.33222$$

4) *Expand* $(1+8x)^{\frac{1}{2}}$ *as far as* x^2*. Hence find* $\sqrt{3}$ *to* $3.p.$

   Solution

   $$(1+8x)^{\frac{1}{2}}\cong1+\frac{1}{2}(8x)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(8x)^2$$
   $$\cong1+4x-8x^2$$

   Since the expansion is valid for $|8x|<1\ or\ -1<8x<+1\ or\ -\frac{1}{8}<x<\frac{1}{8}$, x' s suitable substitution should be selected to find $\sqrt{3}$ but with in this interval.

   From $\sqrt{3}=\sqrt{(4-1)}=2\sqrt{\left(1-\frac{1}{4}\right)}$

   Comparing $1+8x\equiv1-\frac{1}{4}\Rightarrow x=-\frac{1}{32}$

   From $(1+8x)^{\frac{1}{2}}\cong1+4x-8x^2$, for $x=-\frac{1}{32}$

   $$\sqrt{\left(1-\frac{1}{4}\right)}\cong1-\frac{1}{8}-\frac{8}{32^2}$$

   $$\therefore\frac{\sqrt{3}}{2}\cong\frac{111}{128}\ or\ \sqrt{3}\cong\frac{111}{64}$$

   Finding suitable value of x for substitution

   Let $x=\frac{1}{a}$, where 'a' has an exact square root

   $\Rightarrow 1+8x=1+\frac{8}{a}=\frac{a+8}{a}=3N$, N = no. also with exact sq. root

   Using try and error method

For $a = 4,\ 1 + 8x = \frac{4+8}{4} = \frac{12}{4} = \frac{3\times4}{4}$. This is convenient but $-\frac{1}{8} < x < \frac{1}{8}$, hence $x = \frac{1}{4}$ is not in the range hence rejected.

For $a = 16,\ 1 + 8x = \frac{16+8}{16} = \frac{24}{16} = \frac{3\times8}{16}$, 8 has no exact square root

For $a = 25, 1 + 8x = \frac{25+8}{25} = \frac{33}{25} = \frac{3\times11}{25}$, 25 has no exact square root

For $a = 100, 1 + 8x = \frac{100+8}{100} = \frac{108}{100} = \frac{3\times36}{100}$, 36 has an exact square root

$\therefore x = \frac{1}{100}\ for -\frac{1}{8} < x < \frac{1}{8}$ can be use.

From $(1 + 8x)^{\frac{1}{2}} \cong 1 + 4x - 8x^2\ , x = \frac{1}{100}$

$\left(1 + \frac{8}{100}\right)^{\frac{1}{2}} \cong 1 + \frac{4}{100} - \frac{8}{100^2}$

$\frac{\sqrt{108}}{10} \cong 1.0392 \therefore \sqrt{3} \cong \frac{10.392}{\sqrt{36}} \cong 1.732$

5) *Expand* $(1 - 3x)^{\frac{1}{4}}$ *in ascending powers of x up to the term in* x^3*. Hence evaluate* $\sqrt[4]{13}$

Solution

$$(1 - 3x)^{\frac{1}{4}} \cong 1 + \frac{1}{4}(-3x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-3x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(-3x)^3$$
$$\cong 1 - \frac{3}{4}x - \frac{27}{32}x^2 - \frac{819}{128}x^3$$

Now to find $\sqrt[4]{13}$, we need a suitable value of x-value in the range $|-3x| < 1\ or -\frac{1}{3} < x < \frac{1}{3}$

Br try and error method

Let $x = \frac{1}{a}\ , a = no.with\ exact\ 4^{th} root$

$\Rightarrow 1 - 3x = 1 - \frac{3}{a} = \frac{a-3}{a} = \frac{13N}{a}\ .N = no.with\ exact\ 4^{th} root$

For $a = 16\ , 1 - 3x = \frac{16-3}{16} = \frac{13}{16} = \frac{13(1)^4}{16}$

Hence $x = \frac{1}{16}$ can be used

$$(1 - 3x)^{\frac{1}{4}} \cong 1 + \frac{1}{4}(-3x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-3x)^2 + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(-3x)^3$$

$$\left(1 - \frac{3}{16}\right)^{\frac{1}{4}} \cong 1 - \frac{3}{4}\left(\frac{3}{16}\right) - \frac{27}{32}\left(\frac{3}{16}\right)^2 - \frac{819}{128}\left(\frac{3}{16}\right)^3$$
$$\cong 0.9495$$

$\therefore \sqrt[4]{13} \cong 0.9495 \times \sqrt[4]{16} \cong 1.899$

6) *Expand* $(1 - x)^{\frac{1}{3}}$ *in ascending powers of x as far as* 4th *term. By taking the first two terms of x find the value of* $\sqrt[3]{37}$ *, correct to* $6\ s.f$.

Solution

$$(1 - x)^{\frac{1}{3}} \cong 1 + \frac{1}{3}(-x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(-x)^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(-x)^3$$

$\cong 1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3$

Finding a suitable value of x for approximation of $\sqrt[3]{37}$

Let $x=\frac{1}{a}$, $a = no.with\ exact\ cube\ root$

$\Rightarrow 1-x=1-\frac{1}{a}=\frac{a-1}{a}=\frac{37N}{a}$. $N = no.with\ exact\ cube\ root$

For $a=8$, $1-x=\frac{8-1}{8}=\frac{7}{8}$, this does not contain 37

For $a=64$, $1-x=\frac{63}{64}$, which does not contain 37

⋮

For $a=1000$, $1-x=\frac{999}{1000}=\frac{27\times37}{1000}$, this holds

Taking $x=\frac{1}{1000}$ in the interval $-1<x<1$

$(1-x)^{\frac{1}{3}}\cong 1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3$

$\left(1-\frac{1}{1000}\right)^{\frac{1}{3}}\cong 1-\frac{1}{3}\left(\frac{1}{1000}\right)\cong\frac{2999}{3000}\ \cong\frac{\sqrt[3]{999}}{10}\cong\frac{2999}{3000}$

$\sqrt[3]{37\times27}\cong\frac{2999}{300}$

$\therefore\sqrt[3]{37}\cong\frac{2999}{300}\times\frac{1}{3}\cong 3.33222$

NOTE:

Examples (4), (5) and (6) illustrate how you can use try and error method to obtain a suitable value of x for approximation of nth root using binomial expansion.

See example below.

7) *Expand* $(1+x)^{\frac{1}{3}}$ *up to term in* x^2. *Hence evaluate* $\sqrt[3]{9}$ *correct to* 3 $d.p.$

Solution

$(1+x)^{\frac{1}{3}}\cong 1+\frac{1}{3}x+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}x^2\ \cong 1+\frac{1}{3}x-\frac{1}{9}x^2$

To find $\sqrt[3]{9}$, a suitable x value must be got.

Let $x=\frac{1}{a}$, $a = no.with\ exact\ cube\ root$

$\Rightarrow 1+x=1+\frac{1}{a}=\frac{a+1}{a}=\frac{9N}{a}$. $N = no.with\ exact\ cube\ root$

For $a=8$, $1+x=\frac{8+1}{8}=\frac{9}{8}=\frac{9(1)^3}{8}$, hence $x=\frac{1}{8}$ can be used

$(1+x)^{\frac{1}{3}}\cong 1+\frac{1}{3}x-\frac{1}{9}x^2$

$\left(1+\frac{1}{8}\right)^{\frac{1}{3}}\cong 1+\frac{1}{3}\left(\frac{1}{8}\right)-\frac{1}{9}\left(\frac{1}{8}\right)^2\cong\frac{601}{576}$

$\therefore\sqrt[3]{9}\cong\frac{601}{576}\times 2\cong 2.087$

Task:

Expand $(1-3x)^{\frac{1}{3}}$ in ascending powers of x a far as the term in x^3. Hence evaluate $\sqrt[3]{5}$

Hint: $x=\frac{1}{8}$

More examples on Binomial theorem

1. *Use binomial theorem to find*
   a) $\sqrt{1.001}$ *to* $6.p$
   b) $\frac{1}{(1.02)^2}$ *to* $4d.p$
   c) $\sqrt{0.998}$ *to* $6.p$
   d) $\frac{1}{\sqrt{0.98}}$ *to* $4d.p$

Solution

a) $\sqrt{1.001} = (1+0.001)^{\frac{1}{2}}$; *let* $x = 0.001$

$(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

Now $x = 0.001$

$\Rightarrow (1+0.001)^{\frac{1}{2}} \cong 1 + \frac{1}{2}(0.001) - \frac{1}{8}(0.001)^2 + \frac{1}{16}(0.001)^3 \cong 1.000500$

b) $\frac{1}{(1.02)^2} = \frac{1}{(1.02)^{-2}} = (1+0.02)^{-2}$; *let* $x = 0.02$

$(1+x)^{-2} \cong 1 - 2x + \frac{-2(-3)}{2!}x^2 + \frac{-2(-3)(-4)}{3!}x^3 \cong 1 - 2x + 3x^2 - 4x^3$

Now $x = 0.02$

$\Rightarrow (1+0.02)^{-2} \cong 1 - 2(0.02) + 3(0.02)^2 - 4(0.02)^3$

$\therefore \frac{1}{(1.02)^2} \cong 0.9612$

c) $\sqrt{0.998} = \sqrt{1-0.002} = (1-0.002)^{\frac{1}{2}}$

$(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

Now $x = -0.002$

$\Rightarrow (1-0.002)^{\frac{1}{2}} \cong 1 + \frac{1}{2}(-0.002) - \frac{1}{8}(-0.002)^2 + \frac{1}{16}(-0.002)^3$

$\therefore \sqrt{0.998} \cong 0.998999$

d) $\frac{1}{\sqrt{0.98}} = (0.98)^{-\frac{1}{2}} = (1-0.02)^{-\frac{1}{2}}$; let x $= -0.02$

$(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

Now x $= -0.02$

$\Rightarrow (1-0.02)^{\frac{1}{2}} \cong 1 + \frac{1}{2}(-0.02) - \frac{1}{8}(-0.02)^2 + \frac{1}{16}(-0.02)^3$

$\therefore \frac{1}{\sqrt{0.98}} \cong 1.0102$

Task:

Use binomial theorem to find $\sqrt[3]{1.03}$ *correct to 5s.f*

2. *Use Binomial theorem to evaluate*
   a) $\sqrt{23}$ to 3 d. p
   b) $\sqrt{37}$ to 5 s. f
   c) $\sqrt[3]{8.01}$ to 5 s. f
   d) $\sqrt[3]{64.08}$ to 5 s. f

Solution

a) $\sqrt{23} = \sqrt{25-2} = \sqrt{25\left(1-\frac{2}{25}\right)} = 5\left(1-\frac{2}{25}\right)^{\frac{1}{2}}$

From $(1+x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

$\therefore \sqrt{23} \cong 5\left[1+\frac{1}{2}\left(-\frac{2}{25}\right)-\frac{1}{8}\left(-\frac{2}{25}\right)^2\right]$ $[higher\ power\ terms\ ignored]$

$\cong 4.796$

NOTE:

$\left|-\frac{2}{25}\right| < 1\ .i.e. -1 < \frac{2}{25} < 1$

b) $\sqrt{37} = \sqrt{36+1} = \sqrt{36\left(1+\frac{1}{36}\right)} = 6\left(1+\frac{1}{36}\right)^{\frac{1}{2}}$

$\therefore \sqrt{37} \cong 6\left[1+\frac{1}{2}\left(\frac{1}{36}\right)-\frac{1}{8}\left(\frac{1}{36}\right)^2\right]$ $[higher\ power\ terms\ ignored] \cong 6.0828$

NOTE:

$\sqrt{36+1}$, could not be expanded directly by binomial because $|36| > 1$ yet the required interval of $(1+x)^n$ expansion is $|x| < 1$.

c) $\sqrt[3]{8.01} = \sqrt[3]{8+0.01} = \sqrt[3]{8\left(1+\frac{0.01}{8}\right)} = 2\left(1+\frac{0.01}{8}\right)^{\frac{1}{3}}$

$\cong 2\left[1+\frac{1}{3}\left(\frac{0.01}{8}\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{0.01}{8}\right)^2\right]$

$\therefore \sqrt[3]{8.01} \cong 2\left[1+\frac{0.01}{24}\right] \cong 2.0008$

d) $\sqrt[3]{64.08} = \sqrt[3]{64+0.08} = \sqrt[3]{64\left(1+\frac{0.08}{64}\right)} = 4\left(1+\frac{0.08}{64}\right)^{\frac{1}{3}}$

$\therefore \sqrt[3]{64.08} \cong 4\left[1+\frac{0.08}{64\times 3}\right]$ $[higher\ power\ terms\ ignored] \cong 4.0017$

Proofs in Binomial expansion

1. *Show that, if x is small enough for terms in x^3 and higher power terms neglected,*

$\sqrt{\left(\frac{1-x}{1+x}\right)} \cong 1-x+\frac{x^2}{2}$

By putting $x = \frac{1}{8}$, show that $\sqrt{7} \cong 2\frac{83}{128}$

Solution

$\sqrt{\left(\frac{1-x}{1+x}\right)} = [(1-x)(1+x)^{-1}]^{\frac{1}{2}} = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$

$(1-x)^{\frac{1}{2}} \cong 1+\frac{1}{2}(-x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-x)^2$

$\cong 1+\frac{1}{2}x-\frac{1}{8}x^2$

$(1+x)^{-\frac{1}{2}} \cong 1-\frac{1}{2}x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2$

$\cong 1-\frac{1}{2}x+\frac{3}{8}x^2$

$\therefore \sqrt{\left(\frac{1-x}{1+x}\right)} \cong \left[1+\frac{1}{2}x-\frac{1}{8}x^2\right]\left[1-\frac{1}{2}x+\frac{3}{8}x^2\right]$

$\cong 1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^2 \cong 1-x+\frac{x^2}{2}$

$\left[\begin{array}{l} expansion\ of\ brackets\ yield\ terms\ in\ x^3\ and \\ higher\ power\ terms\ which\ have\ been\ ignored \end{array}\right]$

Now $x = \frac{1}{8}$

$\sqrt{\left(\frac{1-\frac{1}{8}}{1+\frac{1}{8}}\right)} \cong 1-\frac{1}{8}+\frac{1}{2}\left(\frac{1}{8}\right)^2 \cong \sqrt{\left(\frac{7}{9}\right)} \cong \frac{113}{128}$

$\therefore \sqrt{7} \cong 3 \times \frac{113}{1128} = 2\frac{83}{128}$

Task:

If x is so small that its cube and higher powers can be neglected, show that $\sqrt{\left(\frac{1+x}{1-x}\right)} \cong 1+x+\frac{1}{2}x^2$ *. By putting* $x=\frac{1}{17}$*, show that* $\sqrt{2} \cong 1\frac{359}{867}$

2. *Show that the first* $3-$ *terms in the expansion in ascending powers of x of* $(1+8x)^{\frac{1}{4}}$ *are the same as the first* $3-$ *terms in the expansion* $\frac{1+5x}{1+3x}$*. Use the corresponding approximation* $(1+8x)^{\frac{1}{4}} \cong \frac{1+5x}{1+3x}$ *to obtain approximation to* $(1.16)^{\frac{1}{4}}$ *as a rational fraction in its lowest form.*

Solution

$(1+8x)^{\frac{1}{4}} \cong 1+\frac{1}{4}(8x)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}(8x)^2$

$\cong 1+2x-6x^2$

$\frac{1+5x}{1+3x} \cong (1+5x)(1+3x)^{-1}$

$\cong \left[1+(-1)(3x)+\frac{(-1)(-2)}{2!}(3x)^2\right](1+5x)$

$\cong (1-3x+9x^2)(1+5x)$

$\cong 1-3x+9x^2+5x-15x^2$

$\cong 1+2x-6x^2$

$\therefore (1+8x)^{\frac{1}{4}} \cong \frac{1+5x}{1+3x} \cong 1+2x-6x^2$

Using $(1+8x)^{\frac{1}{4}} \cong \frac{1+5x}{1+3x} \cong 1+2x-6x^2$

$(1.16)^{\frac{1}{4}} = (1+0.16)^{\frac{1}{4}}$

By comparison, $8x = 0.16 \ \therefore x = 0.02$

$\Rightarrow (1+0.16)^{\frac{1}{4}} \cong \frac{1+5(0.02)}{1+3(0.02)}$

$\cong \frac{1.1}{1.06} = \frac{59}{53}$

3. *a) It is given that* $f(x)=\frac{1}{(1+x)^2}+\sqrt{4+x}$*. Show that if* x^3 *and higher powers are ignored,* $f(x) \cong a+bx+cx^2$ *, and find the values of a, b and c*

b) Show that $3^n = \binom{n}{0}+2\binom{n}{1}+4\binom{n}{2}+\cdots\ldots\ldots+2^n\binom{n}{n}$*. Hence evaluate* 3^6

c) Given that the first three terms in the expansion in ascending powers of x of $(1-8x)^{\frac{1}{4}}$ *are the same as the first 3-terms in the expansion of* $\frac{1+ax}{1+bx}$*. Find the values of a and b. hence find an approximation to* $(0.6)^{\frac{1}{4}}$ *in the form* $\frac{p}{q}$*.*

Solution

a) $f(x)=\frac{1}{(1+x)^2}+\sqrt{4+x}=(1+x)^{-2}+\sqrt{4\left(1+\frac{x}{4}\right)}$

$(1+x)^{-2} \cong 1+(-2)x+\frac{(-2)(-3)}{2!}x^2 \cong 1-2x+3x^2$

$\sqrt{4\left(1+\frac{x}{4}\right)} = 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}} \cong 2\left[1+\frac{1}{2}\left(\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2\right]$

$\cong 2\left[1+\frac{x}{8}-\frac{1}{128}x^2\right] \cong 2+\frac{x}{4}-\frac{1}{64}x^2$

$\therefore\ f(x) \cong 1-2x+3x^2+2+\frac{x}{4}-\frac{1}{64}x^2$

$\cong 3-x+\frac{191}{64}x^2$

$3-x+\frac{191}{64}x^2 \equiv a+bx+cx^2$, *comparing corresponding coefficients;*

$\therefore a=3\,, b=-1\,, c=\frac{191}{64}$

b) $3^n = \binom{n}{0}+2\binom{n}{1}+4\binom{n}{2}+\cdots\ldots\ldots+2^n\binom{n}{n}$

$3^n = (1+2)^n$; By binomial theorem

$3^n = 1+{}^{n}C_1.(2)^1+{}^{n}C_2.(2)^2+\ldots\ldots\ldots\ldots+{}^{n}C_n(2)^n$

$3^n = \binom{n}{0}+2\binom{n}{1}+4\binom{n}{2}+\cdots\ldots\ldots+2^n\binom{n}{n}$

For $n=6, 3^6 = \binom{6}{0}+(2)^1\binom{6}{1}+(2)^2\binom{n}{2}+\cdots\ldots\ldots+2^6\binom{6}{6}$

$3^6 = 1+12+60+8.{}^{6}C_3+16.{}^{6}C_4+32.{}^{6}C_5+64.{}^{6}C_6$

$= 729$

c) $(1-8x)^{\frac{1}{4}} \cong \frac{1+ax}{1+bx}$

$(1-8x)^{\frac{1}{4}} \cong 1+\frac{1}{4}(-8x)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}(-8x)^2$

$\cong 1-2x-6x^2$

$\frac{1+ax}{1+bx} \cong (1+ax)(1+bx)^{-1}$

$\cong \left[1+(-1)(bx)+\frac{(-1)(-2)}{2!}(bx)^2\right](1+ax)$

$\cong (1-bx+b^2x^2)(1+ax)$

$\cong 1-bx+b^2x^2+ax-abx^2$

$[terms\ in\ x^3\ ignored\ \because do\ not\ make\ the\ 1^{st}\ 3-terms]$

$\cong 1+(a-b)x+(b^2-ab)x^2$

$\therefore 1-2x-6x^2 \equiv 1+(a-b)x+(b^2-ab)x^2$

By comparison of coefficients of corresponding terms,

For x: $-2=a-b$ ..... (1)

For x^2: $-6=b^2-ab=-b(a-b)$..... (2)

$Eqn.(1) \div eqn.(2)$

$-\frac{2}{-6}=\frac{(a-b)}{-b(a-b)}=\frac{1}{-b}\ \therefore b=-3$

From (1) $a=b-2=-3-2=-5$

$\therefore (1-8x)^{\frac{1}{4}} \cong \frac{1-5x}{1-3x} \cong 1-2x-6x^2$

$(0.6)^{\frac{1}{4}} = (1-0.4)^{\frac{1}{4}} = (1-8x)^{\frac{1}{4}}$

$\Rightarrow 0.4 = 8x\ \therefore x=0.05$

From $(1-8x)^{\frac{1}{4}} \cong \frac{1-5x}{1-3x} \cong 1-2x-6x^2, For\ x=0.05$

$(0.6)^{\frac{1}{4}} \cong \frac{1-5(0.05)}{1-3(0.05)}=\frac{0.75}{0.85}=\frac{15}{17}$

Task:

1. *Expand* $\sqrt{\left(\frac{1+2x}{1-2x}\right)}$ *as a series of ascending powers of x up to and including terms in* x^2. *Using* $x=\frac{1}{100}$, *find an approximation of* $\sqrt{51}$ *stating the number of significant figures to which your answer is accurate.* **ANSWER**: $\mathbf{7.141}$ $(4\ s.f)$
2. *Find the first three terms of the binomial expansion* $\frac{2(1+x)}{\sqrt{1-\frac{x}{4}}}$. *Hence evaluate* $\frac{10}{\sqrt{15}}$ *correct to 3d.p.*

   HINT:

   $$\frac{2(1+x)}{\sqrt{1-\frac{x}{4}}}=2(1+x)\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}\cong 2(1+x)\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2\right]$$

   $$\cong 2(1+x)\left[1+\frac{x}{8}+\frac{3}{128}x^2\right]$$

   $$\cong 2\left[1+\frac{x}{8}+\frac{3}{128}x^2+x+\frac{x^2}{8}\right]$$

   $$\cong 2\left(1+\frac{9}{8}x+\frac{19}{128}x^2\right)$$

   Now finding suitable value of x for evaluation of $\frac{10}{\sqrt{15}}$

   $\Rightarrow \frac{2(1+x)}{\sqrt{1-\frac{x}{4}}}=\frac{10}{\sqrt{15}}$ *or* $\frac{(1+x)}{\sqrt{1-\frac{x}{4}}}=\frac{5}{\sqrt{15}}$

   $\Rightarrow \frac{(1+x)^2}{1-\frac{x}{4}}=\frac{25}{15}=\frac{5}{3}$ *or* $\frac{4(1+x)^2}{4-x}=\frac{5}{3}$

   $\Rightarrow 12+24x+12x^2=20-5x$ *or* $12x^2+29x-8=0$

   Solving $x=\frac{1}{4}, -\frac{8}{3}$

   Since the expansion $\sqrt{1-\frac{x}{4}}$ is valid for $\left|\frac{x}{4}\right|<1$ *or* $|x|<4$, then taking $x=\frac{1}{4}$

   $\Rightarrow \frac{2\left(1+\frac{1}{4}\right)}{\sqrt{1-\frac{1}{16}}}\cong 2\left[1+\frac{9}{8}\left(\frac{1}{4}\right)+\frac{19}{128}\left(\frac{1}{4}\right)^2\right]$

   $\Rightarrow \frac{\frac{10}{4}}{\sqrt{\left(\frac{15}{16}\right)}}\cong \frac{264\times 2}{2048}$

   $\therefore \frac{10}{\sqrt{15}}\cong 2.581$
3. *Determine the binomial expansion of* $\left(1+\frac{x}{2}\right)^4$. *Hence evaluate* $(2.1)^4$ *correct to 2d.p* (***Uneb* 2005**)

   HINT:

   $\left(1+\frac{x}{2}\right)^4=1+2x+\frac{3}{2}x^2+\frac{x^3}{2}$

   $(2.1)^4=(2+0.1)^4=\left[2\left(1+\frac{0.1}{2}\right)\right]^4=16\left(1+\frac{0.1}{2}\right)^4$

   But $\frac{x}{2}=\frac{0.1}{2}\therefore x=0.1$

   $\Rightarrow 16\left(1+\frac{0.1}{2}\right)^4\cong 16\left(1+2(0.1)+\frac{3}{2}(0.1)^2+\frac{(0.1)^3}{2}\right)\cong 19.448$
4. *Expand* $\sqrt{\left(\frac{1+5x}{1-5x}\right)}$ *as far as and including the term in* x^3. *Taking the first three terms and* $x=\frac{1}{9}$ *evaluate* $\sqrt{14}$

5. *Show that* $\sqrt{(1+4ax)} \cong 1+2ax-2a^2x^2+4a^3x^3+$ ……, *and deduce the expansion for* $\sqrt{(1-4x)}$ *up to the term in* x^4. *Letting* $x=\frac{1}{10}$, *evaluate* $\sqrt{6}$ *to* $5\ d.p.$

   HINT: $compairing \sqrt{(1+4ax)}\ with\ \sqrt{(1-4x)}, a=-1$

6. *Expand* $(1-x)^{\frac{1}{3}}$ *in ascending powers of x up to term in* x^3. *Hence evaluate* $\sqrt[3]{998}$.

   Hint: $\sqrt[3]{998}=\sqrt[3]{(1000-2)}=\sqrt[3]{1000(1-0.002)}=10\sqrt[3]{(1-0.002)}$

7. *If x is sufficiently small and allow any terms in* x^5 *or higher power terms be neglected, show that* $(1+x)^6(1-2x^3)^{10} \cong 1+6x+15x^2-105x^4$

8. *Expand using binomial theorem* $(1+4x)^{\frac{1}{2}}$ *up to the fourth term. Hence evaluate* $\sqrt{6}$ *to* $3\ d.p$ *using* $x=\frac{1}{100}$

9. *a) Expand* $\sqrt{\left(\frac{1+x}{1-x}\right)}$ *in ascending powers of x to a term in* x^2.

   b) (i) Using the expansion of $\sqrt{(1+x)}$ *up to the term in* x^3, *find the value of* $\sqrt{1.08}$ *to* $4\ d.p$

   (ii) Express $\sqrt{1.08}$ *in the form* $\frac{a}{b}\sqrt{c}$. *Hence evaluate* $\sqrt{3}$ *to* $3\ s.f.$ (***Uneb* 2010**)

10. *show that if x is so small in comparison with unity that* x^3 *and higher power terms can be neglected,* $\frac{(1-4x)^{\frac{1}{2}}.(1+3x)^{\frac{1}{3}}}{\sqrt{1+x}} \cong 1-\frac{3}{2}x-\frac{33}{8}x^2$

PERCENTAGES AND ERRORS IN BINOMIAL APPROXIMATION

The knowledge of measuration is applicable.

Examples

1) *The radius of a cylinder is reduced by 3% and its height increased by 4%. Determine the appropriate percentage change in;*
   a) Its volume
   b) In curved surface area
   (Neglect product of small changes)

Solution

a) Let r and h be the original values of radius and height respectively

Volume of cylinder, $V=\pi r^2h$

New values of r and h are respectively; $\left(\frac{100-3}{100}\right)r$, $\left(\frac{100+4}{100}\right)h$

New volume, $V' = \pi\left[\left(\left(\frac{100-3}{100}\right)r\right)^2\left(\frac{100+4}{100}\right)h\right]$

$= \pi(1-0.03)^2r^2.(1+0.04)h$

$= \pi r^2h.(1-0.03)^2.(1+0.04)$

But $(1-0.03)^2 \cong 1+2(-0.03)$

$\cong 1-0.06\ (Neglecting\ product\ of\ small\ changes)$

$\therefore V' = \pi r^2h.(1-0.06).(1+0.04)$

$= \pi r^2h.[1+0.04-0.06]\ (Neglecting\ product\ of\ small\ changes)$

$= 0.98\pi r^2h$

$\%ge\ change\ in\ V = \left(\frac{V-V'}{V}\right)\times 100$

$= \left(\frac{\pi r^2h-0.98\pi r^2h}{\pi r^2h}\right)\times 100 = 2\%$

$\therefore Volume\ reduced\ by\ 2\%$

b. Curved surface area of a cylinder, $A = 2\pi rh$

New curved surface area, $A' = 2\pi\left[\left(\frac{100-3}{100}\right)r.\left(\frac{100+4}{100}\right)h\right]$

$\Rightarrow A' = 2\pi rh.(1-0.03).(1+0.04)$

$= 2\pi rh[1+0.04-0.03]$ $[Products\ of\ higher\ smaller\ terms\ ignnored]$

$\therefore A' = 2\pi rh.(1.01)$

$\%ge\ change\ in\ A = \left(\frac{A-A'}{A}\right)\times 100$

$= \left(\frac{2\pi rh.(1.01)-2\pi rh}{2\pi rh}\right)\times 100 = 1\%$

$\therefore Curved\ surface\ area\ increased\ by\ 1\%$

2) *Pressure P and volume V are elated by the expression $PV^3 = C$ where C is a constant. Find the appropriate percentage change in C when P is increased by 2% and volume decreased by 0.8%.*

Solution

Let P and V be original values of pressure and volume respectively.

New value of $P = \left(\frac{100+2}{100}\right)P = (1+0.02)P$

New value of $V = \left(\frac{100-0.8}{100}\right)V = (1-0.008)V$

Now new value of $C = C' = (1+0.02)P[(1-0.008)V]^3$

$\Rightarrow C' = PV^3.(1+0.02).(1-0.008)^3$

$But\ (1-0.008)^3 \cong 1-3(0.008), higher\ power\ terms\ ignored$

$\therefore C' = PV^3.(1+0.02).\big(1-3(0.008)\big)$

$= PV^3.[1-3(0.008)+0.02]$ $, products\ of\ smaller\ terms\ ignored$

$= PV^3.(0.996)$

$\therefore \%ge\ change\ in\ C = \left(\frac{C-C'}{C}\right)\times 100 = \left(\frac{PV^3 - PV^3.(0.996)}{PV^3}\right)\times 100$

$\therefore \%ge\ change\ in\ C\ (Reduction) = 0.4\%$

3) *An error is made in measuring the radius of a sphere. Find the percentage error in surface area.*

Solution

Let the error lead to increase in radius.

Area of sphere, $A = 4\pi r^2$ where A and r are original values.

New value of $r = \left(1+\frac{2}{100}\right)r = (1+0.02)r$

New value of $A = A' = 4\pi[(1+0.02)r]^2$

$= 4\pi r^2(1+0.02)^2$

$= 4\pi r^2\big(1+2(0.02)\big) = 4\pi r^2(1.04)$

$\therefore \%ge\ change\ in\ A = \left(\frac{A-A'}{A}\right)\times 100$

$= \left(\frac{4\pi r^2(1.04)-4\pi r^2}{4\pi r^2}\right)\times 100 = 4\%$

NOTE:

If error was to lead to a decrease in r you get the same percentage change 4%.

4) *An error of $2\frac{1}{2}\%$ is made in measurement of the area of a circle. What percentage error results in the radius and circumference?*

Solution

Area $A = \pi r^2$, where A and r are original values of area and radius of circle respectively.

New value of $A = \left(\frac{100+2.5}{100}\right)A = (1 + 0.025)A$ $[let\ error\ lead\ to\ increase\ in\ A]$

From $A = \pi r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}}$

Now new value of $r = \sqrt{\frac{(1+0.025)A}{\pi}} = \sqrt{\frac{A}{\pi}}.(1 + 0.025)^{\frac{1}{2}}$

$= \sqrt{\frac{A}{\pi}}.\left[1 + \frac{1}{2}(0.025)\right]$, $higher\ power\ terms\ neglected = 1.0125\sqrt{\frac{A}{\pi}}$

$\%ge\ change\ in\ r = \left(\frac{1.0125\sqrt{\frac{A}{\pi}} - \sqrt{\frac{A}{\pi}}}{\sqrt{\frac{A}{\pi}}}\right) \times 100 = 1.25\%$

Circumference $C = 2\pi r$, where C and r are original values of circumference and radius respectively. Since the error of $2\frac{1}{2}\%$ leads to an increase in r by 1.25%, then;

New value of $r = \left(\frac{100+1.25}{100}\right)r = (1 + 0.0125)r$

$\therefore$ New value of $C = 2\pi(1 + 0.0125)r = 2\pi r(1 + 0.0125)$

$\therefore \%ge\ change\ in\ C = \left(\frac{2\pi r(1+0.0125) - 2\pi r}{2\pi r}\right) \times 100 = 1.25\%$

Alternatively

$A = \pi r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}}$

$C = 2\pi r \Rightarrow C = 2\pi\sqrt{\frac{A}{\pi}} = 2\sqrt{\pi}.\sqrt{A} = 2\sqrt{\pi A}$

New value of A due to error $= (1 + 0.025)A$

New value of $C = C' = 2\sqrt{\pi}.\sqrt{(1 + 0.025)A}$

$= 2\sqrt{\pi A}.(1 + 0.025)^{\frac{1}{2}}$

$\cong 2\sqrt{\pi A}.\left(1 + \frac{0.025}{2}\right) = 2\sqrt{\pi A}(1 + 0.0125)$, $higher\ power\ terms\ neglected$

$\therefore \%ge\ change\ in\ C = \left(\frac{2\sqrt{\pi A}(1+0.0125) - 2\sqrt{\pi A}}{2\sqrt{\pi A}}\right) \times 100 = 1.25\%\ as\ before$

5) *One side of a rectangle is 3-times the other. If the perimeter increases by 2%, what is the percentage increase in area?*

Solution

$length = 3w$

$\boxed{P = perimeter}$ $width = w$ $P = 2(3w + w) = 8w$

New value of $P = P' = \left(\frac{100+2}{100}\right)P = (1 + 0.02)P$

Now area, $A = 3w^2$

From $P = 8w \Rightarrow w = \frac{P}{8}$

$\therefore A = 3\left(\frac{P}{8}\right)^2 = \frac{3}{64}P^2$

New value of $A\ (due\ to\ increase\ in\ P) = A' = \frac{3}{64}[(1 + 0.02)P]^2$

$= \frac{3}{64}P^2\left((1 + 0.02)\right)^2 = \frac{3}{64}P^2[1 + 2(0{,}02)]$, $higher\ power\ terms\ neglected$

$= 1.04\left(\frac{3}{64}P^2\right)$

$\therefore \%ge\ in\ area = \left[\frac{(1.04-1)\frac{3}{64}P^2}{\frac{3}{64}P^2}\right] \times 100 = 4\%$

Alternatively

$P = 8w \Rightarrow w = \frac{P}{8}$; $New\ P = (1+0.02)P$

Finding percentage change in w due to 2% increase in P

New $w = w' = \frac{(1+0.02)P}{8}$

$\%ge\ change\ in\ w = \left(\frac{\frac{(1+0.02)P}{8} - \frac{P}{8}}{\frac{P}{8}} \times\right) 100 = 2\%$

Now area , $A = 3w^2$

New area due to increase in w= $3[(1+0.02)w]^2$

$= 3w^2(1+0.02)^2 \cong 3w^2[1+2(0.02)]$

$= 3w^2.(1.04)$

$\therefore \%ge\ in\ area = \left(\frac{3w^2.(1.04)-3w^2}{3w^2}\right) \times 100 = 4\%$

6) *The height cylinder 10cm and its radius is 4cm. find an approximate increase in volume when the radius increases to 4.02.*

Solution

Volume, $V = \pi r^2 h$; assuming h is constant

$\Rightarrow r = 4cm\ , h = 10cm$

Finding $\%ge$ change in r= $\left(\frac{4.02-4}{4}\right) \times 100 = 0.5\%$

Now new value of $V - V' = \pi\left[\left(\frac{100+0.5}{100}\right)r\right]^2 . h$

$= \pi r^2 h\ (1+0.005)^2 \cong\ \pi r^2 h[1+2(0.005)]$

$V' = 1.01\pi r^2 h$

$\therefore Increase\ in\ volume = 1.01\pi r^2 h - \pi r^2 h$

$= 0.01\pi r^2 h = 0.01\pi \times 4^2 \times 10$

$= 1.6\pi\ cm^3$

7) *The base radius of a right circular cone increases and the volume changes by 2%. If the height remains constant, find the percentage increase in the circumference of the base.*(***Uneb* 2011**)

Solution

Let radius increase by $x\%$

Let $V = \frac{1}{3}\pi r^2 h$, where r and V are original values of radius and volume respectively

$\Rightarrow New\ volume, V' = \frac{1}{3}\pi\left[\left(\frac{100+x}{100}\right)r\right]^2 h$

$= \frac{1}{3}\pi r^2 h(1+0.01x)^2$

$= \frac{\pi r^2 h}{3}(1+0.02x)$ $[Higher\ power\ terms\ neglected]$

Now $\%ge\ change\ in\ volume = \left(\frac{V'-V}{V}\right) \times 100 = 2$

$\therefore \left(\frac{\frac{\pi r^2 h}{3}(1+0.02x) - \frac{\pi r^2 h}{3}}{\frac{\pi r^2 h}{3}}\right) \times 100 = 2$

$\Rightarrow 0.02x = 0.02 \therefore x = 1$

$\therefore Increase\ in\ radius\ is\ by\ 1\%$

Now circumference, $C = 2\pi r$

New value of C after increase in radius by 1% $= C' = 2\pi\left(\frac{100+1}{100}\right)r = 2\pi r(1.01)$

$\therefore \%ge\ increase\ in\ C = \left(\frac{2\pi r(1.01)-2\pi r}{2\pi r}\right)\times 100 = 1\%$

8) *If L is the length of a pendulum and t(s) is the time of one complete swing. It is known that*
$L = kt^2$. *If the length of a pendulum increases by x%, x being small, find the corresponding increase in the time of swing.*

Solution

$L = kt^2$, taking k a constant

Increase in length $= \left(\frac{100+x}{100}\right)L = (1+0.01x)L$

Now $t = \sqrt{\frac{L}{k}}$

$\therefore$ New value of $t = t' = \sqrt{\frac{(1+0.01x)L}{k}} = \sqrt{\frac{L}{k}}.(1+0.01x)^{\frac{1}{2}}$

$$\cong \sqrt{\frac{L}{k}}.\left[1+\frac{1}{2}(0.01x)\right] \cong \sqrt{\frac{L}{k}}.(1+0.005x)$$

$\therefore \%ge\ increase\ in\ t = \left(\frac{t'-t}{t}\right)\times 100$

$$= \left(\frac{\sqrt{\frac{L}{k}}.(1+0.005x)-\sqrt{\frac{L}{k}}}{\sqrt{\frac{L}{k}}}\right)\times 100 = 0.5x\%$$

9) *The period T of a pendulum is calculated from the formula* $T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$ *where l is the length of a pendulum and g is acceleration due to gravity. Find the percentage change in the period caused by lengthening the pendulum by 2%.*

Solution

$T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$, new value of $l = \left(\frac{100+2}{100}\right)l = (1+0.02)l$

$\therefore New\ T = T' = 2\pi\sqrt{\left(\frac{(1+0.02)l}{g}\right)}$

$$= 2\pi\sqrt{\left(\frac{l}{g}\right)}.(1+0.02)^{\frac{1}{2}}$$

$$= 2\pi\sqrt{\left(\frac{l}{g}\right)}.\left(1+\frac{0.02}{2}\right)\ \ [higher\ power\ terms\ neglected]$$

$\therefore T' = 2\pi\sqrt{\left(\frac{l}{g}\right)}.(1.01)$

$\therefore \%ge\ change\ in\ T = \left(\frac{T'-T}{T}\right)\times 100$

$$= \left(\frac{2\pi\sqrt{\left(\frac{l}{g}\right)}.(1.01)-2\pi\sqrt{\left(\frac{l}{g}\right)}}{2\pi\sqrt{\left(\frac{l}{g}\right)}}\right)\times 100 = 1\%$$

10) *Find the approximate percentage change in the square of a quantity when the quantity itself changes by 0.1%. hence find the approximate value for* $(10.01)^2$

Solution

Let the quantity be x

$\Rightarrow y = x^2$

Let percentage change in square of quantity be an increase

$\therefore New\ value\ of\ x = \left(\frac{100+0.1}{100}\right)x = (1+0.001)x$

Now percentage change in square of quantity. $i.e. y = \left(\frac{y^{'}-y}{y}\right) \times 100$

But $y^{'} = [(1+0.001)x]^2 = (1+0.001)^2 x^2$

$\cong [1+2(0.001)]x^2 \cong 1.002x^2$

$\therefore$ Required $\%ge$ change in square of a quantity

$$= \left(\frac{1.002x^2 - x^2}{x^2}\right) \times 100 = 0.2\%$$

Now let $y = (10.01)^2 = (10+0.01)^2$

Hence if $x = 10$, then x increases by $0.01 = 1\%$

Since $y^{'} = x^2 \cong 1.002x^2$, $x = 10$

$\Rightarrow y^{'} = 1.002 \times 10^2 = 100.2$

$$\left[\begin{array}{c} \boldsymbol{Alt}: (10.01)^2 = (10+0.01)^2 = (10.01)^2 = (10+0.01)^2 \\ = \left[10\left(1+\frac{0.01}{10}\right)\right]^2 = 100(1+0.001)^2 = 10^2 \times \left(1+2(0.001)\right) = 100.2 \end{array}\right]$$

NOTE:

The above examples on percentages and errors in binomial expansion can also be done using the knowledge of *differentiation.i.e. Small increments and percentages*

Task:

1. *The modulus of rigidity G is given by* $G = \frac{R^4\theta}{L}$ *where R is radius,* θ *is the angle of twist and L is length. Find the approximate percentage in G when R is measured* 1.5% *too large and* θ *is measured* 5% *too small.* $\boldsymbol{Answer}: \mathbf{1}\%$
2. *The volume of a cone is given by* $V = \frac{1}{3}\pi r^2 h$*. If the volume increases by* $\frac{2\pi}{3}\ cm^3/$ *min and height increase by* $0.03cm/min$ *. Find the rate of change of the radius r when* $r = 10\ cm$ *and* $h = 5\ cm.$ $\boldsymbol{Answer}: \mathbf{5}$
3. *If the radius of a spherical bubble increases from* $1\ cm$ *to* $1.02\ cm$*, find the approximate increase in volume.* $\boldsymbol{Answer}: \mathbf{0.08\pi\ cm^3}$
4. *If* $R = ar^n$ *and an error of x% is made in measuring r, prove that an error of nx% will result in R*
5. *The volume of a sphere is increased by 3%. Find the percentage increase in the radius.* $\boldsymbol{Answer}: \mathbf{1}\%$

VECTORS

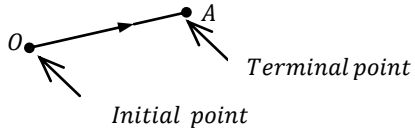
Definition:

A quantity that can be specified with magnitude and direction $e.g.$ force, acceleration, displacement, velocity etc is a vector.

Representation of a vector

A vector is represented symbolically as $\vec{a}$ or **a** or $\underset{\sim}{a}$

Graphically, it is represented by a line



The length of a line is the magnitude and the arrow on the line is the direction of the vector.

Classification of vectors

Vectors are classified depending on their behavior and properties.

a) **Null vector or Zero vector**

A vector with a null/zero magnitude, and is denoted as **O**

b) **Unit vector**

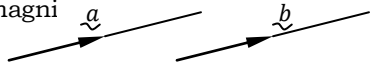
A vector with a unit magnitude is a unit vector. It is denoted with a cap $.i.e.$ for a vector, its unit vector I a

NOTE

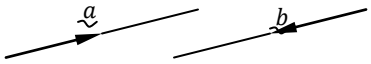
$$\frac{VECTOR}{|VECTOR|} = UNIT\ VECTOR$$

c) **Like/unlike vectors**

Like vectors are vectors with the same direction regardless of their magni

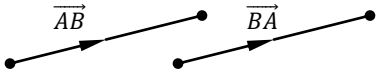


Unlike vectors are vectors with different directions



d) **Negative vectors**

Negative vectors are vectors with same magnitude but opposite in direction. Consider the figure below



$\overrightarrow{AB} = -\overrightarrow{BA}$ **Application:** subtraction of vectors

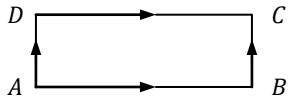
e) **Equal vectors**

Equal vectors arte vectors with same magnitude and direction

Application

- Rhombus
- Parallelogram
- Rectangles
- Squares

Consider the figure below



For a rectangle
$\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

Note

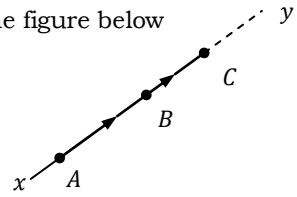
$\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$, $\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$. Thus equal vectors are also parallel but with the same magnitude and direction.

f) **Collinear/ parallel vectors**

Are vectors with the same line of action or having line of action parallel to one another.

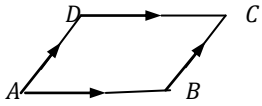
Consider the figure below

a)



$\overrightarrow{AB}$ *and* $\overrightarrow{BC}$, act on point B, but both have xy line of action, hence $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$

b)



ABCD is a parallelogram. $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$ $\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$. Thus $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are collinear and also $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are collinear/parallel.

Condition for collinear/parallel vectors

From fig (a) $\overrightarrow{AB} = k\overrightarrow{BC}$ *or* $\overrightarrow{AB} = \lambda\overrightarrow{AC}$

From fig (b) $\overrightarrow{AB} = k\overrightarrow{DC}$ *or* $\overrightarrow{AD} = \lambda\overrightarrow{BC}$

Where k and λ are scalars or constants

g) **Position vectors**

Are vectors that specify position of a point with respect to a fixed point O $(Origin)$

Application

Used mainly in coordinate system, for locating a point in $x - y$ or $x - y - z$ planes

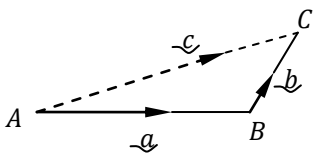
h) **Coplanar vectors**

Are vectors which lie in the same plane or vectors paralleled to the same plane

ADDITION OF VECTORS

Vectors are added by finding their *resultant, a single or equivalent vector.*
Consider the following cases of vector addition.

1) **Case I:** *Addition of **TWO** vectors.*



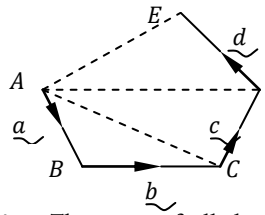
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

or $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c}$

This is called *triangle law* of addition of vectors.
In general, to find the sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, draw them as a chain, starting the second where the first ends; the sum $\boldsymbol{c}$ is given by the single vector joining the start of the first to the end of the second.

2) **Case II:** *Addition of a number of vectors;* $\boldsymbol{a+b+c+d+\ldots\ldots\ldots\ldots}$

A)

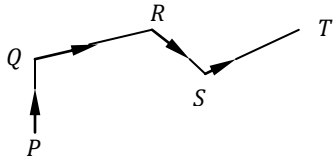


a) Draw the vectors as achain
b) Then;

$$\boldsymbol{a}+\boldsymbol{b}=\overrightarrow{AC},\overrightarrow{AC}+\boldsymbol{c}=\overrightarrow{AD}$$
$$\therefore \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\overrightarrow{AD}$$
$$\overrightarrow{AD}+\boldsymbol{d}=\overrightarrow{AE}$$
$$\therefore \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}=\overrightarrow{AE}$$

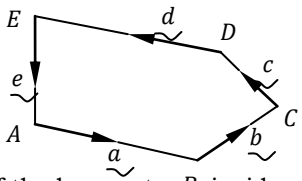
i.e. The sum of all the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ is given by the single vector joining the start of the first to the end of the last . $i.e.\overrightarrow{AE}$

B)



$$\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RS}+\overrightarrow{ST}=\overrightarrow{PT}$$

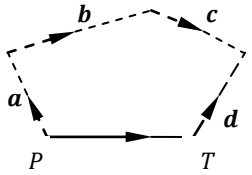
3) **Case III:** *Sum of vectors* $= 0$ *for a closed figure*
Consider the figure below



$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EA}=0$$
$$i.e..\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{e}=\boldsymbol{0}$$

NOTE
The end of the last vecto incides with the beginning of the first, so the resultant vector is zero with no magnitude.

COMPONENTS OF A GIVEN VECTOR
A single vector can be replaced by any number of component vectors so long as they form a chain in the vector diagram, beginning at initial point and ending at the terminal point.

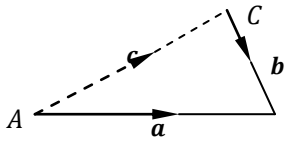


$$\overrightarrow{PT}=\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}$$

See examples immediately after *subtraction of vectors.*

SUBTRACTION OF VECTORS
During the addition of vectors in a specified direction it may be encountered that the direction of a vector needs to be changed in order for it to be added. Once a direction of vector is changed, a *negative vector* is formed, hence addition, subtraction of a vector occurs.

NOTE
Triangle law is still used for subtraction of *two* vectors.

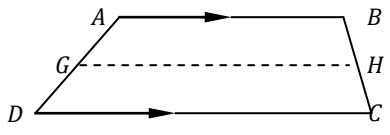


$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
But $\overrightarrow{BC} = -\overrightarrow{CB}$
$\therefore \overrightarrow{AC} = \overrightarrow{AB} - \overrightarrow{CB}$ *or* $\boldsymbol{c = a - b}$

Examples on components of vectors

1. *ABCD is a quadrilateral, with G and H the midpoints of DA and BC respectively. Show that* $\overrightarrow{AB} + \overrightarrow{DC} = 2\overrightarrow{GH}$

Solution

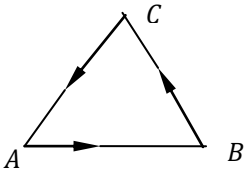


By chain vectors, $\overrightarrow{AB} = \overrightarrow{AG} + \overrightarrow{GH} + \overrightarrow{HB}$
Also $\overrightarrow{DC} = \overrightarrow{DG} + \overrightarrow{GH} + \overrightarrow{HC}$
Now $\overrightarrow{AB} + \overrightarrow{DC} = \overrightarrow{AG} + \overrightarrow{GH} + \overrightarrow{HB} + \overrightarrow{DG} + \overrightarrow{GH} + \overrightarrow{HC}$
$= 2\overrightarrow{GH} + (\overrightarrow{AG} + \overrightarrow{DG}) + (\overrightarrow{HB} + \overrightarrow{HC})$

But G is the midpoint of AD, $\Rightarrow \overrightarrow{AG} = \overrightarrow{GD}$ *or* $\overrightarrow{AG} = -\overrightarrow{DG}$ $(negative\ vector)$
Also H is the midpoint of BC, $\Rightarrow \overrightarrow{BH} = \overrightarrow{HC}$ *or* $-\overrightarrow{HB} = \overrightarrow{BH} = \overrightarrow{HC}$
$\therefore \overrightarrow{AB} + \overrightarrow{DC} = 2\overrightarrow{GH} + (-\overrightarrow{DG} + \overrightarrow{DG}) + (-\overrightarrow{HC} + \overrightarrow{HC}) = 2\overrightarrow{GH}$

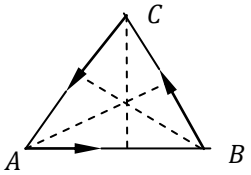
2. *Points L, M, N are midpoints of the sides AB, BC, CA of the triangle ABC. Show that;*
   a. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$
   b. $2\overrightarrow{AB} + 3\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{LC}$
   c. $\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CL} = 0$

Solution



a. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$ $(For\ a\ closed\ figure)$

b. To show that $2\overrightarrow{AB} + 3\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{LC}$



$\overrightarrow{CA} = \overrightarrow{CL} + \overrightarrow{LA}$
$\therefore 2\overrightarrow{AB} + 3\overrightarrow{BC} + \overrightarrow{CA} = 4\overrightarrow{AL} + 3\overrightarrow{BL} + 3\overrightarrow{LC} + \overrightarrow{CL} + \overrightarrow{LA}$
$= 4\overrightarrow{AL} + 3(-\overrightarrow{LB}) + 3\overrightarrow{LC} + (-\overrightarrow{LC}) + (\overrightarrow{-AL})$
$= 3\overrightarrow{AL} - 3\overrightarrow{LB} + 2\overrightarrow{LC}$
Since $AL = LB, 3\overrightarrow{AL} - 3\overrightarrow{LB} = 0$
$\therefore 2\overrightarrow{AB} + 3\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{LC}$

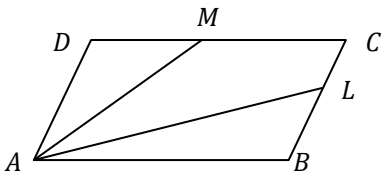
c. To show that $\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CL} = 0$
From the figure (b) above
$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}, \overrightarrow{BN} = \overrightarrow{BC} + \overrightarrow{CN}, \overrightarrow{CL} = \overrightarrow{CA} + \overrightarrow{AL}$
$\overrightarrow{AM} + \overrightarrow{BN} + \overrightarrow{CL} = \overrightarrow{AB} + \overrightarrow{BM} + \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AL} + \overrightarrow{CN}$
$= (\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) + (\overrightarrow{BM} + \overrightarrow{AL} + \overrightarrow{CN})$
$= 0 + \frac{1}{2}(\overrightarrow{BC} + \overrightarrow{AB} + \overrightarrow{CA}) = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) = 0$

Task

Prove that the line joining he midpoint of sides of a triangle is parallel to the third side and half its length.

3. *ABCD is a parallelogram. If L, M are the midpoints of BC and CD, express* $\overrightarrow{AL}$ *and* $\overrightarrow{AM}$ *in terms of* $\overrightarrow{AB}$ *and* $\overrightarrow{AD}$. *Show that* $\overrightarrow{AL} + \overrightarrow{AM} = \frac{3}{2}\overrightarrow{AC}$

Solution



$\overrightarrow{AL} = \overrightarrow{AB} + \overrightarrow{BL}$

But $\overrightarrow{BL} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\overrightarrow{AD}$ $[\because \overrightarrow{AD} = \overrightarrow{BC}]$

$\therefore \overrightarrow{AL} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$

$\overrightarrow{AM} = \overrightarrow{AD} + \overrightarrow{DM}$

But $\overrightarrow{DM} = \frac{1}{2}\overrightarrow{DC}$, $\overrightarrow{DC} = \overrightarrow{AB}$, $\therefore \overrightarrow{DM} = \frac{1}{2}\overrightarrow{AB}$

$\Rightarrow \overrightarrow{AM} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB}$

Now to show: $\overrightarrow{AL} + \overrightarrow{AM} = \frac{3}{2}\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD} + \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB} = \overrightarrow{AL} + \overrightarrow{AM}$

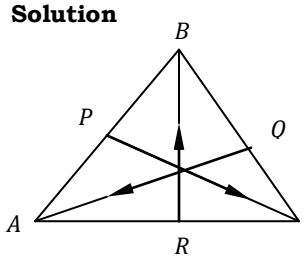
$\Rightarrow \overrightarrow{AL} + \overrightarrow{AM} = \frac{3}{2}\overrightarrow{AB} + \frac{3}{2}\overrightarrow{AD} = \frac{3}{2}(\overrightarrow{AB} + \overrightarrow{AD})$

But $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\therefore \overrightarrow{AL} + \overrightarrow{AM} = \frac{3}{2}\overrightarrow{AC}$

4. *If P, Q, R are midpoints of the sides AB, BC and CA of the triangle ABC and O be a point with in t, prove that* $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}$

Solution



$\overrightarrow{OA} = \overrightarrow{OP} + \overrightarrow{PA}$

$\overrightarrow{OB} = \overrightarrow{OQ} + \overrightarrow{QB}$

$\overrightarrow{OC} = \overrightarrow{OR} + \overrightarrow{RC}$

$\therefore \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{PA} + \overrightarrow{OQ} + \overrightarrow{QB} + \overrightarrow{OR} + \overrightarrow{RC}$

$= (\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}) + (\overrightarrow{PA} + \overrightarrow{QB} + \overrightarrow{RC})$

But $\overrightarrow{PA} = \frac{1}{2}\overrightarrow{BA}$, $\overrightarrow{QB} = \frac{1}{2}\overrightarrow{CB}$, $\overrightarrow{RC} = \frac{1}{2}\overrightarrow{AC}$

$\Rightarrow \overrightarrow{PA} + \overrightarrow{QB} + \overrightarrow{RC} = \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{CB} + \overrightarrow{AC}) = \frac{1}{2} \times 0 = 0$

$\therefore \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}$

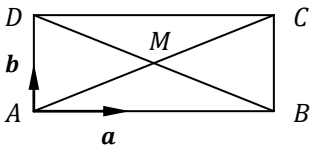
NOTE

Parameters can also be applied in solving the above examples through forming parametric equations.

5. *ABCD is a rectangle. By vectors show that the diagonals bisect each other.*

Solution

Points A, M and C are collinear, and B, M and D are also collinear.



Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$

Now $\overrightarrow{AM} = \lambda\overrightarrow{AC}$ and $\overrightarrow{BM} = \mu\overrightarrow{BD}$ where λ , μ are constants.

$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \overrightarrow{AB} + \mu\overrightarrow{BD}$

But $BD = BA + AD = -\boldsymbol{a} + \boldsymbol{b}$

$\therefore \overrightarrow{AM} = \boldsymbol{a} + \mu(-\boldsymbol{a} + \boldsymbol{b})$

Also $\overrightarrow{AM} = \lambda\overrightarrow{AC}; \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$ $[\because \overrightarrow{AD} = \overrightarrow{BC} = \boldsymbol{b}]$

$\therefore \overrightarrow{AM} = (\boldsymbol{a} + \boldsymbol{b})\lambda$ ..... (1)

Also $\overrightarrow{AM} = \boldsymbol{a} + \mu(-\boldsymbol{a} + \boldsymbol{b}) = (1 - \mu)\boldsymbol{a} + \mu\boldsymbol{b}$ ..... (2)

Since (1) and (2) are the same

$\Rightarrow (\boldsymbol{a}+\boldsymbol{b})\lambda = \boldsymbol{a}+\mu(-\boldsymbol{a}+\boldsymbol{b}) = (1-\mu)\boldsymbol{a}+\mu\boldsymbol{b}$

Equating coefficients of corresponding vectors

For $\boldsymbol{a}$: $\boldsymbol{\lambda} = 1-\mu$ ..... (3)

For $\boldsymbol{b}$: $\lambda = \mu$ ..... (4)

Solving (3) and (4) simultaneously

$\mu = 1-\mu \; or \; \mu = \frac{1}{2} = \lambda$

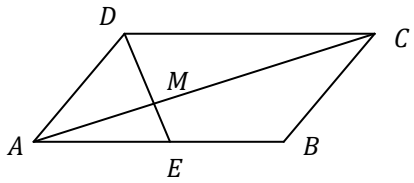
Since $\overrightarrow{AM} = \lambda\overrightarrow{AC}$ and $\overrightarrow{BM} = \mu\overrightarrow{BD}$, then $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$ and $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BD}$

$\therefore M$ is the midpoint of the diagonals, and hence diagonals bisect.

6. *If ABCD is a parallelogram and E is the midpoint of AB, show by vector method that DE trisects and is trisected by AC.*

Solution

Points A, M and Care collinear and D, M and E are also collinear



Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$

$\overrightarrow{AM} = \lambda\overrightarrow{AC}$, $\overrightarrow{DM} = k\overrightarrow{DE}$, where k , λ are constants

$\overrightarrow{AM} = \lambda\overrightarrow{AC}$; $\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC}$

$= \overrightarrow{AB}+\overrightarrow{AD} = \boldsymbol{a}+\boldsymbol{b}$ $\left[\because \overrightarrow{AD} = \overrightarrow{BC} = \boldsymbol{b}\right]$

$\therefore \overrightarrow{AM} = (\boldsymbol{a}+\boldsymbol{b})\lambda$ ..... (1)

$\overrightarrow{DM} = k\overrightarrow{DE}$; $\overrightarrow{DM} = \overrightarrow{DA}+\overrightarrow{AM} = -\boldsymbol{b}+\overrightarrow{AM}$

$\Rightarrow \overrightarrow{DM} = -\boldsymbol{b}+(\boldsymbol{a}+\boldsymbol{b})\lambda$ ..... (2)

Now $\overrightarrow{DE} = \overrightarrow{DA}+\overrightarrow{AE} = -\boldsymbol{b}+\frac{1}{2}\overrightarrow{AB} = -\boldsymbol{b}+\frac{1}{2}\boldsymbol{a}$

$\therefore \overrightarrow{DM} = k\overrightarrow{DE}$ gives;

$-\boldsymbol{b}+(\boldsymbol{a}+\boldsymbol{b})\lambda = k\left(-\boldsymbol{b}+\frac{1}{2}\boldsymbol{a}\right) \; or \; (\lambda-1)\boldsymbol{b}+\lambda\boldsymbol{a} = -k\boldsymbol{b}+\frac{k}{2}\boldsymbol{a}$

Equating coefficients of corresponding vectors

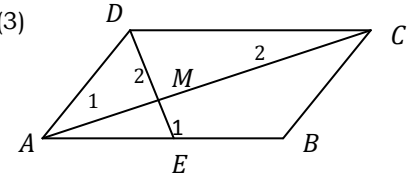
For $\boldsymbol{a}$: $\lambda = \frac{k}{2}$ ..... (3)

For $\boldsymbol{b}$: $\lambda - 1 = -k$ ..... (4)

$Eqn.(3)$ in to $eqn.(4)$ gives

$\frac{k}{2}-1 = -k \Rightarrow \frac{3}{2}k = 1 \; or \; k = \frac{2}{3}$,

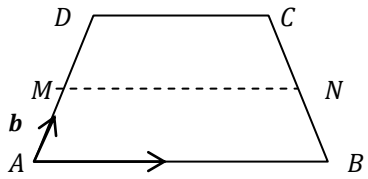
and from (1) $\lambda = \frac{1}{2}\times\frac{2}{3} = \frac{1}{3}$



$\therefore \overrightarrow{AM} = \frac{1}{3}\overrightarrow{AC}$ *and* $\overrightarrow{DM} = \frac{2}{3}\overrightarrow{DE}$, hence DE trisects and is trisected by AC.

7. *Show that the line joining the midpoints of two non parallel sides of a trapezium is parallel to the other sides and half their sum.*

Solution



Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$

Since $\overrightarrow{AB}$ // $\overrightarrow{DC}$, then $\overrightarrow{DC} = \lambda\overrightarrow{AB} = \lambda\boldsymbol{a}$

Now expressing $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

$\overrightarrow{MN} = \overrightarrow{MA}+\overrightarrow{AB}+\overrightarrow{BN}$

$\overrightarrow{MA} = \frac{1}{2}\overrightarrow{DA}$, $\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BC}$

$\Rightarrow \overrightarrow{MN} = \frac{\boldsymbol{a}}{2}\boldsymbol{b}+\boldsymbol{a}+\frac{1}{2}\overrightarrow{BC}$

But $\overrightarrow{BC} = \overrightarrow{BA}+\overrightarrow{AD}+\overrightarrow{DC} = -\boldsymbol{a}+\boldsymbol{b}+\lambda\boldsymbol{a}$

$\Rightarrow \overrightarrow{MN} = -\frac{1}{2}\boldsymbol{b}+\boldsymbol{a}+\frac{1}{2}(-\boldsymbol{a}+\boldsymbol{b}+\lambda\boldsymbol{a})$

$$= -\frac{1}{2}\boldsymbol{b} + \boldsymbol{a} - \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{\mathbf{1}}{\mathbf{2}}\lambda\boldsymbol{a} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\lambda\boldsymbol{a} = \frac{1}{2}(1+\lambda)\boldsymbol{a}$$

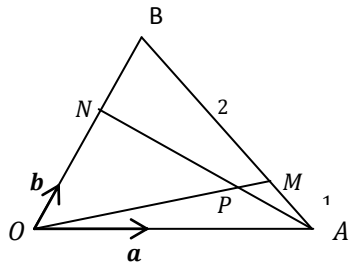
Since $\frac{1}{2}(1+\lambda)$ is a scalar, let $\frac{1}{2}(1+\lambda) = \mu$

$\therefore \overrightarrow{MN} = \mu\boldsymbol{a}$, which shows that $\overrightarrow{MN} \,//\, \overrightarrow{AB}$ and $\overrightarrow{DC}$

From $\overrightarrow{MN} = \frac{1}{2}(1+\lambda)\boldsymbol{a} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\lambda\boldsymbol{a} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{DC} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{DC}\right)$, as required.

8. *OAB is a triangle M and N are points on AB and OB respectively, such that* $\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AB}$ *and* $\overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB}$. *If* $\overrightarrow{OA} = \boldsymbol{a}$ *and* $\overrightarrow{OB} = \boldsymbol{b}$, *find the position vector of P, the intersection of AN and OM, in terms of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$

Solution:



$\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AB} \Rightarrow \frac{AM}{AB} = \frac{1}{3}$ *or* $\overrightarrow{AM}:\overrightarrow{AB} = 1:3$

$\overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB} \Rightarrow \frac{ON}{OB} = \frac{3}{4}$

Points O, P, M and A, P, N are collinear.

Let $\overrightarrow{OP} = \lambda\overrightarrow{OM}$; $\overrightarrow{AP} = \mu\overrightarrow{AN}$

Now expressing $\overrightarrow{OP}$ in terms of $\lambda, \mu, \boldsymbol{a}, and\ \boldsymbol{b}$

$\overrightarrow{OP} = \lambda\overrightarrow{OM}$; $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$

$\Rightarrow \overrightarrow{OP} = \lambda\left(\overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}\right)$ *But* $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \boldsymbol{b} - \boldsymbol{a}$

$\therefore \overrightarrow{OP} = \lambda\left(\boldsymbol{a} + \frac{1}{3}(\boldsymbol{b} - \boldsymbol{a})\right) = \frac{2}{3}\lambda\boldsymbol{a} + \frac{1}{3}\lambda\boldsymbol{b}$ ..... (1)

Also $\overrightarrow{AP} = \mu\overrightarrow{AN}$; $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = \overrightarrow{OP} - \boldsymbol{a}$

$\overrightarrow{AN} = \overrightarrow{AO} + \overrightarrow{ON}$; $\overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB}$

$\therefore \overrightarrow{AN} = \overrightarrow{AO} + \frac{3}{4}\overrightarrow{OB} = -\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$

$\Rightarrow \overrightarrow{OP} - \boldsymbol{a} = \mu\left(-\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) = (1-\mu)\boldsymbol{a} + \frac{3}{4}\mu\boldsymbol{b}$

$\therefore OP = (1-\mu)\boldsymbol{a} + \frac{3}{4}\mu\boldsymbol{b}$ ..... (2)

$Eqn.1 = eqn.2$

$\frac{2}{3}\lambda\boldsymbol{a} + \frac{1}{3}\lambda\boldsymbol{b} = (1-\mu)\boldsymbol{a} + \frac{3}{4}\mu\boldsymbol{b}$

$\frac{2}{3}\lambda = 1 - \mu$ ..... (3)

[$Equating\ coefficients\ of\ \boldsymbol{a}$]

$\frac{1}{3}\lambda = \frac{3}{4}\mu$ ..... (4)

[$Equating\ coefficients\ of\ \boldsymbol{b}$]

From *eqn. (3)* $\mu = 1 - \frac{2}{3}\lambda$ ..... (4)

Eqn. (5) in to eqn. (4) gives

$\frac{1}{3}\lambda = \frac{3}{4}\left(1 - \frac{2}{3}\lambda\right) = \frac{3}{4} - \frac{1}{2}\lambda \Rightarrow \frac{1}{3}\lambda + \frac{1}{2}\lambda = \frac{3}{4}$

$\therefore \frac{5}{6}\lambda = \frac{3}{4} \Rightarrow \lambda = \frac{9}{10}$

From eqn. (4), $\mu = 1 - \frac{2}{3} \times \frac{9}{10} = \frac{2}{5}$

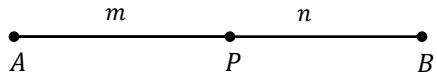
$\overrightarrow{OP} = \frac{2}{3}\lambda\boldsymbol{a} + \frac{1}{3}\lambda\boldsymbol{b} = \frac{2}{3}\left(\frac{9}{10}\right)\boldsymbol{a} + \frac{1}{3}\left(\frac{9}{10}\right)\boldsymbol{b} = \frac{3}{5}\boldsymbol{a} + \frac{3}{10}\boldsymbol{b}$

Task

Show that the diagonals of a square, rhombus and parallelogram bisect each other.

POSITION OF A POINT DIVIDING A LINE *AB* IN THE RATIO $m:n$ (Ratio Theorem)

Let the line AB be divided by a point P in the ratio $m:n$



Ratio $\overrightarrow{AP}:\overrightarrow{PB} = m:n$ *or* $\frac{m}{n} = \frac{\overrightarrow{AP}}{\overrightarrow{PB}} \Rightarrow n\overrightarrow{AP} = m\overrightarrow{PB}$

By position vectors,

$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\overrightarrow{OA} + \overrightarrow{OP}, \overrightarrow{PB} = \overrightarrow{PO} + \overrightarrow{OB} = \overrightarrow{OP} - \overrightarrow{OB}$

$\Rightarrow n(-\overrightarrow{OA} + \overrightarrow{OP}) = m(\overrightarrow{OP} - \overrightarrow{OB})$

$\Rightarrow n\overrightarrow{OP} + m\overrightarrow{OP} = m\overrightarrow{OB} + n\overrightarrow{OA} \ \therefore \overrightarrow{OP}(m+n) = n\overrightarrow{OA} + m\overrightarrow{OB}$

$\Rightarrow \overrightarrow{OP} = \frac{n}{m+n}\overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{OB}$

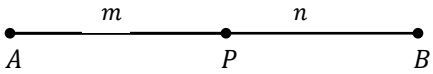
If the position vectors of A and B are respectively $\boldsymbol{a}$ and $\boldsymbol{b}$, then $OA = \boldsymbol{a}, OB = \boldsymbol{b}$. The ratio theorem becomes,

$$\boxed{\overrightarrow{OP} = \frac{n}{m+n}\boldsymbol{a} + \frac{m}{m+n}\boldsymbol{b}}$$

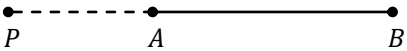
NOTE

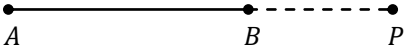
1. P is a point that divides AB either *internally* or *externally* in the ratio the ratio $m:n$

a) For *internal division*, the ratio is represented as
$\overrightarrow{AP}:\overrightarrow{PB} = m:n[Both\ ratio\ values + ve]$



b) For *external division*, the ratio is represented as
$\overrightarrow{AP}:\overrightarrow{PB} = m:-n\ or\ -m:n\ [Both\ ratio\ values of\ opposite\ sign]$

i)


ii)


$$\boxed{\overrightarrow{AP}:\overrightarrow{PB} = m:n}$$

2. The knowledge of position vectors is applicable in ratio theorem.

Examples

1. *Find the position vector (p.v) of a point which divides the line segment AB in the ratio*
   a) $8:3$
   b) $5:-4$

Solution

a) Ratio $8:3 = m:n\ [Internal\ division]$

$\overrightarrow{AP}:\overrightarrow{PB} = 8:3\ or\ \frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{8}{3} \Rightarrow 3\overrightarrow{AP} = 8\overrightarrow{PB}$

By position vectors, $3(\overrightarrow{OP} - \overrightarrow{OA}) = 8(\overrightarrow{OB} - \overrightarrow{OP}) \Rightarrow 3\overrightarrow{OP} - 3\overrightarrow{OA} = 8\overrightarrow{OB} - 8\overrightarrow{OP}$

$11\overrightarrow{OP} = 3\overrightarrow{OA} + 8\overrightarrow{OB} \Rightarrow \overrightarrow{OP} = \frac{3}{11}\overrightarrow{OA} + \frac{8}{11}\overrightarrow{OB}$

Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$ then $\overrightarrow{OP} = \frac{3}{11}\boldsymbol{a} + \frac{8}{11}\boldsymbol{b}$

b) Ratio $5:-4 = m:n$ $[External\ division]$

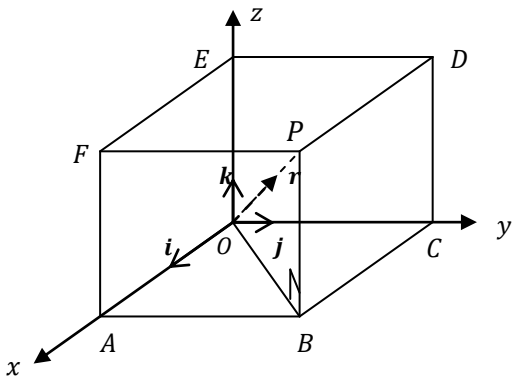
$\overrightarrow{AP}:\overrightarrow{PB} = 5:-4$ *or* $\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{5}{-4} \Rightarrow -4\overrightarrow{AP} = 5\overrightarrow{PB}$

By position vectors, $-4(\overrightarrow{OP} - \overrightarrow{OA}) = 5(\overrightarrow{OB} - \overrightarrow{OP}) \Rightarrow -4\overrightarrow{OP} + 4\overrightarrow{OA} = 5\overrightarrow{OB} - 5\overrightarrow{OP}$

$\overrightarrow{OP} = 5\overrightarrow{OB} - 4\overrightarrow{OA} \ \ \therefore\ \overrightarrow{OP} = -4\boldsymbol{a} + 5\boldsymbol{b}$

VECTOR *"r"* IN TERMS OF UNIT VECTORS i, j AND k

Consider the figure below



$\overrightarrow{OP} = \boldsymbol{r} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$

Let $|\overrightarrow{OA}| = x\ units,$

$|\overrightarrow{AB}| = y\ units, |\overrightarrow{BP}| = z\ units$

From $vector = |vector| \times unit\ vector,$

$\overrightarrow{OA} = x.\boldsymbol{i}, \overrightarrow{AB} = z.\boldsymbol{j}\ and\ \overrightarrow{BP} = z.\boldsymbol{k}$

$\therefore \overrightarrow{OP} = x.\boldsymbol{i} + y.\boldsymbol{j} + z.\boldsymbol{k}$

A point $P(x, y, z)$ has a p.v

$\overrightarrow{OP} = \boldsymbol{r} = x.\boldsymbol{i} + y.\boldsymbol{j} + z.\boldsymbol{k}$

or $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

MODULUS/MAGNITUDE/LENGTH OF A VECTOR r

From ΔOPB, by Pythagoras theorem,

$|\overrightarrow{OP}|^2 = |\boldsymbol{r}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{BP}|^2$; But $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = x.\boldsymbol{i} + y.\boldsymbol{j}$

$|\overrightarrow{OB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{AB}|^2$ $[\because \Delta OAB\ is\ right\ angled]$

$= x^2 + y^2$

$\therefore |\overrightarrow{OP}|^2 = x^2 + y^2 + z^2 \Rightarrow$ $\boxed{|\overrightarrow{OP}| = |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}}$

DISTANCE BETWEEN TWO VECTORS

If $\boldsymbol{r_1} = x_1.\boldsymbol{i} + y_1.\boldsymbol{j} + z_1.\boldsymbol{k}$ and $\boldsymbol{r_2} = x_2.\boldsymbol{i} + y_2.\boldsymbol{j} + z_2.\boldsymbol{k}$, distance between $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ is

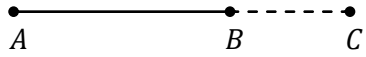
$$\boxed{Distance = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}$$

More examples on Ratio Theorem

1. *Given the position vector* $\overrightarrow{OA} = (3, -2, 5)$ *and* $OB = (9, 1, -1)$. *Find the position vector of point C such that C divides* $\overline{AB}$ *internally in the ratio* $5:-3$ (***Uneb* 2003**)

Solution

Method I



$\overrightarrow{AC}:\overrightarrow{CB} = 5:-3 \Rightarrow \frac{\overrightarrow{AC}}{\overrightarrow{CB}} = \frac{5}{-3}$ *or* $-3\overrightarrow{AC} = 5\overrightarrow{CB}$

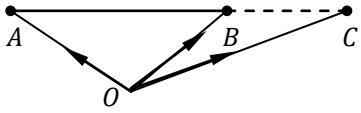
$\Rightarrow -3(\overrightarrow{OC} - \overrightarrow{OA}) = 5(\overrightarrow{OB} - \overrightarrow{OC})$

$\Rightarrow -3\overrightarrow{OC} + 3\overrightarrow{OA} = 5\overrightarrow{OB} - 5\overrightarrow{OC} \;\; \therefore 2\overrightarrow{OC} = 5\overrightarrow{OB} - 3\overrightarrow{OA}$

$$\Rightarrow \overrightarrow{OC} = \frac{5}{2}\overrightarrow{OB} - \frac{3}{2}\overrightarrow{OA} = \frac{5}{2}\begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{45}{2} - \frac{9}{2} \\ \frac{5}{2} + \frac{6}{2} \\ -\frac{5}{2} - \frac{15}{2} \end{pmatrix} = \begin{pmatrix} 18 \\ \frac{11}{2} \\ -10 \end{pmatrix} \;\; \therefore \overrightarrow{OC} = 18\boldsymbol{i} + \frac{11}{2}\boldsymbol{j} - 10\boldsymbol{k}$$

Method II



By Ratio Theorem $\overrightarrow{AC}:\overrightarrow{CB} = 5:-3 = m:n$

$$\overrightarrow{OC} = \frac{n}{m+n}\overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{OB}$$

$$\therefore \overrightarrow{OC} = -\frac{3}{5+-3}\overrightarrow{OA} + \frac{5}{5+-3}\overrightarrow{OB} = \frac{5}{2}\overrightarrow{OB} - \frac{3}{2}\overrightarrow{OA}$$

$$= \frac{5}{2}\begin{pmatrix} 9 \\ 1 \\ -1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

$$\therefore \overrightarrow{OC} = \begin{pmatrix} \frac{45}{2} - \frac{9}{2} \\ \frac{5}{2} + \frac{6}{2} \\ -\frac{5}{2} - \frac{15}{2} \end{pmatrix} = \begin{pmatrix} 18 \\ \frac{11}{2} \\ -10 \end{pmatrix} \Rightarrow \therefore \overrightarrow{OC} = 18\boldsymbol{i} + \frac{11}{2}\boldsymbol{j} - 10\boldsymbol{k}$$

2. *A and B are points whose position vectors are* $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{k}$ *and* $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ *respectively. Determine the p.v of the point that divides AB internally in the ratio* $4:-1$ (***uneb* 1993**)

Solution

Method I

Ratio of division is $\overrightarrow{AP}:\overrightarrow{PB} = 4:-1$ *or* $\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{4}{-1} \;\Rightarrow\; -\overrightarrow{AP} = 4\overrightarrow{PB}$

$\Rightarrow -\overrightarrow{OP} + \overrightarrow{OA} = 4\overrightarrow{OB} - 4\overrightarrow{OP}$

$\Rightarrow 3\overrightarrow{OP} = 4\overrightarrow{OB} - \overrightarrow{OA} \;\; \therefore \overrightarrow{OP} = \frac{4}{3}\overrightarrow{OB} - \frac{1}{3}\overrightarrow{OA} = \frac{4}{3}(\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) - \frac{1}{3}(2\boldsymbol{i} + \boldsymbol{k})$

$\Rightarrow \overrightarrow{OP} = \left(\frac{4}{3} - \frac{2}{3}\right)\boldsymbol{i} - \frac{4}{3}\boldsymbol{j} + \left(\frac{12}{3} - \frac{1}{3}\right)\boldsymbol{k} = \frac{2}{3}\boldsymbol{i} - \frac{4}{3}\boldsymbol{j} + \frac{11}{3}\boldsymbol{k}$

Method II

Using Ratio Theorem

$\overrightarrow{OP} = \frac{n}{m+n}\overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{OB}$, *Ratio*: $\overrightarrow{AP}:\overrightarrow{PB} = 4:-1 = m:n$

$\therefore \overrightarrow{OP} = \frac{-1}{4+-1}\overrightarrow{OA} + \frac{4}{4+-1}\overrightarrow{OB} = -\frac{1}{3}\overrightarrow{OA} + \frac{4}{3}\overrightarrow{OB}$

$= -\frac{1}{3}(2\boldsymbol{i} + \boldsymbol{k}) + \frac{4}{3}(\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) = \frac{2}{3}\boldsymbol{i} - \frac{4}{3}\boldsymbol{j} + \frac{11}{3}\boldsymbol{k}$

NOTE

You can work the vectors in terms of $\boldsymbol{i}, \boldsymbol{j},$ *and* $\boldsymbol{k}$ or write the given vectors in terms of column vectors. *i.e.* For $\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

3. *Given that the position vectors of A, B and C are*

$\overrightarrow{OA} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 7 \\ 10 \\ -7 \end{pmatrix}$

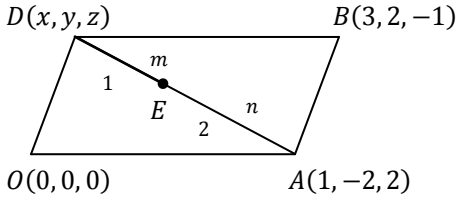
i) Prove that A, B and C are collinear

ii) If OABD is a parallelogram, find the p.v of E and F such that E divides DA in the ratio $1:2$ *and F divides it externally in the ratio* $1:2$ (***Uneb* 2007**)

Solution

i) Shall be tackled later after looking at collinear vectors

ii) For a parallelogram, $\overrightarrow{OD} = \overrightarrow{AB}$ and $\overrightarrow{OA} = \overrightarrow{DB}$



$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$$

$$\therefore D(x, y, z) = D(2, 4, -3)$$

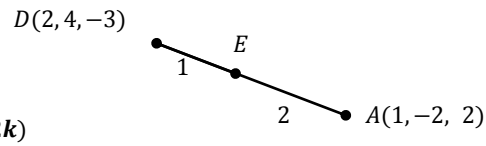
Now finding coordinates of E that divides DA internally in the ratio 1:2

Using Ratio Theorem

$$\overrightarrow{OE} = \frac{n}{m+n}\overrightarrow{OD} + \frac{m}{m+n}\overrightarrow{OA} \quad [state\ from\ D\ to\ A]$$

$$= \frac{2}{2+1}\overrightarrow{OD} + \frac{1}{2+1}\overrightarrow{OA}$$

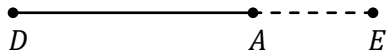
$$= \frac{2}{3}\overrightarrow{OD} + \frac{1}{3}\overrightarrow{OA} = \frac{2}{3}\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$



$$or\ \overrightarrow{OE} = \frac{2}{3}(2\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}) + \frac{1}{3}(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$$

$$\begin{pmatrix} \frac{4}{3} + \frac{1}{3} \\ \frac{8}{3} - \frac{2}{3} \\ -\frac{6}{3} + \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ 2 \\ \frac{4}{3} \end{pmatrix} \therefore \overrightarrow{OE} = \frac{5}{3}\boldsymbol{i} + 2\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$$

Now finding E, for external division



Ratio $\overrightarrow{DE}:\overrightarrow{EA} = 1:-2 = m:n$

Using Ratio Theorem

$$\overrightarrow{OE} = \frac{n}{m+n}\overrightarrow{OD} + \frac{m}{m+n}\overrightarrow{OA}$$

$$\Rightarrow \overrightarrow{OE} = \frac{-2}{1+-2}\overrightarrow{OD} + \frac{1}{1+-2}\overrightarrow{OA} = 2\overrightarrow{OD} - \overrightarrow{OA}$$

$$\overrightarrow{OE} = 2\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4-1 \\ 8+2 \\ -6-2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ -8 \end{pmatrix} \therefore \overrightarrow{OE} = 5\boldsymbol{i} + 10\boldsymbol{j} - 8\boldsymbol{k}$$

Examples on Distance between two points

1. *Prove that the points* $2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{i}-3\boldsymbol{j}-5\boldsymbol{k}$ *and* $3\boldsymbol{i}-4\boldsymbol{j}-4\boldsymbol{k}$ *are vertices of a right angled* Δ.

Solution

Let $P(2,-1,1)$, $Q(1,-3,-5)$, $R(3,-4,-4)$

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}1\\-3\\-5\end{pmatrix}-\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}-1\\2\\-6\end{pmatrix}\Rightarrow\left|\overrightarrow{PQ}\right|=\sqrt{(-1)^2+2^2+(-6)^2}=\sqrt{41}$$

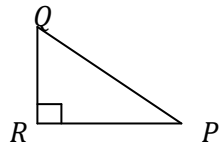
$$\overrightarrow{QR}=\overrightarrow{OR}-\overrightarrow{OQ}=\begin{pmatrix}3\\-4\\-4\end{pmatrix}-\begin{pmatrix}1\\-3\\-5\end{pmatrix}=\begin{pmatrix}2\\-1\\1\end{pmatrix}\Rightarrow\left|\overrightarrow{QR}\right|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{6}$$

$$\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}=\begin{pmatrix}3\\-4\\-4\end{pmatrix}-\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\-3\\-5\end{pmatrix}\Rightarrow\left|\overrightarrow{PR}\right|=\sqrt{1^2+(-3)^2+(-5)^2}=\sqrt{35}$$

For a right angled triangle,

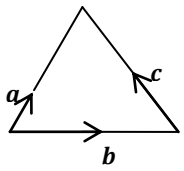
$sum\ of\ squares\ of\ two\ sides = square\ of\ the\ longer\ side$

$$\Rightarrow\left|\overrightarrow{QR}\right|^2+\left|\overrightarrow{PR}\right|^2=\left|\overrightarrow{PQ}\right|^2$$



2. *Prove that the vectors* $2\boldsymbol{i}+3\boldsymbol{j}-6\boldsymbol{k}$, $6\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$ *and* $3\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}$ *form the sides of an equilateral triangle.*

Solution



Let $\boldsymbol{a}=6\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$, $\boldsymbol{b}=2\boldsymbol{i}+3\boldsymbol{j}-6\boldsymbol{k}$ *and* $\boldsymbol{c}=3\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}$

$|\boldsymbol{a}|=\sqrt{6^2+(-2)^2+3^2}=7\ units$

$|\boldsymbol{b}|=\sqrt{2^2+3^2+(-6)^2}=7\ units$

$|\boldsymbol{c}|=\sqrt{3^2+6^2+(-2)^2}=7\ units$

∴ Vectors form sides of an equilateral triangle.

General examples

3. *Show that the points A, B and C with position vectors*
   a) $4\boldsymbol{i}-8\boldsymbol{j}-13\boldsymbol{k}, 3\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k}$ *and* $3\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$
   b) $5\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}, 2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$ *and* $7\boldsymbol{i}-3\boldsymbol{j}+10\boldsymbol{k}$
   c) $7\boldsymbol{i}+10\boldsymbol{k}, -\boldsymbol{i}+6\boldsymbol{j}+\boldsymbol{k}$ *and* $-\boldsymbol{i}+9\boldsymbol{j}+6\boldsymbol{k}$

   , are vertices of a triangle.

Solution

a) $A(4,-8,-13)$, $B(3,-2,-3)$ $C(3,-1,-2)$

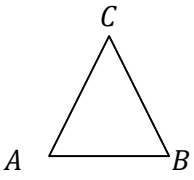
For a closed figure of three sides, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}3\\-2\\-3\end{pmatrix}-\begin{pmatrix}4\\-8\\-13\end{pmatrix}=\begin{pmatrix}-1\\6\\10\end{pmatrix}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}3\\1\\-2\end{pmatrix}-\begin{pmatrix}3\\-2\\-3\end{pmatrix}=\begin{pmatrix}0\\3\\1\end{pmatrix},$$

and $\overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}=\begin{pmatrix}4\\-8\\-13\end{pmatrix}-\begin{pmatrix}3\\1\\-2\end{pmatrix}=\begin{pmatrix}1\\-9\\-11\end{pmatrix}$

Now $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\begin{pmatrix}-1\\6\\10\end{pmatrix}+\begin{pmatrix}0\\3\\1\end{pmatrix}+\begin{pmatrix}1\\-9\\-11\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}=0$

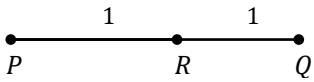


b) *Left as an exercise*

c) *Left as an exercise*

4. *Given that the p.vs of P and Q are* $4\boldsymbol{i}-3\boldsymbol{j}+5\boldsymbol{k}$ *and* $\boldsymbol{i}+2\boldsymbol{k}$ *respectively, find the coordinates of the point R such that* $PQ:PR=2:1$

Solution



Then R is the midpoint of PQ or R divides PQ internally in the ratio 1:1

Method I

$\overrightarrow{PQ}:\overrightarrow{PR}=2:1 \Rightarrow \frac{\overrightarrow{PQ}}{\overrightarrow{PR}}=\frac{2}{1}$ *or* $\overrightarrow{PQ}=2\overrightarrow{PR}$

$\Rightarrow \overrightarrow{OQ}-\overrightarrow{OP}=2\left(\overrightarrow{OR}-\overrightarrow{OP}\right)=2\overrightarrow{OR}-2\overrightarrow{OP}$

$\overrightarrow{OP}=2\overrightarrow{OR}-\overrightarrow{OQ} \Rightarrow \overrightarrow{OR}=\frac{1}{2}\left(\overrightarrow{OP}+\overrightarrow{OQ}\right)$

$$\therefore \overrightarrow{OR}=\frac{1}{2}\left[\begin{pmatrix}4\\-3\\5\end{pmatrix}+\begin{pmatrix}1\\0\\2\end{pmatrix}\right]=\begin{pmatrix}\frac{5}{2}\\-\frac{3}{2}\\\frac{7}{2}\end{pmatrix} \therefore R\left(\tfrac{5}{2},-\tfrac{3}{2},\tfrac{7}{2}\right)$$

Method II

By ratio theorem, $\overrightarrow{PQ}:\overrightarrow{RQ}=1:1\equiv m:n$

$\overrightarrow{OR}=\frac{n}{m+n}\overrightarrow{OP}+\frac{m}{m+n}\overrightarrow{OQ}$

$$=\frac{1}{2}\overrightarrow{OP}+\frac{1}{2}\overrightarrow{OQ}=\frac{1}{2}\left(\overrightarrow{OP}+\overrightarrow{OQ}\right)=\frac{1}{2}\frac{1}{2}\left[\begin{pmatrix}4\\-3\\5\end{pmatrix}+\begin{pmatrix}1\\0\\2\end{pmatrix}\right]=\begin{pmatrix}\frac{5}{2}\\-\frac{3}{2}\\\frac{7}{2}\end{pmatrix}$$

$\therefore R\left(\frac{5}{2},-\frac{3}{2},\frac{7}{2}\right)$

5. *The points A and B have p.vs* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *respectively relative to origin, where* $\boldsymbol{a}=2\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$ *and* $\boldsymbol{b}=-4\boldsymbol{i}+s\boldsymbol{j}+t\boldsymbol{k}$*. Find the possible values of s and t if* $\left|\overrightarrow{AB}\right|=7$ *and* $s=2t$

Solution

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-4\\s\\t\end{pmatrix}-\begin{pmatrix}2\\1\\-3\end{pmatrix}=\begin{pmatrix}-6\\s-1\\t+3\end{pmatrix}$$

$\left|\overrightarrow{AB}\right|=\sqrt{(-6)^2+(s-1)^2+(t+3)^2}=7$

Squaring both sides

$36+(s-1)^2+(t+3)^2=49$

$(s-1)^2+(t+3)^2=13$..... (1)

Now

$s=2t$ .....(2)

$eqn,(2)\ in\ to\ (1)$

$(2t-1)^2+(t+3)^2=13$

$4t^2-4t+1+t^2+6t+9=13 \Rightarrow 5t^2+2t-3=0$

$5t^2+5t-3t-3=0$

$5t(t+1)-3(t+1)=0 \Rightarrow (5t-3)(t+1)=0$

$\therefore t=-1,\frac{3}{5}$

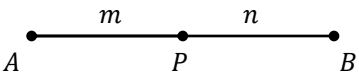
For $t=-1$, $s=2t=-2$ and for $t=\frac{3}{5}$, $s=2t=\frac{6}{5}$

$\therefore s=\frac{6}{5}, t=\frac{3}{5}; s=-2, t=-1$

6. *A and B are points with p.vs* $\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$ *and* $7\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}$ *respectively. Find the coordinates of P and Q which divide AB*
   a) Internally in the ratio 1:2
   b) Externally in the ratio 3:1

Solution

a)

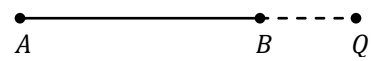


Ratio: $\overrightarrow{AP}:\overrightarrow{PB}=1:2 \Rightarrow \frac{\overrightarrow{AP}}{\overrightarrow{PB}}$
$=\frac{1}{2}$ *or* $2\overrightarrow{AP}=\overrightarrow{PB}$

$\Rightarrow 2\left(\overrightarrow{OP}-\overrightarrow{OA}\right)=\overrightarrow{OB}-\overrightarrow{OP} \quad \therefore 3\overrightarrow{OP}=\overrightarrow{OB}+2\overrightarrow{OA}$

$\Rightarrow OP=\frac{1}{3}(7\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})+\frac{2}{3}(\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k})=3\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k} \quad \therefore P(3,-1,3)$

b)



Ratio: $\overrightarrow{AQ}:\overrightarrow{QB}=3:-1$ (***check ratio values***)

$\frac{\overrightarrow{AQ}}{\overrightarrow{QB}}=\frac{3}{-1} \Rightarrow -\overrightarrow{AQ}=3\overrightarrow{QB}$

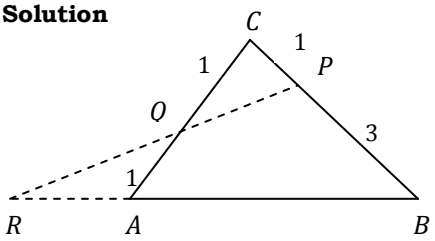
$\Rightarrow 2\left(\overrightarrow{OQ}-\overrightarrow{OA}\right)=3\left(\overrightarrow{OB}-\overrightarrow{OQ}\right)$

$2\overrightarrow{OQ}=3\overrightarrow{OB}-\overrightarrow{OA} \Rightarrow \overrightarrow{OQ}=\frac{3}{2}(7\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})-(\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k})$

$\overrightarrow{OQ}=\frac{19}{2}\boldsymbol{i}-\frac{1}{2}\boldsymbol{j}-7\boldsymbol{k} \quad \therefore Q\left(\frac{19}{2},-\frac{1}{2},-7\right)$

7. *The vertices A, B and C of a triangle have p.vs* ***a, b,*** *and* ***c*** *respectively. The point P divides BC internally in the ratio 3:1, the point Q is the midpoint of CA and the point R divides AB externally in the ratio 1:3. If the p.vs of P, Q and R are* ***p, q*** *and* ***r*** *respectively, express* ***r*** *in terms of* ***p*** *and* ***q****. find also the ratio* $PR:QR$.

Solution



Given: $\overrightarrow{OA}=\boldsymbol{a}, \overrightarrow{OB}=\boldsymbol{b}, \overrightarrow{OC}=\boldsymbol{c}$
$\overrightarrow{BP}:\overrightarrow{PC}=3:1, \overrightarrow{AQ}:\overrightarrow{QC}=1:1$
$\overrightarrow{AR}:\overrightarrow{RB}=-1:3$

$\overrightarrow{OP}=\boldsymbol{p}, \overrightarrow{OQ}=\boldsymbol{q}, \overrightarrow{OR}=\boldsymbol{r}$

From $\overrightarrow{AR}:\overrightarrow{RB}=-1:3$

$\Rightarrow \frac{\overrightarrow{AR}}{\overrightarrow{RB}}=-\frac{1}{3}$ *or* $3\overrightarrow{AR}=-\overrightarrow{RB}$

$\Rightarrow 3\left(\overrightarrow{OR}-\overrightarrow{OA}\right)=-\left(\overrightarrow{OB}-\overrightarrow{OR}\right)=-\overrightarrow{OB}+\overrightarrow{OR}$

$\Rightarrow 2\overrightarrow{OR}=3\overrightarrow{OA}-\overrightarrow{OB} \quad \therefore \overrightarrow{OR}=\frac{3}{2}\boldsymbol{a}-\frac{1}{3}\boldsymbol{b}$ ..... (1)

From $\overrightarrow{BP}:\overrightarrow{PC}=3:1 \Rightarrow \frac{\overrightarrow{BP}}{\overrightarrow{PC}}=\frac{3}{1}$ *or* $\overrightarrow{BP}=3\overrightarrow{PC}$

$\Rightarrow \overrightarrow{OP}-\overrightarrow{OB}=3\left(\overrightarrow{OC}-\overrightarrow{OP}\right)=3\overrightarrow{OC}-3\overrightarrow{OP}$

$\Rightarrow 4\overrightarrow{OP}=\overrightarrow{OB}+3\overrightarrow{OC}=\boldsymbol{b}+3\boldsymbol{c}$

$\therefore \overrightarrow{OP}=\frac{1}{4}\boldsymbol{b}+\frac{3}{4}\boldsymbol{c}$ ..... (2)

From $\overrightarrow{AQ}:\overrightarrow{QC}=1:1 \Rightarrow \overrightarrow{AQ}=\overrightarrow{QC}$

$\Rightarrow \overrightarrow{OQ}-\overrightarrow{OA}=\overrightarrow{OC}-\overrightarrow{OQ}$

$\Rightarrow 2\overrightarrow{OQ}=\overrightarrow{OA}+\overrightarrow{OC} \quad \therefore \overrightarrow{OQ}=\frac{1}{2}\overrightarrow{OA}+\frac{1}{2}\overrightarrow{OC}$

$\therefore \boldsymbol{q} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$ ..... (3)

Solving (1) , (2) and (3) simultaneously by expressing $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$

$\boldsymbol{p} = \frac{1}{4}\boldsymbol{b} + \frac{3}{4}\boldsymbol{c}$ *or* $4\boldsymbol{p} = \boldsymbol{b} + 3\boldsymbol{c}$ ..... (4)

$\boldsymbol{q} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$ *or* $2\boldsymbol{q} = \boldsymbol{a} + \boldsymbol{c}$ ..... (5)

Eliminating **c** from (4) and (5)

$4\boldsymbol{p} = \boldsymbol{b} + 3\boldsymbol{c}$
$2\boldsymbol{q} = \boldsymbol{a} + \boldsymbol{c}$ $\Big| \times 3$

$\Rightarrow 4\boldsymbol{p} = \boldsymbol{b} + 3\boldsymbol{c}$
$6\boldsymbol{q} = 3\boldsymbol{a} + 3\boldsymbol{c}$
$4\boldsymbol{p} - 6\boldsymbol{q} = -3\boldsymbol{a} + \boldsymbol{b}$ ..... (6)

From (1) $\overrightarrow{OR} = \boldsymbol{r} = \frac{3}{2}\boldsymbol{a} - \frac{1}{3}\boldsymbol{b}$ *or* $2\boldsymbol{r} = 3\boldsymbol{a} - \boldsymbol{b}$ ..... (7)

Eliminating **a** and **b** from (6) and (7), adding the equations

$\Rightarrow 4\boldsymbol{p} - 6\boldsymbol{q} + 2\boldsymbol{r} = 0$ *or* $2\boldsymbol{p} - 3\boldsymbol{q} + \boldsymbol{r} = 0$

$\therefore \boldsymbol{r} = 3\boldsymbol{q} - 2\boldsymbol{p}$

Ratio $\overrightarrow{PR}:\overrightarrow{QR} = \frac{\overrightarrow{PR}}{\overrightarrow{QR}}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (3\boldsymbol{q} - 2\boldsymbol{p}) - \boldsymbol{p} = 3(\boldsymbol{q} - \boldsymbol{p})$

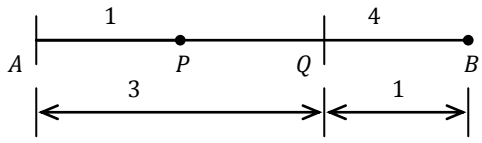
$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = (3\boldsymbol{q} - 2\boldsymbol{p}) - \boldsymbol{q} = 2(\boldsymbol{q} - \boldsymbol{p})$

$\therefore \frac{\overrightarrow{PR}}{\overrightarrow{QR}} = \frac{3}{2}\frac{(\boldsymbol{q}-\boldsymbol{p})}{(\boldsymbol{q}-\boldsymbol{p})} = \frac{3}{2}$

$\therefore \overrightarrow{PR}:\overrightarrow{QR} = 3:2$ $(Internal\ division)$

8. A, P, Q, B are points on a straight line such that $AP:PB = 1:4, AQ:QB = 3:1$. Find;
   a) $AP:PQ:QB$
   b) $AQ:QP$

Solution



Let p.v of A and B be **a** and **b** respectively.

For $AP:PQ = 1:4 \Rightarrow \overrightarrow{OP} = \frac{4}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b}$

For $AQ:QB = 3:1 \Rightarrow \overrightarrow{OQ} = \frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$

a) $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \frac{4}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b} - \boldsymbol{a} = \frac{1}{5}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \left(\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) - \left(\frac{4}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b}\right) = -\frac{11}{20}\boldsymbol{a} + \frac{11}{20}\boldsymbol{b} = \frac{11}{20}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{QB} = \overrightarrow{OB} - \overrightarrow{OQ} = \boldsymbol{b} - \left(\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) = -\frac{1}{4}\boldsymbol{a} + \frac{1}{4}\boldsymbol{b} = \frac{1}{4}(\boldsymbol{b} - \boldsymbol{a})$

$Now\ \overrightarrow{AP}:\overrightarrow{PQ} = \frac{\overrightarrow{AP}}{\overrightarrow{PQ}} = \frac{\frac{1}{5}(\boldsymbol{b}-\boldsymbol{a})}{\frac{11}{20}(\boldsymbol{b}-\boldsymbol{a})} = \frac{4}{11}$ $\quad \therefore \overrightarrow{AP}:\overrightarrow{PQ} = 4:11$

Also $\overrightarrow{PQ}:\overrightarrow{QB} = \frac{\frac{11}{20}(\boldsymbol{b}-\boldsymbol{a})}{\frac{1}{4}(\boldsymbol{b}-\boldsymbol{a})} = \frac{11}{5}$ $\quad \therefore \overrightarrow{PQ}:\overrightarrow{QB} = 11:5$

Combining the ratios using the ratio algebra,

$AP:PQ = 4:11$; $PQ:QB = 11:5$ $\therefore AP:PQ:QB = 4:11:5$

b) $\overrightarrow{AQ}:\overrightarrow{QP} = \frac{\overrightarrow{AQ}}{\overrightarrow{QP}}$

$\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = \left(\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) - \boldsymbol{a} = \frac{3}{4}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \left(\frac{4}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b}\right) - \left(\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}\right) = -\frac{11}{20}(\boldsymbol{b} - \boldsymbol{a})$

$\therefore \overrightarrow{AQ}:\overrightarrow{QP} = \frac{\overrightarrow{AQ}}{\overrightarrow{QP}} = \frac{\frac{3}{4}(\boldsymbol{b}-\boldsymbol{a})}{-\frac{11}{20}(\boldsymbol{b}-\boldsymbol{a})} = \frac{3}{4}\times\frac{20}{-11} = \frac{15}{-11}$ $\therefore AQ:QP = 15:-11$

NOTE:

Vectors $\overrightarrow{OP} = \frac{4}{5}\boldsymbol{a} + \frac{1}{5}\boldsymbol{b}$ and $\overrightarrow{OQ} = \frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$ have been deduced using *Ratio Theorem*

Task

1. If the points A and B have p.vs **a** and **b** respectively, find in terms of **a** and **b** the p.vs of
   a) The point P which divides AB internally in the ratio 5:3
   b) The point Q which divides AB externally in the ratio 3:1
   c) The point R such that A is the midpoint of BR

   ANSWER: a) $\frac{1}{8}(3\boldsymbol{a}+5\boldsymbol{b})$,b) $\frac{1}{2}(3\boldsymbol{b}-\boldsymbol{a})$ c) $2\boldsymbol{a}-\boldsymbol{b}$
2. Write down, in terms of **I , j** and **k** the position vectors of the points $A(2,-5,3)$ and $B(7,0,-2)$. Hence find the coordinates of C which divides AB internally in the ratio 2:3 and point D which divides it externally in the ratio 3:8.

   ANSWER: $2\boldsymbol{i}-5\boldsymbol{j}+3\boldsymbol{k}; 7\boldsymbol{i}-2\boldsymbol{k};\ (4,-3,1),(-1,-8,6)$
3. The points A, B, C and D have coordinates $A(-7,\ 9), B(3,4), C(1,2)$ and $D(-2.-9)$. Find the position vector of a point P that divides AB in the ratio $2:3$ and Q that divides it in the ratio $1:-4$
4. **ANSWER**:Show that the points A, B and C with p.vs $3\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}, 8\boldsymbol{i}+7\boldsymbol{j}+4\boldsymbol{k}$ and $11\boldsymbol{i}+4\boldsymbol{j}+5\boldsymbol{k}$ respectively are vertices of a triangle. (***Uneb* 2011**)
5. Given that $\boldsymbol{a} = 2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}, \boldsymbol{b} = \boldsymbol{i}+2\boldsymbol{k}, \boldsymbol{c} = \boldsymbol{i}+2\boldsymbol{j}$ and $\boldsymbol{d} = -4\boldsymbol{j}+8\boldsymbol{k}$. Express **d** in terms of **a, b** and **c.**

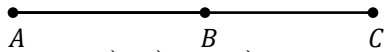
   Answer: *Left as an exercise*

   [$HINT: Let\ \boldsymbol{d} =$
   $\lambda\boldsymbol{a}+\mu\boldsymbol{b}+k\boldsymbol{c}, Form\ and\ solve\ 3\ simulutaneous\ equations\ for\ the\ scalars$]

COLLINEARITY OF VECTORS

Vectors are either collinear or parallel.

Consider three collinear points A, B and C



vectors $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{AC}$ are collinear if $\boxed{\lambda\overrightarrow{AB}+\mu\overrightarrow{BC}+\gamma\overrightarrow{AC}=0}$ where $\lambda, \mu\ and\ \gamma$ are Scalars/constants and $\boxed{\lambda+\mu+\gamma=0}$

When vectors are given in the form $\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$, 3-equations are formulated containing only scalars. The 3 equations are solved for the scalars using cross multiplication, and when put in , $\boxed{\lambda+\mu+\gamma=0}$ and satisfies then the vectors are collinear.

Alternatively, (commonly used)

For $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{AC}$ to be collinear/parallel, then $\boxed{\overrightarrow{AB}=\lambda\overrightarrow{BC}=\mu\overrightarrow{AC}}$, where λ, μ are scalars.

Examples

1. *Show that the points with the position vectors*

a) $\boldsymbol{i}+2\boldsymbol{j}+5\boldsymbol{k}\,,3\boldsymbol{i}+2j+\boldsymbol{k}$ *and* $2\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$

b) $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}\,,4\boldsymbol{i}+3\boldsymbol{j}$ *and* $10\boldsymbol{i}+7\boldsymbol{j}-2\boldsymbol{k}$

, are collinear.

Solution

a) Let $A(1\,,2\,,5)\,,B(3\,,2\,,1)\,,C(2\,,2\,,3)$

For A, B and C to be collinear, $\overrightarrow{AB}=\lambda\overrightarrow{AC}$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}3\\2\\1\end{pmatrix}-\begin{pmatrix}1\\2\\5\end{pmatrix}=\begin{pmatrix}2\\0\\-4\end{pmatrix}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}2\\2\\3\end{pmatrix}-\begin{pmatrix}1\\2\\5\end{pmatrix}=\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

Now $\overrightarrow{AB}=\lambda\overrightarrow{AC}$ gives

$$\begin{pmatrix}2\\0\\-4\end{pmatrix}=\lambda\begin{pmatrix}1\\0\\-2\end{pmatrix}\Rightarrow 2=\lambda \ldots\ldots\ldots\ldots\ldots (1)$$

$$\Rightarrow -4=-2\lambda \;\; or\; \lambda=2 \ldots\ldots\ldots\ldots\ldots (2)$$

Since the $\lambda=2$ is consistent for the two equations formulated, the points are collinear.

Alternatively (Not commonly used)

Let $A(1\,,2\,,5)\,,B(3\,,2\,,1)\,,C(2\,,2\,,3)$ have position vectors $\overrightarrow{OA},\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.

For the vectors to be collinear, then $\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}+\gamma\overrightarrow{OC}=0$

$$\Rightarrow \lambda\begin{pmatrix}1\\2\\5\end{pmatrix}+\mu\begin{pmatrix}3\\2\\1\end{pmatrix}+\gamma\begin{pmatrix}2\\2\\3\end{pmatrix}=0$$

$\Rightarrow \lambda+3\mu+2\gamma=0$ ..... (1)

$\Rightarrow 2\lambda+2\mu+2\gamma=0$ ..... (2)

$\Rightarrow 5\lambda+\mu+\gamma=0$..... (3)

Also $\lambda+\mu+\gamma=0$ ..... (4)

Expressing $\lambda\,,\mu$ in terms of γ only

Considering eqn. (1) and eqn. (2), eliminating λ

$$\begin{array}{l|l}\lambda+3\mu+2\gamma=0 & \times 2\\ \underline{2\lambda+2\mu+2\gamma=0} & \underline{\times 1}\end{array}$$

$4\mu+2\gamma=0\Rightarrow\mu=-\frac{1}{2}\gamma$..... (5)

Eqn. (5) in to *eqn. (1)* gives

$\lambda+3\left(-\frac{1}{2}\gamma\right)+2\gamma=0\;\Rightarrow\lambda=-\frac{1}{2}\gamma$ ..... (6)

Eqn. (5), (6) in to (3) and (4) for confirmation

From eqn. (3),

$5\lambda+\mu+\gamma=0\Rightarrow-\frac{5}{2}\gamma-\frac{1}{2}\gamma+3\gamma=0\;\; correct$

From eqn. (4) $\lambda+\mu+\gamma=0$

$\Rightarrow-\frac{1}{2}\gamma-\frac{1}{2}\gamma+\gamma=0\;\; correct$. Hence points are collinear.

b) *Left as an exercise*

2. *Show that the points R, P and T with position vectors are*

a) $\frac{1}{2}(3\boldsymbol{a}-\boldsymbol{b})\,,\frac{1}{70}(53\boldsymbol{a}+17\boldsymbol{b})$ *and* $\frac{1}{5}(\boldsymbol{a}+4\boldsymbol{b})$

b) $7\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c},10a+2\boldsymbol{b}+9\boldsymbol{c}$ *and* $\boldsymbol{a}-\boldsymbol{b}+3\boldsymbol{c}$

c) $\frac{1}{2}(17a+20\boldsymbol{b}+3\boldsymbol{c})\,,10\boldsymbol{a}+11\boldsymbol{b}+2\boldsymbol{c}$ *and* $8\boldsymbol{a}+\frac{29}{3}\boldsymbol{b}+\frac{4}{3}\boldsymbol{c}$, are collinear.

Solution

a) For R, P and T to be collinear, $\overrightarrow{RP} = \lambda\overrightarrow{RT}$

$\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR} = \frac{1}{70}(53\boldsymbol{a} + 17\boldsymbol{b}) - \frac{1}{2}(3\boldsymbol{a} - \boldsymbol{b})$

$= -\frac{26}{35}\boldsymbol{a} + \frac{26}{35}\boldsymbol{b} = \frac{26}{35}(\boldsymbol{b} - \boldsymbol{a})$

$\overrightarrow{RT} = \overrightarrow{OT} - \overrightarrow{OR} = \frac{1}{5}(\boldsymbol{a} + 4\boldsymbol{b}) - \frac{1}{2}(3\boldsymbol{a} - \boldsymbol{b})$

$= -\frac{13}{10}\boldsymbol{a} + \frac{13}{10}\boldsymbol{b} = \frac{13}{10}(\boldsymbol{b} - \boldsymbol{a})$

Now $\overrightarrow{RP} = \lambda\overrightarrow{RT}$ yields

$\frac{26}{35}(\boldsymbol{b} - \boldsymbol{a}) = \lambda\left[\frac{13}{10}(\boldsymbol{b} - \boldsymbol{a})\right] \Rightarrow \frac{26}{35} = \frac{13}{10}\lambda \quad \therefore \lambda = \frac{2}{3}(scalar)$

$\therefore \overrightarrow{RP} = \frac{2}{3}\overrightarrow{RT}$, if R, P and T are collinear

b) $\overrightarrow{RP} = \lambda\overrightarrow{RT}$

$\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR} = (10\boldsymbol{a} + 11\boldsymbol{b} + 2\boldsymbol{c}) - (7\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) = 3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c}$

$\overrightarrow{RT} = \overrightarrow{OT} - \overrightarrow{OR} = (\boldsymbol{a} - \boldsymbol{b} + 3\boldsymbol{c}) - (7\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$

$= -6a - 2\boldsymbol{b} - 4\boldsymbol{c} = -2(3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c})$

$\Rightarrow (3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c}) = -2(3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c})\lambda$

$\Rightarrow 1 = -2\lambda \quad \therefore \lambda = -\frac{1}{2}\ (scalar)$

$\therefore \overrightarrow{RP} = -\frac{1}{2}\overrightarrow{RT}$, if R, P and T are collinear

3. *Given that* $\boldsymbol{a} = 2\boldsymbol{i} + 6\boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} + t\boldsymbol{j}$ *and* $\boldsymbol{c} = 2\lambda\boldsymbol{i} - \boldsymbol{j}$. *Find the value of* λ *and* t *if*
   a) $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are parallel*
   b) $\boldsymbol{a}$ *and* $\boldsymbol{c}$ *are collinear*

Solution

a) For **a** and **b** to be parallel, $\boldsymbol{a} = \mu\boldsymbol{b}$

$\Rightarrow (2\boldsymbol{i} + 6\boldsymbol{j}) = \mu(3\boldsymbol{i} + t\boldsymbol{j}) = 3\mu\boldsymbol{i} + t\mu\boldsymbol{j}$

Equating coefficients of unit vectors

For $\boldsymbol{i}$: $2 = 3\mu \Rightarrow \mu = \frac{2}{3}$

For $\boldsymbol{j}$: $6 = t\mu \Rightarrow 6 = \frac{2}{3}t \Rightarrow t = 9$

b) **a** and **c** are collinear/parallel if $\boldsymbol{a} = k\boldsymbol{c}$

$\Rightarrow 2\boldsymbol{i} + 6\boldsymbol{j} = k(2\lambda\boldsymbol{i} - \boldsymbol{j}) = 2\lambda k\boldsymbol{i} - k\boldsymbol{j}$

Equating coefficients of unit vectors

For $\boldsymbol{i}$: $2 = 2k\lambda$ ..... (1)

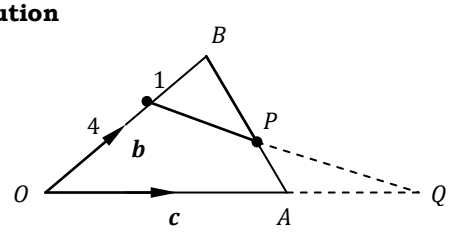
For $\boldsymbol{j}$: $6 = -k \Rightarrow k = -6$ ..... (2)

Eqn. (2 in to (1)

$2 = -12\lambda \Rightarrow \lambda = -\frac{1}{6} \quad \therefore k = -6\ , \lambda = -\frac{1}{6}$

4. *Given that* $\overrightarrow{OA} = \boldsymbol{a}$ *and* $\overrightarrow{OB} = \boldsymbol{b}$, *point R is on* $\overrightarrow{OB}$ *such that* $\overrightarrow{OR}:\overrightarrow{RB} = 4:1$. *Point P is on BA such that* $\overrightarrow{BP}:\overrightarrow{PA} = 2:3$, *and when RP and OA are both produced they meet at point Q. Find*
   a) $\overrightarrow{OR}$ and $\overrightarrow{OP}$ in terms of **a** and b
   b) $\overrightarrow{OQ}$ in terms of **a** (***Uneb* 2006**)

Solution



a) From $\overrightarrow{OR}:\overrightarrow{RB}=4:1 \Rightarrow \frac{\overrightarrow{OR}}{\overrightarrow{RB}}=\frac{4}{1}$

$\overrightarrow{OR}=4\overrightarrow{RB}$

$\Rightarrow \overrightarrow{OR}=4\left(\overrightarrow{OB}-\overrightarrow{OR}\right) \Rightarrow \overrightarrow{OR}=\frac{4}{5}\overrightarrow{OB}$

$\therefore \overrightarrow{OR}=\frac{4}{5}\boldsymbol{b}$

From $\overrightarrow{BP}:\overrightarrow{PA}=2:3 \Rightarrow \frac{\overrightarrow{BP}}{\overrightarrow{PA}}=\frac{2}{3} \Rightarrow 3\overrightarrow{BP}=2\overrightarrow{PA}$

$\Rightarrow 3\left(\overrightarrow{OP}-\overrightarrow{OB}\right)=2\left(\overrightarrow{OA}-\overrightarrow{OP}\right) \Rightarrow 5\overrightarrow{OP}=2\overrightarrow{OA}+3\overrightarrow{OB}$

$\Rightarrow \overrightarrow{OP}=\frac{2}{5}\overrightarrow{OA}+\frac{3}{5}\overrightarrow{OB} \quad \therefore \overrightarrow{OP}=\frac{2}{5}\boldsymbol{a}+\frac{3}{5}\boldsymbol{b}$

b) Since O,A and Q are collinear, then

$\overrightarrow{OQ}=\lambda\overrightarrow{OA}=\lambda\boldsymbol{a}$ ..... (1)

Since R, P and Q are also collinear, then

$\overrightarrow{RQ}=\mu\overrightarrow{RP} \Rightarrow \left(\overrightarrow{OQ}-\overrightarrow{OR}\right)=\mu\left(\overrightarrow{OP}-\overrightarrow{OR}\right)$

$\therefore \overrightarrow{OQ}=\overrightarrow{OR}+\mu\left(\overrightarrow{OP}-\overrightarrow{OR}\right)=(1-\mu)\overrightarrow{OR}+\mu\overrightarrow{OP}$

$=(1-\mu).\frac{4}{5}\boldsymbol{b}+\mu\left(\frac{2}{5}\boldsymbol{a}+\frac{3}{5}\boldsymbol{b}\right)=\left(\frac{4-4\mu}{5}+\frac{3}{5}\mu\right)\boldsymbol{b}+\frac{2}{5}\mu\,\boldsymbol{a}$

$\therefore \overrightarrow{OQ}=\left(\frac{4-\mu}{5}\right)\boldsymbol{b}+\frac{2}{5}\mu\,\boldsymbol{a}$ ..... (2)

$eqn.(2)=eqn.(1)$

$\Rightarrow \lambda\boldsymbol{a}=\frac{2}{5}\mu\,\boldsymbol{a}+\left(\frac{4-\mu}{5}\right)\boldsymbol{b}$

Equating corresponding vectors

$\lambda=\frac{2}{5}\mu$ ..... (3)

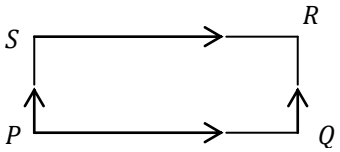
$0=\frac{4-\mu}{5} \Rightarrow \mu=4$..... (4)

$\therefore \lambda=\frac{8}{5}$

From eqn. (1), $\overrightarrow{OQ}=\lambda\overrightarrow{OA}=\lambda\boldsymbol{a}=\frac{8}{5}\boldsymbol{a}$

5. *The vertices of a triangle are $P(2,-1,5), Q(7,1,-3)$ and $R(2\,,-2,0)$. Find the coordinates of S if PQRS is a rectangle.*

Solution



using $\overrightarrow{PQ}=\overrightarrow{SR}$ or $\overrightarrow{PS}=\overrightarrow{QR}$

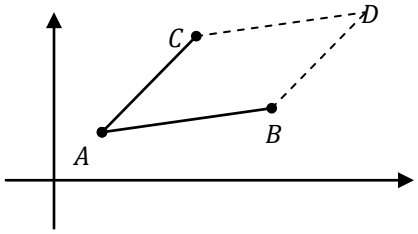
$\overrightarrow{PQ}=\overrightarrow{SR}$, gives $\overrightarrow{OQ}-\overrightarrow{OP}=\overrightarrow{OR}-\overrightarrow{OS}$

$\therefore \overrightarrow{OS}=\overrightarrow{OR}+\overrightarrow{OP}-\overrightarrow{OQ}$

$$=\begin{pmatrix}2\\-2\\0\end{pmatrix}+\begin{pmatrix}2\\-1\\5\end{pmatrix}-\begin{pmatrix}7\\1\\-3\end{pmatrix}=\begin{pmatrix}-3\\-4\\8\end{pmatrix} \quad \therefore S(-3\,,-4,\ 8)$$

6. *The vertices of a parallelogram are $A(1\,,1), B(5\,,3), C(3\,,7)$ and D. find the coordinates of D.*

Solution



For a parallelogram, $\overrightarrow{AB}=\overrightarrow{CD}$ or $\overrightarrow{AC}=\overrightarrow{BD}$

Using $\overrightarrow{AB}=\overrightarrow{CD}$

$\overrightarrow{OB}-\overrightarrow{OA}=\overrightarrow{OD}-\overrightarrow{OC} \Rightarrow \overrightarrow{OD}=\overrightarrow{OC}+\overrightarrow{OB}-\overrightarrow{OA}$

$\overrightarrow{OD}=\begin{pmatrix}3\\7\end{pmatrix}+\begin{pmatrix}5\\3\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}7\\9\end{pmatrix} \quad \therefore D(7\,,9)$

PRODUCT OF TWO VECTORS

There are two types of products

- Scalar or Dot product
- Vector or Cross product

SCALAR/DOT PRODUCT

Consider two position vectors $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$ making and angle of θ with each other. The scalar product is defined as

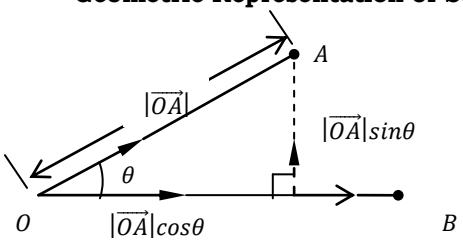
$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta \ \ where\, 0 \leq \theta \leq \pi$$

The product will either be $+ve\ or - ve\ or\ zero$ depending on the value of θ.

NOTE

Two vectors can be dot multiplied if they are in *same direction.*

Geometric Representation of Scalar product



$|\overrightarrow{OA}|cos\theta$ is the projection of $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$\overrightarrow{OA}cos\theta$ and $\boldsymbol{b}$ are in the same direction.

Multiplying their scalars

$\Rightarrow |\overrightarrow{OA}|cos\theta \times |\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|cos\theta = \boldsymbol{a}.\boldsymbol{b}$

$$\therefore \boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta$$

PROPERTIES OF SCALAR PRODUCT

The properties below are very useful in the computations involving product of vectors.

1) $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$ commutative property
2) $(\boldsymbol{\lambda a}).\boldsymbol{b} = \lambda(\boldsymbol{a}.\boldsymbol{b})$ where $\lambda = scalar$
3) $(\boldsymbol{\lambda a}).(\mu \boldsymbol{b}) = \mu\lambda(\boldsymbol{a}.\boldsymbol{b}) = (\mu \boldsymbol{a}.\lambda \boldsymbol{b})$
4) $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$ Distributive property
5) $(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} + \boldsymbol{b}) = (\boldsymbol{a} + \boldsymbol{b})^{\boldsymbol{2}} = \boldsymbol{a}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$
   $= a^{\boldsymbol{2}} + \boldsymbol{2a}.\boldsymbol{b} + b^{\boldsymbol{2}}$ ($a\ and\ b$ are now $scalars$)
6) $(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a} - \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$
   $= a^2 - b^2$ $[\boldsymbol{a} \neq a]$. Here a and b are scalars
7) $(\boldsymbol{a} - \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b}) = (\boldsymbol{a} - \boldsymbol{b})^{\boldsymbol{2}} = \boldsymbol{a}.\boldsymbol{a} - \boldsymbol{a}.\boldsymbol{b} - \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$
   $= a^{\boldsymbol{2}} + \boldsymbol{2a}.\boldsymbol{b} + b^{\boldsymbol{2}}$ ($a\ and\ b$ are now $scalars$)
8) For $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta$
   a) If $\theta = 90^0\ i.e.\ \boldsymbol{a}\ is\ perpendicular\ to\ \boldsymbol{b}$, then $\boldsymbol{a}.\boldsymbol{b} = 0$
      $[\because cos90^0 = 0]$. Hence **a** and **b** are perpendicular if

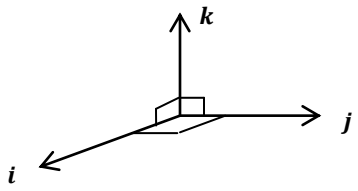
      $$\boldsymbol{a}.\boldsymbol{b} = 0$$

   b) If $\theta = 180^0.i.e.\ \boldsymbol{a}\ and\ \boldsymbol{b}\ are\ in\ opposide\ direction$, then
      $\boldsymbol{a}.\boldsymbol{b} = -|\boldsymbol{a}||\boldsymbol{b}|$ $[\because cos180^0 = -1]$
   c) Angle between **a** and **b** is given from

      $$cos\theta = \frac{|\boldsymbol{a}||\boldsymbol{b}|}{\boldsymbol{a}.\boldsymbol{b}}$$

MULTIPLICATION OF UNIT VECTORS $\boldsymbol{i, j, k}$

Unit vectors are mutually perpendicular



$\boldsymbol{i}.\boldsymbol{i} = \boldsymbol{i}^2 = |\boldsymbol{i}||\boldsymbol{i}|cos0 = 1$
$\boldsymbol{j}.\boldsymbol{j} = \boldsymbol{j}^2 = |\boldsymbol{j}||\boldsymbol{j}|cos0 = 1$
$\boldsymbol{k}.\boldsymbol{k} = \boldsymbol{k}^2 = |\boldsymbol{k}||\boldsymbol{k}|cos0 = 1$
$\boldsymbol{k}.\boldsymbol{i} = |\boldsymbol{k}||\boldsymbol{i}|cos90 = 0$
$\boldsymbol{i}.\boldsymbol{j} = |\boldsymbol{i}||\boldsymbol{j}|cos90 = 0$
$\boldsymbol{k}.\boldsymbol{j} = |\boldsymbol{k}||\boldsymbol{j}|cos90 = 0$

The above results in conjunction with the above properties of dot product are useful in vector product computations.

Example

1. *Given that* $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ *and* $\boldsymbol{c} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$. *Find*
   a) $\boldsymbol{a}.\boldsymbol{b}$
   b) $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c})$
   c) $(\boldsymbol{a}.\boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b})$

Solution

a) $\boldsymbol{a}.\boldsymbol{b} = (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}).(\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) = 2\boldsymbol{i}.\boldsymbol{i} + 2\boldsymbol{i}.\boldsymbol{j} - 6\boldsymbol{i}.\boldsymbol{k}$

$$+\boldsymbol{j}.\boldsymbol{i} + \boldsymbol{j}.\boldsymbol{j} - 3\boldsymbol{j}.\boldsymbol{k} + \boldsymbol{k}.\boldsymbol{i} + \boldsymbol{k}.\boldsymbol{j} - 3\boldsymbol{k}.\boldsymbol{k}$$

$$= 2 + 0 - 0 + 0 + 1 + 0 + 0 + 0 - 3 = 2 + 1 - 3 = 0$$

b) $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c})\ ; \boldsymbol{b} + \boldsymbol{c} = (\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + (\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}) = 2\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k}$

$\Rightarrow \boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}).(2\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k})$

$= \boldsymbol{i}.\boldsymbol{i} + 2\boldsymbol{i}.\boldsymbol{j} - 3\boldsymbol{i}.\boldsymbol{k} + \boldsymbol{j}.\boldsymbol{i} + \boldsymbol{j}.\boldsymbol{j} - 3\boldsymbol{j}.\boldsymbol{k} - 3\boldsymbol{k}.\boldsymbol{i} - 6\boldsymbol{k}.\boldsymbol{j} + 9\boldsymbol{k}.\boldsymbol{k}$

$= 2 + 0 + 0 + 0 + 2 - 0 + 0 + 0 - 6 = 2 + 2 - 6 = -2$

c) $(\boldsymbol{a}.\boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b});\ \boldsymbol{a} - \boldsymbol{b} = (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) - (\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) = \boldsymbol{i} + 0\boldsymbol{j} + 4\boldsymbol{k}$

But $\boldsymbol{a}.\boldsymbol{b} = 0 \Rightarrow (\boldsymbol{a}.\boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b}) = 0.(\boldsymbol{i} + 0\boldsymbol{j} + 4\boldsymbol{k}) = 0$

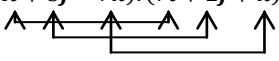
NOTE

When doting two vectors in multiplication always dot the corresponding vectors only. $i.e.\ i^2, j^2$ *and* k^2, since other products of not similar unit vectors yield zero results

2. *Given that* $\boldsymbol{a} = 4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}, \boldsymbol{b} = 7\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ *and* $\boldsymbol{c} = 3\boldsymbol{i} - 4\boldsymbol{j} + 9\boldsymbol{k}$. *Find*
   a) $\boldsymbol{a}.\boldsymbol{b}$
   b) $\boldsymbol{b}.\boldsymbol{c}$
   c) $\boldsymbol{a}.(\boldsymbol{a} + \boldsymbol{b})$
   d) $(\boldsymbol{a}.\boldsymbol{b}).(\boldsymbol{b} + \boldsymbol{c})$

Solution

a) $\boldsymbol{a}.\boldsymbol{b} = (4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}).(7\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$



$= 28 + 10 - 7 = 31$

b) $\boldsymbol{b}.\boldsymbol{c} = (7\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}).(3\boldsymbol{i} - 4\boldsymbol{j} + 9\boldsymbol{k})$

$= 21 - 8 + 9 = 22$

c) $\boldsymbol{a}.(\boldsymbol{a} + \boldsymbol{b})\ ; \boldsymbol{a} + \boldsymbol{b} = (4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}) + (7\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = 11\boldsymbol{i} + 7\boldsymbol{j} - 6\boldsymbol{k}$

$\Rightarrow \boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}) = (4\boldsymbol{i}+5\boldsymbol{j}-7\boldsymbol{k}).(11\boldsymbol{i}+7\boldsymbol{j}-6\boldsymbol{k}) = 44+35+42 = 121$

Alternatively

Using $\boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a}+\boldsymbol{a}.\boldsymbol{b}$

$\boldsymbol{a}.\boldsymbol{a} = (4\boldsymbol{i}+5\boldsymbol{j}-7\boldsymbol{k}).(4\boldsymbol{i}+5\boldsymbol{j}-7\boldsymbol{k}) = 16+25+49 = 90$

$\therefore\ \boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}) = 90+31 = 121$

d) $(\boldsymbol{a}.\boldsymbol{b}).(\boldsymbol{b}+\boldsymbol{c}) = 31\times[(7\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})+(3\boldsymbol{i}-4\boldsymbol{j}+9\boldsymbol{k})]$

$= 31(10\boldsymbol{i}-\mathbf{2}\boldsymbol{j}+10\boldsymbol{k}) = 310\boldsymbol{i}-62\boldsymbol{j}+310\boldsymbol{k}$

NOTE

When two vectors are dot multiplied the answer is a constant value **not** a vector.

3. *Given that* $|\boldsymbol{a}| = 7, |\boldsymbol{b}| = 4$ *and* $\boldsymbol{a}.\boldsymbol{b} = \mathbf{8}$**,** *calculate:*
   a) $(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b})$ *and* $(\boldsymbol{a}-\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b})$
   b) Hence find $|\boldsymbol{a}+\boldsymbol{b}|$ *and* $|\boldsymbol{a}-\boldsymbol{b}|$

Solution

a) $(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a}+2\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{b} = \boldsymbol{a}.\boldsymbol{a}+\boldsymbol{b}.\boldsymbol{b}+2(8)$

But if $\boldsymbol{a} = x_1\boldsymbol{i}+x_2\boldsymbol{j}+x_3\boldsymbol{k}$, $|\boldsymbol{a}| = \sqrt{{x_1}^2+{x_2}^2+{x_3}^2}$

, and $\boldsymbol{a}.\boldsymbol{a} = (x_1\boldsymbol{i}+x_2\boldsymbol{j}+x_3\boldsymbol{k}).(x_1\boldsymbol{i}+x_2\boldsymbol{j}+x_3\boldsymbol{k}) = {x_1}^2+{x_2}^2+{x_3}^2$

$\therefore |\boldsymbol{a}|^2 = \boldsymbol{a}.\boldsymbol{a} = 49$, $|\boldsymbol{b}|^2 = \boldsymbol{b}.\boldsymbol{b} = 16$

$\Rightarrow (\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b}) = 49+16+16 = 81$

b) $(\boldsymbol{a}-\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a}-2\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{b}$

$= \boldsymbol{a}.\boldsymbol{a}-2\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{b} = 49-2\times 8+16 = 49$

From $|\boldsymbol{a}|^2 = \boldsymbol{a}.\boldsymbol{a} \Rightarrow |\boldsymbol{a}+\boldsymbol{b}|^{\mathbf{2}} = (\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b})$

$\therefore |\boldsymbol{a}+\boldsymbol{b}| = \sqrt{(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b})} = \sqrt{81} = 9$

Also $|\boldsymbol{a}-\boldsymbol{b}| = \sqrt{(\boldsymbol{a}-\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b})} = \sqrt{49} = 7$

Task

Given that $|\boldsymbol{x}| = 5$, $|\boldsymbol{y}| = 10$ *and* $\boldsymbol{x}.\boldsymbol{y} = 22$, calculate $|\boldsymbol{x}+\boldsymbol{y}|$ and $|\boldsymbol{x}-\boldsymbol{y}|$

Answer: $\mathbf{13}, \mathbf{9}$

4. *Vectors* $\boldsymbol{a}$**,** $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *are such that* $\boldsymbol{a}.\boldsymbol{c} = 3$, $\boldsymbol{b}.\boldsymbol{c} = 4$*. Given that the vector* $\boldsymbol{d} = \boldsymbol{a}+\lambda\boldsymbol{b}$ *is perpendicular to c, find the value of* λ*.*

Solution

$\boldsymbol{d}$ *is perpendicular to* $\boldsymbol{c}$ *if* $\boldsymbol{d}.\boldsymbol{c} = 0$ $[see\ properties]$

$\boldsymbol{d} = \boldsymbol{a}+\lambda\boldsymbol{b} = (\boldsymbol{a}+\lambda\boldsymbol{b}).\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{c}+\lambda\boldsymbol{b}.\boldsymbol{c} = 0$ $[see\ properties]$

$\Rightarrow 3+4\lambda = 0\ \therefore \lambda = -\frac{3}{4}$

5. *Given that* $\boldsymbol{a} = 2\boldsymbol{i}+6\boldsymbol{j}$ *and* $\boldsymbol{b} = 3\boldsymbol{i}+\lambda\boldsymbol{j}$**.** *Find the value of* λ *such that* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are perpendicular.*

Solution

For **a** and **b** to be perpendicular $\boldsymbol{a}.\boldsymbol{b} = 0$

$\Rightarrow (2\boldsymbol{i}+6\boldsymbol{j}).(3\boldsymbol{i}+\lambda\boldsymbol{j}) = 0$

$\Rightarrow 6+6\lambda = 0\ \therefore \lambda = 1$

6. *Find the angle between vectors*
   a) $8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$ *and* $2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$
   b) $\boldsymbol{j}+7\boldsymbol{k}$ *and* $-5\boldsymbol{i}+4\boldsymbol{j}+3\boldsymbol{k}$
   c) $2\boldsymbol{i}-3\boldsymbol{j}+8\boldsymbol{k}$ *and* $6\boldsymbol{i}-4\boldsymbol{j}-3\boldsymbol{k}$

Solution

a) Let $\boldsymbol{a} = 8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$

From dot product, $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta$

$\boldsymbol{a}.\boldsymbol{b} = (8\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}).(2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}) = 16-2-4 = 10$

$|\boldsymbol{a}| = \sqrt{8^2 + (-1)^2 + 4^2} = 9\,,\ |\boldsymbol{b}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$

$\therefore 10 = 9 \times 3cos\theta \ \ or\ cos\theta = \frac{10}{27}$

$\Rightarrow \theta = cos^{-1}\left(\frac{10}{27}\right) = 68.3^0$

b) $Left\ as\ an\ exercise$ $\quad$ ***Answer***: 60^0

c) Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + 8\boldsymbol{k}\,, \boldsymbol{b} = 6\boldsymbol{i} - 4\boldsymbol{j} - 3\boldsymbol{k}$

From dot product, $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta$

$\boldsymbol{a}.\boldsymbol{b} = (2\boldsymbol{i} - 3\boldsymbol{j} + 8\boldsymbol{k}).(6\boldsymbol{i} - 4\boldsymbol{j} - 3\boldsymbol{k}) = 12 + 12 - 24 = 0$

$\therefore 0 = |\boldsymbol{a}||\boldsymbol{b}|cos\theta \ \ or\ cos\theta = 0 \Rightarrow \theta = cos^{-1}(0) = 90^0$

Proofs involving dot product

1. *Given that* $|\boldsymbol{a}| = 3, |\boldsymbol{b}| = 8$ *and* $|\boldsymbol{a} - \boldsymbol{b}| = 7$*, show that the angle between* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is* 60^0*.*

Solution

By dot product, $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta$

But $|\boldsymbol{a}|^2 = \boldsymbol{a}.\boldsymbol{a}\,, |\boldsymbol{b}|^2 = \boldsymbol{b}.\boldsymbol{b}$ *and* $|\boldsymbol{a} - \boldsymbol{b}|^2 = (\boldsymbol{a} - \boldsymbol{b})^2$

From $|\boldsymbol{a} - \boldsymbol{b}|^2 = 49 = (\boldsymbol{a} - \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{a}.\boldsymbol{a} - 2\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{b}.\boldsymbol{b}$

$\Rightarrow 49 = 3^2 - 2\boldsymbol{a}.\boldsymbol{b} + 8^2 = 73 - 2\boldsymbol{a}.\boldsymbol{b}$

$\Rightarrow -24 = -2\boldsymbol{a}.\boldsymbol{b} \ \ \therefore \boldsymbol{a}.\boldsymbol{b} = 12$

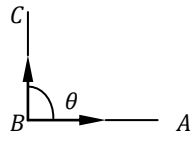
$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|cos\theta = 12$

$\Rightarrow 12 = 3 \times 8\ cos\theta \ \ or\ cos\theta = \frac{1}{2}$

$\therefore cos^{-1}\left(\frac{1}{2}\right) = 60^0$

2. *The points A, B and C have p.vs* $\boldsymbol{a} = 5\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}\,, \boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ *and* $\boldsymbol{c} = 7\boldsymbol{i} - 3\boldsymbol{j} + 10\boldsymbol{k}$ *respectively. Show that* $< ABC$ *is a right angle.*

Solution



NOTE: indicated vectors originate from B

Given: $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}\,, \overrightarrow{OC} = \boldsymbol{c}$

$\overrightarrow{BA} = \boldsymbol{d} = \overrightarrow{OA} - \overrightarrow{OB}$

$= (5\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}\,) - (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$

$= 3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$

$\overrightarrow{BC} = \boldsymbol{e} = \overrightarrow{OC} - \overrightarrow{OB} = (7\boldsymbol{i} - 3\boldsymbol{j} + 10\boldsymbol{k}\,) - (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$

$= 5\boldsymbol{i} - 2\boldsymbol{j} + 7\boldsymbol{k}$

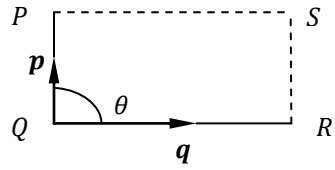
For $\boldsymbol{d}$ and $\boldsymbol{e}$ to be perpendicular, $\boldsymbol{d}.\boldsymbol{e} = \boldsymbol{0}$, by dot product $\boldsymbol{d}.\boldsymbol{e} = |\boldsymbol{d}||\boldsymbol{e}|cos\theta$

$\Rightarrow \boldsymbol{d}.\boldsymbol{e} = (3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}).(5\boldsymbol{i} - 2\boldsymbol{j} + 7\boldsymbol{k}) = 15 - 8 - 7 = 0$

$\therefore \boldsymbol{d}.\boldsymbol{e} = 0 \Rightarrow cos\theta = 0 \ \ \therefore \theta = 90^0$

3. *The vertices of a* Δ *are* $P(2, -1, 5)\,, Q(7, 1, -3)$ *and* $R(13, -2, 0)$*. Show that* $< PQR = 90^0$*. Find the coordinates of S if PQRS is a rectangle.* (***Uneb* 2002**)

Solution



Let $\overrightarrow{PQ} = \boldsymbol{p} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix}$

$\overrightarrow{QR} = \boldsymbol{q} = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{pmatrix} 13 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix}$

For **p** and **q** to be perpendicular, $\boldsymbol{p}.\boldsymbol{q} = \mathbf{0}$

$\Rightarrow \boldsymbol{p}.\boldsymbol{q} = \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix}.\begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} = 30 - 6 - 24 = 0$

$or\ \boldsymbol{p}.\boldsymbol{q} = (5\boldsymbol{i} + 2\boldsymbol{j} - 8\boldsymbol{k}).(6\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}) = 30 - 6 - 24 = 0$

$\therefore cos\theta = 0\ \ or\ \ \theta = 90^0$

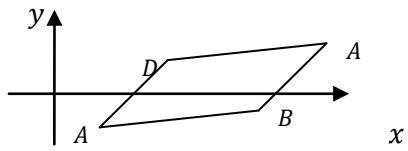
For a rectangle, $\overrightarrow{QR} = \overrightarrow{PS}$

$\Rightarrow \boldsymbol{q} = \overrightarrow{OS} - \overrightarrow{OP}\ \ \therefore\ \ \overrightarrow{OS} = \boldsymbol{q} + \overrightarrow{OP}$

$\Rightarrow \overrightarrow{OS} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 8 \end{pmatrix} \therefore S(8, -4, 8)$

4. *ABCD is a quadrilateral with* $A(2, -2), B(5, -1), C(6, 2)$ *and* $D(3, 1)$. *Show that the quadrilateral is a rhombus.*

Solution



For a rhombus
- All sides are equal
- Diagonals meet at 90^0
- All opposite angles are equal but not right angles

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$|\overrightarrow{AB}| = |\overrightarrow{DC}| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$|\overrightarrow{AD}| = |\overrightarrow{BC}| = \sqrt{3^2 + 1^2} = \sqrt{10}$

Hence all sides are equal= $\sqrt{10}\ units$

Now diagonals are $\overrightarrow{BD}$ and $\overrightarrow{AC}$

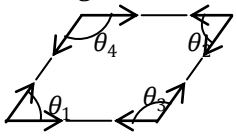
$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

For $\overrightarrow{BD}$ and $\overrightarrow{AC}$ to be perpendicular, $\overrightarrow{BD}.\overrightarrow{AC} = 0$

$\overrightarrow{BD}.\overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.\begin{pmatrix} 4 \\ 4 \end{pmatrix} = 8 - 8 = 0$. Hence diagonals meet at 90^0

Now angles are as in the figure.



$< \theta_1 = angle\ between\ \overrightarrow{AD}\ and\ \overrightarrow{AB}$

$\Rightarrow \overrightarrow{AD}.\overrightarrow{AB} = |\overrightarrow{AD}||\overrightarrow{AB}|cos\theta_1$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix}.\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \sqrt{10} \times \sqrt{10}\ cos\theta$

$cos\theta_1 = \frac{3+3}{10} \Rightarrow \theta_1 = 53.1^0$

$< \theta_2 = angle\ between\ \overrightarrow{CB}\ and\ \overrightarrow{CD}$

$\Rightarrow \overrightarrow{CB}.\overrightarrow{CD} = |\overrightarrow{CB}||\overrightarrow{CD}|cos\theta_2$

But $\overrightarrow{CB} = -\overrightarrow{BC}, \overrightarrow{CD} = -\overrightarrow{DC}$

$\therefore |\overrightarrow{CB}| = |\overrightarrow{BC}| = |\overrightarrow{CD}| = |\overrightarrow{DC}| = \sqrt{10}$

$\therefore \overrightarrow{CB}.\overrightarrow{CD} = \overrightarrow{BC}.\overrightarrow{DC} = |\overrightarrow{BC}||\overrightarrow{DC}|cos\theta_2$

$\Rightarrow \begin{pmatrix}1\\3\end{pmatrix}.\begin{pmatrix}3\\1\end{pmatrix} = \sqrt{10}\times\sqrt{10}\,cos\theta_2$

$\therefore \theta_1 = \theta_2 = 53.1^0$

$< \theta_3 = angle\ between\ \overrightarrow{AD}\ and\ \overrightarrow{AB}$

$\Rightarrow \overrightarrow{BC}.\overrightarrow{BA} = |\overrightarrow{BC}||\overrightarrow{BA}|cos\theta_3$

$\Rightarrow \begin{pmatrix}1\\3\end{pmatrix}.\begin{pmatrix}-3\\-1\end{pmatrix} = \sqrt{10}\times\sqrt{10}\,cos\theta_3\ [\because |\overrightarrow{BA}| = |\overrightarrow{AB}|]$

$-3-3 = 10cos\theta_3\ \ or\ 10cos\theta_3 = -0.6\ \ \therefore \theta_3 = 126.9^0$

$< \theta_3 = angle\ between\ \overrightarrow{DA}\ and\ \overrightarrow{DC} = 126.9^0$

Hence this is a rhombus.

5. *The points $P(4,-6,1), Q(2,8,4)$ and $R(3,7,14)$ lie in the same plane. Show that the angle formed between $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ is $cos^{-1}\left(\frac{14}{\sqrt{209\times102}}\right)$.*

Solution

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}2\\8\\4\end{pmatrix} - \begin{pmatrix}4\\-6\\1\end{pmatrix} = \begin{pmatrix}-2\\14\\5\end{pmatrix}$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{pmatrix}3\\7\\14\end{pmatrix} - \begin{pmatrix}2\\8\\4\end{pmatrix} = \begin{pmatrix}1\\-1\\10\end{pmatrix}$

Using dot product $\overrightarrow{PQ}.\overrightarrow{QR} = |\overrightarrow{PQ}||\overrightarrow{QR}|cos\theta$

$\overrightarrow{PQ}.\overrightarrow{QR} = \begin{pmatrix}-2\\14\\5\end{pmatrix}.\begin{pmatrix}1\\-1\\10\end{pmatrix} = -2-14+30 = 14$

$|\overrightarrow{PQ}| = \sqrt{(-2)^2+14^2+3^2} = \sqrt{209}$

$|\overrightarrow{QR}| = \sqrt{1^2+(-1)^2+10^2} = \sqrt{102}$

$\therefore cos\theta = \left(\frac{14}{\sqrt{209}\times\sqrt{102}}\right) = \left(\frac{14}{\sqrt{209\times102}}\right) \Rightarrow \theta = cos^{-1}\left(\frac{14}{\sqrt{209\times102}}\right)$

6. *Show that A, B and C with p.vs respectively are vertices of a triangle of a ΔABC*

a) $2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}, \boldsymbol{i}-3\boldsymbol{j}-5\boldsymbol{k}\ and\ 3\boldsymbol{i}-4\boldsymbol{j}-4\boldsymbol{k}$

b) $4\boldsymbol{i}-5\boldsymbol{j}-2\boldsymbol{k}, 8\,\boldsymbol{i}-5\boldsymbol{j}+6\boldsymbol{k}\ and\ origin$

c) $7\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}, \boldsymbol{i}+5\boldsymbol{j}+2\boldsymbol{k}\ and\ 3\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$

Solution

a) Let $A(2,-1,1), B(1,-3,-5), C(3,-4,-4)$

For A, B and C to be vertices of a ΔABC, then

$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = 0\ (for\ a\ closed\ figure)$

or

Sum of angles at A, B and C is 180^0

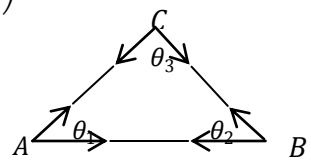
Considering the **first case**

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\-3\\-5\end{pmatrix} - \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-6\end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}3\\-4\\4\end{pmatrix} - \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-3\\-5\end{pmatrix}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}3\\-4\\-4\end{pmatrix} - \begin{pmatrix}1\\-3\\-5\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$

Now from $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\begin{pmatrix}-1\\-2\\-6\end{pmatrix}+\begin{pmatrix}1\\-3\\-5\end{pmatrix}+\begin{pmatrix}2\\-1\\1\end{pmatrix}=0$. Hence the vectors are vertices of a triangle.



Now considering **second case**

$< \theta_1 = angle\ between\ \overrightarrow{AC}\ and\ \overrightarrow{AB}$

By dot product $\overrightarrow{AC}.\overrightarrow{AB}=|\overrightarrow{AC}|.|\overrightarrow{AB}|cos\theta_1$

$\Rightarrow \begin{pmatrix}1\\-3\\-5\end{pmatrix}.\begin{pmatrix}-1\\-2\\-6\end{pmatrix}=\sqrt{1^2+(-3)^2+(-5)^2}.\sqrt{(-1)^2+(-2)^2+(-6)^2}\ cos\theta_1$

$-1+6+30=\sqrt{35}\times\sqrt{41}.cos\theta_1 \Rightarrow\ cos\theta_1=\frac{35}{\sqrt{35\times41}}\ \ \therefore \theta_1=22.49^0$

$< \theta_2 = angle\ between\ \overrightarrow{BA}\ and\ \overrightarrow{BC}$

$\Rightarrow \overrightarrow{BA}.\overrightarrow{BC}=|\overrightarrow{BA}||\overrightarrow{BC}|cos\theta_2\ , but\ \overrightarrow{BA}=-\overrightarrow{AB}$

$\Rightarrow \begin{pmatrix}1\\2\\6\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix}=\sqrt{1^2+2^2+6^2}\times\sqrt{2^2+(-1)^2+1^2}.cos\theta_2$

$\Rightarrow 2-2+6=\sqrt{41}\times\sqrt{6}.cos\theta_2 \Rightarrow\ cos\theta_2=\frac{6}{\sqrt{41}\times\sqrt{6}}\ \ \therefore \theta_2=67.51^0$

$< \theta_3 = angle\ between\ \overrightarrow{CA}\ and\ \overrightarrow{CB}$

But $\overrightarrow{CA}=-\overrightarrow{AC}\ and\ \overrightarrow{CB}=-\overrightarrow{BC}$

$\Rightarrow (-\overrightarrow{AC}).(-\overrightarrow{BC})=|\overrightarrow{AC}|.|\overrightarrow{BC}|\ cos\theta_3\ \ [\because |\overrightarrow{AC}|=|\overrightarrow{-AC}|, |\overrightarrow{BC}|=|\overrightarrow{-BC}|]$

$\Rightarrow \overrightarrow{AC}.\overrightarrow{BC}=|\overrightarrow{AC}|.|\overrightarrow{BC}|\ cos\theta_3$

$\begin{pmatrix}1\\-3\\-5\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix}=\sqrt{1^2+(-3)^2+(-5)^2}\times\sqrt{2^2+(-1)^2+1^2}.cos\theta_3$

$2+3-5=\sqrt{35}\times\sqrt{6}.cos\theta_3\ \ or\ cos\theta_3=0\ \ \ \therefore \theta=90^0$

$Now\ \theta_1+\theta_2+\theta_3=22.49^0+67.51^0+90^0=180^0$, which satisfy the condition.

Hence points are vertices of a triangle.

b) $Left\ as\ an\ exercise$

c) $Left\ as\ an\ exercise$

NOTE

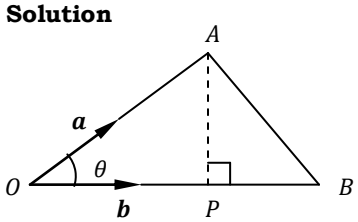
The question can also be done using $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$

Other Scalar Product Proofs

1. *The points A and B have p.vs* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *with respect to the origin O. show that the area of a triangle OAB is given by*

$\Delta=\frac{1}{2}\sqrt{\{a^2b^2-(\boldsymbol{a}.\boldsymbol{b})^2\}}$, where $a=|\boldsymbol{a}|, b=|\boldsymbol{b}|$

Solution



$Area, \Delta=\frac{1}{2}\times|\overrightarrow{OB}|\times\overrightarrow{AP}$

From dot product, $\boldsymbol{a}.\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\ cos\theta$

From ΔOAP, $\overrightarrow{AP}=|\overrightarrow{OA}|sin\theta=asin\theta$

$\therefore \Delta=\frac{1}{2}\times b\times asin\theta=\frac{1}{2}absin\theta$

But $sin\theta=\sqrt{1-cos^2\theta}\ \ [\because \boldsymbol{cos^2\theta+sin^2\theta=1}]$

From $\boldsymbol{a}.\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|cos\theta=abcos\theta$

$\Rightarrow \frac{\boldsymbol{a}.\boldsymbol{b}}{ab}=cos\theta$

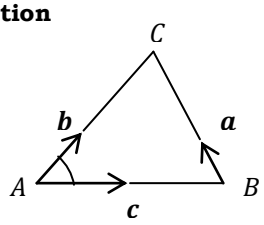
$$\Rightarrow sin\theta = \sqrt{1-\left(\frac{\boldsymbol{a.b}}{ab}\right)^2} = \frac{1}{ab}\sqrt{a^2b^2-(\boldsymbol{a.b})^2}$$

$$\therefore Area, \Delta = \frac{1}{2}absin\theta = \frac{1}{2}ab \times \frac{1}{ab}\sqrt{a^2b^2-(\boldsymbol{a.b})^2}$$

$$\therefore \Delta = \frac{1}{2}\sqrt{\{a^2b^2-(\boldsymbol{a.b})^2\}} \; As\ required$$

2. *In a ΔABC with vectors* $AC = \boldsymbol{b}$, $AB = \boldsymbol{c}$ *and* $BC = \boldsymbol{a}$, *then* $a^2 = b^2 + c^2 - 2bc.cosA$, *the cosine rule where* $a = |\boldsymbol{a}|, b = |\boldsymbol{b}|$

Solution



$$a^2 = \boldsymbol{a}.\boldsymbol{a} = (\boldsymbol{b}-\boldsymbol{c}).(\boldsymbol{b}-\boldsymbol{c})$$
$$= \boldsymbol{b}.\boldsymbol{b} - 2\boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{c}$$
$$= b^2 - 2\boldsymbol{b}.\boldsymbol{c} + c^2$$

But $\boldsymbol{b}.\boldsymbol{c} = |\boldsymbol{b}||\boldsymbol{c}|cosA = bc\ cosA$

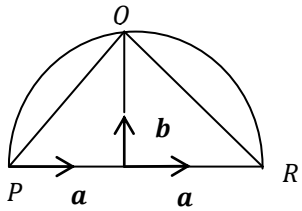
$$\therefore a^2 = b^2 - 2bc.cosA + c^2$$

or $a^2 = b^2 + c^2 - 2bc.cosA\ As\ required$

Note: the figure is very important in the derivation.

3. *Show that any angle inscribed in a semi circle is a right angle.*

Solutior



Let O be the centre of the semi circle,

$\overrightarrow{PO} = \overrightarrow{OR} = \boldsymbol{a}$ and

$OQ = \boldsymbol{b}$

Required: $\overrightarrow{QP}.\overrightarrow{QR} = 0\ .i.e. < P\hat{Q}R = 90^0$

$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = -\boldsymbol{a} - \boldsymbol{b} = -(\boldsymbol{a}+\boldsymbol{b})$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \boldsymbol{a} - \boldsymbol{b}$

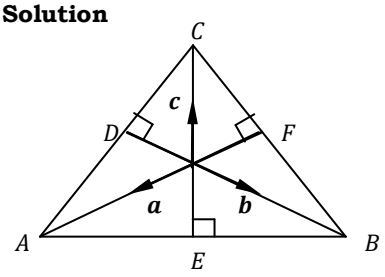
Now $\overrightarrow{QP}.\overrightarrow{QR} = -(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b}) = -(\boldsymbol{a}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{b})$

But $\boldsymbol{a}.\boldsymbol{a} = a^2, \boldsymbol{b}.\boldsymbol{b} = b^2\ and\ a^2 = b^2 = radius$

$\Rightarrow \overrightarrow{QP}.\overrightarrow{QR} = -(a^2 - b^2) = (a^2 - a^2) = 0 \therefore < P\hat{Q}R = 90^0$

4. *In ΔABC, the altitudes through A, B and C intersect at point O. show that if* $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$ *and* $\overrightarrow{OC} = \boldsymbol{c}$, *then* $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{a}.\boldsymbol{c}$ *and* $\boldsymbol{b}.\boldsymbol{a} = \boldsymbol{b}.\boldsymbol{c}$. *show also further that* $\boldsymbol{a}.\boldsymbol{c} = \boldsymbol{b}.\boldsymbol{c}$

Solution



From the figure, points A, O, F; B, O, D and C, O, E are collinear.

$\Rightarrow \overrightarrow{AF} = \lambda\overrightarrow{OA}, \overrightarrow{BD} = \mu\overrightarrow{OB}\ and\ \overrightarrow{OE} = k\overrightarrow{OC}$, where $\lambda, \mu\ and\ k$ are either positive/negative constants. Since

$\overrightarrow{AF} \perp \overrightarrow{BC}, then\ \overrightarrow{AF}.\overrightarrow{BC} = 0$ ..... (1)

$\overrightarrow{AD} \perp \overrightarrow{AC}$, then $\overrightarrow{AD}.\overrightarrow{AC} = 0$..... (2)

$\overrightarrow{CE} \perp \overrightarrow{AB}$, then $\overrightarrow{CE}.\overrightarrow{AB} = 0$ ..... (3)

From eqn. (1)

$\overrightarrow{AF}.\overrightarrow{BC} = 0, \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \boldsymbol{c} - \boldsymbol{b}$

$\Rightarrow \lambda OA.(\boldsymbol{c}-\boldsymbol{b}) = 0\ or\ \boldsymbol{a}.(\boldsymbol{c}-\boldsymbol{b}) = 0$

$\therefore \boldsymbol{a}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{b} = 0 \Rightarrow \boldsymbol{a}.\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{b}$

From eqn. (2)

$\overrightarrow{AD}.\overrightarrow{AC} = 0, \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \boldsymbol{c} - \boldsymbol{a}$

$\Rightarrow \mu OB.(\boldsymbol{c}-\boldsymbol{a}) = 0\ or\ \boldsymbol{b}.(\boldsymbol{c}-\boldsymbol{a}) = 0$

$\therefore \boldsymbol{b}.\boldsymbol{c} - \boldsymbol{b}.\boldsymbol{a} = 0 \Rightarrow \boldsymbol{b}.\boldsymbol{a} = \boldsymbol{b}.\boldsymbol{c}$

From $\boldsymbol{a}.\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{b} \Rightarrow \boldsymbol{a}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{b} = 0$ ..... (4)

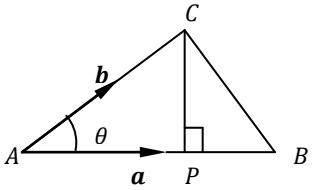
From $\boldsymbol{b}.\boldsymbol{a} = \boldsymbol{b}.\boldsymbol{c} \Rightarrow \boldsymbol{b}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{c} = 0$ ..... (5)

$Eqn.(4) + eqn.(5)$

$\boldsymbol{a}.\boldsymbol{c} - \boldsymbol{b}.\boldsymbol{c} = 0 \ \therefore \boldsymbol{b}.\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{c}$

AREA OF A TRIANGLE, BY DOT PRODUCT

Consider A, B and C be vertices of a triangle ABC



Let $\overrightarrow{AB} = \boldsymbol{a}, \overrightarrow{AC} = \boldsymbol{b}$

$|\overrightarrow{AC}| = |\boldsymbol{b}|, |\overrightarrow{AB}| = |\boldsymbol{a}|$

$Area, \Delta = \frac{1}{2} \times base \times height = \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{AC}|$

But $\overrightarrow{CP} = |\overrightarrow{AC}| sin\theta = |\boldsymbol{b}| sin\theta$

Now finding $sin\theta$

By dot product, $\overrightarrow{AC}.\overrightarrow{AB} = |\overrightarrow{AC}||\overrightarrow{AB}| cos\theta$

$\Rightarrow cos\theta = \frac{\boldsymbol{b}.\boldsymbol{a}}{|\boldsymbol{b}||\boldsymbol{a}|}$

$\Rightarrow Area, \Delta = \frac{1}{2} \times |\boldsymbol{a}| \times |\boldsymbol{b}| \times sin\theta = \frac{1}{2} \times |\boldsymbol{a}| \times \sqrt{(1 - cos^2\theta)}$ [$\because \boldsymbol{cos^2\theta + sin^2\theta = 1}$]

$= \frac{1}{2} \times |\boldsymbol{a}| \times |\boldsymbol{b}| \times \sqrt{\left(1 - \left(\frac{\boldsymbol{b}.\boldsymbol{a}}{|\boldsymbol{b}||\boldsymbol{a}|}\right)^2\right)} = \frac{1}{2}\sqrt{\{|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}.\boldsymbol{b})^2\}}$

$$\therefore Area, \Delta = \frac{1}{2}\sqrt{\{|\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}.\boldsymbol{b})^2\}}$$

Examples

Find the area of the triangle with vertices

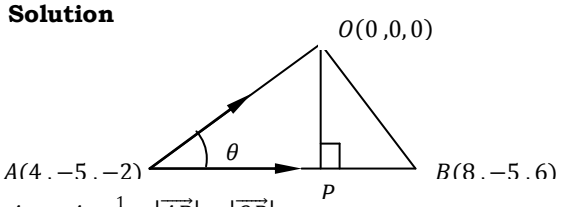
a) $A(4, -5, -2), B(8, -5, 6)$ *and origin*

b) $OA = \boldsymbol{i} - \boldsymbol{j}, OB = 5\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}, OC = \boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$

c) $A(0, -2, 1), B(1, -1, -2)$ *and* $C(-1, 1, 0)$

Solution

a)



$Area, \Delta = \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{OP}|$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 8 \\ -5 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{4^2 + 0^2 + 8^2} = \sqrt{80}$

$|\overrightarrow{OP}| = |\overrightarrow{AO}| sin\theta; \ |\overrightarrow{AO}| = \sqrt{4^2 + (-5)^2 + (-2)^2} = \sqrt{45}$

From dot product, $\overrightarrow{AO}.\overrightarrow{AB} = |\overrightarrow{AO}||\overrightarrow{AB}| cos\theta$

$\overrightarrow{AO}.\overrightarrow{AB} = -\begin{pmatrix} 4 \\ -5 \\ -2 \end{pmatrix}.\begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} = -(16 + 0 - 16) = 0$

$\therefore \overrightarrow{AO}.\overrightarrow{AB} = 0 \Rightarrow cos\theta = 0 \ \therefore \theta = 90^0$

$\therefore Area, \Delta = \frac{1}{2} \times |\overrightarrow{AB}| \times |\overrightarrow{OP}| = \frac{1}{2} \times \sqrt{80} \times \sqrt{45} \times sin90^0$

$= \frac{1}{2} \times \sqrt{16 \times 5} \times \sqrt{9 \times 5} = \frac{1}{2} \times 60 = 30 \ sq.units$

b) $2\sqrt{2}\ sq.units$

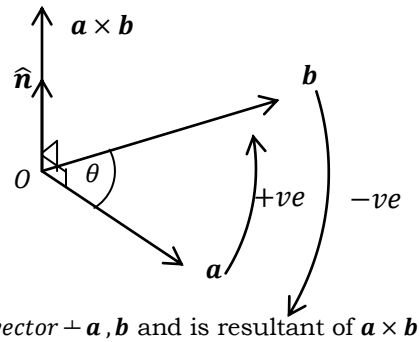
c) ***Answer***: $\sqrt{30}\ sq.units$

VECTOR/CROSS PRODUCT

For two vectors **a** and **b**, the cross product of **a** and **b** is such that

$$\boldsymbol{a}\times\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|sin\theta.\hat{\boldsymbol{n}}\ ,where\ 0\le\theta\le 360^0$$

Consider the figure below

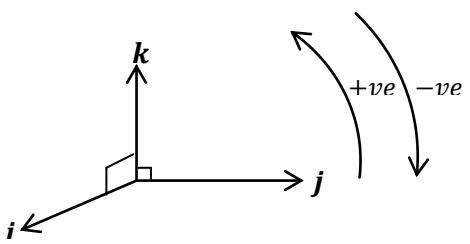


$\hat{\boldsymbol{n}}=Normal\ unit\ vector\perp\boldsymbol{a}\ ,\boldsymbol{b}$ and is resultant of $\boldsymbol{a}\times\boldsymbol{b}$

Properties of cross product

1. $\boldsymbol{a}\times\boldsymbol{b}\neq\boldsymbol{b}\times\boldsymbol{a}$ [$\because\ \boldsymbol{a}\times\boldsymbol{b}\ is\ in\ anticlockwise\ while\ \boldsymbol{a}\times\boldsymbol{b}\ is\ in\ clokwise\ direction$]
2. $m(\boldsymbol{a}\times\boldsymbol{b})=m\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{a}\times m\boldsymbol{b}$
3. From the definition $\boldsymbol{a}\times\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|sin\theta.\hat{\boldsymbol{n}}$
   a) For $\theta=0^0,\boldsymbol{a}\times\boldsymbol{b}=0\ .i.e.\underline{\boldsymbol{a}\ and\ \boldsymbol{b}\ are\ parallel}.$ Thus for **a** and **b** to be parallel or collinear, $\boxed{\boldsymbol{a}\times\boldsymbol{b}=0}$
   b) For $\theta=90^0,\boldsymbol{a}\times\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|.\hat{\boldsymbol{n}}\ ,\ \boldsymbol{a}\perp\boldsymbol{b}$. Thus for perpendicular vectors $\boldsymbol{a}\perp\boldsymbol{b}$ $\boxed{\boldsymbol{a}\times\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|.\hat{\boldsymbol{n}}}$
4. $\boldsymbol{a}\times\boldsymbol{a}=\boldsymbol{b}\times\boldsymbol{b}=\boldsymbol{c}\times\boldsymbol{c}=0,since\ \theta=0$
5. $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})\neq(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$

MULTIPLICATION OF UNIT VECTORS $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$



$\left.\begin{matrix}\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}\\ \boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i}\\ \boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}\end{matrix}\right\}$ *Crossed in Anticlockwise*

$\left.\begin{matrix}\boldsymbol{j}\times\boldsymbol{i}=-\boldsymbol{k}\\ \boldsymbol{i}\times\boldsymbol{k}=-\boldsymbol{j}\\ \boldsymbol{k}\times\boldsymbol{j}=-\boldsymbol{i}\end{matrix}\right\}$ *Crossed in Clockwise*

Examples

Given that $\boldsymbol{a}=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}\ ,\boldsymbol{b}=4\boldsymbol{i}+5\boldsymbol{j}-6\boldsymbol{k}\ and\ \boldsymbol{c}=7\boldsymbol{i}+8\boldsymbol{j}+9\boldsymbol{k}$ **,** *find;*

a) $\boldsymbol{a}\times\boldsymbol{b}$

b) $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$

Solution

a) $\boldsymbol{a}\times\boldsymbol{b}=(\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})\times(4\boldsymbol{i}+5\boldsymbol{j}-6\boldsymbol{k})$

$$= 4\boldsymbol{i}\times\boldsymbol{i} + 5\boldsymbol{i}\times\boldsymbol{j} - 6\boldsymbol{i}\times\boldsymbol{k} +$$
$$8\boldsymbol{j}\times\boldsymbol{i} + 10\boldsymbol{j}\times\boldsymbol{j} - 12\boldsymbol{j}\times\boldsymbol{k} +$$
$$12\boldsymbol{k}\times\boldsymbol{i} + 15\boldsymbol{k}\times\boldsymbol{j} - 18\boldsymbol{k}\times\boldsymbol{k}$$
$$= 0 + 5\boldsymbol{k} + 6\boldsymbol{j} +$$
$$-8\boldsymbol{k} + 0 - 12\boldsymbol{i} +$$
$$12\boldsymbol{j} - 15\boldsymbol{i} - 0$$
$$= -27\boldsymbol{i} + 18\boldsymbol{j} - 3\boldsymbol{k}$$

b) $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = (-27\boldsymbol{i} + 18\boldsymbol{j} - 3\boldsymbol{k})\times(7\boldsymbol{i} + 8\boldsymbol{j} + 9\boldsymbol{k})$

$= 186\boldsymbol{i} + 222\boldsymbol{j} - 342\boldsymbol{k}$ ***Left as an exercise***

NOTE:

The same results can still be obtained using multiplication of vectors using *Rectangular Components.*

MULTIPLICATION OF VECTORS BY RECTANGULAR COMPONENTS

Consider $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ *and* $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

$\Rightarrow \boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$, where the elements in | | give a matrix whose determinant is of a 2×2 square matrix

Example

Referring to the above example

$\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}, \boldsymbol{b} = 4\boldsymbol{i} + 5\boldsymbol{j} - 6\boldsymbol{k}$ *and* $\boldsymbol{c} = 7\boldsymbol{i} + 8\boldsymbol{j} + 9\boldsymbol{k}$

a) $\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 4 & 5 & -6 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 2 & 3 \\ 5 & -6 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 3 \\ 4 & -6 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$

$= (-12 - 15)\boldsymbol{i} + (-6 - 12)\boldsymbol{j} + (5 - 8)\boldsymbol{k}$

$= -27\boldsymbol{i} + 18\boldsymbol{j} - 3\boldsymbol{k}$ *As before*

b) $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ -27 & 18 & -3 \\ 7 & 8 & 9 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 18 & -3 \\ 8 & 9 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -27 & -3 \\ 7 & 9 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} -27 & 18 \\ 7 & 8 \end{vmatrix}$

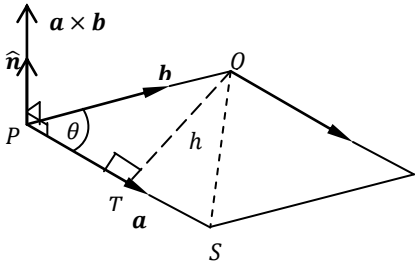
$= 186\boldsymbol{i} + 222\boldsymbol{j} - 342\boldsymbol{k}$ *As before*

NOTE:

Using Rectangular Components in multiplication of two vectors is easier to handle to handle than the first method.

GEOMETRIC INTERPRETATION OF VECTOR PRODUCT

The magnitude of a vector product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the *area* of a parallelogram whose adjacent sides are represented by $\boldsymbol{a}$ and $\boldsymbol{b}$.



From the definition;

$\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|sin\theta.\hat{\boldsymbol{n}}$,

where $\hat{\boldsymbol{n}}$ *is a unit vector* $\perp$ *to* $\boldsymbol{a}$ *and* $\boldsymbol{b}$

$\therefore |\boldsymbol{a}\times\boldsymbol{b}| = \left||\boldsymbol{a}||\boldsymbol{b}|sin\theta.\hat{\boldsymbol{n}}\right|$; *since* $|\hat{\boldsymbol{n}}| = 1$

$\Rightarrow |\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}|\{|\boldsymbol{b}|sin\theta\}$

From $\Delta PQT, h = |\boldsymbol{b}|sin\theta = height\ of\ parallerogram$

$\Rightarrow |\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}|\times h$, where $|\boldsymbol{a}| = base\ of\ parallelogram$

$\therefore |\boldsymbol{a}\times\boldsymbol{b}| = base \times height = Area\ of\ parallellogram$

But $\Delta s\ PQS\ and QRS\ are\ equal$

$$\therefore |\boldsymbol{a}\times\boldsymbol{b}| = 2\times Area\ of\ \Delta PQS\ or\ Area\ of\ \Delta = \frac{1}{2}\times|\boldsymbol{a}\times\boldsymbol{b}|$$

Examples

1. *Find the unit vector perpendicular to each vector* $\boldsymbol{a} = 2\boldsymbol{i}-2\boldsymbol{j}+4\boldsymbol{k}$ *and* $\boldsymbol{b} = 3\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$ *. find also the angle between* $\boldsymbol{a}$ *and* $\boldsymbol{b}$

Solution

Using $\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|sin\theta.\hat{\boldsymbol{n}} \Rightarrow \hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|sin\theta}$

But $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|sin\theta\ [\because |\hat{\boldsymbol{n}}| = 1]$

$$\Rightarrow \hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|} = unit\ vector\ perpendicular\ to\ \boldsymbol{a}\ and\ \boldsymbol{b}$$

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\ 2 & -2 & 4\\ 3 & 1 & 2\end{vmatrix} = \boldsymbol{i}\begin{vmatrix}-2 & 4\\ 1 & 2\end{vmatrix} - \boldsymbol{j}\begin{vmatrix}2 & 4\\ 3 & 2\end{vmatrix} + \boldsymbol{k}\begin{vmatrix}2 & -2\\ 3 & 1\end{vmatrix} = -8\boldsymbol{i}+8\boldsymbol{j}+8\boldsymbol{k}$$

$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{(-8)^2+8^2+8^2} = 8\sqrt{3}$

$\therefore \hat{\boldsymbol{n}} = \frac{-8\boldsymbol{i}+8\boldsymbol{j}+8\boldsymbol{k}}{8\sqrt{3}} = \frac{1}{\sqrt{3}}(=-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}\ )$

$Now\ finding\ \theta$

$|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|sin\theta\ , |\boldsymbol{a}| = \sqrt{2^2+(-2)^2+4^2} = \sqrt{24}\ ,\ |\boldsymbol{b}| = \sqrt{3^2+1^2+2^2} = \sqrt{14}$

$\Rightarrow sin\theta = \frac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{8\sqrt{3}}{\sqrt{24}\times\sqrt{14}} = 8\sqrt{\left(\frac{3}{24\times14}\right)} = \frac{8}{\sqrt{112}}$

$\therefore \theta = sin^{-1}\left(\frac{8}{\sqrt{112}}\right) = 4.1^0$

2. *Find the unit vector perpendicular to vectors* $\boldsymbol{a} = 6\boldsymbol{i}-3\boldsymbol{j}-14\boldsymbol{k}$ *and* $\boldsymbol{b} = 3\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}$*. Hence also find the angle between them.*

Solution

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\ 6 & -3 & -14\\ 3 & 2 & -3\end{vmatrix} = \boldsymbol{i}\begin{vmatrix}-3 & -14\\ 2 & -3\end{vmatrix} - \boldsymbol{j}\begin{vmatrix}6 & -14\\ 3 & -3\end{vmatrix} + \boldsymbol{k}\begin{vmatrix}6 & -3\\ 3 & 2\end{vmatrix} = 37\boldsymbol{i}-24\boldsymbol{j}+21\boldsymbol{k}$$

$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{37^2+24^2+21^2} = \sqrt{2386}$

$\therefore \hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|} = \frac{1}{\sqrt{2386}}(37\boldsymbol{i}-24\boldsymbol{j}+21\boldsymbol{k}\ )$

$|\boldsymbol{a}| = \sqrt{6^2+(-3)^2+(-14)^2} = \sqrt{241}\ ,\ |\boldsymbol{b}| = \sqrt{3^2+2^2+(-3)^2} = \sqrt{22}$

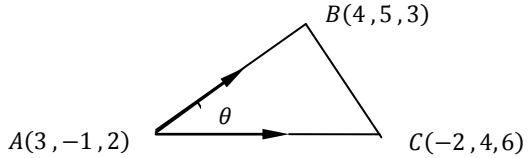
$From\ sin\theta = \frac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{\sqrt{2386}}{\sqrt{241}\times\sqrt{22}}$

$\therefore \theta = 26.7^0$

3. *Find the area of the triangle with vertices*
   a) $(3\ ,-1\ ,2)\ ,(4,\ 5\ ,3)\ and\ (-2\ ,\ 4\ ,\ 6)$
   b) $i-2j+3k\ ,-3i-2j+k\ and\ origin$
   c) $(1\ ,-1\ ,2)\ ,(2,\ 1\ ,-1)\ and\ (3\ ,-1\ ,\ 2)$

Solution

a) Let $A(3\ ,-1\ ,2)\ ,B(4,\ 5\ ,3)\ and\ C(-2\ .\ 4\ .\ 6)$



Let $\boldsymbol{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}4\\5\\3\end{pmatrix} - \begin{pmatrix}3\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\6\\1\end{pmatrix}$

$\boldsymbol{b} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}-2\\4\\6\end{pmatrix} - \begin{pmatrix}4\\5\\3\end{pmatrix} = \begin{pmatrix}-5\\5\\4\end{pmatrix}$

$Area\ of \Delta = \frac{1}{2}\,|\boldsymbol{a} \times \boldsymbol{b}|$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 6 & 1 \\ -5 & 5 & 4 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix}6 & 1\\5 & 4\end{vmatrix} - \boldsymbol{j}\begin{vmatrix}1 & 1\\-5 & 4\end{vmatrix} + \boldsymbol{k}\begin{vmatrix}1 & 6\\-5 & 5\end{vmatrix} = 19\boldsymbol{i} - 9\boldsymbol{j} + 35\boldsymbol{k}$

$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{19^2 + (-9)^2 + 35^2} = \sqrt{1667}$

$\therefore Area = \frac{1}{2} \times \sqrt{1667} = 20.41\ sq.units$

b) $3\sqrt{5}\ sq.units$Let

c) Left as an exercise ***Answer***: $\sqrt{\mathbf{13}}\ sq.units$

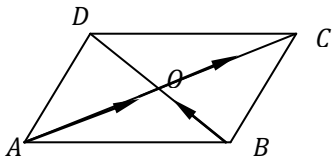
4. *Calculate the area of a parallelogram having diagonals;*

   a) $\overrightarrow{AC} = 3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}\ and\ \overrightarrow{BD} = \boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$

   b) $\overrightarrow{AB} = 2\boldsymbol{i} + 4\boldsymbol{j} - 6\boldsymbol{k}\ and\ \overrightarrow{DC} = 4\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}$

Solution

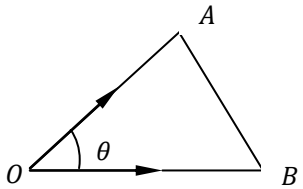
a) Let $O = midpoint\ of\ \overline{AC}\ and\ \overline{BD}$



Now there are $4 - triangles$ of the same area

$\therefore Area\ of\ ABCD = 4 \times area\ of\ one\ triangle$

Consider ΔOCD



$\overrightarrow{OB} = -\overrightarrow{BO}$

Let $\overrightarrow{OC} = \boldsymbol{c}\ and\ \overrightarrow{OB} = \boldsymbol{b}$

But $\overrightarrow{BO} = \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}(\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k})$

$\Rightarrow \overrightarrow{OB} = \boldsymbol{b} = \frac{1}{2}(-\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k})$

Also $\overrightarrow{OC} = \boldsymbol{c} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$\boldsymbol{b} \times \boldsymbol{c} = \frac{1}{2}(-\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}) \times \frac{1}{2}(3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$

$= \frac{1}{4}[(-\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}) \times (3\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})]\ [\because (\boldsymbol{a}m) \times (\boldsymbol{b}n) = mn(\boldsymbol{a} \times \boldsymbol{b})]$

$\boldsymbol{b} \times \boldsymbol{c} = \frac{1}{4}\begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ -1 & 3 & -4 \\ 3 & 1 & -2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix}3 & -4\\1 & -2\end{vmatrix} - \boldsymbol{j}\begin{vmatrix}-1 & -4\\3 & -2\end{vmatrix} + \boldsymbol{k}\begin{vmatrix}-1 & 3\\3 & 1\end{vmatrix}$

$= \frac{1}{4}(-2\boldsymbol{i} + 14\boldsymbol{j} - 10\boldsymbol{k}) = \frac{1}{2}(-\boldsymbol{i} + 7\boldsymbol{j} - 5\boldsymbol{k})$

$Area\ of \Delta = \frac{1}{2}\,|\boldsymbol{b} \times \boldsymbol{c}|$

$|\boldsymbol{b} \times \boldsymbol{c}| = \frac{1}{2}\sqrt{(-1)^2 + 7^2 + (-5)^2} = \frac{1}{2}\sqrt{75} = \frac{1}{2} \times 5\sqrt{3}$

$\therefore Area\ of \Delta = \frac{1}{2}\,|\boldsymbol{b} \times \boldsymbol{c}| = \frac{1}{2} \times \frac{1}{2} \times 5\sqrt{3} = \frac{5}{4}\sqrt{3}\ sq.units$

$\therefore Area\ of\ parallelogram = 4 \times \frac{5}{4}\sqrt{3} = 5\sqrt{3}\ sq.units$

b) *Left as an exercise.* ***Answer***: $30\ sq.units$

Task

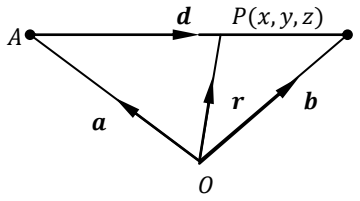
Find the unit vector perpendicular to vectors;

a) $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j}\ and\ \boldsymbol{b} = \boldsymbol{i}$; ***Answer***: $\frac{\sqrt{2}}{2}$

b) $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}\ and\ \boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$; ***Answer***: $\frac{\sqrt{2}}{3}$

VECTOR EQUATION OF A STRAIGHT LINE

a) **To find the vector equation of a line passing through a given point A and parallel to a given vector.**



$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$; *but* $\overrightarrow{AP}$ *is* // to $\boldsymbol{d}$

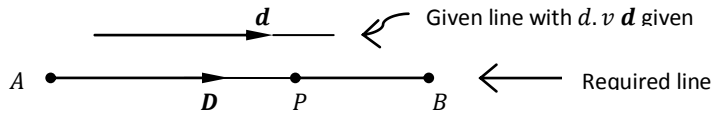
$\therefore \overrightarrow{AP} = \lambda \boldsymbol{d}$

$\Rightarrow \vec{P} = \boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$

$$\therefore \boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$$

, which is in *parametric* form of equation of a line

b) **To find the Cartesian equation of a line through a given point** $A(a_1, a_2, a_3)$ **and parallel to the given line whose direction cosines are directly proportional to** $B(b_1, b_2, b_3)$



From (a) above, the required equation is

$\therefore \boldsymbol{r} = \boldsymbol{a} + k\,\boldsymbol{d}$

But $\boldsymbol{d}$ *is* // $\boldsymbol{D} \Rightarrow \boldsymbol{D} = \mu \boldsymbol{d}$

$\therefore \boldsymbol{r} = \boldsymbol{a} + k\,\mu \boldsymbol{d}$, replacing $k\mu$ *with* λ

$$\Rightarrow \boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$$

Where $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \boldsymbol{d} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\Rightarrow x = a_1 + \lambda\, b_1 \quad \therefore \lambda = \frac{x - a_1}{b_1}$

$\Rightarrow y = a_2 + \lambda\, b_2 \quad \therefore \lambda = \frac{y - a_2}{b_2}$

$\Rightarrow z = a_3 + \lambda\, b_3 \quad \therefore \lambda = \frac{z - a_3}{b_3}$

$\therefore$ *The Cartesian equation of a line is*

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

c) **To find the equation of a line passing through two given points A and B**

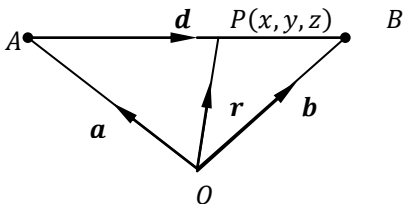
Let $A(a_1, a_2, a_3)$ *and* $B(b_1, b_2, b_3)$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$; *but* $\overrightarrow{AP}$ *is* // $\overrightarrow{AB}$

$\therefore \overrightarrow{AP} = \lambda \overrightarrow{AB}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

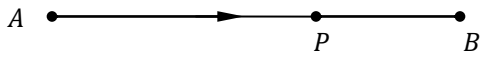
$\therefore \overrightarrow{OP} = \boldsymbol{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$



$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \left[\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right]$$

VECTOR EQUATION OF A LINE IN NON-PARAMETRIC FORM

Let $A(a_1, a_2, a_3)$ *and* $B(b_1, b_2, b_3)$



$\overrightarrow{AP}$ *is* // $\overrightarrow{AB} \Rightarrow \overrightarrow{AP} \times \overrightarrow{AB} = 0$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \boldsymbol{a}, \boldsymbol{a} = \overrightarrow{OA}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \boldsymbol{b} - \boldsymbol{a} = \boldsymbol{d}$

$$(\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{d} = 0$$

General examples

1. *Find the vector equation of a line through a point* $4\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$ *and parallel to the vector* $-2\boldsymbol{i} + \boldsymbol{j} + 6\boldsymbol{k}$. *Express the equation in;*
   a) Cartesian form
   b) Non-parametric form

Solution

Let the point be $A(4, 2, -3), \boldsymbol{d} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$

$\overrightarrow{AP}$ *is* // $\boldsymbol{d}$, $\overrightarrow{AP} = \lambda \boldsymbol{d}$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$

$\Rightarrow \boldsymbol{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$

$\therefore \boldsymbol{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$

Let $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{r}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$

$x = 4 - 2\lambda \Rightarrow \lambda = \frac{x-4}{-2}$

$y = 2 + \lambda \Rightarrow \lambda = \frac{y-2}{1}$

$z = -3 + 6\lambda \Rightarrow \lambda = \frac{z+3}{6}$

$\therefore \frac{x-4}{-2} = \frac{y-2}{1} = \frac{z+3}{6}$, $cartesian\ form$

Now expressing in Non-parametric form

$\overrightarrow{AP}$ *is* // $\boldsymbol{d} \Rightarrow \overrightarrow{AP} \times \boldsymbol{d} = 0$ [For parallel vectors]

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}, \ \boldsymbol{d} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}$

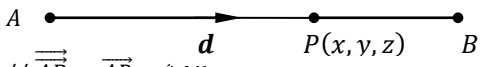
$\therefore \left[\boldsymbol{r} - \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}\right] \times \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix} = 0 \ \ or \ \ [\boldsymbol{r} - (4\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})] \times (-2\boldsymbol{i} + \boldsymbol{j} + 6\boldsymbol{k}) = 0$

2. *Find the equation of a straight line through points;*
   a) $A(1, -1, -1)$ *and* $B(3, 4, 2)$
   b) $\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ *and* $4\boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}$
   c) $A(1, 2, 5)$ *and origin*

   Express the equation in Cartesian form

Solution

a) $A(1, -1, -1)$ *and* $B(3, 4, 2)$



$\overrightarrow{AP}$ *is* $// \overrightarrow{\overrightarrow{AB}} \Rightarrow \overrightarrow{AP} = \lambda AB$

$\overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$

$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \lambda(\overrightarrow{OB} - \overrightarrow{OA})$

$\therefore \boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \left[\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\right]$

$\therefore \boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$

Now expressing in Cartesian form

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$

$\Rightarrow x = 1 + 2\lambda \ \Rightarrow \lambda = \frac{x-1}{2}$

$\Rightarrow y = -1 + 5\lambda \ \Rightarrow \lambda = \frac{y+1}{5}$

$\Rightarrow z = -1 + 3\lambda \ \Rightarrow \lambda = \frac{z+1}{3}$

$\therefore \frac{x-1}{2} = \frac{y+1}{5} = \frac{z+1}{3}$

b) *Left as an exercise.* **ANSWER**: $\frac{x-1}{3} = \frac{y-2}{-5} = \frac{z-3}{-8}$

c) *Left as an exercise.* **ANSWER**: $\frac{x-1}{-1} = \frac{y-2}{-2} = \frac{z-5}{-5}$

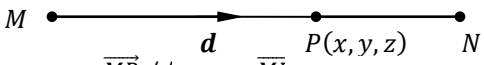
3. *Given the points* $A(-3, 3, 4)$, $B(5, 7, 2)$ *and* $C(1, 1, 4)$. *Find the vector equation of a line which joins the midpoints of* $\overline{AB}$ *and* $\overline{BC}$.

Solution

Let M and N be the points of $\overline{AB}$ *and* $\overline{BC}$ respectively.

$M\left(\frac{-3+5}{2}, \frac{3+7}{2}, \frac{4+2}{2}\right) = M(1, 5, 3)$

$N\left(\frac{5+1}{2}, \frac{1+7}{2}, \frac{4+2}{2}\right) = N(3, 4, 3)$



$\overrightarrow{MP} \ // MN \ \Rightarrow \overrightarrow{MP} = \lambda MN$

$\overrightarrow{MP} = \overrightarrow{OP} - \overrightarrow{OM} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} x-1 \\ y-5 \\ z-3 \end{pmatrix}$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \begin{pmatrix}3\\4\\3\end{pmatrix} - \begin{pmatrix}1\\5\\3\end{pmatrix} = \begin{pmatrix}2\\-1\\0\end{pmatrix}$

$\therefore \begin{pmatrix}x-1\\y-5\\z-3\end{pmatrix} = \lambda\begin{pmatrix}2\\-1\\0\end{pmatrix}$ *or* $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\5\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\0\end{pmatrix}$

4. *Given that* $A(1\,,2\,,3)\,, B(4\,,3\,,2)$, $C(-1\,,-4\,,-5)$ *and* $D(-5\,,-2\,,-1)$. *Find the vector equation of a line joining points M and N such that M is the midpoint of* $\overline{AB}$ *and N divides* $\overline{CD}$ *in the ratio* $-4:1$.

Solution

$M\left(\frac{1+4}{2}\,,\frac{2+3}{2}\,,\frac{3+2}{2}\right) = M\left(\frac{5}{2},\frac{5}{2},\frac{5}{2}\right)$

Finding the coordinates of N

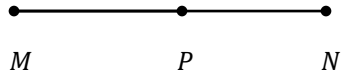
Ratio of division: $CN:ND = -4:1$

$\frac{\overrightarrow{CN}}{\overrightarrow{ND}} = -\frac{4}{1} \Rightarrow \overrightarrow{CN} = -4\overrightarrow{ND}$

$\overrightarrow{ON} - \overrightarrow{OC} = -4\left(\overrightarrow{OD} - \overrightarrow{ON}\right) = -4\overrightarrow{OD} + 4\overrightarrow{ON}$

$-3\overrightarrow{ON} = \overrightarrow{OC} - 4\overrightarrow{OD} \Rightarrow \overrightarrow{ON} = -\frac{1}{3}\left[\begin{pmatrix}-1\\-4\\-5\end{pmatrix} - 4\begin{pmatrix}-5\\-2\\-1\end{pmatrix}\right] = -\frac{1}{3}\begin{pmatrix}19\\4\\-1\end{pmatrix}$

$\therefore N\left(-\frac{19}{3}\,,-\frac{4}{3}\,,\frac{1}{3}\right)$



$\overrightarrow{MP} = \lambda\overrightarrow{MN} \Rightarrow \overrightarrow{OP} - \overrightarrow{OM} = \lambda\left(\overrightarrow{ON} - \overrightarrow{OM}\right)$

$\Rightarrow \boldsymbol{r} - \begin{pmatrix}\frac{5}{2}\\\frac{5}{2}\\\frac{5}{2}\end{pmatrix} = \lambda\begin{pmatrix}-\frac{19}{3}\\-\frac{4}{3}\\\frac{1}{3}\end{pmatrix} - \begin{pmatrix}\frac{5}{2}\\\frac{5}{2}\\\frac{5}{2}\end{pmatrix} = \begin{pmatrix}-\frac{53}{6}\\-\frac{23}{6}\\-\frac{13}{6}\end{pmatrix} = -\frac{\lambda}{6}\begin{pmatrix}53\\23\\13\end{pmatrix}$

Let $-\frac{\lambda}{6} = k\,, \Rightarrow \boldsymbol{r} - \begin{pmatrix}2.5\\2.5\\2.5\end{pmatrix} = k\begin{pmatrix}53\\23\\13\end{pmatrix}$ *or* $\boldsymbol{r} = \begin{pmatrix}2.5\\2.5\\2.5\end{pmatrix} + k\begin{pmatrix}53\\23\\13\end{pmatrix}$

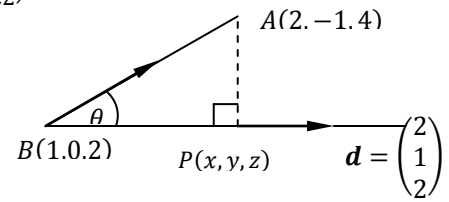
EQUATION OF A PERPENDICULAR AND PERPENDICULAR DISTANCE OF A POINT FROM A GIVEN LINE:

Dot or vector product may be used.

1. *Find the equation of perpendicular line from point* $\boldsymbol{A} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ *on to the line* $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$. *Find also perpendicular distance of* $\boldsymbol{A}$ *from* $\boldsymbol{r}$

Solution

The line $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ passes through the point $B(1,0,2)$ with direction vector $\boldsymbol{d} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$



Finding the equation of perpendicular AP:

Vectors $\overrightarrow{AP} \perp \boldsymbol{d}$

$\Rightarrow \overrightarrow{AP}.\boldsymbol{d} = 0$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} x-2 \\ y+1 \\ z-4 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x-2 \\ y+1 \\ z-4 \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0 \; or \; 2x-4+y+1+2z-8=0$

$2x + y + 2z = 11$ ..... (1)

But $P(x,y,z)$ lies on the line

$\Rightarrow \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\Rightarrow x = 1 + 2\lambda \, , y = \lambda \, , z = 2 + 2\lambda$ ..... (2)

$Eqn.(2) \; in \; to \; eqn.(1)$

$\Rightarrow 2(1 + 2\lambda) + \lambda + 2(2 + 2\lambda) = 11$

$\Rightarrow 2 + 4\lambda + \lambda + 4 + 4\lambda = 11 \; \therefore \lambda = \frac{5}{9}$

$P(x,y,z) = P(1 + 2\lambda \, , \lambda \, , 2 + 2\lambda) \Rightarrow P\left(1 + \frac{10}{9} \, , \frac{5}{9} \, , 2 + \frac{10}{9}\right) = P\left(\frac{19}{9} \, , \frac{5}{9} \, , \frac{28}{9}\right)$

Now finding equation of $\overline{AP}$

$P\left(\frac{19}{9} \, , \frac{5}{9} \, , \frac{28}{9}\right)$

$AQ = \lambda AP \Rightarrow \overrightarrow{OQ} - OA = \lambda\left(\overrightarrow{OP} - \bar{C}\right.$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \lambda\left[\left(\frac{19}{9}\boldsymbol{i} + \frac{5}{9}\boldsymbol{j} + \frac{28}{9}\boldsymbol{k}\right) - (2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k})\right]$

$= \lambda\left(\frac{1}{9}\boldsymbol{i} + \frac{14}{9}\boldsymbol{j} - \frac{8}{9}\boldsymbol{k}\right) = \frac{\lambda}{9}(\boldsymbol{i} + 14\boldsymbol{j} - 8\boldsymbol{k}) \; ; let \; \frac{\lambda}{9} = \mu$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}$

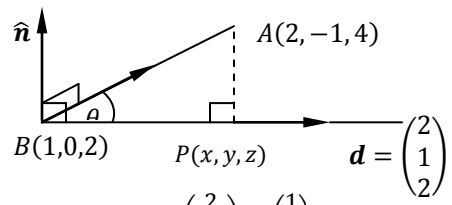
$\therefore \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}$

Now finding $|\overrightarrow{AP}| = perpendicular\ distance$

$\overrightarrow{AP} = \frac{1}{9}\boldsymbol{i} + \frac{14}{9}\boldsymbol{j} - \frac{8}{9}\boldsymbol{k}$

$\therefore |\overrightarrow{AP}| = \sqrt{\left(\frac{1}{9}\right)^2 + \left(\frac{14}{9}\right)^2 + \left(-\frac{8}{9}\right)^2} = \frac{\sqrt{29}}{3} units$

Alternative method of finding perpendicular distance of a point from a line:



$|\overrightarrow{AP}| = |\overrightarrow{BA}| sin\theta$

By cross product,

$\boldsymbol{d} \times \overrightarrow{BA} = |\boldsymbol{d}||\overrightarrow{BA}| sin\theta . \hat{\boldsymbol{n}}$

$|\boldsymbol{d} \times \overrightarrow{BA}| = |\boldsymbol{d}||\overrightarrow{BA}| sin\theta \ [\because |\hat{\boldsymbol{n}}| = 1]$

$\therefore |\overrightarrow{AP}| = |\overrightarrow{BA}| sin\theta = \frac{|\boldsymbol{d} \times \overrightarrow{BA}|}{|\boldsymbol{d}|}$

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\boldsymbol{d} \times \overrightarrow{BA} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 4\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$ [*See cross product above how toget this answer*]

$|\boldsymbol{d} \times \overrightarrow{BA}| = \sqrt{4^2 + (-2)^2 + (-3)^2} = \sqrt{29}\ ; |\boldsymbol{d}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

$\therefore |\overrightarrow{AP}| = \frac{\sqrt{29}}{3} units$

2. *Find the equation and length of perpendicular drawn from a point* $R(-2\ ,2\ ,-3)$*to the line* $x - 3 = \frac{y+1}{2} = \frac{z-2}{-4}$

Solution

Expressing the equation in parametric form

Let $x - 3 = \frac{y+1}{2} = \frac{z-2}{-4} = \lambda$

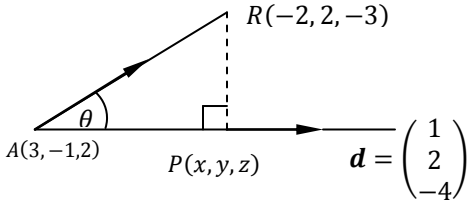
$\Rightarrow x - 3 = \lambda\ , x = 3 + \lambda$

$\Rightarrow \frac{y+1}{2} = \lambda\ , y = -1 + 2\lambda$

$\Rightarrow \frac{z-2}{-4}\lambda\ , z = 2 - 4\lambda$

$\therefore \boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

This line passes through point $A(3\ ,-1\ ,2)$ and has a direction vector $\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$



Finding the equation of perpendicular PR:

Vectors $\overrightarrow{PR} \perp \boldsymbol{d}$

$\Rightarrow \overrightarrow{PR}.\boldsymbol{d} = 0$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix}-2\\2\\-3\end{pmatrix} - \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2-x\\2-y\\-3-z\end{pmatrix}$

$\Rightarrow \begin{pmatrix}-2-x\\2-y\\-3-z\end{pmatrix}.\begin{pmatrix}1\\2\\-4\end{pmatrix} = 0 \; or \; -2-x+4-2y+12+4z = 0$

$-x-2y+4z = -14$ ..... (1)

But $P(x,y,z)$ lies on the line

$\Rightarrow \boldsymbol{r} = \begin{pmatrix}3\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-4\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$\Rightarrow x = 3+\lambda\,, y = -1+2\lambda\,, z = 2-4\lambda$ ..... (2)

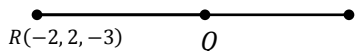
$Eqn.(2)\ in\ to\ eqn.(1)$

$\Rightarrow -(3+\lambda) - 2(-1+2\lambda) + 4(2-4\lambda) = -14$

$\Rightarrow -3-\lambda+2-4\lambda+8-16\lambda = -14\ or\ -21\lambda = -21\ \therefore \lambda = 1$

$P(x,y,z) = P(3+\lambda\ ,-1+2\lambda\ ,2-4\lambda) \Rightarrow P(3+1\,,-1+2\,,2-4) = P(4\,,1\,,-2)$

Now finding equation of $\overline{RP}$



$\overrightarrow{RQ} = \lambda\overrightarrow{RP} \Rightarrow \overrightarrow{OQ} - \overrightarrow{OR} = \lambda(\overrightarrow{OP} - \bar{O}R)$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}-2\\2\\-3\end{pmatrix} = \lambda\left[\begin{pmatrix}4\\1\\-2\end{pmatrix} - \begin{pmatrix}-2\\2\\-3\end{pmatrix}\right] = \lambda\begin{pmatrix}6\\-1\\1\end{pmatrix}$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}-2\\2\\-3\end{pmatrix} = \lambda\begin{pmatrix}6\\-1\\1\end{pmatrix} \; or \; \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2\\2\\-3\end{pmatrix} + \lambda\begin{pmatrix}6\\-1\\1\end{pmatrix}$

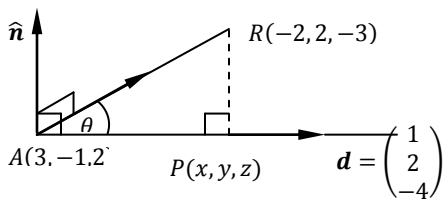
$\therefore \boldsymbol{r} = \begin{pmatrix}-2\\2\\-3\end{pmatrix} + \lambda\begin{pmatrix}6\\-1\\1\end{pmatrix}$

Now finding $|\overrightarrow{PR}| = perpendicular\ distance$

$\overrightarrow{PR} = \begin{pmatrix}6\\-1\\1\end{pmatrix}$

$\therefore |\overrightarrow{PR}| = \sqrt{(6)^2+(-1)^2+(1)^2} = \sqrt{38}\,units$

Alternative method of finding perpendicular distance of a point from a line:



$|\overrightarrow{PR}| = |\overrightarrow{AR}|sin\theta$

By cross product,

$\boldsymbol{d} \times \overrightarrow{AR} = |\boldsymbol{d}||\overrightarrow{AR}|sin\theta.\hat{\boldsymbol{n}}$

$|\boldsymbol{d} \times \overrightarrow{AR}| = |\boldsymbol{d}||\overrightarrow{AR}|sin\theta\ [\because |\hat{\boldsymbol{n}}| = 1]$

$\therefore |\overrightarrow{PR}| = |\overrightarrow{AR}|sin\theta = \frac{|\boldsymbol{d}\times\overrightarrow{AR}|}{|\boldsymbol{d}|}$

$\overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA} = \begin{pmatrix}-2\\2\\-3\end{pmatrix} - \begin{pmatrix}3\\-1\\2\end{pmatrix} = \begin{pmatrix}-5\\3\\-5\end{pmatrix}; \boldsymbol{d} = \begin{pmatrix}1\\2\\-4\end{pmatrix}$

$\boldsymbol{d} \times \overrightarrow{AR} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -4 \\ -5 & 3 & -5 \end{vmatrix} = 2\boldsymbol{i} + 25\boldsymbol{j} + 13\boldsymbol{k}$

$\left|\boldsymbol{d} \times \overrightarrow{AR}\right| = \sqrt{2^2 + (25)^2 + (13)^2} = \sqrt{798}\,; |\boldsymbol{d}| = \sqrt{1^2 + 2^2 + (-4)^2} = \sqrt{21}$

$\therefore \left|\overrightarrow{AP}\right| = \frac{\sqrt{798}}{\sqrt{21}} = \sqrt{38}\, units$

NOTE:

A student must know how to identify the point through which the line is passing If the equation is in Cartesian form. This can easily be deduced from the parametric form.

Examples

Express the following equations in parametric form.

a) $x - 1 = \frac{y-2}{0} = \frac{z-1}{3}$

b) $\frac{x-1}{2} = \frac{4-y}{3} = \frac{3-z}{0}$

c) $2x + 1 = \frac{y-2}{4} = \frac{3-z}{2}$

Solution

a) $x - 1 = \frac{y-2}{0} = \frac{z-1}{3}$

Let $x - 1 = \lambda \Rightarrow x = 1 + \lambda$

$For\ \frac{y-2}{0},\ y - 2 = 0 \Rightarrow y = 2 = 2 + 0\lambda$

$\frac{z-1}{3} = \lambda \Rightarrow z = 1 + 3\lambda$

$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix} \Rightarrow \boldsymbol{r} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$

b) $\frac{x-1}{2} = \frac{4-y}{3} = \frac{3-z}{0}$

Let $\frac{x-1}{2} = \lambda \Rightarrow x = 1 + 2\lambda$

$\frac{4-y}{3} = \lambda \Rightarrow y = 4 - 3\lambda$

$For\ \frac{3-z}{0}, 3 - z = 0 \Rightarrow z = 3 = 3 + 0\lambda$

$\therefore \boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\4\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\0\end{pmatrix}$

c) $2x + 1 = \frac{y-2}{4} = \frac{3-z}{2}$

Let $2x + 1 = \lambda \Rightarrow x = -\frac{1}{2} + \frac{1}{2}\lambda$

$\frac{y-2}{4} = \lambda \Rightarrow y = 2 + 4\lambda$

$\frac{3-z}{2} = \lambda \Rightarrow z = 3 - 2\lambda$

$\therefore \boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}\frac{1}{2}\\4\\-2\end{pmatrix}$

$or\ \boldsymbol{r} = \begin{pmatrix}-0.5\\2\\3\end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix}1\\8\\-4\end{pmatrix}\ or\ \boldsymbol{r} = \begin{pmatrix}-0.5\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\8\\-4\end{pmatrix}$

INTERSECTION OF TWO LINES

Let the lines be;

$\boldsymbol{r}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ *and* $\boldsymbol{r}_2 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

For two lines to intersect,

$$\boxed{\boldsymbol{r}_1 = \boldsymbol{r}_2}$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$\Rightarrow a_1 + \lambda b_1 = c_1 + \mu d_1$ ..... (1)

$\Rightarrow a_2 + \lambda b_2 = c_2 + \mu d_2$ ..... (2)

$\Rightarrow a_3 + \lambda b_3 = c_3 + \mu d_3$ ..... (3)

Solving for the constants λ *and* μ in any two equations, the values should satisfy the third equation *. i. e.* the three equations must be consistent for λ *and* μ.

Examples

Show that the pair of lines intersect, and find the coordinates of point of intersection.

a) $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ *and* $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$

b) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$ *and* $\boldsymbol{r} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

c) $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ and the line passing through $A(2, 2, 5)$ and $B(1, 2, 3)$.

Solution

a) $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$; $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$

$\boldsymbol{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$; $\boldsymbol{r}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -10 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$

For intersection $\boldsymbol{r}_1 = \boldsymbol{r}_2$

$$\Rightarrow \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -10 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 8 \end{pmatrix}$$

$\Rightarrow 4 + \lambda = 1 + 2\mu$ *or* $\lambda - 2\mu = -3$ ..... (1)

$\Rightarrow -3 - 4\lambda = -1 - 3\mu$ *or* $-4\lambda + 3\mu = 2$..... (2)

$\Rightarrow -1 + 7\lambda = -10 + 8\mu$ *or* $7\lambda - 8\mu = -9$ ..... (3)

Solving eqn. (1) and eqn. (2) simultaneously

$4 \times eqn.(1) + eqn.(2)$

$\Rightarrow \quad 4\lambda - 8\mu = -12$

$\Rightarrow \quad -4\lambda + 3\mu = 2$

From eq $\quad -5\mu = -10 \, , \mu = 2 \qquad 4 = 1$

$\therefore \lambda = 1 \, , \mu - 2$ ..... (4)

$subst.eqn.(4) in\ to\ eqn.(3)$

$7 - 16 = -9 = \boldsymbol{RHS}$. Hence lines intersect.

From $\boldsymbol{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$, $for\ \lambda = 1\ then\ x = 5 \, , y = -7 \, , z = 6$

$\therefore (5, -7, 6)$ is the point of intersection.

b) $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\5\\2\end{pmatrix}$ *and* $\boldsymbol{r} = \begin{pmatrix}3\\7\\5\end{pmatrix} + \mu\begin{pmatrix}2\\1\\2\end{pmatrix}$

$\begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\5\\2\end{pmatrix} = \begin{pmatrix}3\\7\\5\end{pmatrix} + \mu\begin{pmatrix}2\\1\\2\end{pmatrix}$

$\Rightarrow 1 + 2\lambda = 3 + 2\mu \ or \ 2\lambda - 2\mu = 2$ ..... (1)

$\Rightarrow 2 + 5\lambda = 7 + \mu \ or \ 5\lambda - \mu = 5$..... (2)

$\Rightarrow 3 + 2\lambda = 5 + 2\mu \ or \ 2\lambda - 2\mu = 2$..... (3)

Solving eqn. (1) and eqn. (2) simultaneously

$eqn.(1) - eqn.(2)$

$\Rightarrow \quad \lambda - \mu = 1$

$\Rightarrow \ 5\lambda - \mu = 5$

$-4\lambda = -4\ ,\ \lambda = 1$

From eqn. (1) $\lambda - \mu = 1, \lambda = 1 \Rightarrow \mu = 0$ ..... (4)

Eqn. (4) in to eqn. (3) $\Rightarrow 2 - 0 = 2$. Hence lines intersect.

From $\boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\5\\2\end{pmatrix}, \lambda = 1 \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\7\\5\end{pmatrix}$

$\therefore (3, 7, 5)$ is the point of intersection of the lines.

c) $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3} \Rightarrow \boldsymbol{r}_1 = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\3\end{pmatrix}$

$AP = \lambda\vec{AB}$

$\Rightarrow \overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$

$\Rightarrow \boldsymbol{r} - \begin{pmatrix}2\\2\\5\end{pmatrix} = \lambda\left[\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\2\\5\end{pmatrix}\right] = \begin{pmatrix}-1\\0\\-2\end{pmatrix}$

$\therefore \boldsymbol{r}_2 = \begin{pmatrix}2\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\-2\end{pmatrix}$ Is the equation through the two given points.

For intersection of lines, $\boldsymbol{r}_1 = \boldsymbol{r}_2$

$\Rightarrow \begin{pmatrix}1\\2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\3\end{pmatrix} = \begin{pmatrix}2\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\-2\end{pmatrix}$

$\Rightarrow 1 + \mu = 2 - \lambda \ or \ \mu + \lambda = 1$ ..... (1)

$\Rightarrow 1 + 3\mu = 5 - 2\lambda \ or \ 3\mu + 2\lambda = 4$ ..... (2)

Solving equations simultaneously

$2 \times eqn.(1) - eqn.(2)$

$2\lambda + 2\mu = 2$

$-\ \ 2\lambda + 3\mu = 4$

$-\mu = -2 \Rightarrow \mu = 2$

From eqn. (1) , $\lambda = 1 - 2 = -1$

From $\boldsymbol{r}_1 = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\3\end{pmatrix}, \mu = 2 \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}2\\0\\6\end{pmatrix} = \begin{pmatrix}3\\2\\7\end{pmatrix}$

$\therefore (3, 2, 7)$ is the point of intersection.

Alternative approach to these questions:

Expressing the equations in Cartesian form;

a) $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ ..... (1)

$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ ..... (2)

Splitting the equations in to two other equations;

For eqn. (1) $\frac{x-4}{1}=\frac{y+3}{-4}$, $\frac{y+3}{-4}=\frac{z+1}{7}$

For eqn. (2) $\frac{x-1}{2}=\frac{y+1}{-3}$, $\frac{y+1}{-3}=\frac{z+10}{8}$

$\Rightarrow \frac{x-4}{1}=\frac{y+3}{-4}$ or $-4x+16=y+3$

$\therefore 4x+y=13$ ..... (3)

$\Rightarrow \frac{y+3}{-4}=\frac{z+1}{7}$ or $7y+21=-4z-4$

$\therefore 7y+4z=-25$ ..... (4)

$\Rightarrow \frac{x-1}{2}=\frac{y+1}{-3}$ or $-3x+3=2y+2$

$\therefore 3x+2y=1$ ..... (5)

$\Rightarrow \frac{y+1}{-3}=\frac{z+10}{8}$ or $8y+8=-3z-30$

$\therefore 8y+3z=-38$ ..... (6)

Solving any *three* equations simultaneously for point of intersection;

Solving eqn. (3) and eqn. (5)

$2\times\left|\begin{matrix}4x+y=13\Rightarrow 8x+2y=26\\ 3x+2y=1\Rightarrow\ 3x+2y=1\end{matrix}\right.$

Subtracting the equations; $5x=25\Rightarrow x=5$

From $4x+y=13$; $x=5\Rightarrow 20+y=13\therefore y=-7$

From eqn. (4); $7y+4z=-25$, $y=-7\Rightarrow -49+4z=-25\therefore z=6$

$\therefore (5,-7,6)$ is the point of intersection.

This must satisfy the fourth equation if the lines intersect.

$8y+3z=-38$; $y=-7$, $z=6$

$\Rightarrow 8(-7)+3(6)=-38=\boldsymbol{RHS}$. Hence lines intersect at $(5,-7,6)$

b) $\boldsymbol{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\5\\2\end{pmatrix}\Rightarrow\frac{x-1}{2}=\frac{y-2}{5}=\frac{z-3}{2}$ ..... (1)

$\boldsymbol{r}=\begin{pmatrix}3\\7\\5\end{pmatrix}+\mu\begin{pmatrix}2\\1\\2\end{pmatrix}\Rightarrow\frac{x-3}{2}=\frac{y-7}{1}=\frac{z-5}{2}$ ..... (2)

From eqn. (1): $\frac{x-1}{2}=\frac{y-2}{5}\Rightarrow 5x-2y=1$..... (3)

$\frac{y-7}{1}=\frac{z-5}{2}\Rightarrow 2y-5z=-11$..... (4)

From eqn. (2): $\frac{x-3}{2}=\frac{y-7}{1}\Rightarrow x-2y=-11$ ..... (5)

$\frac{y-7}{1}=\frac{z-5}{2}\Rightarrow 2y-z=9$..... (6)

Solving any *three* equations simultaneously for point of intersection;

Solving eqn. (3) and eqn. (5)

$$\begin{array}{rl} & 5x-2y=1\\ -\ & x-2y=-11\\ \hline & 4x\quad\ =12\Rightarrow x=3\end{array}$$

From $x-2y=-11$, $x=3\Rightarrow 3-2y=-11\therefore y=7$

From eqn. (4): $2y-5z=-11, y=7\Rightarrow 14-5z=-11\therefore z=5$

$\therefore (3\,,7\,,5)$ is the point of intersection.

This must satisfy eqn. (6)

$2y - z = 9 \Rightarrow 14 - 5 = 9 = \boldsymbol{RHS}$. Hence lines intersect at $(3\,,7\,,5)$

2. *Prove that the lines* $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ *and* $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ *intersect. Find the point of intersection of the lines.* ***Answer***: $(5\,,-7\,,6)$

3. *Find the equation of a straight line perpendicular to both lines* $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3}$ *and* $\frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2}$ *and passing through their point of intersection.*

Solution

$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3}$, $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2}$, $\boldsymbol{r} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

For intersection to occur, $\boldsymbol{r_1} = \boldsymbol{r_2}$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$\Rightarrow 1 + \lambda = -2 + 2\mu \;\; or \;\; \lambda - 2\mu = -3$ ..... (1)

$\Rightarrow 1 + 2\lambda = 5 - \mu \;\; or \; 2\lambda + \mu = 4$..... (2)

$\Rightarrow -2 + 3\lambda = -3 + 2\mu \;\; or \;\; 3\lambda - 2\mu = -1$..... (3)

Solving (1) and (2) simultaneously

$2 \times eqn.(1) - eqn.(2)$

$2\lambda - 4\mu = -6$

$-$

$2\lambda + \mu = 4$

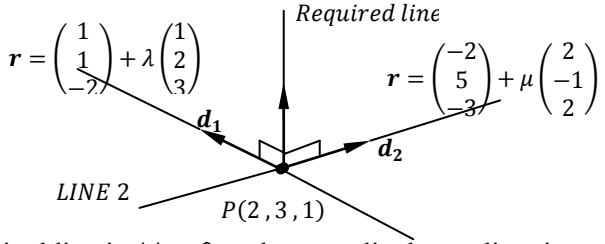
$-5\mu = -10 \Rightarrow \mu = 2$

From eqn. (1) $\lambda - 2\mu = -3\,, \mu = 2 \; solving \;\; \lambda = 1$

$[Confirming, from\ (3), 3\lambda - 2\mu = -1 \Rightarrow 3 - 4 = -1, hence\ lines\ intersect]$

From $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, for\ \lambda = 1 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$\therefore (2\,,3\,,1)$ is the point of intersection.



Required line is // to $\hat{\boldsymbol{n}}$ and perpendicular to direction vectors of the given lines.

$\hat{\boldsymbol{n}} = \frac{\boldsymbol{d_1} \times \boldsymbol{d_2}}{|\boldsymbol{d_1} \times \boldsymbol{d_2}|}$

$\boldsymbol{d_1} \times \boldsymbol{d_2} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = 7\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$

$\therefore \hat{\boldsymbol{n}} = \frac{\boldsymbol{d_1} \times \boldsymbol{d_2}}{|\boldsymbol{d_1} \times \boldsymbol{d_2}|} = \frac{7\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}}{\sqrt{90}}$

$\therefore \boldsymbol{d} = d.v\ of\ required\ line = \frac{7\boldsymbol{i}+4\boldsymbol{j}-5\boldsymbol{k}}{\sqrt{90}}.\gamma$ $[\because for\ \boldsymbol{d}\ and\ \hat{\boldsymbol{n}}\ to\ be \quad \boldsymbol{d} = \gamma\hat{\boldsymbol{n}}]$ //

Required line: $\boldsymbol{r} = \boldsymbol{a} + \alpha\,\boldsymbol{d}$

$\Rightarrow \boldsymbol{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \frac{\alpha\gamma}{\sqrt{90}}.\begin{pmatrix}7\\4\\-5\end{pmatrix}$ $or\ \boldsymbol{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \beta\begin{pmatrix}7\\4\\-5\end{pmatrix}, where\ \frac{\alpha\gamma}{\sqrt{90}} = \beta$

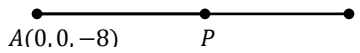
NOTE:

To avoid over manipulation of constants when finding the direction vector of the required line after crossing $\boldsymbol{d}_1 and\ \boldsymbol{d}_2$, then $\boldsymbol{d}_1 = \hat{\boldsymbol{n}}$ since for parallel lines the ratio of direction vectors for both lines must be the same. (***see notes on parallel lines***)

3. *Find the equation of the line through the point* $(1\,,3\,,-2)$ *and perpendicular to the line passing through points* $A(0\,,0\,,-8)$, $B(1\,,2\,,-3)$ *and* $\boldsymbol{r} = 4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k} + \lambda(-2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$

Solution

Equation of line through A and B;



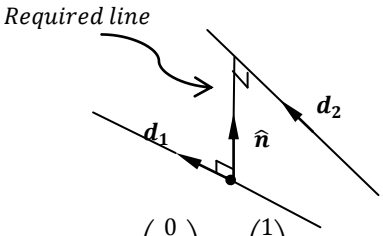
$\overrightarrow{AP} = \mu\overrightarrow{AB}\,, \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \begin{pmatrix}1\\2\\-3\end{pmatrix} - \begin{pmatrix}0\\0\\-8\end{pmatrix} = \begin{pmatrix}1\\2\\5\end{pmatrix}$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \begin{pmatrix}0\\0\\-8\end{pmatrix}$

$\therefore\ \boldsymbol{r_1} = \begin{pmatrix}0\\0\\-8\end{pmatrix} + \mu\begin{pmatrix}1\\2\\5\end{pmatrix}$

$\boldsymbol{r_2} = \begin{pmatrix}4\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\5\end{pmatrix}$



$\boldsymbol{r_2} = \begin{pmatrix}4\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\5\end{pmatrix}$

$\boldsymbol{r_1} = \begin{pmatrix}0\\0\\-8\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\5\end{pmatrix}$

For $\boldsymbol{r_1}\,, \boldsymbol{d_1} = \begin{pmatrix}0\\0\\-8\end{pmatrix}, for\ \boldsymbol{r_2}\,, \boldsymbol{d_2} = \begin{pmatrix}2\\5\end{pmatrix}$

Direction vector or perpendicular is

$\boldsymbol{d_1} \times \boldsymbol{d_2} = \begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\1 & 2 & 5\\-2 & 3 & -1\end{vmatrix} = -17\boldsymbol{i} - 9\boldsymbol{j} + 4\boldsymbol{k}$

Required line: $\boldsymbol{r} = \begin{pmatrix}1\\3\\-2\end{pmatrix} + \mu\begin{pmatrix}-17\\-9\\4\end{pmatrix}$

4. *Given the two lines are parametrically given by:*

$\boldsymbol{r_1} = -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$ *and*

$\boldsymbol{r_2} = -3\boldsymbol{i} + p\boldsymbol{j} + 7\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

If the lines intersect, find the

a. the value of $t\,, s\ and\ p$.
b. Coordinates of the points of intersection.

Solution

For lines to intersect, $\boldsymbol{r_1} = \boldsymbol{r_2}$

$\Rightarrow -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) = -3\boldsymbol{i} + p\boldsymbol{j} + 7\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$\Rightarrow \begin{pmatrix}-1\\2\\1\end{pmatrix} + t\begin{pmatrix}1\\-2\\3\end{pmatrix} = \begin{pmatrix}-3\\p\\7\end{pmatrix} + s\begin{pmatrix}1\\-1\\2\end{pmatrix}$

$\Rightarrow -1 + t = -3 + s \ \ or \ \ t - s = -2$ ..... (1)

$\Rightarrow 2 - 2t = p - s \ \ or \ \ 2t - s = 2 - p$ ..... (2)

$\Rightarrow 1 + 3t = 7 + 2s \ \ or \ 3t - 2s = 6$ ..... (3)

Solving eqn. (1) and eqn. (3) simultaneously

$2 \times \left| \begin{matrix} t - s = -2 \Rightarrow 2t - 2s = -4 \\ 3t - 2s = 6 \Rightarrow 3t - 2s = 6 \end{matrix} \right.$

Subtracting; $-t = -10 \ \therefore \ t = 10$

From $t - s = -2 \, , t = 10 \Rightarrow s + 10 + 2 = 12$

$\therefore s = 12 \, , t = 10$

$From \ \ 2t - s = 2 - p \Rightarrow 21 - 12 = 2 - p \ \therefore p = -6$

From $\boldsymbol{r_1} = \begin{pmatrix}-1\\2\\1\end{pmatrix} + t\begin{pmatrix}1\\-2\\3\end{pmatrix}, t = 10 \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\2\\1\end{pmatrix} + 10\begin{pmatrix}1\\-2\\3\end{pmatrix} = \begin{pmatrix}9\\-18\\31\end{pmatrix}$

$\therefore (9 \, , -18 \, , 31)$ is the point of intersection.

5. *Given that the lines* $\boldsymbol{r_1} = \begin{pmatrix}p\\2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\5\\3\end{pmatrix}$ *and* $\boldsymbol{r_2} = \begin{pmatrix}2\\12\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\p\end{pmatrix}$ *intersect, find the value of p. hence find the point of intersection.*

Solution

For point of intersection, $\boldsymbol{r_1} = \boldsymbol{r_2}$

$\Rightarrow \begin{pmatrix}p\\2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\5\\3\end{pmatrix} = \begin{pmatrix}2\\12\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\p\end{pmatrix}$

$\Rightarrow p + \mu = 2 + \lambda$ ..... (1)

$\Rightarrow 2 + 5\mu = 12 \ \Rightarrow \mu = 2$ ..... (2)

$\Rightarrow 1 + 3\mu = 5 + \lambda p$ ..... (3)

$eqn.(2) \ into \ eqn,(1)$

$\Rightarrow 1 + 6 = 5 + \lambda p \, , \lambda p = 1 \ \ or \ \lambda = \frac{1}{p}$ ..... (4)

$eqn.(2),(4) into \ eqn.(1)$

$\Rightarrow p + 2 = 2 + \lambda \, , p = \lambda$ ..... (5)

$eqn.(5) \ into \ eqn,(4)$

$\Rightarrow \lambda = \frac{1}{\lambda} \, , \lambda^2 = 1 \ \therefore \lambda = \pm 1$

$\therefore p = \pm 1$

6. *Show that the lines* $\boldsymbol{r_1} = \begin{pmatrix}3\\5\\7\end{pmatrix} + m\begin{pmatrix}1\\2\\1\end{pmatrix}$ *and* $\boldsymbol{r_2} = \begin{pmatrix}1\\2\\3\end{pmatrix} + n\begin{pmatrix}2\\3\\5\end{pmatrix}$ *do not intersect and not parallel.* [$i.e. are \ skew$]

Solution

For lines to intersect, $\boldsymbol{r_1} = \boldsymbol{r_2}$

$\begin{pmatrix}3\\5\\7\end{pmatrix} + m\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + n\begin{pmatrix}2\\3\\5\end{pmatrix}$

$\Rightarrow 3 + m = 1 + 2n \ \ or \ \ m - 2n = -2$ ..... (1)

$\Rightarrow 5 + 2m = 2 + 3n \ \ or \ \ 2m - 3n = -3$ ..... (2)

$\Rightarrow 7 + m = 3 + 5n \ \ or \ \ m - 5n = -4$ ..... (3)

Solving eqn. (1) and eqn. (2) simultaneously

$m = 0 \, , n = 1$ ..... (4)

$eqn.(4) in\ to\ eqn.(5)$

$0-5 \neq -4$. Hence lines do not intersect.

Considering a case of parallel lines

For parallel lines, t*he direction ratio values are similar.*

For $\boldsymbol{r_1} = \begin{pmatrix}3\\5\\7\end{pmatrix} + m\begin{pmatrix}1\\2\\1\end{pmatrix}, direction\ ratio: 1:2:1$

For $\boldsymbol{r_2} = \begin{pmatrix}1\\2\\3\end{pmatrix} + n\begin{pmatrix}2\\3\\5\end{pmatrix}, direction\ ratio: 2:3:5$

Since the direction ratios are not the same, lines are not parallel. Thus are s*kew.*

7. *Show that the lines are parallel*

   a) $\boldsymbol{r_1} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\5\end{pmatrix}\ and\ \boldsymbol{r_2} = \begin{pmatrix}3\\5\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-5\end{pmatrix}$

   b) $\boldsymbol{r_1} = \begin{pmatrix}3\\5\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\-5\end{pmatrix}\ and\ \boldsymbol{r_2} = \begin{pmatrix}1\\0\\7\end{pmatrix} + \mu\begin{pmatrix}-2\\3\\5\end{pmatrix}$

Solution

a) For $\boldsymbol{r_1} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\5\end{pmatrix}, direction\ ratio: -2:1:5$

For $\boldsymbol{r_2} = \begin{pmatrix}3\\5\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-5\end{pmatrix}, direction\ ratio: 2:-1:-5, dividing\ through\ by -1$

$\Rightarrow direction\ ratio\ is -2:1:5$

Since the direction ratios are the same the lines are parallel.

b) For $\boldsymbol{r_1} = \begin{pmatrix}3\\5\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\-5\end{pmatrix}, direction\ ratio: 2:-3:-5$

For $\boldsymbol{r_2} = \begin{pmatrix}1\\0\\7\end{pmatrix} + \mu\begin{pmatrix}-2\\3\\5\end{pmatrix}, direction\ ratio: -2:3:5, dividing\ through\ by -1$

$\Rightarrow direction\ ratio\ is\ 2:-3:-5$

The ratios are the same, hence lines are parallel.

NOTE:

Also for parallel lines $\boldsymbol{d_1} \times \boldsymbol{d_2} = \boldsymbol{0}$ can be applied.

5. *Given the equation of the lines in the form*

   $\frac{x-2}{3} = \frac{y-4}{5} = \frac{z-7}{2}$, show that $(8, 14, 11)$ lie on this line.

Solution

Let $\frac{x-2}{3} = \frac{y-4}{5} = \frac{z-7}{2} = \lambda$; $\Rightarrow \frac{x-2}{3} = \lambda, \frac{y-4}{5} = \lambda, \frac{z-7}{2} = \lambda$

For $(8, 14, 11)$, λ must be consistent

$\frac{x-2}{3} = \frac{8-2}{3} = 2 = \lambda$

$\frac{y-4}{5} = \frac{14-4}{5} = 2 = \lambda$

$\frac{z-7}{2} == \frac{11-7}{2} = 2 = \lambda$

Since $\lambda = 2$, consistent for $(8, 14, 11)$, *hence point lies on line*

6. *Given the equation of line:* $\boldsymbol{r} = (2+3\lambda)\boldsymbol{i} + (4+5\lambda)\boldsymbol{j} + (7+2\lambda)\boldsymbol{k}$. *Show that* $(-4, -6, 3)$ *lie on the line.*

Solution

$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$\Rightarrow x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (2 + 3\lambda)\boldsymbol{i} + (4 + 5\lambda)\boldsymbol{j} + (7 + 2\lambda)\boldsymbol{k}$

Equating unit vectors,

$x = 2 + 3\lambda\,,\ y = 4 + 5\lambda\,,\ z = 7 + 2\lambda$

Point $(-4\,,-6\,,3)$ must give the same value of λ in the equations formulated.

$\Rightarrow -4 = 2 + 3\lambda \quad \therefore \lambda = -2$

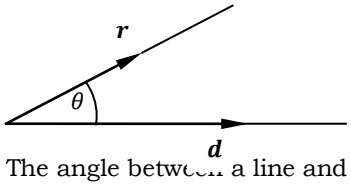
$\Rightarrow -6 = 4 + 5\lambda \quad \therefore \lambda = -2$

$\Rightarrow 3 = 7 + 2\lambda \quad \therefore \lambda = -2$

Since $\lambda = 2$, consistent for$(-4\,,-6\,,3)$, *hence point lies on line*

ANGLE BETWEEN A VECTOR AND A LINE

Consider the line $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$ and the vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$



The angle between a line and a vector is the angle between *vector* and *direction vector* of the line. *i.e.*

By dot product, $\boldsymbol{d}.\boldsymbol{r} = |\boldsymbol{d}||\boldsymbol{r}|cos\theta$

Let $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$\boldsymbol{d}.\boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = a_1d_1 + a_2d_2 + a_3d_3$

$|\boldsymbol{d}| = \sqrt{{d_1}^2 + {d_2}^2 + {d_3}^2}\,, |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\therefore cos\theta = \frac{\boldsymbol{d}.\boldsymbol{r}}{|\boldsymbol{d}||\boldsymbol{r}|}$$

Example

1. *Find the angle between*
   a. *Vector* $\boldsymbol{r} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ *and line* $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$
   b. *Vector* $\boldsymbol{r} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ *and line* $\frac{x-1}{2} = \frac{4-y}{1} = \frac{3-z}{2}$
   c. *Vector* $\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$ *and line* $\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

Solution

a. By dot product, $\boldsymbol{d}.\boldsymbol{r} = |\boldsymbol{d}||\boldsymbol{r}|cos\theta$

Let $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\boldsymbol{r}.\boldsymbol{d} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2 + 6 + 3 = 11$

$|\boldsymbol{d}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$, $|\boldsymbol{r}| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$

$\therefore cos\theta = \frac{\boldsymbol{d}.\boldsymbol{r}}{|\boldsymbol{d}||\boldsymbol{r}|} = \frac{11}{\sqrt{14}\times\sqrt{14}} = \frac{11}{14} \Rightarrow \theta = 38.2^0$

b. $\boldsymbol{r} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{d} = ?$

Let $\frac{x-1}{2} = \frac{4-y}{1} = \frac{3-z}{2} = \lambda$

$\frac{x-1}{2} = \lambda \Rightarrow x = 1 + 2\lambda$

$\frac{4-y}{1} = \lambda \Rightarrow y = 4 - \lambda$

$\frac{3-z}{2} = \lambda \Rightarrow z = 3 - 2\lambda$

$\therefore \boldsymbol{r} = \begin{pmatrix}1\\4\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\-2\end{pmatrix} \Rightarrow \boldsymbol{d} = \begin{pmatrix}2\\-1\\-2\end{pmatrix}$

$\Rightarrow \boldsymbol{r}.\boldsymbol{d} = \begin{pmatrix}3\\-2\\1\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix} = 6 + 2 - 2 = 6$

$\Rightarrow |\boldsymbol{d}| = \sqrt{2^2+(-1)^2+(-2)^2} = \sqrt{9} = 3$, $|\boldsymbol{r}| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{14}$

$\therefore cos\theta = \frac{\boldsymbol{d}.\boldsymbol{r}}{|\boldsymbol{d}||\boldsymbol{r}|} = \frac{6}{3\times\sqrt{14}} = \frac{11}{14} \Rightarrow \theta = 57.7^0$

c. *Left as an exercise*

2. *Find the value of λ if*

a. Vector $\boldsymbol{r} = 2\boldsymbol{i} + \lambda\boldsymbol{j} + 3\boldsymbol{k}$ *and line* $\boldsymbol{r} = \begin{pmatrix}3\\1\\5\end{pmatrix} + k\begin{pmatrix}1\\2\\3\end{pmatrix}$ *are perpendicular*

b. Vector $\boldsymbol{r} = \boldsymbol{i} + \lambda\boldsymbol{j} + \boldsymbol{k}$ *and line* $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\1\end{pmatrix}$ *make an angle of* 60^0

Solution

a. $\boldsymbol{r} = 2\boldsymbol{i} + \lambda\boldsymbol{j} + 3\boldsymbol{k} = \begin{pmatrix}2\\\lambda\\3\end{pmatrix}$, $\boldsymbol{r} = \begin{pmatrix}3\\1\\5\end{pmatrix} + k\begin{pmatrix}1\\2\\3\end{pmatrix} \Rightarrow \boldsymbol{d} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

By dot product,

$\boldsymbol{d}.\boldsymbol{r} = |\boldsymbol{d}||\boldsymbol{r}|cos\theta$, *but* $\theta = 90^0$ [$\because$ *vector is perpendicular to line*]

$\therefore \boldsymbol{r}.\boldsymbol{d} = 0 \Rightarrow \begin{pmatrix}2\\\lambda\\3\end{pmatrix}.\begin{pmatrix}1\\2\\3\end{pmatrix} = 0$ *or* $2 + 2\lambda + 9 = 0$ $\therefore \lambda = -\frac{11}{2}$

b. $\boldsymbol{r} = \boldsymbol{i} + \lambda\boldsymbol{j} + \boldsymbol{k} = \begin{pmatrix}1\\\lambda\\1\end{pmatrix}$, $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-2\\1\end{pmatrix} \Rightarrow \boldsymbol{d} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$

By dot product, $\boldsymbol{d}.\boldsymbol{r} = |\boldsymbol{d}||\boldsymbol{r}|cos\theta$, *but* $\theta = 60^0$

$\Rightarrow \boldsymbol{d}.\boldsymbol{r} = |\boldsymbol{d}||\boldsymbol{r}|cos60^0 = |\boldsymbol{d}||\boldsymbol{r}| \times \frac{1}{2}$ $\left[\because cos60^0 = \frac{1}{2}\right]$

$\therefore \begin{pmatrix}1\\-2\\1\end{pmatrix}.\begin{pmatrix}1\\\lambda\\1\end{pmatrix} = \sqrt{1^2+(-2)^2+1^2} \times \sqrt{1^2+\lambda^2+1^2} \times \frac{1}{2}$

$\Rightarrow 1 - 2\lambda + 1 = \frac{\sqrt{6}}{2} \times \sqrt{2+\lambda^2}$ *or* $4 - 4\lambda = \sqrt{12+6\lambda^2}$

squaring both sides

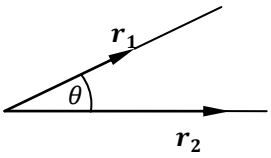
$16 - 32\lambda + 16\lambda^2 = 12 + 6\lambda^2 \Rightarrow 10\lambda^2 - 32\lambda + 4 = 0$

Solving, $\lambda = 3.07, 0.13$

ANGLE BETWEEN TWO LINES

Consider the lines;

$\boldsymbol{r}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ *and* $\boldsymbol{r}_2 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$



The angle between two lines is the angle between the *direction vectors of the lines.*

For $\boldsymbol{r}_1$, *direction vector* $\boldsymbol{d}_1 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ *and for* $\boldsymbol{r}_2$, *direction vector* $\boldsymbol{d}_2 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$$\therefore \boldsymbol{d}_1.\boldsymbol{d}_2 = |\boldsymbol{d}_1||\boldsymbol{d}_2|cos\theta$$

Examples

1. *Find the angle between the pair of lines*

   a) $\frac{x+1}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ *and* $\boldsymbol{r} = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$

   b) $\boldsymbol{r} = (1+2\lambda)\boldsymbol{i} + (3+\lambda)\boldsymbol{j} + (5+2\lambda)\boldsymbol{k}$ *and* $\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 16 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$

   c) $\frac{10-x}{2} = \frac{4-y}{3} = \frac{8-z}{4}$ *and* $\boldsymbol{r} = \begin{pmatrix} 7 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$

Solution

a) $\frac{x+1}{1} = \frac{y-2}{2} = \frac{z+3}{3} \Rightarrow \boldsymbol{r} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + k \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \therefore \boldsymbol{d_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$\boldsymbol{r} = \begin{pmatrix} 4 \\ 8 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \boldsymbol{d_2} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = -3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

From dot product $\boldsymbol{d}_1.\boldsymbol{d}_2 = |\boldsymbol{d}_1||\boldsymbol{d}_2|cos\theta$

$\boldsymbol{d}_1.\boldsymbol{d}_2 = (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}).(-3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = -3 - 4 + 3 = -4$

$|\boldsymbol{d}_1| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, |\boldsymbol{d}_2| = \sqrt{(-3)^2 + (-2)^2 + 1^2} = \sqrt{14}$

$\therefore cos\theta = \frac{-4}{\sqrt{14} \times \sqrt{14}} \Rightarrow \theta = 106.6^0$

b) $\boldsymbol{r} = (1+2\lambda)\boldsymbol{i} + (3+\lambda)\boldsymbol{j} + (5+2\lambda)\boldsymbol{k}$

$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \therefore \boldsymbol{d_1} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

$\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 16 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \Rightarrow \boldsymbol{d_2} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = 2\boldsymbol{i} + 0\boldsymbol{j} + 5\boldsymbol{k}$

$\boldsymbol{d}_1.\boldsymbol{d}_2 = (2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).(2\boldsymbol{i} + 0\boldsymbol{j} + 5\boldsymbol{k}) = 4 + 0 + 10 = 14$

$|\boldsymbol{d}_1| = \sqrt{2^2 + 1^2 + 2^2} = 3, |\boldsymbol{d}_2| = \sqrt{2^2 + 0^2 + 5^2} = \sqrt{29}$

$\therefore cos\theta = \frac{14}{3 \times \sqrt{29}} \Rightarrow \theta = 29.9^0$

c) Left as an exercise

Task

1. *Find the angle between the lines:*

   a) $x - 1 = y - 3 = z - 7$ *and* $\frac{x-5}{2} = \frac{y+1}{3} = \frac{z}{5}$

   ANSWER: $\theta = 120^0$

   b) $x = 2 - 4t, x = 1 + 5t$

$y = 1 - 2t\ , y = -2 - t$

$z = 1 + 2t\ , z = 1 - 2t$

c) $\frac{x-3}{2} = \frac{y-3}{5}\ , z = 0\ ;\ \frac{x+1}{3} = \frac{y+2}{1}\ ,\ z = 0$ **ANSWER**: $\theta = 35^0$

[***HINT***: $\boldsymbol{d}_1 = 2\boldsymbol{i} + 5\boldsymbol{j} + 0\boldsymbol{k}\ ,\ \boldsymbol{d}_2 = 3\boldsymbol{i} + \boldsymbol{j} + 0\boldsymbol{k}$] **ANSWER**: $\theta = 50^0$

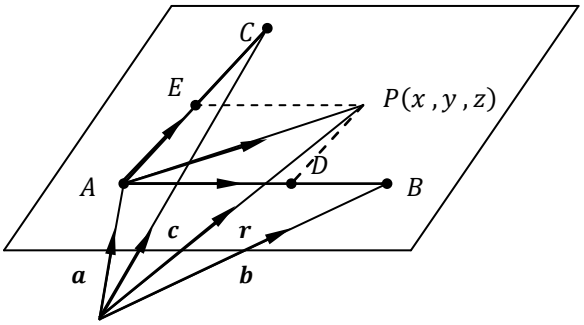
2. *Prove that the vectors* $\boldsymbol{a} = 7\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}\ , \boldsymbol{b} = 3\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ *and* $\boldsymbol{c} = 6\boldsymbol{i} - 16\boldsymbol{j} - 15\boldsymbol{k}$ *are mutually perpendicular.*
3. *a) Find the equation of the line through* $A(2\ , 2, 5)$ *and* $B(1\ , 2\ , 3)$
   b) If the line in (a) above meets the line $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ *at P, find*
   i. coordinates of P
   ii. angle between the lines (***Uneb* 2004**)

   ANSWER: $\boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}\ , P(3\ , 2\ , 7)\ , \theta = 171.9^0$
4. *Show that vector* $2\boldsymbol{i} - 5\boldsymbol{j} + 3.5\boldsymbol{k}$ *is perpendicular to the line* $\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} + \lambda(4\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k})$
   (***Uneb* 2000**)
5. *Find the equation of a line through points* $A(2\ , 2, 5)\ , B(1\ , 2\ , 3)$ *and* $C(0\ , 2\ , 1)$
   [***HINT***: 1st *check whether A, B and C are collinear, if not different equations must be formed*]
6. *Show that the lines* $\boldsymbol{r_1} = (1 + 4\lambda)\boldsymbol{i} + (1 - \lambda)\boldsymbol{j} + 2\lambda\boldsymbol{k}$ *and* $\boldsymbol{r_1} = (5 + 3\mu)\boldsymbol{i} + 2\mu\boldsymbol{j} + (2 - 5\mu)\boldsymbol{k}$ cut at right angles. Find the position vector of point of intersection. **ANSWER**: $5\boldsymbol{i} + 2\boldsymbol{k}$
7. *The points A, B, C and D have coordinates* $(3\ , -2\ , 0)\ , (-1\ , 2\ , 4)\ , (0\ , 1\ , -1)$ *and* $(5\ , -4\ , 4)$ *respectively. Find*
   a) The point of intersection of the lines AB and CD
   b) The angle between the lines in (a) above

   ANSWER: $(2\ , -1\ , 1)\ ;\ 70.5^0$
8. *The points A, B and C have p.vs* $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}\ , \boldsymbol{b} = \boldsymbol{j} - 4\boldsymbol{k}$ *and* $\boldsymbol{c} = 6\boldsymbol{i} + \mathbf{4}\boldsymbol{j} + 5\boldsymbol{k}$ *respectively. Find the p.v of a point R on* $\overline{BC}$ *such tat* $\overline{AR}$ *is perpendicular to* $\overline{BC}$*. Hence find the perpendicular distance of A from BC.* ***Answer***: $4\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}\ ;\ \overline{AR} = \sqrt{21}$
9. *The points A, B and C have p.vs* $3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}\ ,\ \boldsymbol{i} + 2\boldsymbol{k}$ *and* $4\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ *respectively. Find the vector perpendicular to both vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$*.*
   ANSWER: $\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$
10. *Show that the lines* $\boldsymbol{r_1} = 2\lambda\boldsymbol{i} - 3\boldsymbol{j} + (\lambda - 2)\boldsymbol{k}$ *and* $\boldsymbol{r_1} = (1 + \mu)\boldsymbol{i} + (2 - \mu)\boldsymbol{j} + (2\mu - 5)\boldsymbol{k}$ *do not intersect.*

VECTOR EQUATION OF A PLANE

a) **To find the vector equation of a plane through three non-collinear points A, B and C**



Let the $p.vs$ of A, B, C and P be $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *and* $\boldsymbol{r}$ respectively.
$\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP}$. But $\overrightarrow{AD} // \overrightarrow{AB}, \overrightarrow{DP} // \overrightarrow{AC}$
$\therefore \overrightarrow{AP} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$
But $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \boldsymbol{a}$

$\overrightarrow{AB} = \overrightarrow{O}$ O $\overrightarrow{OA} = \boldsymbol{b} - \boldsymbol{a}$
$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \boldsymbol{c} - \boldsymbol{a}$
$\therefore \boldsymbol{r} - \boldsymbol{a} = \boldsymbol{\lambda}(\boldsymbol{b} - \boldsymbol{a}) + \mu(\boldsymbol{c} - \boldsymbol{a})$

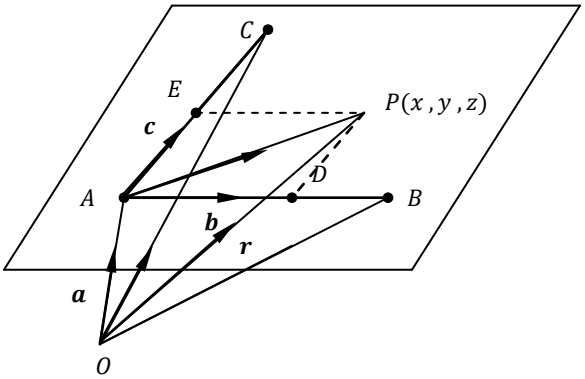
$$\therefore \boldsymbol{r} - \boldsymbol{a} = \boldsymbol{\lambda}(\boldsymbol{b} - \boldsymbol{a}) + \mu(\boldsymbol{c} - \boldsymbol{a})$$

NOTE:
If A is the origin, then the equation of a plane will be given by
$\overrightarrow{AP} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC} \Rightarrow \overrightarrow{OP} = \lambda\overrightarrow{OB} + \mu\overrightarrow{OC}$
$\therefore \boldsymbol{r} = \boldsymbol{\lambda b} + \mu\boldsymbol{c}$

$$\therefore \boldsymbol{r} = \boldsymbol{\lambda b} + \mu\boldsymbol{c}$$

This equation represents an equation of a plane passing through the origin and parallel to *two* given vectors **b** and **c.**
If given two vectors, then the equation of plane containing the two vectors is of the form above.

b) **To find vector equation of a plane passing through a given point A and parallel to two given vectors $\vec{b}$ and $\vec{c}$.**



Let $p.v$ of P be $\boldsymbol{r}$ and A be $\boldsymbol{a}$
From $\overrightarrow{AP} = \overrightarrow{AE} + \overrightarrow{EP}$; But $\overrightarrow{AE} = \lambda\boldsymbol{c}$, $\overrightarrow{EP} = \mu\boldsymbol{b}$
$\Rightarrow \overrightarrow{AP} = \lambda\boldsymbol{c} + \mu\boldsymbol{b}$

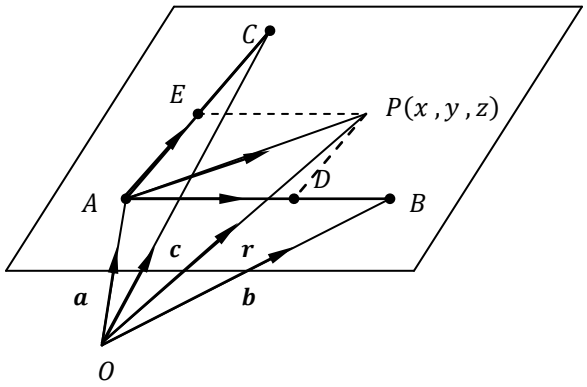
$\Rightarrow \overrightarrow{OP} - \overrightarrow{OA} = \lambda\boldsymbol{c} + \mu\boldsymbol{b}$

$$\therefore \boldsymbol{r} = \boldsymbol{a} + \mu\boldsymbol{b} + \lambda\boldsymbol{c}$$

If A is the Origin. $i.e.\ A(0\,,0\,,0)$, then the equation of the plane containing *two* vectors **b** and **c** is

$$\therefore \boldsymbol{r} = \mu\boldsymbol{b} + \lambda\boldsymbol{c}$$

c) **To find the vector equation of a plane through two given points A and B and parallel to a given vector** $\vec{c}$.



Let the $p.v$ of P, A and B be $\boldsymbol{r}, \boldsymbol{a}$ *and* $\boldsymbol{b}$ respectively.
As above, the equation of the plane is
$\overrightarrow{AP} = \lambda\overrightarrow{AB} + \mu\boldsymbol{c}$
$\therefore \boldsymbol{r} - \boldsymbol{a} = \boldsymbol{\lambda}(\boldsymbol{b} - \boldsymbol{a}) + \mu\boldsymbol{c}$

or $\boxed{\therefore \boldsymbol{r} = \boldsymbol{a} + \boldsymbol{\lambda}(\boldsymbol{b} - \boldsymbol{a}) + \mu\boldsymbol{c}}$

NOTE:
The above rectangular enclosed equations are called *parametric equations* of a plane.
When parameters λ *and* μ are eliminated from the equation of a plane, the resulting equation of a plane is called *Cartesian equation of a plane* of the form $Ax + By + Cy + D = 0$.

Examples

1. *Find the Cartesian equation of a plane*

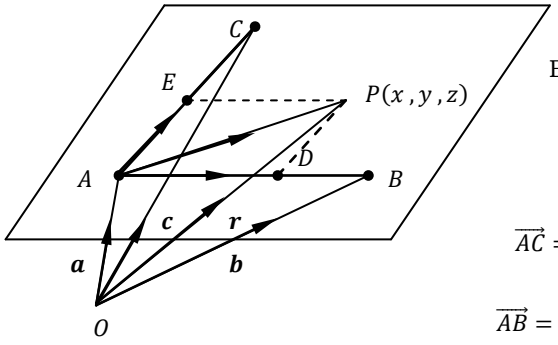
a) *Passing through the points* $A(-1\,,1\,,2)\,, B(1\,,-2\,,1)$ *and* $C(2\,,2\,,4)$

b) *Containing vectors* $\begin{pmatrix}0\\3\\0\end{pmatrix}$ *a nd* $\begin{pmatrix}1\\0\\2\end{pmatrix}$

c) *Passing through point* $(1\,,2\,,3)$ *and parallel to vectors* $\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ *and* $2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

Solution

a)



Let $\overrightarrow{OP} = \boldsymbol{r}$
$\overrightarrow{AP} = \overrightarrow{AE} + \overrightarrow{EP}$
But $\overrightarrow{AE}$ // $\overrightarrow{AC}$ and $\overrightarrow{EP}$ // $\overrightarrow{AB}$
$\therefore \overrightarrow{AP} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$

$$\therefore \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \begin{pmatrix}-1\\1\\2\end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}2\\2\\4\end{pmatrix} - \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\-2\\1\end{pmatrix} - \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\-3\\-1\end{pmatrix}$$

$$\therefore \boldsymbol{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\-3\\-1\end{pmatrix}$$

Expressing the equation in Cartesian form

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

$\Rightarrow x = -1 + 3\lambda + 2\mu$ ..... (1)

$y = 1 + \lambda - 3\mu$ ..... (2)

$z = 2 + 2\lambda - \mu$ ..... (3)

Considering (1) and (2), eliminating λ and μ

From (1) $x + 1 = 3\lambda + 2\mu$ ..... (4)

From (2) $y - 1 = \lambda - 3\mu$ ..... (5)

$Eqn.(4) \times 3 - eqn.(3)$

$3y - 3 = 3\lambda - 9\mu$

$x + 1 = 3\lambda + 2\mu$

$3y - x - 4 = -11\mu \quad \Rightarrow \mu = \frac{x-3y+4}{11}$ ..... (6)

$eqn.(6) in\ to\ eqn.(4)$

$\Rightarrow \lambda = y - 1 + 3\mu = y - 1 + 3\left[\frac{x-3y+4}{11}\right] = \frac{11y-11+3x-9y+12}{11}$

$\therefore \lambda = \frac{3x+2y+1}{11}$ ..... (7)

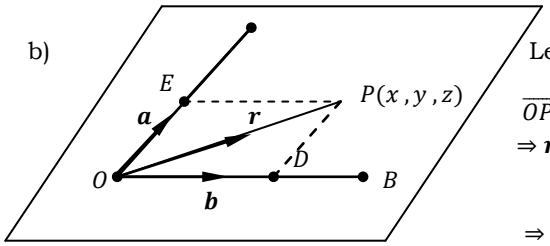
$eqn.(6),(7) in\ to\ (3) to\ conclude\ the\ Cartesian\ plane\ equation$

$\Rightarrow z = 2 + 2\left(\frac{3x+2y+1}{11}\right) - \left(\frac{x-3y+4}{11}\right)$

$11z = 22 + 6x + 4y + 2 - x + 3y - 4$

$\therefore 5x + 7y - 11z + 20 = 0$

b)



Let $\overrightarrow{OP} = \boldsymbol{r}$, $\boldsymbol{a} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\overrightarrow{OP} = \overrightarrow{OD} + \overrightarrow{DP}$, $\overrightarrow{OD}$ // $\boldsymbol{b}$ and $\overrightarrow{DP}$ // $\boldsymbol{a}$

$\Rightarrow \boldsymbol{r} = \lambda\boldsymbol{b} + \mu\boldsymbol{a}$

$$\Rightarrow \boldsymbol{r} = \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

Expressing this equation in Cartesian form

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$\Rightarrow x = \lambda$ ..... (1)

$\Rightarrow y = 3\mu$ ..... (2)

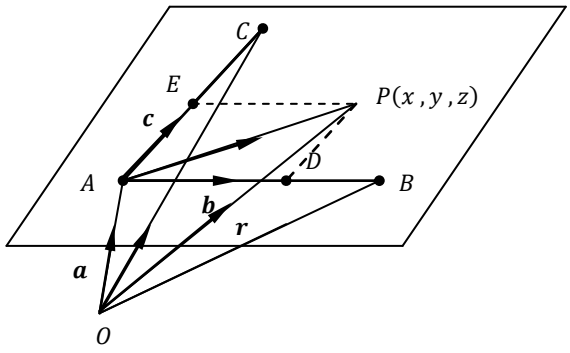
$\Rightarrow z = 2\lambda$ ..... (3)

$eqn\ (1) in\ to\ eqn.(3)$

$\therefore z = 2x\ \ or\ \ 2x - z = 0$ is the required equation of plane.

Note: The eqn. (2) cannot be used in any way to find the required equation of plane.

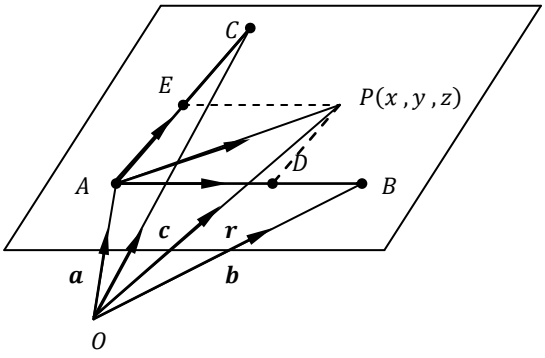
c)



Let $\boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$,
$\boldsymbol{r} = \overrightarrow{OP}$
and $\overrightarrow{OA} = \boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP} = \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \boldsymbol{a} = \lambda\boldsymbol{b} + \mu\boldsymbol{c} \Rightarrow \boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
$\Rightarrow \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$
$\Rightarrow x = 1 + \lambda + 2\mu \ or \ x - 1 = \lambda + 2\mu$ ..... (1)
$\Rightarrow y = 2 - 2\lambda - 3\mu \ or \ y - 2 = -2\lambda - 3\mu$ ..... (2)
$\Rightarrow z = 3 + 3\lambda + \mu \ or \ z - 3 = 3\lambda + \mu$ ..... (3)
Considering eqn. (1) and eqn. (2)
$eqn.(1) \times 2 + eqn.(2)$
$2x - 2 = 2\lambda + 4\mu$
$y - 2 = -2\lambda - 3\mu$
$2x - 2 + y - 2 = \mu \Rightarrow \mu = 2x + y - 4$ ..... (4)
$eqn.(4) \ in \ to \ eqn.(1)$
$x - 1 = \lambda + 2\mu \Rightarrow \lambda = x - 1 - 2\mu$
$\therefore \lambda = x - 1 - 2(2x + y - 4) = -3x - 2y + 7$..... (5)
$eqn.(4),(5) in \ to \ (3) to \ conclude \ equation \ of \ plane$
$\Rightarrow z - 3 = 3(-3x - 2y + 7) + 2x + y - 4$
$z - 3 = -9x - 6y + 21 + 2x + y - 4 = -7x - 5y + 17$
$\therefore 7x + 5y + z - 20 = 0$

2. *Find the Cartesian equation of a plane passing through* $A(0,3,-4), B(2,-1,2)$ *and* $C(7,4,-1)$. *Show that* $Q(10,13,-10)$ *lies on the plane*
***Uneb* (2001)**
Solution



Let the $p.vs$ of A, B, C and P be $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *and* $\boldsymbol{r}$ respectively.
$\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP}$. But $\overrightarrow{AD} // \overrightarrow{AB}, \overrightarrow{DP} // \overrightarrow{AC}$
$\therefore \overrightarrow{AP} = \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$
But $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$
$= \boldsymbol{r} - \boldsymbol{a}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \boldsymbol{b} - \boldsymbol{a}$ $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \boldsymbol{c} - \boldsymbol{a}$

$\therefore \boldsymbol{r} - \boldsymbol{a} = \boldsymbol{\lambda}(\boldsymbol{b} - \boldsymbol{a}) + \mu(\boldsymbol{c} - \boldsymbol{a})$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \lambda\left[\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}\right] + \mu\left[\begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}\right]$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} + \mu\begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$$

$\Rightarrow x = 2\lambda + 7\mu$ ..... (1)

$\Rightarrow y = 3 - 4\lambda + \mu \ or \ y - 3 = -4\lambda + \mu$ ..... (2)

$\Rightarrow z = -4 + 6\lambda + 3\mu \ or \ z + 4 = 6\lambda + 3\mu$ ..... (3)

Considering eqn. (1) and (2)

$eqn.(1) \times 2 + eqn.(2)$

$2x = 4\lambda + 14\mu$

$y - 3 = -4\lambda + \mu$

$2x + y - 3 = 15\mu \Rightarrow \mu = \frac{2x+y-3}{15}$ ..... (4)

$eqn.(4) \ in \ to \ eqn.(1)$

$x = 2\lambda + 7\mu \Rightarrow \lambda = \frac{x-7\mu}{2}$

$\therefore \lambda = \frac{x-7\left(\frac{2x+y-3}{15}\right)}{2} = \frac{15x-14x-7y+21}{30} = \frac{x-7y+21}{30}$ ..... (5)

$eqn.(4),(5) in \ to \ (3) to \ conclude \ equation \ of \ plane$

$\Rightarrow z + 4 = 6\left(\frac{x-7y+21}{30}\right) + 3\left(\frac{2x+y-3}{15}\right)$

$\Rightarrow z + 4 = \left(\frac{x-7y+21}{5}\right) + \left(\frac{2x+y-3}{5}\right)$

$\Rightarrow 5z + 20 = x - 7y + 21 + 2x + y - 3$

$\therefore 3x - 6y - 5z = 2$

For $Q(10, 13, -10)$ to lie on the plane, it must satisfy the equation

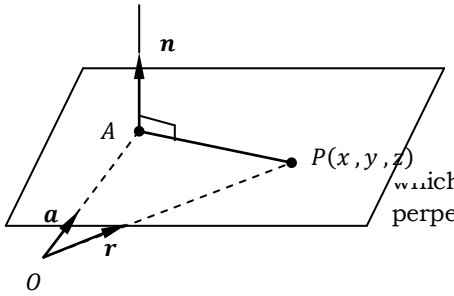
$\Rightarrow 3(10) - 6(13) - 5(-10) = 2 = \boldsymbol{RHS}$ **,** hence point lies on the plane.

VETOR EQUATION OF A PLANE IN NORMAL FORM *i.e.* Non-parametric/Scalar/Dot-product form

Consider a plane which passes through a point A with $p.v \ \boldsymbol{a}$ and which is perpendicular to vector $\boldsymbol{n}$. Let $p.v$ of point $P(x, y, z)$.

$\overrightarrow{AP} \perp \boldsymbol{n}, \therefore \overrightarrow{AP}.\boldsymbol{n} = 0$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \boldsymbol{a} \ \therefore (\boldsymbol{r} - \boldsymbol{a}).\boldsymbol{n} = 0$



$\Rightarrow \boldsymbol{r}.\boldsymbol{n} - \boldsymbol{a}.\boldsymbol{n} = 0$

$\Rightarrow \boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$,

$let \ \boldsymbol{a}.\boldsymbol{n} = d \ (scalar)$

$\therefore \boldsymbol{r}.\boldsymbol{n} = d$

which is the equation of a plane perpendicular to vector **n.**

NOTE:

From $vector = |vector| \times unit \ vector \Rightarrow \boldsymbol{n} = |\boldsymbol{n}| \times \hat{\boldsymbol{n}}, \hat{\boldsymbol{n}} = unit \ vector.$

$\therefore \boldsymbol{r}.\boldsymbol{n} = \boldsymbol{r}.\hat{\boldsymbol{n}}.\ |\boldsymbol{n}| = d \ \ or \ \ \boldsymbol{r}.\hat{\boldsymbol{n}} = D$,where $D = \frac{d}{|\boldsymbol{n}|} = scalar.$

$\boxed{\therefore \ \boldsymbol{r}.\hat{\boldsymbol{n}} = D}$, also represents equation of a plane perpendicular to $\hat{\boldsymbol{n}}$.

CARTESIAN EQUATION OF A PLANE

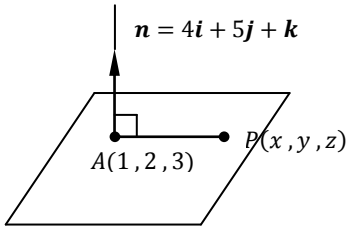
From $\boldsymbol{r}.\boldsymbol{n} = d$, $\boldsymbol{r} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $let\ \boldsymbol{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} A \\ B \\ C \end{pmatrix} = d$

The Cartesian equation of a plane is $\boxed{\therefore Ax + By + Cz = d}$

Examples:

1. *Find the equation of a plane through the point* $(1\,,2\,,3)$ *and perpendicular to* $\boldsymbol{r} = 4\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$.
(***Uneb* 2004**)
Solution



Let $\boldsymbol{r} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $let\ \boldsymbol{n} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

$\overrightarrow{AP} \perp \boldsymbol{n}\,, \therefore \overrightarrow{AP}.\boldsymbol{n} = 0$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \boldsymbol{r} - \boldsymbol{a} = \boldsymbol{r} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\Rightarrow \left[\boldsymbol{r} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right].\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 0$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 0$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} - (4 + 10 + 3) = 0 \ \ or \ \ \boldsymbol{r}.\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 17$

$or \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 17 \ \ \therefore 4x + 5y + z = 17$

2. *Show that the equation of the plane through point A with* $p.v\ -2\boldsymbol{i} + 4\boldsymbol{k}$ *perpendicular to vector* $\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ *is* $x + 3y - 2z + 10 = 0$. (***Uneb* 2000**)
Solution
Using $\boldsymbol{r}.\boldsymbol{n} = d$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = d \ \ But\ \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = d \ \ or\ x + 3y - 2z = d$

But $A(-2\,,0\,,4)$ lie on plane

$\Rightarrow (-2) + 3(0) - 2(4) = d \ \ \therefore d = -10$

$\therefore x + 3y - 2z + 10 = 0$

3. *Find the equation of the plane through point A with* $p.v\ 5\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ *perpendicular to vector* $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$. (***Uneb* 1998**)

Solution

Using $\boldsymbol{r}.\boldsymbol{n} = d$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix}1\\3\\-2\end{pmatrix} = d \;\; But\; \boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}3\\4\\-1\end{pmatrix} = d \;\; or\; 3x + 4y - z = d$

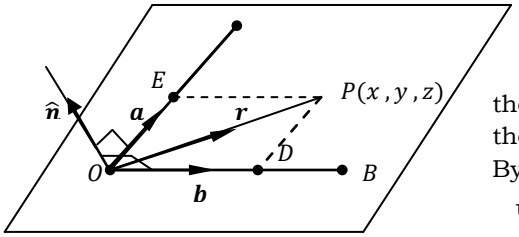
But $(5\,,-2\,,3)$ lie on plane

$\Rightarrow 3(5) + 4(-2) - (3) = d \;\; \therefore d = 4$

$\therefore 3x + 4y - z - 4 = 0$

4. *Given that* $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}$ *and* $\boldsymbol{b} = -\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$. *Determine the equation containing* $\boldsymbol{a}$ *and* $\boldsymbol{b}$.

Solution



Since only two vectors are given , and no other point is given, taking the point as the *origin* through which the plane passes, then finding the unit vector normal to the vectors, By cross product,

using $\hat{\boldsymbol{n}} = \frac{\boldsymbol{b}\times\boldsymbol{a}}{|\boldsymbol{b}\times\boldsymbol{a}|}$

$$\boldsymbol{b}\times\boldsymbol{a} = \begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\ -1 & -3 & 2\\ 1 & -3 & 3\end{vmatrix} = -3\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}$$

$|\boldsymbol{b}\times\boldsymbol{a}| = \sqrt{9 + 25 + 36} = \sqrt{70}$

$\therefore \hat{\boldsymbol{n}} = \frac{1}{\sqrt{70}}(3\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k})$

Using $\boldsymbol{r}.\hat{\boldsymbol{n}} = D \Rightarrow \boldsymbol{r}.\frac{1}{\sqrt{70}}(3\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}) = D$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix}3\\5\\6\end{pmatrix} = \sqrt{70}D = k\;(scalar)$

$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}3\\5\\6\end{pmatrix} = k\,, -3x + 5y + 6z = k$

But $(0\,,0\,,0)$ lies on the plane

$\Rightarrow -3(0) + 5(0) + 6(0) = k$

$\therefore -3x + 5y + 6z = 0$

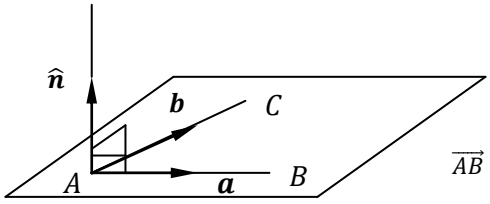
5. *Find the Cartesian equation of a plane through;*

a) $A(0\,,3\,,-4)\,, B(2\,,-1\,,2)$ *and* $C(7\,,4\,,-1)$

b) $C(-1\,,1\,,2)\,, D(1\,,-2\,,1)$ *and* $E(2\,,2,4)$

Solution

a)



Let $\overrightarrow{AB} = \boldsymbol{a}\,, \overrightarrow{AC} = \boldsymbol{b}$

$\hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}2\\-1\\2\end{pmatrix} - \begin{pmatrix}0\\3\\-4\end{pmatrix} = \begin{pmatrix}2\\-4\\6\end{pmatrix}$$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ -1 & -3 & 2 \\ 1 & -3 & 3 \end{vmatrix} = -18\boldsymbol{i} + 36\boldsymbol{j} + 30\boldsymbol{k}$

$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{(-18)^2 + 36^2 + 30^2} = \sqrt{2520}$

$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{2520}}(-18\boldsymbol{i} + 36\boldsymbol{j} + 30\boldsymbol{k})$

Using $\boldsymbol{r}.\hat{\boldsymbol{n}} = D \Rightarrow \boldsymbol{r}.\frac{1}{\sqrt{2520}}(-18\boldsymbol{i} + 36\boldsymbol{j} + 30\boldsymbol{k}) = D$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix} -18 \\ 36 \\ 30 \end{pmatrix} = \sqrt{2520}D = k\ (scalar)$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} -18 \\ 36 \\ 30 \end{pmatrix} = k\ , -18x + 36y + 30z = k$

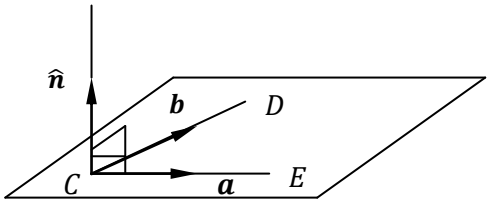
But $A\ , B$ *and* C lie on the plane. Using point $A(0\ , 3\ , -4)$

$\Rightarrow -18(0) + 36(3) + 30(-4) = k \therefore k = -12$

$\therefore -18x + 36y + 30z = -12$ ..... $\div (-6)$

$\therefore 3x - 6y - 5z - 2 = 0$

b) $\boldsymbol{Let}\ \overrightarrow{\boldsymbol{CE}} = \boldsymbol{a}\ , \overrightarrow{\boldsymbol{CD}} = \boldsymbol{b}$



$\hat{\boldsymbol{n}} = \frac{\boldsymbol{a} \times \boldsymbol{b}}{|\boldsymbol{a} \times \boldsymbol{b}|}$

$\boldsymbol{a} = \overrightarrow{\boldsymbol{CE}} = \overrightarrow{\boldsymbol{OE}} - \overrightarrow{\boldsymbol{OC}}$

$= \begin{pmatrix} \boldsymbol{2} \\ \boldsymbol{2} \\ \boldsymbol{4} \end{pmatrix} - \begin{pmatrix} \boldsymbol{-1} \\ \boldsymbol{1} \\ \boldsymbol{2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{3} \\ \boldsymbol{1} \\ \boldsymbol{2} \end{pmatrix}$

$\boldsymbol{b} = \overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & 2 \\ 2 & -3 & -1 \end{vmatrix} = 5\boldsymbol{i} + 7\boldsymbol{j} - 11\boldsymbol{k}$

$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{(5)^2 + 7^2 + (-11)^2} = \sqrt{195}$

$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{195}}(5\boldsymbol{i} + 7\boldsymbol{j} - 11\boldsymbol{k})$

Using $\boldsymbol{r}.\hat{\boldsymbol{n}} = D \Rightarrow \boldsymbol{r}.\frac{1}{\sqrt{195}}(5\boldsymbol{i} + 7\boldsymbol{j} - 11\boldsymbol{k}) = D$

$\Rightarrow \boldsymbol{r}.\begin{pmatrix} 5 \\ 7 \\ -11 \end{pmatrix} = \sqrt{195}D = k\ (scalar)$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 5 \\ 7 \\ -11 \end{pmatrix} = k\ , 5x + 7y - 11z = k$

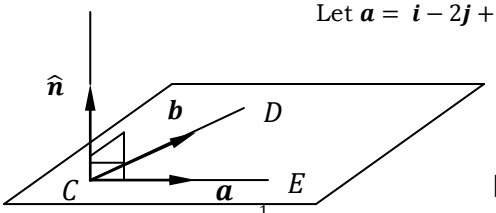
But $C\ , D$ *and* E lie on the plane. Using point $C(-1\ , 1\ , 2)$

$\Rightarrow 5(-1) + 7(1) - 11(2) = k \therefore k = -20$

$\therefore 5x + 7y - 11z = -20$

6. *Find the equation of a plane passing through point* $(1\ .2\ , 3)$ *and parallel to vectors* $\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ *and* $2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$.

Solution



Let $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}, \boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}, C(1.2, 3)$

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|}$$

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & -2 & 3 \\ 2 & -3 & 1 \end{vmatrix} = 7\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$$

$$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{(7)^2 + (5)^2 + (1)^2} = \sqrt{75}$$

$Using\ \boldsymbol{r}.\hat{\boldsymbol{n}} = D \Rightarrow \boldsymbol{r}.\frac{1}{\sqrt{75}}(7\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}) = D$

$$\Rightarrow \boldsymbol{r}.\begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} = \sqrt{75}D = k\ (scalar)$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} = k, 7x + 5y + z = k$$

$But\ A, B\ and\ C$ lie on the plane. Using point $C(1.2, 3)$

$\Rightarrow 7(1) + 5(2) + 3 = k \therefore k = 20$

$\therefore 7x + 5y + z = 20$

Deduction from Cartesian equation of a plane $Ax + By + Cz = d$

From $\boldsymbol{r}.\boldsymbol{n} = d, \boldsymbol{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} A \\ B \\ C \end{pmatrix} = d,\ Ax + By + Cz = d$$

From the Cartesian equation of a plane, there exists a *NORMAL VECTOR* to plane whose direction ratios are obtained from coefficients of x, y, z in the equation of plane. $i.e. \boldsymbol{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

Example

State the normal vector to the plane;

a) $2x + 3y + 5z = 7$

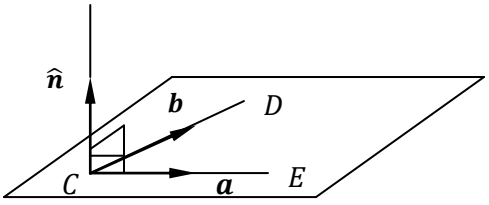
b) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix} + \mu\begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}$

c) $\boldsymbol{r} = (3 + 4\lambda - 3\mu)\boldsymbol{i} + 2\lambda\boldsymbol{j} + (3 - 3\mu)\boldsymbol{k}$

Solution

a) $2x + 3y + 5z = 7, \boldsymbol{n} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

b) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix} + \mu\begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}$



Let $\boldsymbol{a} = \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix}$

Using $\hat{\boldsymbol{n}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|}$

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 4 & -5 & 6 \\ 7 & 0 & 5 \end{vmatrix} = -25\boldsymbol{i} + 29\boldsymbol{j} + 35\boldsymbol{k}$$

$$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{(-25)^2 + (29)^2 + (35)^2} = \alpha\ (Scalar)$$

$\Rightarrow \hat{\boldsymbol{n}} = \frac{1}{\alpha}(-25\boldsymbol{i} + 29\boldsymbol{j} + 35\boldsymbol{k})$

Direction ratio of normal vector is $-25:29:35$ in simplified form

$\therefore \boldsymbol{n} = \begin{pmatrix} -25 \\ 29 \\ 35 \end{pmatrix}$

c) $\boldsymbol{r} = (3 + 4\lambda - 3\mu)\boldsymbol{i} + 2\lambda\boldsymbol{j} + (3 - 3\mu)\boldsymbol{k}$

$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}$

Let $\boldsymbol{a} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 4 & 2 & 0 \\ -3 & 0 & -3 \end{vmatrix} = -6\boldsymbol{i} - 12\boldsymbol{j} + 6\boldsymbol{k}$

Normal vector ratio is $-6:-12:6 = 1:2:-1$

$\therefore Normal\ vector\ is\ \boldsymbol{n} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Alternative approach of obtaining normal vector to plane expressed in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$

Find in scalar product form the equation of the plane

$\boldsymbol{r} = (1 + 3\lambda + 2\mu)\boldsymbol{i} + (1 + \lambda + 4\mu)\boldsymbol{j} + (\mu - \lambda)\boldsymbol{k}$

Solution

$\Rightarrow \boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$, Let $\boldsymbol{a} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$

Let the normal vector $\boldsymbol{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

Since **n** is perpendicular to **a** and **b**, then by Dot product

$\boldsymbol{n}.\boldsymbol{a} = 0 \Rightarrow (A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}).(3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) = 0$

$\therefore 3A + B - C = 0$ ..... (1)

$\boldsymbol{n}.\boldsymbol{b} = 0 \Rightarrow (A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}).(2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) = 0$

$\therefore 2A + 4B + C = 0$ ..... (2)

Expressing B and C in terms of A only

$eqn.(1) + eqn.(2)$

$\Rightarrow 5A + 5B = 0 \ \therefore A = -B$ ..... (3)

$eqn.(3)\ in\ to\ eqn.(1)$

$\Rightarrow 3A - A - C = 0 \ \therefore C = 2A$

$Now\ \boldsymbol{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, Direction\ ratio: A:B:C$

$A:-A:2A = 1:-1:2$

$\therefore \boldsymbol{n} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

From $\boldsymbol{r}.\boldsymbol{n} = d \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = d$. But $(1, 1, 0)$ lies on the plane

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = d \ \therefore d = 1 - 1 + 0 = 0$

Equation of plane in dot product form is $\boldsymbol{r}.(\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})=0$

NOTE:

The above approach can also be used to determine the normal vector to *two* given vectors.

INTERSECTION OF A LINE AND A PLANE

Consider the line $\boldsymbol{r}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$ and plane in Cartesian form

$Ax+By+Cz=d$

From the line, $\boldsymbol{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$

$\Rightarrow x=a_1+\lambda d_1\,, y=a_2+\lambda d_2\,, z=a_3+\lambda d_3$ ..... (1)

$Substitute\ eqn.(1) in\ to\ eqn.of\ plane$

$A(a_1+\lambda d_1)+B(a_2+\lambda d_2)+C(a_3+\lambda d_3)=d$

$$\Rightarrow (Aa_1+Ba_2+Ca_3)+\lambda(Ad_1+Bd_2+Cd_3)=0$$

From which the value of λ can be obtained. The point of intersection of the line and plane can be obtained from $eqn.(1)$

$.i.e.\left[\left((a_1+\lambda d_1)\,,(a_2+\lambda d_2)\,.\right)(a_3+\lambda d_3)\right]$

Deduction from the rectangular enclosed result

The result above is true for every value of λ hence it is an identity. $i.e.$

$$Aa_1+Ba_2+Ca_3=0$$

$$Ad_1+Bd_2+Cd_3=0$$

, which are the conditions for a line to lie on a plane.

Remark:

1. The condition $\boldsymbol{Ad_1+Bd_2+Cd_3=0}$ shows that the line is perpendicular to the normal of the plane, hence parallel to the plane.
2. The condition $\boldsymbol{Aa_1+Ba_2+Ca_3=0}$ **s**hows that the point $(a_1\,,a_2\,,a_3)$ lies on the plane.

Conclusion

For a line to lie in a plane the line must be parallel to the plane and *one* point on the line must also be on plane.

Examples

1. *Find the point of intersection of the given line and plane.*
   a) $\boldsymbol{r}=(2\boldsymbol{i}-\boldsymbol{k})+\lambda(i+3\boldsymbol{j})\,, plane: \boldsymbol{r}.(5\boldsymbol{i}-\boldsymbol{j}+7\boldsymbol{k})=9$
   b) $\boldsymbol{r}=\lambda\boldsymbol{i}-2\boldsymbol{j}+(2\lambda-1)\boldsymbol{k}\,, plane: \boldsymbol{r}.\begin{pmatrix}2\\1\\-3\end{pmatrix}=5$
   c) $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}\,, -4x+4y-5z-3=0$

Solution

a) $\boldsymbol{r}=(2\boldsymbol{i}-\boldsymbol{k})+\lambda(i+3\boldsymbol{j})=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\0\end{pmatrix}\,;\ \boldsymbol{r}.(5\boldsymbol{i}-\boldsymbol{j}+7\boldsymbol{k})=9\ \therefore 5x-y+7z=9$

From the line, $\Rightarrow x=2+\lambda\,, y=3\lambda\,, z=-1$ ..... (1)

$Substitute\ eqn.(1) in\ to\ eqn.of\ plane$

$5(2+\lambda)-(3\lambda)+7(-1)=9 \Rightarrow 10+5\lambda-3\lambda-7=9$

$\therefore \lambda=3$

$From\ eqn.(1), point\ of\ intersection\ is;$

$x=2+3=5\ , y=3(3)=9\ , z=-1$

$(5\ ,9\ ,-1)$ is the point of intersection of the line and plane.

b) $\boldsymbol{r}=\lambda\boldsymbol{i}-2\boldsymbol{j}+(2\lambda-1)\boldsymbol{k} \Rightarrow \boldsymbol{r}=\begin{pmatrix}0\\-2\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\Rightarrow\ x=\lambda\ , y=-2\ , z=-1+2\lambda$ ..... (1)

$\boldsymbol{r}.\begin{pmatrix}2\\1\\-3\end{pmatrix}=5 \Rightarrow 2x+y-3z=5$ ..... (2)

$eqn.(1) in\ to\ eqn.(2)$

$\Rightarrow 2\lambda+(-2)-3(-1+2\lambda)=5\ \ \therefore 2\lambda-2+3-6\lambda=5$

$\therefore \lambda=-1$

$From\ eqn.(1), point\ of\ intersection\ is;$

$x=-1\ , y=-2\ , z=-1-2=-3$

$(-1\ ,-2\ ,-3)$ is the point of intersection of the line and plane.

c) Let $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}=\lambda$

$\Rightarrow\ x=3+2\lambda\ , y=4+3\lambda\ , z=5+4\lambda$ ..... (1)

$\Rightarrow -4x+4y-5z-3=0$ ..... (2)

$eqn.(1) in\ to\ eqn.(2)$

$\Rightarrow -4(3+2\lambda)+4(4+3\lambda)-5(5+4\lambda)-3=0$

$\Rightarrow -12-8\lambda+16+12\lambda-25-20\lambda-3=0\ \ or -16\lambda=24$

$\therefore \lambda=-\frac{3}{2}$

$From\ eqn.(1), point\ of\ intersection\ is;$

$x=3-3=0\ , y=4-\frac{9}{2}=-\frac{1}{2}\ , z=-1$

$\left(0\ ,-\frac{1}{2}\ ,-1\right)$ is the point of intersection of the line and plane.

2. *Prove that the line(s) lie on the given plane(s)*

a) $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}\ ,\ 4x+4y-5z-3=0$

b) $\boldsymbol{r}=\begin{pmatrix}2\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\0\end{pmatrix}\ and\ 2x+2y+z=5$

Solution

a) $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}\ , \boldsymbol{r}=\begin{pmatrix}3\\4\\5\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\4\end{pmatrix} \Rightarrow \boldsymbol{d}=\begin{pmatrix}2\\3\\4\end{pmatrix}$

$Plane: 4x+4y-5z-3=0 \Rightarrow \boldsymbol{n}=\begin{pmatrix}4\\4\\-5\end{pmatrix}$

For a line to lie in a plane,

- The line is perpendicular to the normal of the plane $.i.e.\boldsymbol{n}.\boldsymbol{d}=0$
- One point on the line lies on the plane

$\Rightarrow \boldsymbol{n}.\boldsymbol{d}=(4\boldsymbol{i}+4\boldsymbol{j}-\mathbf{5}\boldsymbol{k}).(2\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k})=8+12-20=0$

From the line $(3\ ,4\ ,5)$ lies on the line

$\Rightarrow : 4(3)+4(4)-5(5)-3=0$, which satisfies the plane. Hence point lies on the plane.

b) $\boldsymbol{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\0\end{pmatrix} \Rightarrow \boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j}$

$2x + 2y + z = 5 \Rightarrow \boldsymbol{n} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$\Rightarrow \boldsymbol{n}.\boldsymbol{d} = (= -\boldsymbol{i} + \boldsymbol{j} + 0\boldsymbol{k}).(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = -2 + 2 + 0 = 0$. Hence line is parallel to the plane .i.e. normal $\perp$ plane.

From the line $(2\,,0\,,1)$ lies on the line

$\Rightarrow 2(2) + 2(0) + 1 = 5$, which satisfies the plane. Hence point lies on the plane.

EQUATION OF A PLANE CONTAINING A GIVEN LINE $\boldsymbol{r = a + \lambda d}$ AND SATISFYING ANY OTHER GIVEN CONDITION

Let the line be expressed in the Cartesian form $\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$ and let the plane be $Ax + By + Cz = d$.

Since the plane is to pass through the point on the line $.i.e.(a_1\,,a_2\,,a_3)$, then

$Ax + By + Cz = d$ ..... (1)

$Aa_1 + Ba_2 + Ca_3 = d$ ..... (2)

$eqn.(1) - eqn.(2)$ give the equation of a plane without d

$\Rightarrow A(x - a_1) + B(y - a_2) + C(z - a_3) = 0$ ..... (3)

NOTE:

Plane in equations (3) and (1) are the same.

Equation of plane contains the line if;

$Ad_1 + Bd_2 + Cd_3 = 0$ ..... (4)

Case I:

From equations (1) and (3)

Let the other condition satisfying the plane be that it also passes through point $(\alpha\,,\alpha\,,\gamma)$

From eqn. (3),

$A(\alpha - a_1) + B(\beta - a_2) + C(\gamma - a_3) = 0$ ..... (5)

Eliminating constants A, B and C from the zeroed equations

$$\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ d_1 & d_2 & d_3 \\ \alpha - a_1 & \beta - a_2 & \gamma - a_3 \end{vmatrix} = 0$$

Case II

Let the given condition be that the required plane is perpendicular to plane $ax + by + cz = k$

Planes $Ax + By + Cz = d$ and $ax + by + cz = k$ are perpendicular if their normal are perpendicular

$\Rightarrow aA + Bb + Cc = 0$ ..... (6)

Now eliminating constants A, B and C from the zeroed equations (6) , (3) and (4), the required equation of plane is;

$$\begin{vmatrix} x - a_1 & y - a_2 & z - a_3 \\ d_1 & d_2 & d_3 \\ a & b & c \end{vmatrix} = 0$$

Example

1. *Find the equation of a plane containing the line* $\boldsymbol{r} = (2\boldsymbol{i} + \boldsymbol{k}) + \lambda(-\boldsymbol{i} + \boldsymbol{j})$ *and passing through the point with position vector* $\boldsymbol{i} + 3\boldsymbol{k}$.

Solution

Let the equation of the plane be $Ax + By + Cz = d$…… (1)

From the line: $\boldsymbol{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\0\end{pmatrix}$, $(2\,,0\,,1)$ lies on the plane.

$\Rightarrow 2A + B(0) + C = d$

…… (2)

$eqn.(1) - eqn.(2)$ to obtain a zeroed equation

$\Rightarrow A(x-2) + B(y-0) + C(z-1) = 0$ …… (3)

Plane (3) passes through $(1\,,0\,,3)$

$\Rightarrow A(1-2) + B(0-0) + C(3-1) = 0 \;\; or \;\; -A + 0B + 2C = 0$ …… (4)

Also eqn. (1) contains line if $\boldsymbol{n}.\boldsymbol{d} = 0$

$\Rightarrow \begin{pmatrix}A\\B\\C\end{pmatrix}.\begin{pmatrix}-1\\1\\0\end{pmatrix} = 0 \;\; or - A + B + 0C = 0$ …… (5)

Eliminating constants A, B and C from the zeroed equations (3), (4) and (5)

$\Rightarrow \begin{vmatrix} x-2 & y & z-1 \\ -1 & 0 & 2 \\ -1 & 1 & 0 \end{vmatrix} = 0$

$\Rightarrow (x-2)\begin{vmatrix}0 & 2\\1 & 0\end{vmatrix} - y\begin{vmatrix}-1 & 2\\-1 & 0\end{vmatrix} + (z-1)\begin{vmatrix}-1 & 0\\-1 & 1\end{vmatrix} = 0$

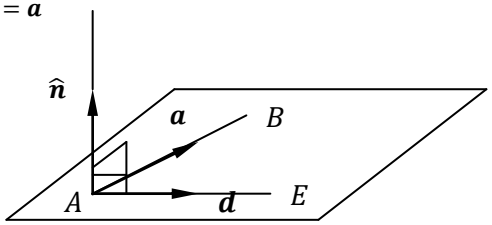
$\Rightarrow -2(x-2) - y(2) + (z-1)(-1) = 0$

$\therefore 2x + 2y + z = 5$

Alternatively

Let $A(2\,,0\,,1)\,, B(1\,,0\,,3)$

$\overrightarrow{AB} = \boldsymbol{a}$



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\0\\3\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix}$

$= \begin{pmatrix}-1\\0\\-2\end{pmatrix}$

Finding normal vector to plane

$\hat{\boldsymbol{n}} = \frac{\boldsymbol{d}\times\boldsymbol{a}}{|\boldsymbol{d}\times\boldsymbol{a}|}$

$\boldsymbol{d}\times\boldsymbol{a} = \begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\-1 & 1 & 0\\-1 & 0 & 2\end{vmatrix} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} \Rightarrow |\boldsymbol{d}\times\boldsymbol{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

$\therefore \hat{\boldsymbol{n}} = \frac{\boldsymbol{d}\times\boldsymbol{a}}{|\boldsymbol{d}\times\boldsymbol{a}|} = \frac{1}{3}(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

But normal vector direction ratio is $2:2:1$

$\therefore \boldsymbol{n} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

Using $\boldsymbol{r}.\boldsymbol{n} = d$

$\Rightarrow \boldsymbol{r}.(2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) = d \;\; or \;\; 2x + 2y + z = d$

This equation is satisfied by either $A(2\,,0\,,1) \;\; or \;\; B(1\,,0\,,3)$

$\Rightarrow 4 + 1 = d \;\; \therefore d = 5$

The equation of plane now is $2x + 2y + z = 5$

2. *Find the equation of a plane through the line* $\frac{x+1}{2}=\frac{y+2}{3}=\frac{z+3}{4}$ *and perpendicular to the plane* $x-2y+3z=4$.

Solution

Let the plane be $Ax+By+Cz=d$ ..... (1)

$\Rightarrow \boldsymbol{n}=A\boldsymbol{i}+B\boldsymbol{j}+C\boldsymbol{k}$

Plane contains the line if the point $(-1,-2,-3)$ on line lies on it

$\Rightarrow A(-1)+B(-2)+C(-3)=d$ ..... (2)

$eqn.(1)-eqn.(2)$ to obtain a zeroed equation

$A(x+1)+B(y+2)+C(z+3)=0$ ..... (3)

Also eqn. (1) contains line if $\boldsymbol{n}.\boldsymbol{d}=0$

$\Rightarrow \begin{pmatrix}A\\B\\C\end{pmatrix}.\begin{pmatrix}2\\3\\4\end{pmatrix}=0$ *or* $2A+3B+4C=0$ ..... (4)

For planes to be perpendicular, their normals are at right angles.

From $eqn.(1)$, $\boldsymbol{n}_1=A\boldsymbol{i}+B\boldsymbol{j}+C\boldsymbol{k}$

From $x-2y+3z=4$, $\boldsymbol{n}_2=\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$

$\therefore \boldsymbol{n}_1.\boldsymbol{n}_2=0\Rightarrow A-2B+3B=0$ ..... (5)

Eliminating constants A, B and C from the zeroed equations (3), (4) and (5)

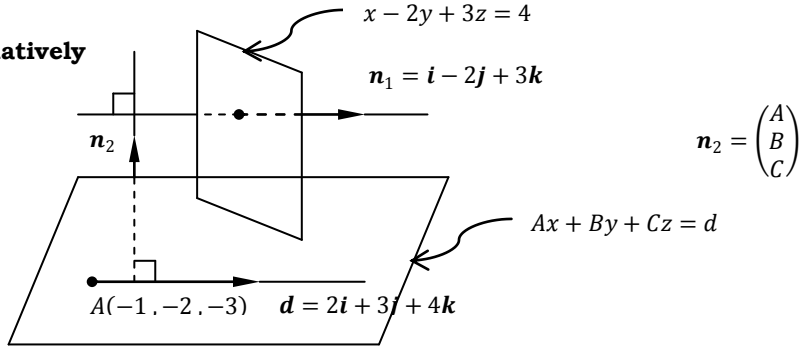
$$\Rightarrow \begin{vmatrix}x+1 & y+2 & z+3\\ 2 & 3 & 4\\ 1 & -2 & 3\end{vmatrix}=0$$

$$\Rightarrow (x+1)\begin{vmatrix}3 & 4\\ -2 & 3\end{vmatrix}-(y+2)\begin{vmatrix}2 & 4\\ 1 & 3\end{vmatrix}+(z+3)\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix}=0$$

$\Rightarrow 17(x+1)-2(y+2)+-7(z+3)=0$

$\therefore 17x-2y-7z=8$

Alternatively



From the figure, $\boldsymbol{n}_1//\boldsymbol{d}$ and $\boldsymbol{n}_2$ is perpendicular to the line and $\boldsymbol{n}_1$

Finding $\boldsymbol{n}_2$ using cross product

$$\Rightarrow \boldsymbol{n}_2=\begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\ 2 & 3 & 4\\ 1 & -2 & 3\end{vmatrix}=17\boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k}$$

Using $\boldsymbol{r}.\boldsymbol{n}=k(scalar)$

$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}17\\-2\\-7\end{pmatrix}=k, 17x-2y-7z=k$

But $(-1,-2,-3)$ *lies on the plane*

$\Rightarrow -17+4+21=k \ \therefore k=8$

$\therefore 17x-2y-7z=8$ as before.

3. *Find the equation of a plane which passes through the point* $(4,5,-6)$ *and is;*
   a) *Parallel to the plane* $2x+3y+5z+6=0$
   b) *Parallel to the line joining points* $(5,2,3)$ *and* $(1,6,4)$
   c) *Perpendicular to planes* $3x+5y-6z=0$ *and* $7x+y+2z=8$

Solution

a) Let the plane be $Ax+By+Cz=d$ ..... (1)
Eqn. (1) passes through $(4,5,-6)$
$\Rightarrow 4A+5B-6C=d$ ..... (2)
$eqn.(1)-eqn.(2)$
$A(x-4)+B(y-5)+C(z+6)=0$ ..... (3)
Plane (1) is parallel to $2x+3y+5z+6=0$ if their normals are parallel
$\Rightarrow \boldsymbol{n}_1=\begin{pmatrix}A\\B\\C\end{pmatrix}, \boldsymbol{n}_2=\begin{pmatrix}2\\3\\5\end{pmatrix} \therefore \boldsymbol{n}_1=\lambda\boldsymbol{n}_2$
$\boldsymbol{n}_1=\begin{pmatrix}2\lambda\\3\lambda\\5\lambda\end{pmatrix} \Rightarrow Direction\ ratio\ of\ \boldsymbol{n}_2\ is\ 2\lambda:3\lambda:5\lambda=2:3:5$
$\therefore \boldsymbol{n}_1=2\boldsymbol{i}+3\boldsymbol{j}+5\boldsymbol{k}=\boldsymbol{A}i+B\boldsymbol{j}+C\boldsymbol{k}$ ..... (4)
$eqn.(4)\ in\ to\ eqn.(3)$
$2(x-4)+3(y-5)+5(z+6)=0\ \ or\ 2x+3y+5z+7=0$

b) Direction vector of line joining $(5,2,3)$ *and* $(1,6,4)$ *is*
$\begin{pmatrix}7\\6\\4\end{pmatrix}-\begin{pmatrix}5\\2\\3\end{pmatrix}=\begin{pmatrix}-4\\4\\1\end{pmatrix}=\boldsymbol{d}$
Plane (1) or (3) above is parallel to the line ($i.e.\ Direction\ vector\ \boldsymbol{d}$) if the normal of plane and **d** are parallel
$\Rightarrow \boldsymbol{n}=\lambda\boldsymbol{d}=\begin{pmatrix}-4\lambda\\4\lambda\\\lambda\end{pmatrix};\ \begin{pmatrix}-4\lambda\\4\lambda\\\lambda\end{pmatrix} \Rightarrow Direction\ ratio\ of\ \boldsymbol{n}\ is -4\lambda:4\lambda:\lambda=-4:4:1$
$\therefore \boldsymbol{n}=-4\boldsymbol{i}+4\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{A}i+B\boldsymbol{j}+C\boldsymbol{k}$ ..... (5)
$eqn.(5)\ in\ to\ eqn.(3)$
$-4(x-4)+4(y-5)+1(z+6)=0\ \ or\ 4x-4y-z+2=0$

c) $\boldsymbol{n_1}\perp\boldsymbol{n_2}, \boldsymbol{n_2}\perp\boldsymbol{n_3}$
$\Rightarrow (3\boldsymbol{i}+5\boldsymbol{j}-6\boldsymbol{k}).(\boldsymbol{A}i+B\boldsymbol{j}+C\boldsymbol{k})=0$
$\therefore 3A+5B-6C=0$ ..... (6)
$\Rightarrow (7\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}).(\boldsymbol{A}i+B\boldsymbol{j}+C\boldsymbol{k})=0$
$\therefore 3A+5B-6C=0$ ..... (7)
Eliminating A, B and C from (3), (6) and (7)
By cross product,
$\Rightarrow \begin{vmatrix}x-4 & y-5 & z+6\\3 & 5 & -6\\7 & 1 & 2\end{vmatrix}=0$
$\Rightarrow (x-4)\begin{vmatrix}5 & -6\\1 & 2\end{vmatrix}-(y-5)\begin{vmatrix}3 & -6\\7 & 2\end{vmatrix}+(z+6)\begin{vmatrix}3 & 5\\7 & 1\end{vmatrix}=0$
$\Rightarrow 16(x-4)-48(y-5)+-32(z+6)=0$
$\therefore x-3y-2z=1$

4. *Find the equation of a plane which passes through points* $(6,2,-4)$ *and* $(3,4,1)$ *and is;*

a) Perpendicular to the plane $x+5y-2z=6$

b) Parallel to the line joining $(1,0,3)$ *and* $(-1,2,4)$

Solution

a) Let plane be: $Ax + By + Cz = d$ ..... (1)
For $(6, 2, -4), 6A + 2B - 4C = d$ ..... (2)
For $(3, 4, 1), 3A + 4C + C = d$ ..... (3)
$eqn.(1) - eqn.(2)$
$A(x-6) + B(y-2) + C(z+4) = 0$ ..... (4)
$eqn.(2) - eqn.(3)$
$3A + 6B - 5C = 0$ ..... (5)
For planes to be perpendicular, their normals are also perpendicular
$\Rightarrow (\boldsymbol{Ai} + B\boldsymbol{j} + C\boldsymbol{k}).(\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}) = 0$
$\Rightarrow A + 5B - 2C = 0$ ..... (6)
Eliminating A, B and C from (3), (6) and (7)
By cross product,

$$\Rightarrow \begin{vmatrix} x-6 & y-2 & z+4 \\ 3 & 6 & -5 \\ 1 & 5 & -2 \end{vmatrix} = 0$$

Simplifying, the plane is
$\therefore 13x + y + 9z = 44$

b) Direction vector of the line is;

$$\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \boldsymbol{d}$$

For a line to be parallel to the plane, the normal vector to plane and direction vector of the line must be perpendicular.

$$\Rightarrow \boldsymbol{n}.\boldsymbol{d} = 0 \quad \therefore \begin{pmatrix} A \\ B \\ C \end{pmatrix}.\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = 0, -2A + 2B + C = 0 \text{ ..... (7)}$$

From (4), (5) and (7), eliminating constants A, B and C
By cross product,

$$\Rightarrow \begin{vmatrix} x-6 & y-2 & z+4 \\ 3 & 6 & -5 \\ -2 & 2 & 1 \end{vmatrix} = 0$$

Simplifying, the plane is
$\therefore 16x + 7y + 18z = 38$

NOTE:
The above question can be done using another alternative as in example (2) above.

5. *Find the equation of a plane containing the line* $\frac{x+1}{-3} = \frac{y-3}{-2} = \frac{z+2}{1}$ *and the point* $(0, 7, -7)$. *Prove that the line* $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ *also lies on the plane.*

Solution

Let plane be: $Ax + By + Cz = d$ ..... (1)
$(0, 7, -7)$ lies on the plane
$\Rightarrow A(0) + B(7) + C(-7) = d$ ..... (2)
$eqn.(1) - eqn.(2)$
$A(x-0) + B(y-7) + C(z+7) = 0$ ..... (3)
$eqn.(1) contains\ line\ if$
$-3A - 2B + C = 0$ ..... (4)
The point on line also lies on the plane (3)
$\Rightarrow A(-1-0) + B(3-7) + C(-2+7) = 0$

$\therefore -A - 4B + 5C = 0$ ..... (3)

From (3), (4) and (5), eliminating constants A, B and C

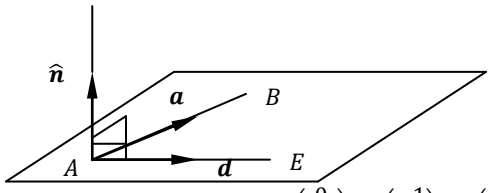
By cross product,

$$\Rightarrow \begin{vmatrix} x & y-2 & z+4 \\ -3 & -2 & 1 \\ -1 & -4 & 5 \end{vmatrix} = 0$$

Simplifying, the plane is

$\therefore 3x - 7y - 5z + 14 = 0$

Alternatively



Let $\overrightarrow{AB} = \boldsymbol{a} = \overrightarrow{OB} - \overrightarrow{OA}$

$A(-1, 3, -2), B(0, 7, -7)$

Let $\boldsymbol{d} = -3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

$$\Rightarrow \boldsymbol{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 7 \\ -7 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}$$

Finding normal vector to plane

$\hat{\boldsymbol{n}} = \frac{\boldsymbol{d} \times \boldsymbol{a}}{|\boldsymbol{d} \times \boldsymbol{a}|}$

$$\boldsymbol{d} \times \boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ -3 & -2 & 1 \\ 1 & 4 & -5 \end{vmatrix} = 6\boldsymbol{i} - 14\boldsymbol{j} - 10\boldsymbol{k} \Rightarrow |\boldsymbol{d} \times \boldsymbol{a}| = \sqrt{6^2 + (-14)^2 + (-10)^2} = \sqrt{332}$$

$\therefore \hat{\boldsymbol{n}} = \frac{\boldsymbol{d} \times \boldsymbol{a}}{|\boldsymbol{d} \times \boldsymbol{a}|} = \frac{1}{\sqrt{332}}(6\boldsymbol{i} - 14\boldsymbol{j} - 10\boldsymbol{k})$

But normal vector direction ratio is $\frac{6}{\sqrt{332}} : -\frac{14}{\sqrt{332}} : -\frac{10}{\sqrt{332}} = 3 : -7 : -5$

$\therefore \boldsymbol{n} = 3\boldsymbol{i} - 7\boldsymbol{j} - 5\boldsymbol{k}$

Using $\boldsymbol{r}.\boldsymbol{n} = k$

$\Rightarrow \boldsymbol{r}.(6\boldsymbol{i} - 14\boldsymbol{j} - 10\boldsymbol{k}) = k \; or \; 6x - 14y - 10z = k$

This equation is satisfied by either $A(-1, 3, -2), B(0, 7, -7)$

$\Rightarrow -3 - 21 + 10 = -14 = k = d \;\; \therefore d = 5$

The equation of plane now is $\therefore 3x - 7y - 5z + 14 = 0$

COPLANAR LINES

Aare lines which lie on the same plane. They can meet/do not meet or parallel.

Consider $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ ..... (1)

$\frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$ ..... (2)

Let the plane be $Ax + By + Cz = d$ ..... (3)

Eqn. (3) contains (1) if (α, β, γ) lie on the plane

$\Rightarrow A\alpha + B\beta + C\gamma = d$ ..... (4)

$eqn.(3) - eqn.(4)$

$A(x - \alpha) + B(y - \beta) + C(z - \gamma) = 0$ ..... (5)

Also Eqn. (3) contains (1) if direction vector of line is perpendicular to plane.

$\Rightarrow Al + Bm + Cn = d$ ..... (6)

Now eqn. (5) contains (2) if

$A(\alpha' - \alpha) + B(\beta' - \beta) + C(\gamma' - \gamma) = 0$ ..... (7)

Also Eqn. (5) contains (2) if direction vector of line is perpendicular to plane
$\Rightarrow Al' + Bm' + Cn' = d$ ..... (8)
From (6), (7) and (8), eliminating constants A, B and C
By cross product,

$$\therefore \begin{vmatrix} \alpha' - \alpha & \beta' - \beta & \gamma' - \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$$

, which is the condition for two lines to be coplanar.
From (5), (6) and (8), eliminating constants A, B and C

$$\therefore \begin{vmatrix} x - \alpha & y - \beta & z - \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$$

, which is the equation of plane containing the two lines.

Example

1. *Prove that the lines* $\frac{x}{1} = \frac{y-2}{2} = \frac{z+3}{3}$ *and* $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$ *are coplanar and lie in the plane* $x - 2y + z + 7 = 0$

Solution

Equation of a plane containing the first line and parallel to the second line is

$\begin{vmatrix} x & y-2 & z+3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0 \, . i.e. x - 2y + z + 7 = 0$, clearly this plane is satisfied by the point $(2, 6, 3)$ which lies on the second line.

2. *Show that the lines* $\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$ *and* $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ *are coplanar and find the equation of plane containing them.*

Solution

Equation of a plane containing the first line and parallel to the second line is

$\begin{vmatrix} x & y-2 & z+3 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0 \, . i.e. 6x - 5y - z = 0$. This plane passes through point $(-1, -1, -1)$ on the second line so that it contains the second line. Thus the two lines are coplanar and the equation of plane containing them is $6x - 5y - z = 0$.

CONDITION FOR 4-POINTS TO BE COPLANAR

Four points will be coplanar if the plane passing through any 3-points passes through the fourth point.

Example

Show that the points below are coplanar

a) $(0, -1, 0), (2, 1, -1), (1, 1, 1)$ *and* $(3, 3, 0)$
b) $(0, 1, 3), (1, 0, -4), (1, 1, -1)$ *and* $(1, 2, 2)$

Solution

a) Let $A(0, -1, 0), B(2, 1, -1), C(1, 1, 1)$ *and* $D(3, 3, 0)$
Finding the equation of plane through A, B and C
Using $\boldsymbol{r}.\boldsymbol{n} = d$

Let $\boldsymbol{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

$\boldsymbol{b} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$

Ratio of normal direction vector $= 4 : -3 : 2 \Rightarrow Normal\ vector\ \boldsymbol{n} = 4\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = d \ \ or\ 4x - 3y + 2z = d$

But $A, B\ and\ C$ lie on this plane

For $A(0\,,-1\,,0)\,, 4(0) - 3(-1) + 2(0) = d \ \ \therefore d = 3$

The equation of the plane is $4x - 3y + 2z = 3$

Now for $D(3\,,3\,,0)\,, 4(3) - 3(3) + 2(0) = 3 = \boldsymbol{RHS}$ **.** Hence points are coplanar.

b) Let $A(0\,,1\,,3)\,, B(1\,,0\,,-4)\,, C(1\,,1\,,-1)\ and\ D(1\,,2\,,2)$

Finding the equation of plane through A, B and C

Using $\boldsymbol{r}.\boldsymbol{n} = d$

Let $\boldsymbol{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -7 \end{pmatrix}$

$\boldsymbol{b} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$

$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & -7 \\ 1 & 0 & -4 \end{vmatrix} = 4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

Ratio of normal direction vector $= 4 : -3 : 1 \Rightarrow Normal\ vector\ \boldsymbol{n} = 4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} = d \ \ or\ 4x - 3y + z = d$

But $A, B\ and\ C$ lie on this plane

For $A(0\,,1\,,3)\,, 4(0) - 3(1) + (3) = d \ \ \therefore d = 0$

The equation of the plane is $4x - 3y + 2z = 0$

Now for $D(1\,,2\,,2)\,, 4(1) - 3(2) + (2) = 3 = \boldsymbol{RHS}$. Hence points are coplanar.

COPLANAR VECTORS

Vectors lying in the same plane are coplanar vectors.

Let vectors **a, b** and **c** lie on the same plane.

For coplanar vectors $\boxed{\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = 0}$

Examples

Show that the vectors are coplanar

a) $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{k}\,, \boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - 7\boldsymbol{k}\ and\ \boldsymbol{c} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

b) $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}\,, \boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}\ and\ \boldsymbol{c} = 3\boldsymbol{i} + 4\boldsymbol{j}$

Solution

a) $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -4 \\ 1 & -1 & -7 \end{vmatrix} = -4\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$

$\boldsymbol{c}.(\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}).(-4\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}) = -4+6-2 = 0$. Thus vectors are coplanar.

b) $\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4\boldsymbol{i}-3\boldsymbol{j}+0\boldsymbol{k}$

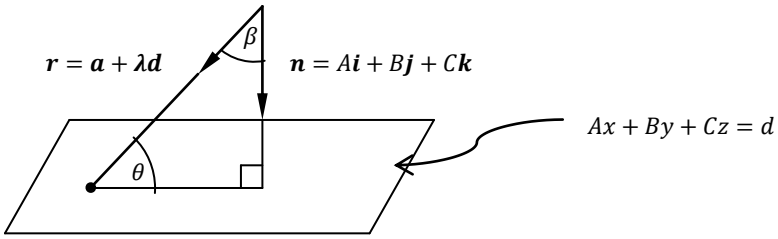
Now $\boldsymbol{c}.(\boldsymbol{a}\times\boldsymbol{b}) = (3\boldsymbol{i}+4\boldsymbol{j}+0\boldsymbol{k}).(4\boldsymbol{i}-3\boldsymbol{j}+0\boldsymbol{k}) = 12-12+0 = 0$. Thus vectors are coplanar.

NOTE:

A correct combination of vectors can reduce on the unnecessary calculations to satisfy the condition for coplanar vectors.

ANGLE BETWEEN A LINE AND A PLANE

Consider the line $\boldsymbol{r} = \boldsymbol{a}+\boldsymbol{\lambda d}$ and the plane $Ax+By+Cz = d$



Required angle $= \theta = 90-\beta$

Finding angle β between the line and the normal vector to plane

$\boldsymbol{n}.\boldsymbol{d} = |\boldsymbol{n}|.|\boldsymbol{d}|cos\beta$

But $\beta = 90-\theta$

$\Rightarrow \boldsymbol{n}.\boldsymbol{d} = |\boldsymbol{n}|.|\boldsymbol{d}|cos(90-\theta)$

$\Rightarrow \boldsymbol{n}.\boldsymbol{d} = |\boldsymbol{n}|.|\boldsymbol{d}|sin\theta$ $[\because cos(90-\theta) = sin\theta]$

$$\therefore \theta = sin^{-1}\left(\frac{\boldsymbol{n}.\boldsymbol{d}}{|\boldsymbol{n}|.|\boldsymbol{d}|}\right)$$

In general the angle between a line and a plane is the complement of the angle between the normal vector of plane and direction vector of the line.

Examples

Find the angle between the following line and plane

a) $\boldsymbol{r} = (\boldsymbol{i}-3\boldsymbol{j})+\lambda(2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k})$, $plane: \boldsymbol{r}.(\boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k}) = 10$

b) $\boldsymbol{r} = (2+\lambda)\boldsymbol{i}-3\boldsymbol{j}+(1-\lambda)\boldsymbol{k}$, $plane: \boldsymbol{r}.\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 6$

c) $\frac{x}{4} = \frac{y-1}{-1} = \frac{z+3}{-5}$, $x-2y+4z+3 = 0$

d) $\boldsymbol{r} = 3\lambda\boldsymbol{i}+2\lambda\boldsymbol{j}-6\lambda\boldsymbol{k}$, $\boldsymbol{r}.(4\boldsymbol{i}-3\boldsymbol{k}) = 20$

Solution

a) $\boldsymbol{r} = (\boldsymbol{i}-3\boldsymbol{j})+\lambda(2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}) \Rightarrow \boldsymbol{d} = 2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}$

$plane: \boldsymbol{r}.(\boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k}) = 10 \Rightarrow \boldsymbol{n} = \boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k}$

Using $\boldsymbol{n}.\boldsymbol{d} = |\boldsymbol{n}||\boldsymbol{d}|sin\theta$

$\boldsymbol{n}.\boldsymbol{d} = (2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}).(\boldsymbol{i}-2\boldsymbol{j}-7\boldsymbol{k}) = 2+2+7 = 11$

$|\boldsymbol{n}| = \sqrt{1+4+49} = \sqrt{54}$, $|\boldsymbol{d}| = \sqrt{4+1+1} = \sqrt{6}$

$\therefore sin\theta = \frac{\boldsymbol{n}.\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|} = \frac{11}{\sqrt{54}\times\sqrt{6}} = \frac{11}{18} \Rightarrow \theta = 37.7^0$

b) $\boldsymbol{r} = (2+\lambda)\boldsymbol{i} - 3\boldsymbol{j} + (1-\lambda)\boldsymbol{k}, plane: \boldsymbol{r}.\begin{pmatrix}4\\1\\-1\end{pmatrix} = 6$

$\boldsymbol{r} = \begin{pmatrix}2\\-3\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-1\end{pmatrix} \Rightarrow \boldsymbol{d} = \begin{pmatrix}1\\0\\-1\end{pmatrix}, plane: \boldsymbol{r}.\begin{pmatrix}4\\1\\-1\end{pmatrix} = 6 \Rightarrow \boldsymbol{n} = \begin{pmatrix}4\\1\\-1\end{pmatrix}$

$\boldsymbol{n}.\boldsymbol{d} = (4\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}).(\boldsymbol{i} + 0\boldsymbol{j} - \boldsymbol{k}) = 4 + 0 + 1 = 5$

$|\boldsymbol{n}| = \sqrt{16+1+1} = \sqrt{18},\ |\boldsymbol{d}| = \sqrt{1+0+1} = \sqrt{2}$

$\therefore sin\theta = \frac{\boldsymbol{n}.\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|} = \frac{5}{\sqrt{18}\times\sqrt{2}} = \frac{5}{6} \Rightarrow \theta = 54.4^0$

c) $\frac{x}{4} = \frac{y-1}{-1} = \frac{z+3}{-5} \Rightarrow \boldsymbol{r} = \begin{pmatrix}0\\1\\-3\end{pmatrix} + \lambda\begin{pmatrix}4\\-1\\-5\end{pmatrix} \therefore \boldsymbol{d} = \begin{pmatrix}4\\-1\\-5\end{pmatrix},\ x - 2y + 4z + 3 = 0 \Rightarrow \boldsymbol{n} = \begin{pmatrix}1\\-2\\4\end{pmatrix}$

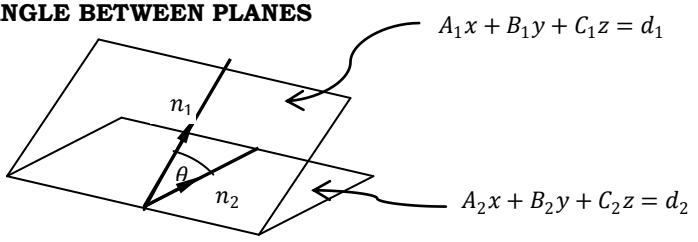
$\boldsymbol{n}.\boldsymbol{d} = (4\boldsymbol{i} - \boldsymbol{j} - 5\boldsymbol{k}).(\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}) = 4 + 2 - 20 = -14$

$|\boldsymbol{n}| = \sqrt{16+1+25} = \sqrt{42},\ |\boldsymbol{d}| = \sqrt{1+4+16} = \sqrt{21}$

$\therefore sin\theta = \frac{\boldsymbol{n}.\boldsymbol{d}}{|\boldsymbol{n}||\boldsymbol{d}|} = \frac{-14}{\sqrt{21}\times\sqrt{42}} \Rightarrow \theta = -28.1^0 \ \therefore \theta = 28.1^0\ (Numerical\ value)$

d) Left as an exercise $\boldsymbol{Answer}: sin\theta = \frac{6}{7} \ \therefore \theta = 59^0$

ANGLE BETWEEN PLANES



The angle between planes is the angle between their *normals*.

Using dot-product

$\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|cos\theta$

$\therefore \theta = cos^{-1}\left(\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right)$

Example

Find the angle between the pair of planes

a) $\boldsymbol{r}.(\boldsymbol{i} - \boldsymbol{j}) = 4\ and\ \boldsymbol{r}.(\boldsymbol{j} + \boldsymbol{k}) = 1$

b) $\boldsymbol{r}.(\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}) = 5\ and\ \boldsymbol{r}.\boldsymbol{j} = 0$

c) $\boldsymbol{r}.(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 1\ and\ \boldsymbol{r}.(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = 0$

d) $2x + 3y - 5z = 1\ and\ x - 2y - z = 0$

Solution

a) $\boldsymbol{r}.(\boldsymbol{i} - \boldsymbol{j}) = 4 \Rightarrow \boldsymbol{n_1} = \boldsymbol{i} - \boldsymbol{j} = \boldsymbol{i} - \boldsymbol{j} + 0\boldsymbol{k}\ ;\ \boldsymbol{r}.(\boldsymbol{j} + \boldsymbol{k}) = 1 \Rightarrow \boldsymbol{n_2} = \boldsymbol{j} + \boldsymbol{k} = 0\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

Using dot-product

$\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|cos\theta$

$\boldsymbol{n_1}.\boldsymbol{n_2} = (\boldsymbol{i} - \boldsymbol{j} + 0\boldsymbol{k}).(0\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 0 - 1 + 0 = -1$

$|\boldsymbol{n_1}| = \sqrt{0+1+1} = \sqrt{2}\ ; |\boldsymbol{n_2}| = \sqrt{0+1+1} = \sqrt{2}$

$\therefore \theta = cos^{-1}\left(\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right) = cos^{-1}\left(\frac{-1}{\sqrt{2}\times\sqrt{2}}\right) = 120^0$

In terms of acute angles, $\theta = 180 - 120 = 60^0$

b) $\boldsymbol{r}.(\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}) = 4 \Rightarrow \boldsymbol{n_1} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}\ ;\ \boldsymbol{r}.\boldsymbol{j} = 0 \Rightarrow \boldsymbol{n_2} = 0\boldsymbol{i} + \boldsymbol{j} + 0\boldsymbol{k}$

Using dot-product

$\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|cos\theta$

$\boldsymbol{n_1}.\boldsymbol{n_2} = (\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}).(0\boldsymbol{i} + \boldsymbol{j} + 0\boldsymbol{k}) = 0 - 2 + 0 = -2$

$|\boldsymbol{n_1}| = \sqrt{1+4+4} = 3\ ; |\boldsymbol{n_2}| = \sqrt{0+1+0} = 1$

$\therefore \theta = cos^{-1}\left(\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right) = cos^{-1}\left(\frac{-2}{3}\right) = 131.8^0$

In terms of acute angles, $\theta = 180 - 131.8^0 = 48.2^0$

c) $\boldsymbol{r}.(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) = 1 \Rightarrow \boldsymbol{n_1} = \boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}\,;\ \boldsymbol{r}.(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}) = 0 \Rightarrow \boldsymbol{n_2} = \boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$

Using dot-product

$\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|cos\theta$

$\boldsymbol{n_1}.\boldsymbol{n_2} = (\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}).(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}) = 1-1+1 = 1$

$|\boldsymbol{n_1}| = \sqrt{1+1+1} = \sqrt{3}\,; |\boldsymbol{n_2}| = \sqrt{1+1+1} = \sqrt{3}$

$\therefore \theta = cos^{-1}\left(\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right) = cos^{-1}\left(\frac{1}{\sqrt{3}\times\sqrt{3}}\right) = cos^{-1}\left(\frac{1}{3}\right) = 70.5^0$

d) $2x+3y-5z = 1 \Rightarrow \boldsymbol{n_1} = 2\boldsymbol{i}+3\boldsymbol{j}-5\boldsymbol{k}\,;\ x-2y-z = 0 \Rightarrow \boldsymbol{n_2} = \boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}$

$\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|cos\theta$

$\boldsymbol{n_1}.\boldsymbol{n_2} = (2\boldsymbol{i}+3\boldsymbol{j}-5\boldsymbol{k}).(\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}) = 2-6+5 = 1$

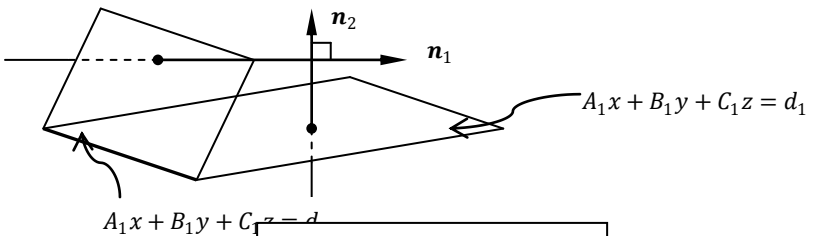
$|\boldsymbol{n_1}| = \sqrt{4+9+25} = \sqrt{38}\,; |\boldsymbol{n_2}| = \sqrt{1+4+1} = \sqrt{6}$

$\therefore \theta = cos^{-1}\left(\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right) = cos^{-1}\left(\frac{1}{\sqrt{38}\times\sqrt{6}}\right) = 86.2^0$

CONDITION FOR PERPENDICULARITY OF 2-PLANES

If two planes are perpendicular, then their normals are also perpendicular.

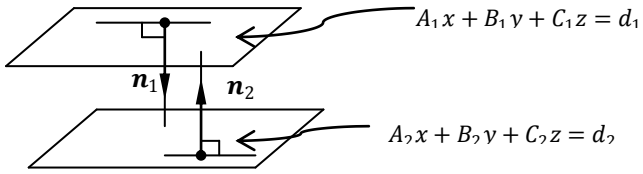
Consider two planes $A_1x+B_1y+C_1z = d_1$ and $A_2x+B_2y+C_2z = d_2$



$\Rightarrow \boldsymbol{n_1}.\boldsymbol{n_2} = \upsilon$ $\quad\boxed{\Rightarrow A_1A_2+B_1B_2+C_1C_2 = 0}$

CONDITION FOR PARALLELLISM OF 2-PLANES

If two planes are parallel, then their *normals* are also parallel.



$\Rightarrow \boldsymbol{n_1} = \lambda\,\boldsymbol{n_2}$

$\Rightarrow A_1\boldsymbol{i}+B_1\boldsymbol{j}+C_1\boldsymbol{k} = \lambda(A_2\boldsymbol{i}+B_2\boldsymbol{j}+C_2\boldsymbol{k})$

$\Rightarrow\ A_1 = \lambda\,A_2 \Rightarrow \lambda = \frac{A_1}{A_2}$

$\Rightarrow\ B_1 = \lambda\,B_2 \Rightarrow \lambda = \frac{B_1}{B_2}$

$\Rightarrow\ C_1 = \lambda\,C_2 \Rightarrow \lambda = \frac{C_1}{C_2}$

$$\Rightarrow \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \lambda$$

EQUATION OF A PLANE PARALLEL TO A GIVEN PLANE $ax+by+cz = d$

The required plane is generally given as $\quad\boxed{ax+by+cz = k(constant)}$

, where$k$ = *constant*, that can be obtained from any other condition satisfying the plane.

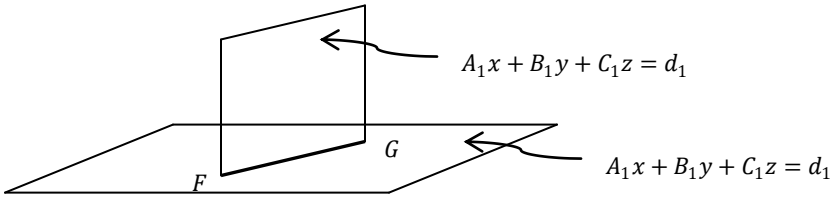
Example

1. *Find the equation of a plane passing through point* $(1\,,2\,,3)$ *and parallel to the plane*
   $4x-5y+6z=7$.
   Solution
   Required plane: $4x-5y+6z=k$
   But $(1\,,2\,,3)$ lies on the plane
   $\Rightarrow 4-10+18=k\ \therefore k=12$
   The plane is $4x-5y+6z=12$
2. *Find the equation of a plane parallel to plane* $2x+3y-4z=1$ *and passing through* $(1\,,2\,,3)$. ***Answer***: $2x+3y-4z+4=0$

INTERSECTION OF PLANES

Intersection of two planes:

Consider two planes $A_1x+B_1y+C_1z=d_1$ and $A_2x+B_2y+C_2z=d_2$



Two planes intersect in a straight *common line*. FG is the common line of intersection

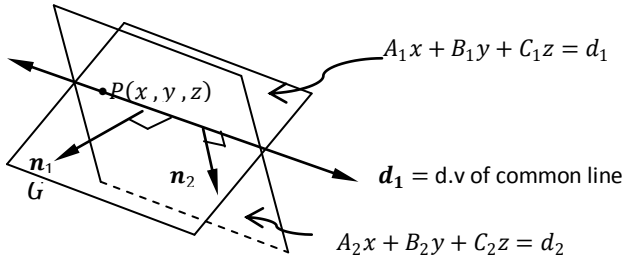
Equation of a plane containing the common line of intersection of two planes:

$$(A_1x+B_1y+C_1z-d_1)+\lambda(A_2x+B_2y+C_2z-d_2)=0$$

The equation is of the form $U+\lambda V=0$ *where* $U=V=0$ and λ = constant to be obtained from the other condition satisfying the plane.

Equation of the common line to 2-intersecting planes:

This is obtained as below.



Direction vector, $\boldsymbol{d}$ is perpendicular to $\boldsymbol{n}_1$ *and* $\boldsymbol{n}_2$

$$\therefore \boldsymbol{d} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$$

Using vector equation of a line $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$
This line intersects at least one of the xy , xz *or the* $yz - plane$. Thus one point on this line has *a zero* $x-$ *or* $y-$ *or* $z-$ *coordinate. i. e.* Let x= 0 *or* $y = 0$ *or* $z = 0$
Now solving for x and y if $z = 0$ in the equations of planes, yields the coordinate on the common line. Hence, equation of line FG is of the form;

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \lambda(\boldsymbol{n}_1 \times \boldsymbol{n}_2) \; for \; z = 0$$

Example

1. *Find the equation of a plane passing through the line of intersection of planes* $x + y + z = 6$ *and* $2x + 3y + 4z = -5$, *and is perpendicular to the plane* $4x + 5y - 5z = 8$. *Find also the equation of line of intersection of the planes* $x + y + z = 6$ *and* $2x + 3y + 4z = -5$.

Solution

$x + y + z = 6$ ..... (1)
$2x + 3y + 4z = -5$ ..... (2)
Required plane: $(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0$
$\Rightarrow x(1 + 2\lambda) + y(1 + 3\lambda) + z(1 + 4\lambda) + (5\lambda - 6) = 0$
This plane is perpendicular to $4x + 5y - 5z = 8$ if:
$4(1 + 2\lambda) + 5(1 + 3\lambda) - 5(1 + 4\lambda) = 0$
$\Rightarrow 4 + 8\lambda + 5 + 15\lambda - 5 - 20 = 0$ *or* $3\lambda + 4 = 0 \; \therefore \lambda = -\frac{4}{3}$
Equation of plane: $\Rightarrow x\left(1 - \frac{8}{3}\right) + y(1 - 4) + z\left(1 - \frac{16}{3}\right) + \left(-\frac{20}{3} - 6\right) = 0$
$-5x - 3y - 13z - 38 = 0$ *or* $5x + 3y + 13 + 38 = 0$
Now finding the equation of the common line:
$x + y + z = 6$, $\boldsymbol{n}_1 = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, and $2x + 3y + 4z = -5$, $\boldsymbol{n}_2 = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$
Direction vector of line: $\boldsymbol{d} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$

$$\boldsymbol{d} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$$

Finding a point on the required line:
Fixing $z = 0$
$\Rightarrow x + y + z = 6 \; becomes \; x + y = 6$ ..... (3)
$\Rightarrow \; 2x + 3y + 4z = -5 \; becomes \; 2x + 3y = -5$ ..... (4)
Solving eqns. (3) and (4) simultaneously,
$x = 23 \; and \; y = -17$
$\Rightarrow (23 , -17 , 0)$ is the point on the required line.

$$\therefore \boldsymbol{r} = \begin{pmatrix} 23 \\ -17 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

2. *Find the parametric equations of the line which is the intersection of* $x + 2y + 3z - 8 = 0$ *and* $3x + 3y + z + 3 = 0$. *Find the equation of the plane containing this line and perpendicular to the plane* $2x + 3y - 4z = -2$.

Solution

Fixing $z = 0$ and solving the equations of planes simultaneously,
$x + 2y + 3z - 8 = 0 \Rightarrow x + 2y = 8$ ..... (1)

$3x + 3y + z + 3 = 0 \Rightarrow 3x + 3y = -3 \; or \; x - y = -1$ ..... (2)
Solving eqns. (1) and (2) simultaneously,
$x = 2 \; and \; y = 3$
$\Rightarrow (2\,,3\,,0)$ is the point on the required line.
Direction vectors of the lines are respectively $\boldsymbol{n}_1 = \boldsymbol{i} + \boldsymbol{2j} + 3\boldsymbol{k}\,, \boldsymbol{n}_2 = 3\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$
Direction vector of line: $\boldsymbol{d} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$

$$\boldsymbol{d} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 3 & -3 & 1 \end{vmatrix} = 11\boldsymbol{i} + 8\boldsymbol{j} - 9\boldsymbol{k}$$

The required line is;

$$\therefore \boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ 8 \\ -9 \end{pmatrix}$$

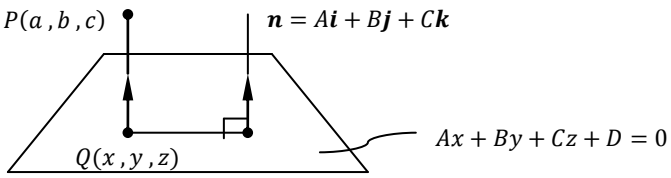
Required plane:
$(x + 2y + 3z - 8) + k(3x + 3y + z + 3) = 0$
$x(1 + 3k) + y(2 - 3k) + z(3 + k) + 3k - 8 = 0$
This plane is perpendicular to plane $2x + 3y - 4z = -2$ if:
$2(1 + 3k) + 3(2 - 3k) - 4(3 + k) = 0 \; [\because \boldsymbol{n}_1 \,.\boldsymbol{n}_2 = 0]$
$\Rightarrow 2 + 6k + 6 - 9k - 12 - 4k = 0 \;\; \therefore k = -\frac{4}{7}$
$\therefore$ Equation of plane is:
$x\left(1 - \frac{17}{7}\right) + y\left(2 + \frac{12}{7}\right) + z\left(3 - \frac{4}{7}\right) - \frac{12}{7} - 8 = 0$
$\therefore -5x + 26y + 17z + 68 = 0$

Task

1. *Find the equation of the plane which contains the line of intersection of the planes*
   $6x + 4y - 5z = 2 \; and \; x - 2y + 3z = 0$ and;
   a) *Perpendicular to the plane* $3x - 2y + z = 5$
   b) *Parallel to the line with direction cosines proportional to* $(1\,,3\,,2)$.
   Answer: $11x + 10y - 13z - 4 = 0, 2x - 20y + 29z + 2 = 0$
2. *Show that the lines* $\frac{1}{3}(x + 4) = \frac{1}{5}(y + 6) = -\frac{1}{2}(z - 1); 3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$ *are coplanar. Find also the coordinates of their point of intersection and the equation of the plane that contains their line.*
   Answer: $(2\,,4\,,-3); 45x - 17y + 25z + 53 = 0$
3. *Show that the lines* $\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{2}\,, 3x + 2y + z - 2 = 0 = x - 3y + 2z - 3$*are coplanar and find the equation of the plane in which they lie.*
   $[HINT: Required: (3x + 2y + z - 2) + \lambda(x - 3y + 2z - 3) = 0]$. This plane is parallel to the line if its normal is perpendicular to d.v. of the line. $i.e.\, 3(3 + \lambda) + (2 - 3\lambda) - 2(1 + 2\lambda) = 0 \;\; \therefore \lambda = \frac{9}{4} \Rightarrow (-5\,,-4\,,7)\,, 21x - 19y + 22z - 125 = 0$

DISTANCE OF A POINT FROM A PLANE

Consider a point $P(a\,,b\,,c)$ and the plane $Ax + By + Cz + D = 0$.
Let $\overline{QP} = d$



Vector $\overrightarrow{QP}$ *is* // to $\boldsymbol{n}$

$\Rightarrow \overrightarrow{QP} = \lambda \boldsymbol{n}$

$But\ \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \begin{pmatrix} a-x \\ b-y \\ c-z \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a-x \\ b-y \\ c-z \end{pmatrix} = \lambda \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$\Rightarrow a - x = \lambda A \ \ or\ x = a - \lambda A$

$\Rightarrow b - y = \lambda b \ \ or\ y = b - \lambda B$

$\Rightarrow c - z = \lambda A \ \ or\ z = c - \lambda C$

$\therefore Q(x\,,y\,,z) \equiv Q(a - \lambda A\,, b - \lambda B\,, c - \lambda C)$ lies on the plane $Ax + By + Cz + d = 0$

$\Rightarrow A(a - \lambda A) + B(b - \lambda B\,) + C(c - \lambda C) + d = 0$

$or\ Aa + Bb + Cc + d = \lambda(A^2 + B^2 + C^2)\ \therefore \lambda = \frac{Aa+Bb+Cc+d}{A^2+B^2+C^2}$ ..... (1)

But $\overrightarrow{QP} = perpendicular\ distance = \left|\overrightarrow{QP}\right| = \lambda|\boldsymbol{n}|$

$\therefore \overrightarrow{QP} = \lambda \times \sqrt{A^2 + B^2 + C^2}$ ..... (2)

$subst.\,eqn.\,(1)\ in\ eqn.\,(2)$

$\Rightarrow Perpendicular\ distance = d = \frac{Aa+Bb+Cc+d}{A^2+B^2+C^2} \times \sqrt{A^2 + B^2 + C^2}$

$\therefore d = \frac{Aa+Bb+Cc+d}{\sqrt{A^2+B^2+C^2}}$ =Pperpendicular distance of point $P(a\,,b\,,c)$ from the plane $Ax + By + Cz + d = 0$

This distance can sometimes be negative, thus the absolute value (numerical value) is taken to ensure $+ve\ distance$.

> Let $Ax + By + Cz + d = 0$ be the plane and $P(a\,,b\,,c)$ be a point. The distance d from the point is given by;
>
> $$d = \left|\frac{Aa + Bb + Cc + d}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Rule to find perpendicular distance of a point from a plane with equation given in general form $Ax + By + Cz + d = 0\ (\boldsymbol{R.H.S} = 0)$

"Substitute the coordinates of the point in the equation of plane with $\boldsymbol{RHS} = 0$ and divide the result by $\sqrt{(coeff.of\ x)^2 + (coeff.of\ y)^2 + (coeff.of\ z)^2}$ "

Examples

1. *A plane passes through* $3 - points\ A(0\,,1,3)\,, B(1\,,0\,,-4)\ and\ C(1\,,1\,,-1)$. *Find its equation and length of a perpendicular to it from* $(2\,,4\,,-3)$.

   Solution

   Equation of plane is left for the student as an exercise. **Answer:** $\boldsymbol{4x - 3y + z = 0}$

   Using the formula,

   Perpendicular distance $= \frac{4(2)-3(4)-3}{\sqrt{4^2+(-3)^2+1^2}} = -\frac{7}{\sqrt{26}} = \frac{7}{\sqrt{26}}(numerically)$

2. *Show that the distance between the parallel planes* $2x - 2y + z + 3 = 0\ and\ 4x - 4y + 2z + 5 = 0\ is\ \frac{1}{6}$.

Solution

Find any point on any plane.

Let $x = y = 0$, now finding z

For $2x - 2y + z + 3 = 0 \Rightarrow z = -3$

$\therefore (0\,,0\,,-3)$ lies on this plane.

Now perpendicular distance of $(0\,,0\,,-3)$ from $4x - 4y + 2z + 5 = 0$ is;

$d = \frac{4(0)-4(0)+2(-3)+5}{\sqrt{4^2+4^2+4}} = -\frac{1}{6} = \frac{1}{6}(Numerically)$

3. *Find the equations of the planes parallel to the plane* $x - 2y + 2z - 3 = 0$ *whose perpendicular distance from* $(1\,,2\,,3)$ *is* 1 *unit.*

Solution

Given plane: $x - 2y + 2z - 3 = 0$

Any plane parallel to it is: $x - 2y + 2z + k = 0$ ..... (1)

Since perpendicular distance of eqn. (1) from $(1\,,2\,,3)$ $is = 1\ unit$

$\Rightarrow \pm\frac{1(1)-2(2)+2(3)+k}{\sqrt{1+4+4}} = 1$

[***use complete formul whenever perpendicular distance is given***]

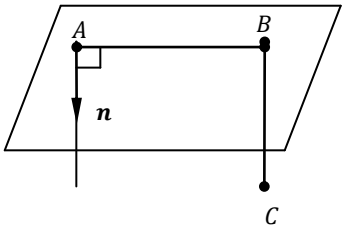
$\Rightarrow \pm\frac{3+k}{3} = 1\ or \pm(3+k) = 3\ \therefore k = 0\,,-6$

From eqn. (1), the required planes are;

$\therefore x - 2y + 2z = 0\,, x - 2y + 2z - 6 = 0$

4. *A plane contains the points* $A(-4\,,9\,,-9)$ *and* $B(5\,,-9\,,6)$ *and is perpendicular to the line which joins* $C(4\,,-6\,,k)$ *and B. Evaluates* k *and find the equation of the plane.*

Solution



$\overrightarrow{BC}\ //\boldsymbol{n} \Rightarrow \boldsymbol{n} = \lambda\overrightarrow{BC}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 4-5 \\ -6+9 \\ k-6 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ k-6 \end{pmatrix}$

Also $\overrightarrow{AB}\ is\ perpendicular\ to\ \overrightarrow{BC}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 5+4 \\ -9-9 \\ 6+9 \end{pmatrix} = \begin{pmatrix} 9 \\ -18 \\ 15 \end{pmatrix}$

$\Rightarrow \overrightarrow{AB}.\overrightarrow{BC} = 0\ \therefore -1(9) + 3(-18) + (k-6)(15) = 0$ ..... $(\div 3)$

$\Rightarrow -51 + 5k = 0 \therefore k = \frac{51}{5}$

$Direction\ ratio\ of\ normal\ is -1 : 3 : k-6\,, k = \frac{51}{5}$

$\Rightarrow \boldsymbol{n}\ has\ direction\ ratio\ of -1 : 3 : \left(\frac{51}{5} - 6\right) = -5 : 15 : 21$

Using $\boldsymbol{r}.\boldsymbol{n} = d$

$\boldsymbol{r}.\begin{pmatrix} -5 \\ 15 \\ 21 \end{pmatrix} = d$, this plane contains A and B.

For $A(-4\,,9\,,-9)\,, d = \begin{pmatrix} -4 \\ 9 \\ -9 \end{pmatrix}.\begin{pmatrix} -5 \\ 15 \\ 21 \end{pmatrix} = 34$

$\therefore -5x + 15y + 21z - 34 = 0\ \ or\ \boldsymbol{r}.\begin{pmatrix} -5 \\ 15 \\ 21 \end{pmatrix} = 34$

COMPLEX NUMBERS

Introduction:

A complex number is that can be expressed in the form:

$z = (Real\ part) + i(Imaginary\ part)\ .i.e.$

$z = a + ib\ ; a\ , b\ \in \mathbb{R}$

Examples

$z = 3 + 4i\ \ , z = 4i = 0 + 4i\ , z = 4 = 4 + 0i$ where $i = \sqrt{-1}$

Origin of symbol $i = \sqrt{-1}$

This originates from finding square root of negative numbers.

Consider

a) $\sqrt{4}$

b) $\sqrt{-4}$

Solution

a) $\sqrt{4} = \pm 2$

b) $\sqrt{-4} = \sqrt{-1 \times 4} = \sqrt{-1} \times \sqrt{4}\ \ \left[By\ surds: \sqrt{ab} = \sqrt{a} \times \sqrt{b}\right]$

$= i \times \pm 2 = \pm 2i\ \ \left[Let\ \sqrt{-1} = i\ (Mathematical\ notation)\right]$

Note:

There is no exact square root of a negative number that yields real numbers that are negative or positive.

Algebra of $i = \sqrt{-1}$

1) Simplify

a) i^2 b) i^3 c) i^5 d) i^4

Solution

a) $i^2 = [i]^2 = \left[\sqrt{-1}\right]^2 = \left[(-1)^{\frac{1}{2}}\right]^2 = -1$

$\therefore i^2 = -1$

b) $i^3 = i^2 \times i = -1 \times i = -i$

$\therefore i^3 = -i$

c) $i^5 = i^2 \times i^3 = -1 \times -i = i$

$\therefore i^5 = i$

d) $i^4 = (i^2)^2 = (-1)^2 = 1$

$\therefore i^4 = 1$

Note:

a. $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{-1 \times -1} = 1$

b. The result $i^2 = -1\ , i^3 = -i\ and\ i^4 = 1$ are important in the algebra of $i = \sqrt{-1}$.

2) Simplify

a) i^{20} b) i^{39} c) i^{40} d) i^{15}

Solution

a) $i^{20} = (i^2)^{10} = (-1)^{10} = 1$

$\therefore i^{20} = 1$

b) $i^{39} = (i^3)^{13} = (-i)^{13} = [-1 \times i]^{13} = -[i]^{13}$

$= -[i^4]^3 \times i = -[1]^3 \times i = -i$

$\therefore i^{39} = -i$

c) $i^{40} = [i^4]^{10} = [1]^{10} = 1$

$\therefore i^{40} = 1$

d) $i^{15} = [i^4]^3 \times i^3 = 1 \times -i = -i$
$\therefore i^{15} = -i$

Trial questions

Simplify

a) i^6 b) i^8 c) i^7 d) i^{17}

Answer:

a) $i^6 = (i^4)(i^2) = 1 \times -1 = -1$
b) $i^8 = [i^4]^2 = 1$
c) $i^7 = (i^4)(i^3) = -i$
d) $i^{17} = [i^4]^4.i = i$

OPERATION IN COMPLEX NUMBERS

The following shall be considered:

i) Addition
ii) Subtraction
iii) Multiplication
iv) Division

Methods used

- Analytical method
- Graphical method

ADDITION OF COMPLEX NUMBERS

A. **Analytical method**

Given that $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ *and* $z_3 = x_3 + iy_3$
Find: i) $z_1 + z_2$ *ii)* $z_1 + z_2 + z_3$

Solution

i. $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$
$= (x_1 + x_2) + i(y_1 + y_2)$

ii. $z_1 + z_2 + z_3 = (x_1 + iy_1) + (x_2 + iy_2) + (x_3 + iy_3)$
$= (x_1 + x_2 + x_3) + i(y_1 + y_2 + y_3)$

Deduction

When adding complex umbers, add the real parts and imaginary parts separately, and write the result in complex number form.

Examples

1) *Given that* $z_1 = 4 + i$, $z_2 = -3 - i$ *and* $z_3 = 3 + 2i$
Find i) $z_1 + z_2$ *ii)* $z_1 + z_2 + z_3$

Solution

i. $z_1 + z_2 = 4 + i + -3 - i$
$= (4 - 3) + i(1 - 1) = 1 + 0i = 1$

ii. $z_1 + z_2 + z_3 = 4 + i + -3 - i + 3 + 2i$
$= (4 - 3 + 3) + i(1 - 1 + 2) = 3 + 2i$

2) *Given that* $z_1 = 4$, $z_2 = 3i$ *and* $z_3 = 3 + 2i$
Find i) $z_1 + z_2$ *ii)* $z_1 + z_2 + z_3$

Solution

i. $z_1 + z_2 = 4 + 3i$

ii. $z_1 + z_2 + z_3 = 4 + 3i + 3 + 2i = 7 + 5i$

B. **Graphical method**

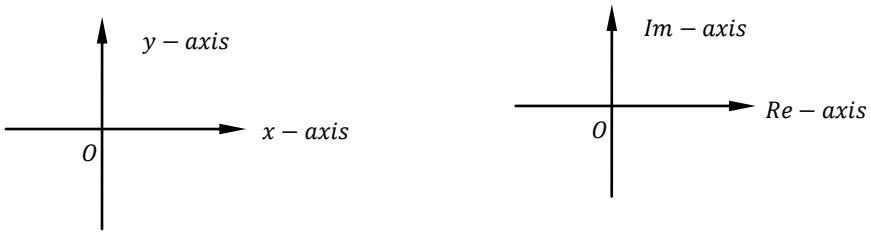
Before we look at this method, let us first look at Argand diagram.

Argand diagram

This is a graph having two axes.

i.e. Real (Re) axis as $x-axis$ and Imaginary(Im) axis as $y-axis$

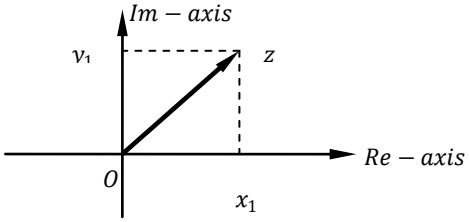
Cartesian graph **Argand diagram**



Representation of a complex number on Argand diagram

Given that $z_1 = x_1 + iy_1$, z has a point (x_1, y_1) in general form. Plotting (x_1, y_1)

From the figure, a complex number is represented using an arrow from the origin to the point representing complex number.

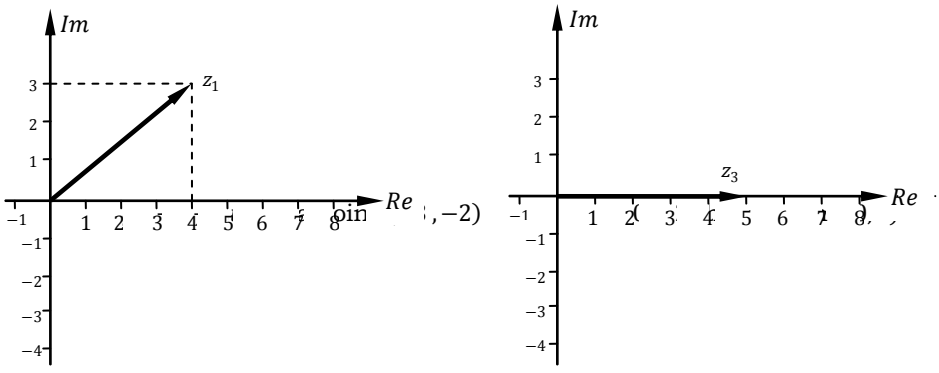


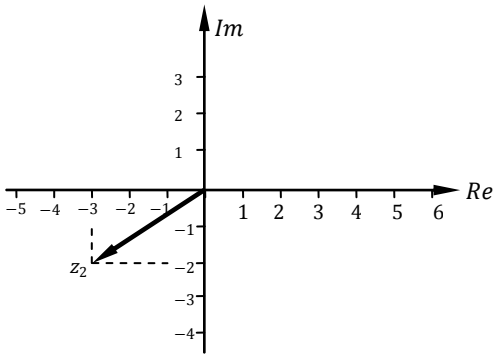
Example

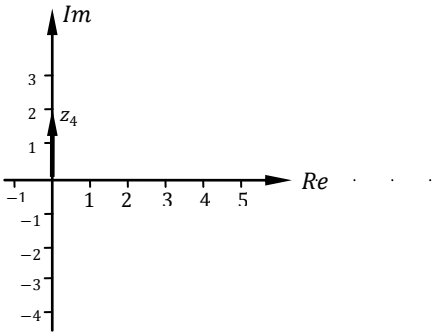
1) *Given that $z_1 = 4 + 3i$, $z_2 = -3 - 2i$, $z_3 = 5$ and $z_4 = 2i$*
Represent the complex numbers on different Argand diagram.

Solution

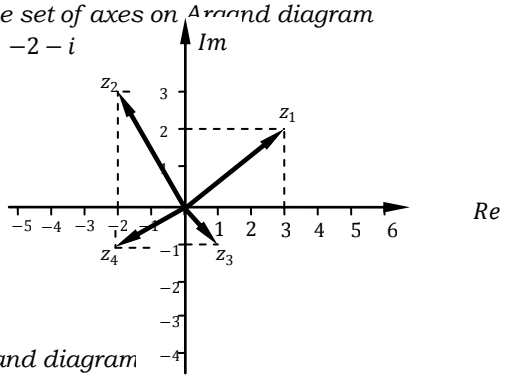
$z_1 = 4 + 3i$ has a point $(4, 3)$ $z_3 = 5 + 0i$ has a point$(5, 0)$







2) *Represent the complex numbers on the same set of axes on Argand diagram* $z_1 = 3 + 2i\ ,\ z_2 = -2 + 3i,\ z_3 = 1 - i$ *and* $z_4 = -2 - i$



Trial questions

Represent the complex numbers on the Argand diagram

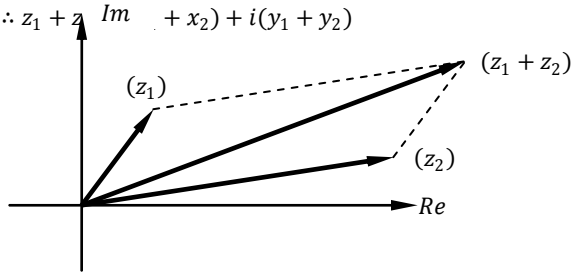
$z_1 = 3 - \frac{1}{2}i\ ,\ z_2 = \frac{3}{2} + \frac{1}{2}i,\ z_3 = 4 - 3i$

Now adding complex numbers by Graphical method

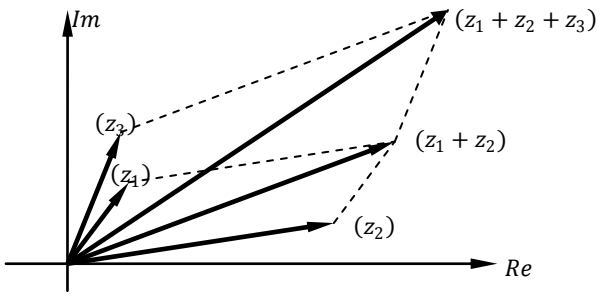
Two complex numbers are added using a parallelogram. Each complex number is represented on the Argand diagram, and the two complex numbers form the adjacent sides of a parallelogram. The addition result is given by the diagonal of a parallelogram.

Consider $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$

$\therefore z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$



Addition of complex numbers by graphical method



Three complex numbers are added as a continuous chain $.i.e. z_1 + z_2 + z_3 = (z_1 + z_2) + z_3$. First add any two $.i.e.\ z_1 + z_2$ to get the result and this result with z_3 give a diagonal of a parallelogram which is the required result.

Example

Given that $z_1 = 4 + i\ ,\ z_2 = -2 + i,\ z_3 = -3 - 2i$ *and* $z_4 = 5 + i$

Find:

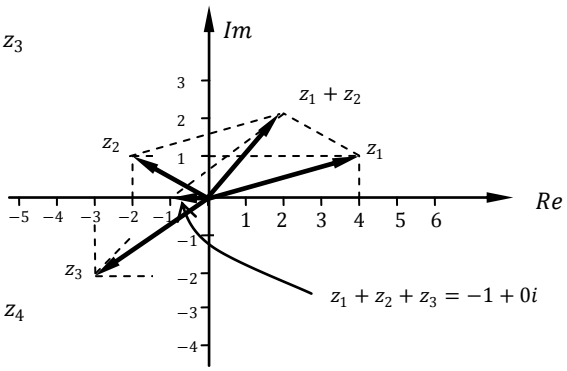
a) $z_1 + z_2 + z_3$

b) $z_2 + z_3 + z_4$

Solution

a) $z_1 + z_2 + z_3 = (z_1 + z_2) + z_3$

Adding $z_1 + z_2$

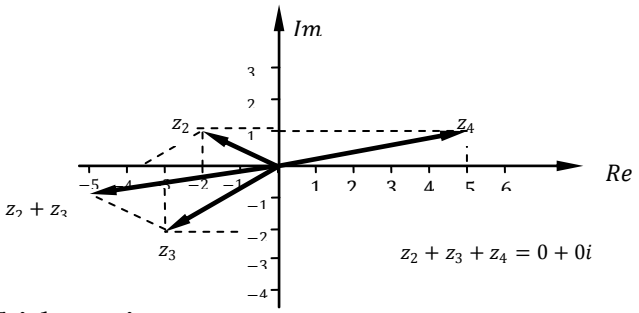


b) $z_2 + z_3 + z_4 = (z_2 + z_3) + z_4$

Adding $z_2 + z_3$

NOTICE:

The diagonal for a parallelogram from z_4 and $z_2 + z_3$ is just a straight line whose midpoint is $(0\ ,0).i.e.$ Parallelogram cannot be drawn.



Trial questions:

1) *Given that* $z_1 = 3 + 4i\ ,\ z_2 = -1 + 2i$ *and* $z_3 = -3i + 2$. *Represent the complex numbers clearly on the Argand diagram.*

   a) $z_1 + z_2$

   b) $z_2 + z_3$

   c) $z_1 + z_2 + z_3$

2) *Given that* $z_1 = 4 + 3i\ ,\ z_2 = -4 - 3i\ ,\ z_3 = 5$ *and* $z_4 = 2i$ *. using Argand diagram find;*

   a) $z_1 + z_2$

   b) $z_2 + z_3$

   c) $z_1 + z_2 + z_3$

Note: $z_3 = 5 = 5 + 0i\ , z_4 = 2i - 0 + 2i$

SUBTRACTION OF COMPLEX NUMBERS

A. **Analytical method**

Given that $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ *and* $z_3 = x_3 + iy_3$

Find: i) $z_1 - z_2$ *ii)* $z_1 - z_2 - z_3$

Solution

i. $z_1 + z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$= (x_1 - x_2) + i(y_1 - y_2)$ $(collecting\ like\ terms)$

ii. $z_1 - z_2 - z_3 = (x_1 + iy_1) - (x_2 + iy_2) - (x_3 + iy_3)$

$= (x_1 - x_2 - x_3) + i(y_1 - y_2 - y_3)$ $(collecting\ like\ terms)$

Deduction

When subtracting complex umbers, subtract the real parts and imaginary parts separately, and write the result in complex number form.

Examples

1) *Given that* $z_1 = 3 + 2i, z_2 = 4 - 3i$ *and* $z_3 = 4 + 3i$

Find i) $z_1 + z_2$ *ii)* $z_1 + z_2 + z_3$

Solution

i. $z_1 + z_2 = 4 + i + \ -3 - i$

$= (4 - 3) + i(1 - 1) = 1 + 0i = 1$

ii. $z_1 + z_2 + z_3 = 4 + i + \ -3 - i + 3 + 2i$

$= (4 - 3 + 3) + i(1 - 1 + 2) = 3 + 2i$

2) *Given that* $z_1 = 3 + 2i$, $z_2 = 4 - 3i$ *and* $z_3 = 4 + 3i$

Find i) $z_1 - z_2$ *ii)* $z_1 - z_2 - z_3$

Solution

i. $z_1 - z_2 = (3 + 2i) - (4 - 3i)$

$= (3 - 4) + i(2 - 3)$ $(collecting\ like\ terms) = -1 + 5i$

ii. $z_1 - z_2 - z_3 = (3 + 2i) - (4 - 3i) - (4 + 3i)$

$= (3 - 4 - 4) + i(2 - 3 - 3) = -5 + 2i$

Task

Given that $z_1 = 4 + 3i$, $z_2 = 3 - i$ *and* $z_3 = -4 - 3i$. *Find:*

a) $z_1 - z_2$ ***Answer*: $\mathbf{1 + 4i}$**

b) $z_2 - z_3$ ***Answer*: $\mathbf{7 + 2i}$**

c) $z_1 - z_2 - z_3$ ***Answer*: $\mathbf{5 + 7i}$**

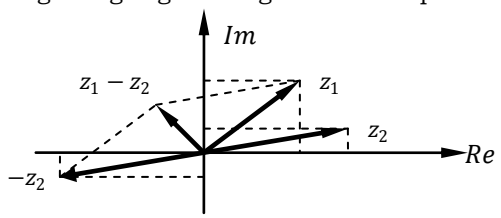
Graphical method

This is done using the Argand diagram just like in addition.

Consider two complex numbers $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$. To find $z_1 - z_2$, proceed as follows.

$z_1 - z_2 = z_1 + (-z_2)$ [***Note this***]

Adding using Argand diagram the complex numbers z_1 *and* $-z_2$

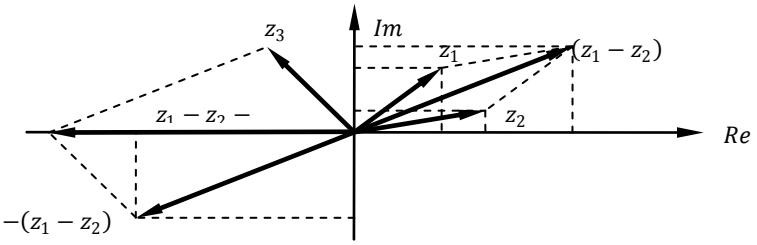


If $z_3 = x_3 + iy_3$, find $z_1 - z_2 - z_3$ if $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$

Solution

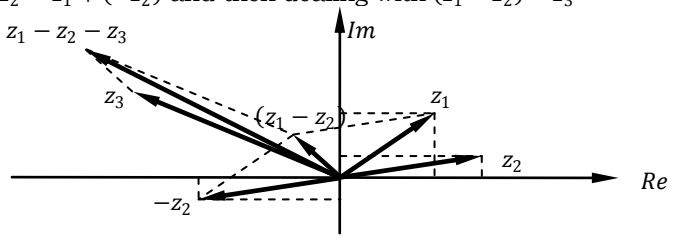
$z_1 - z_2 - z_3 = z_1 - (z_2 + z_3)$ *or* $z_1 - z_2 - z_3 = (z_1 - z_2) - z_3$

Considering $z_1 - z_2 - z_3 = z_1 - (z_2 + z_3)$



Considering second alternative $z_1 - z_2 - z_3 = (z_1 - z_2) - z_3$

Dealing with $z_1 - z_2 = z_1 + (-z_2)$ and then dealing with $(z_1 - z_2) - z_3 = (z_1 - z_2) + (-z_3)$



Note:

Two complex numbers are subtracted by addition of the negative of the second complex number being subtracted from the first.*i.e.* $z_1 - z_2 = z_1 + (-z_2)$. A parallelogram diagonal gives the result, and the answer must be expressed in complex number form.

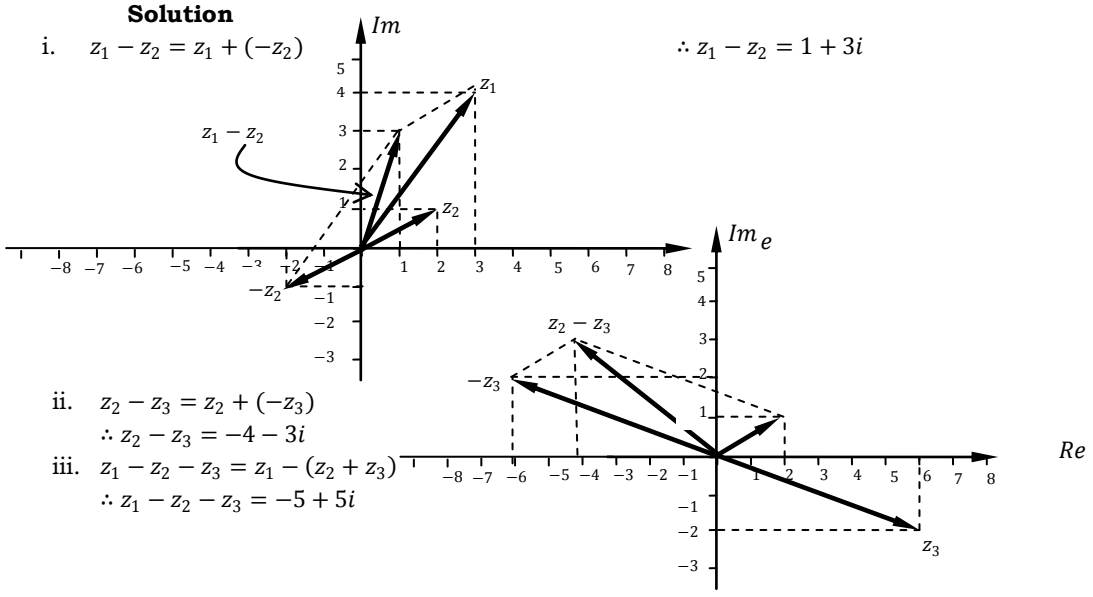
Example

Using the Argand diagram simplify given that $z_1 = 3 + 4i$, $z_2 = 2 + i$ *and* $z_3 = 6 - 2i$

i) $z_1 - z_2$ *ii)* $z_2 - z_3$ *iii)* $z_1 - z_2 - z_3$

Solution

i. $z_1 - z_2 = z_1 + (-z_2)$ $\therefore z_1 - z_2 = 1 + 3i$

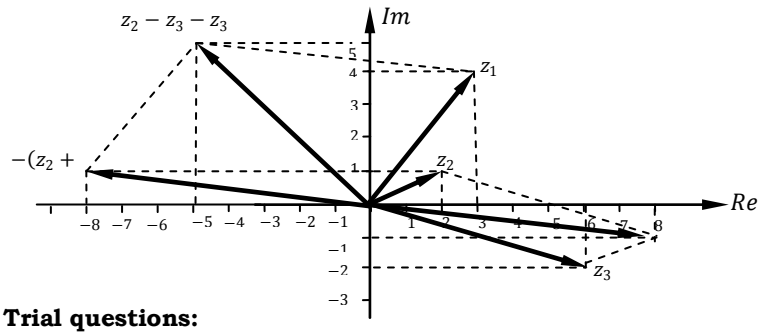


ii. $z_2 - z_3 = z_2 + (-z_3)$

$\therefore z_2 - z_3 = -4 - 3i$

iii. $z_1 - z_2 - z_3 = z_1 - (z_2 + z_3)$

$\therefore z_1 - z_2 - z_3 = -5 + 5i$



Trial questions:

Given that $z_1 = 3 - 2i$, $z_2 = 4$ *and* $z_3 = 2i$. *Using the Argand diagram find;*

i) $z_1 - z_2$ *ii)* $z_2 - z_3$ *iii)* $z_1 + z_2 - z_3$ *iv)* $z_1 - z_2 - z_3$

Answers: $\boldsymbol{(i)\ -1-2i\ \ (ii)\ 4-2i\ \ (iii)\ 7-4i\ \ (iv)\ -1-4i}$

MULTIPLICATION OF COMPLEX NUMBERS

A. **Analytical method**

Given that $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$

To find $z_1.z_2$ proceed as follows:

$z_1.z_2 = (x_1 + iy_1).(x_2 + iy_2)$

$= x_1.x_2 + ix_1.y_2 + iy_1.x_2 + i^2 y_1.y_2$

$= (x_1.x_2 - y_1.y_2) + i(x_1.y_2 + y_1.x_2)$ $[\because i^2 = -1]$

Deduction

When multiplying complex numbers, open the brackets just like in expansion of binomials, simplify the expressions and write the result in complex number form.

Examples

Given that $z_1 = 3 + 2i$, $z_2 = -1 + 2i$ $z_3 = 4 - 3i$ *and* $z_4 = -3 - 4i$. *Find:*

i) $z_1.z_2$ *ii)* $z_2.z_3$ *iii)* $z_3.z_4$ *iv)* $z_1.z_2.z_3$ *iv)* $z_2.z_3.z_4$

Solution

i. $z_1.z_2 = (3 + 2i).(-1 + 2i)$

$= 3(-1 + 2i) + 2i(-1 + 2i)$

$= -3 + 6i - 2i + 4i^2 = -3 - 4 + 4i$ $[\because i^2 = -1]$

$= -7 + 4i$

ii. $z_2.z_3 = (-1 + 2i).(4 - 3i)$

$= -1(4 - 3i) + 2i(4 - 3i)$

$= -4 + 3i + 8i - 6i^2 = -4 + 6 + 11i$ $[\because i^2 = -1]$

$= 2 + 11i$

iii. $z_3.z_4 = (4 - 3i).(-3 - 4i)$

$= 4(-3 - 4i) - 3i(-3 - 4i)$

$= -12 - 16i + 9i + 12i^2 = -12 - 12 - 7i$ $[\because i^2 = -1]$

$= -24 - 7i$

iv. $z_1.z_2.z_3 = [z_1.z_2].z_3$. But from (i) above, $z_1.z_2 = -7 + 14i$

$= [-7 + 14i].(4 - 3i)$

$= -7(4 - 3i) + 4i(4 - 3i)$

$= -28 + 21i + 16i - 12i^2$

$= -28 + 12 + 37i$

$= -16 + 37i$

v. $z_2.z_3.z_4 = [z_2.z_3].z_4$. But $z_2.z_3 = 2 + 11i$

$= [2 + 11i].(-3 - 4i)$

$= 2(-3 - 4i) + 11i(-3 - 4i)$

$= -6 - 8i - 33i - 44i^2$

$= -6 + 44 - 41i$

$= 38 - 41i$

Note:

Graphical multiplication of complex numbers requires Demoivre's theorem which has not been tackled. Graphical multiplication of complex numbers shall be handled later.

Trial questions

1. *Given that* $z_1 = 2 + i$, $z_2 = -2 + i$ *and* $z_3 = 3 - 2i$. *Find:*

   i) $z_1.z_2$ *ii)* $z_2.z_3$ *iii)* $z_3.z_4$ *iv)* $z_1.z_2.z_3$

2. *Given that* $z_1 = 1 + i\sqrt{3}$, $z_2 = \sqrt{3} + i$ *and* $z_3 = 1 + \sqrt{2}i$. *Find:*

   i) $z_1.z_2$ *ii)* $z_2.z_3$ *iii)* $z_3.z_4$ *iv)* $z_1.z_2.z_3$

Answers:

DIVISION OF COMPLEX NUMBERS

Before we look at this, let us first look at a conjugate of a complex number.

Conjugate of a complex number

If $z = a + bi$, then conjugate of z is $\bar{z} = a - ib$

If $z = a - bi$, then conjugate of z is $\bar{z} = a + ib$

Example

Find $\bar{z}$ *if* *i)* $z = 2 + 3i$

ii) $z = -3 - 2i$

Solution

i) $z = 2 + 3i$; $\bar{z} = 2 - 3i$

ii) $z = -3 - 2i$; $\bar{z} = -3 + 2i$

Thus when two complex numbers are to be divided, multiply the numerator and denominator of the quotient being divided by the conjugate of the denominator. In simple terms *rationalize* the denominator.

Consider $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$

$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2}$ ..... $\times \frac{(x_2-iy_2)}{(x_2-iy_2)}$

$= \frac{(x_1+iy_1).(x_2-iy_2)}{(x_2+iy_2).(x_2-iy_2)}$

$= \frac{x_1.x_2+ix_1.y_2+iy_1.x_2+i^2y_1.y_2}{x_1.x_2-ix_1.y_2+iy_1.x_2-i^2y_1.y_2}$

$= \frac{(x_1.x_2-y_1.y_2)+i(x_1.y_2+y_1.x_2)}{{x_2}^2+{y_2}^2}$

Note:

$\frac{z_1}{z_2}$, has now a *real* denominator

Example

1. Given that $z_1 = 2 + 3i$, $z_2 = 4 - 3i$ *and* $z_3 = 1 + i$

   Find i) $\frac{z_1}{z_2}$ ii) $\frac{z_2}{z_3}$

Solution

i. $\frac{z_1}{z_2} = \frac{(2+3i)}{(4-3i)}$ ..... $\times \frac{(4+3i)}{(4+3i)}$

$= \frac{(2+3i).(4+3i)}{(4-3i).(4+3i)} = \frac{8+6i+12i+9i^2}{16+12i-12i-9i^2}$

$= \frac{8-9+18i}{16+9} = \frac{-1+18i}{25}$

$\therefore \frac{z_1}{z_2} = -\frac{1}{25} + \frac{18}{25}i$

ii. $\frac{z_2}{z_3} = \frac{(4-3i)}{(1+i)}$ ..... $\times \frac{(1-i)}{(1-i)}$

$= \frac{(4-3i).(1-i)}{(1+i).(1-i)} = \frac{4-4i-3i+3i^2}{1+i-i-i^2}$

$= \frac{4-3-7i}{1+1} = \frac{1-7i}{2}$

$\therefore \frac{z_2}{z_3} = \frac{1}{2} - \frac{7}{2}i$

2. *Given that* $z_1 = 1 + i\sqrt{3}$, $z_2 = 1 - 2\sqrt{3}i$ *and* $z_3 = -1 + \sqrt{3}i$. *Find:*

i) $\frac{z_1}{z_2}$ *ii)* $\frac{z_2}{z_3}$ *iii)* $\frac{z_3}{z_1 z_2}$ *iv)* $\frac{z_2 z_3}{z_1}$

Solution

i. $\frac{z_1}{z_2} = \frac{(1+i\sqrt{3})}{(1-2\sqrt{3}i)}$ ..... $\times \frac{(1+2\sqrt{3}i)}{(1+2\sqrt{3}i)}$

$= \frac{(1+i\sqrt{3}).(1+2\sqrt{3}i)}{(1-2\sqrt{3}i).(1+2\sqrt{3}i)} = \frac{1+2\sqrt{3}i+\sqrt{3}i+2\sqrt{9}i^2}{1+2\sqrt{3}i-2\sqrt{3}i-4\sqrt{9}i^2}$

$= \frac{1+3\sqrt{3}i-2(3)}{1+4(3)} = \frac{1-6+3\sqrt{3}i}{1+12}$

$\therefore \frac{z_1}{z_2} = -\frac{5}{13} + \frac{3\sqrt{3}}{13}i$

ii. $\frac{z_2}{z_3} = \frac{(1-2\sqrt{3}i)}{(-1+\sqrt{3}i)}$ ..... $\times \frac{(-1-\sqrt{3}i)}{(-1-\sqrt{3}i)}$

$= \frac{(1-2\sqrt{3}i).(-1-\sqrt{3}i)}{(-1+\sqrt{3}i).(-1-\sqrt{3}i)} = \frac{-1-\sqrt{3}i+2\sqrt{3}i+2\sqrt{9}i^2}{1+\sqrt{3}i-\sqrt{3}i-\sqrt{9}i^2}$

$= \frac{-1-2(3)i+\sqrt{3}i}{1+3} = \frac{-7+\sqrt{3}i}{4}$

$\therefore \frac{z_2}{z_3} = -\frac{7}{4} + \frac{\sqrt{3}}{4}i$

iii. $\frac{z_3}{z_1 z_2} = \frac{(-1+\sqrt{3}i)}{(1+i\sqrt{3}).(1-2\sqrt{3}i)} = \frac{(-1+\sqrt{3}i)}{1+i\sqrt{3}-2\sqrt{3}i-2\sqrt{9}i^2}$

$= \frac{(-1+\sqrt{3}i)}{(7-\sqrt{3}i)}$ .Now rationalizing the denominator

$= \frac{(-1+\sqrt{3}i)}{(7-\sqrt{3}i)}$ ..... $\times \frac{(7+\sqrt{3}i)}{(7+\sqrt{3}i)}$

$= \frac{(-1+\sqrt{3}i).(7+\sqrt{3}i)}{(7-\sqrt{3}i).(7+\sqrt{3}i)} = \frac{-7-\sqrt{3}i+7\sqrt{3}i+\sqrt{9}i^2}{49+7\sqrt{3}i-7\sqrt{3}i-\sqrt{9}i^2}$

$= \frac{-7-3+6\sqrt{3}i}{49+3} = \frac{-10+6\sqrt{3}i}{52}$

$= -\frac{10}{52} + \frac{6\sqrt{3}}{52}i = -\frac{5}{26} + \frac{3\sqrt{3}}{26}i$

$\therefore \frac{z_3}{z_1 z_2} = -\frac{5}{26} + \frac{3\sqrt{3}}{26}i$

vi. $\frac{z_2 z_3}{z_1} = \frac{(1-2\sqrt{3}i).(-1+\sqrt{3}i)}{(1+i\sqrt{3})} = \frac{-1+\sqrt{3}i+2\sqrt{3}i-2\sqrt{9}i^2}{(1+i\sqrt{3})}$

$= \frac{-1+2(3)+3\sqrt{3}i}{(1+i\sqrt{3})} = \frac{5+3\sqrt{3}i}{(1+i\sqrt{3})}$

Now rationalizing the denominator

$\frac{z_2 z_3}{z_1} = \frac{(5+3\sqrt{3}i).(1-i\sqrt{3})}{(1+i\sqrt{3}).(1-i\sqrt{3})} = \frac{5-5\sqrt{3}i+3\sqrt{3}i-3\sqrt{9}i^2}{1-i\sqrt{3}+i\sqrt{3}-\sqrt{9}i^2}$

$= \frac{5+3(3)-i2\sqrt{3}}{1+3} = \frac{14-i2\sqrt{3}}{4}$

$= \frac{14}{4} - \frac{2\sqrt{3}}{4}i = \frac{7}{2} - \frac{\sqrt{3}}{2}i \ \ \therefore \frac{z_2 z_3}{z_1} = \frac{7}{2} - \frac{\sqrt{3}}{2}i$

3. *Given that* $z_1 = 1 + i$, $z_2 = 2 - i$ *and* $z_3 = 3 + 2i$. *Find:*

i) $\frac{z_1 + z_2}{z_3}$ *ii)* $\frac{{z_1}^2}{{z_2}^2 . z_3}$ *iii)* $\frac{z_2 + z_3}{z_1 + z_2}$ *iv)* $\frac{z_1 + z_2 + z_3}{z_2 - z_3}$ *v)* $\frac{{z_1}^3 . z_2}{z_1 + z_3}$

Solution

i. $\frac{z_1 + z_2}{z_3}$; $z_1 + z_2 = 1 + i \ + 2 - i \ = 3$

$\therefore \frac{z_1 + z_2}{z_3} = \frac{3}{3 + 2i}$; rationalizing the denominator

$\frac{z_1 + z_2}{z_3} = \frac{3}{3 + 2i} = \frac{3(3 - 2i)}{(3 + 2i).(3 - 2i)}$

$= \frac{9 - 6i}{9 - 6i + 6i - 4i^2} = \frac{9 - 6i}{9 + 4} \ \ \therefore \frac{z_1 + z_2}{z_3} = \frac{9}{13} - \frac{6}{13}i$

ii. $\frac{{z_1}^2}{{z_2}^2 . z_3}$;

${z_1}^2 = z_1 . z_1 = (1 + i).(1 + i) = 1 + i + i + i^2 = 2i$

${z_2}^2 = z_2 . z_2 = (2 - i).(2 - i) = 4 - 2i - 2i + i^2 = 5 - 4i$

$\therefore {z_2}^2 . z_3 = (5 - 4i).(3 + 2i) = 15 + 10i - 12i - 8i^2 = 23 - 2i$

$\therefore \frac{{z_1}^2}{{z_2}^2 . z_3} = \frac{2i}{(23 - 2i)} = \frac{2i(23 + 2i)}{(23 - 2i).(23 + 2i)}$

$= \frac{46i + 4i^2}{529 + 46i - 46i - 4i^2} = \frac{-4 + 46i}{529 + 4} = \frac{-4 + 46i}{533}$

$\therefore \frac{{z_1}^2}{{z_2}^2 . z_3} = -\frac{4}{533} + \frac{46}{533}i$

iii. $\frac{z_2 + z_3}{z_1 + z_2} = \frac{2 - i + 3 + 2i}{1 + i + 2 - i} = \frac{5 + i}{3} \ \ \therefore \frac{z_2 + z_3}{z_1 + z_2} = \frac{5}{3} + \frac{1}{3}i$

iv. $\frac{z_1 + z_2 + z_3}{z_2 - z_3} = \frac{1 + i + 2 - i + 3 + 2i}{2 - i - (3 + 2i)} = \frac{6 + 2i}{-1 - 3i}$; rationalizing denominator

$= \frac{(6 + 2i).(-1 + 3i)}{(-1 - 3i).(-1 + 3i)} = \frac{-6 + 18i - 2i + 6i^2}{1 - 3i + 3i - 9i^2} = \frac{-12 + 16i}{1 + 9} = -\frac{12}{10} + \frac{16}{10}i$

$\therefore \frac{z_1 + z_2 + z_3}{z_2 - z_3} = -\frac{6}{5} + \frac{8}{5}i$

v. $\frac{{z_1}^3 . z_2}{z_1 + z_3}$; ${z_1}^3 = {z_1}^2 . z_1 = 2i.(1 + i) = -2 + 2i$

$\therefore {z_1}^3 . z_2 = (-2 - 2i).(2 - i) = -4 + 2i - 4i + 2i^2 = -2 - 2i$

$z_1 + z_3 = 1 + i + 3 + 2i = 4 + 3i$

$\therefore \frac{{z_1}^3 . z_2}{z_1 + z_3} = \frac{(-2 - 2i)}{(4 + 3i)} = \frac{(-2 - 2i).(4 - 3i)}{(4 + 3i).(4 - 3i)}$

$= \frac{-8 + 6i - 8i + 6i^2}{16 - 12i - 12i - 9i^2} = \frac{-14 - 2i}{16 + 9} = -\frac{14}{25} - \frac{2}{9}i$

$\therefore \frac{{z_1}^3 . z_2}{z_1 + z_3} = -\frac{14}{25} - \frac{2}{9}i$

Trial questions

1) Given that $z = \sqrt{3} + i$; *find i)* z^2 *ii)* $\frac{1}{z}$

2) Given that $z = 3 + 4i$, *find the value of the expression* $z + \frac{25}{z}$

$\left[\boldsymbol{Hint}: 1^{st}\ find \frac{25}{z}\ and\ add\ the\ result\ to\ z\right]$

3) *The total impedance z in an electric circuit with branches* z_1 *and* z_2 *is given by;* $\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$. *Given that* $z_1 = 3 + i4$ *and* $z_2 = 5 + i5$, *where* $i = \sqrt{-1}$, *calculate the total impedance in z in the form* $a + ib$.

$\left[\boldsymbol{Hint}: \frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2} = \frac{z_1 + z_2}{z_1 . z_2}. Now\ proceed\ to\ find\ z_1 + z_2\ and\ z_1 . z_2\ then\ z. \right.$
$\left. Or\ find\ \frac{1}{z_1} and \frac{1}{z_2}\ by\ rationalisation\ and\ then\ z \right]$

4) *Given that* $z_1 = 3 - i\,, z_2 = 3 + i.$ *find*

a) $\frac{{z_1}^2.{z_2}^3}{z_1+z_2}$ b) $\frac{z_2}{z_1}$ c) $\frac{z_2.z_1}{z_2-z_1}$

Note:

Graphical division of complex numbers requires Demoivre's theorem. This shall be tackled later.

DIFFERENT FORMS OF COMPLEX NUMBERS

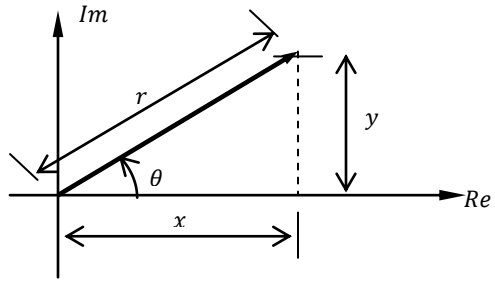
A complex number can mainly be expressed in three forms:

- Standard form $.i.e.z = x + iy$
- Polar form $.i.e.z = r(cos\theta + isin\theta)$
- Exponential form $.i.e.z = re^{\pm i\theta}$

However the most commonly examined at A level are: standard and polar form. Before we look at these in detail, let us first look at modulus and argument of a complex number.

MODULUS AND ARGUMENT OF A COMPLEX NUMBER

Consider the complex number $z = x + iy$ in standard form. Representing z on the Argand diagram



Argument $,\theta$

Argument of a complex number is the angle the complex number makes with *positive Real axis*

Denotion

Argument of z is denoted as $arg(z)$

From the triangle above,

$tan\theta = \frac{y}{x}\ \therefore \theta = tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore arg(z) = arg(x + iy) = \theta = tan^{-1}\left(\frac{y}{x}\right)$$

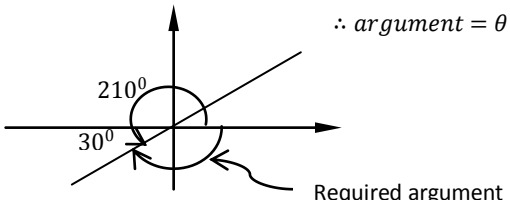
Note:

i. When finding $tan^{-1}\left(\frac{y}{x}\right)$, there is no $i\ on\ y$ from $z = x + iy$ which is the commonest mistake students make $.i.e.tan^{-1}\left(\frac{iy}{x}\right)$, this is not correct.

ii. The range of $\theta\ is - 180^0 < \theta \leq 180^0\ or - \pi < \pi \leq \pi$ in radians $.i.e.principal\ argument$

iii. When θ is being measured, it is done with respect to the$+ve\ real\ axis$ and $\theta\ is + ve$ if measurement is in *anticlockwise direction* where as $-ve$ if measured in a *clockwise direction*

iv. *Measurement of* $\theta = arg(z)$ requires the knowledge of trigonometry about tangents of angles in the four quadrants.
If θ is not in the interval $-\pi < \pi \leq \pi$ then expressing the angle in terms of acute angles must be done to bring the angle in the required interval.
e.g. If $\theta = 210^0$ then $tan210^0 = +tan30^0$



$\therefore argument = \theta = -150^0 = -\frac{5}{6}\pi$

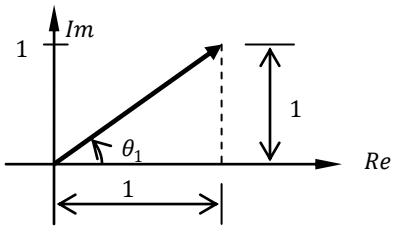
Modulus /Magnitude /length of a complex number
From the right angled triangle above, applying Pythagoras Theorem,
$x^2 + y^2 = r^2 \ \therefore r = \sqrt{x^2 + y^2}$
Denotion
Modulus of z is denoted as $|z| = r$

$$\therefore |z| = r = \sqrt{x^2 + y^2}$$

Note:
In the formula, there is no use of i *on* y *. i.e.* $\therefore |z| = r \neq \sqrt{x^2 + (iy)^2}$
Examples
1. *Given that* $z_1 = 1 + i\,, z_2 = -\sqrt{3} + i\,, z_3 = -5 - 12i$ *and* $z_4 = \sqrt{3} - i$
Find i) $arg(z_1)$ ii) $arg(z_2)$ iii) $arg(z_3)$ iv) $arg(z_4)$
v) $|z_1|$ vi) $|z_2|$ vii) $|z_3|$ viii) $|z_4|$

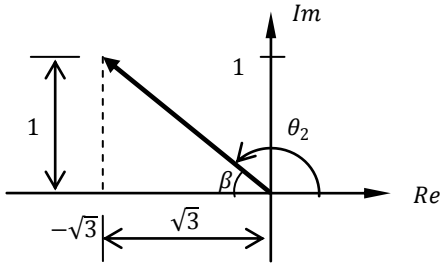
Solution

i. $z_1 = 1 + i$



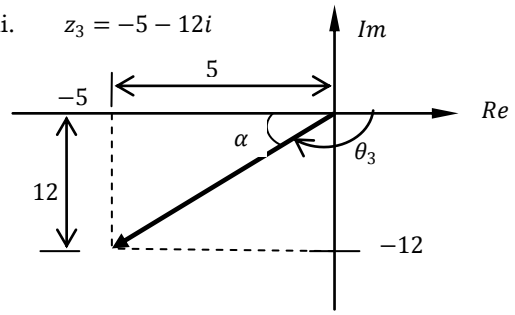
From $|z| = \sqrt{x^2 + y^2}$
$\therefore |z_1| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}\ units$
From $arg(z) = tan^{-1}\left(\frac{y}{x}\right)$
$\therefore arg(z_1) = tan^{-1}\left(\frac{1}{1}\right) = 45^0 = \frac{\pi}{4}$
$\therefore |z_1| = \sqrt{2}\ units\,, arg(z_1) = 45^0 = \frac{\pi}{4}$

ii. $z_2 = -\sqrt{3} + i$



From $|z| = \sqrt{x^2 + y^2}$
$\therefore |z_2| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2\ units$
$arg(z_2) = 180^0 - tan^{-1}\left(\frac{y}{x}\right)$
From $\Delta,\ \beta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^0$
$\therefore arg(z_2) = \theta_2 = 180^0 - 30^0 = 150^0$
$= \frac{5\pi}{6}$
$\therefore |z_2| = 2\ units\,, arg(z_2) = 150^0 = \frac{\pi}{6}$

iii. $z_3 = -5 - 12i$



From $|z| = \sqrt{x^2 + y^2}$

$\therefore |z_2| = \sqrt{(5)^2 + (12)^2} = 13\ units$

$arg(z_3) = \theta_3 = -(180^0 - \alpha)$

From $\Delta,\ \alpha = tan^{-1}\left(\frac{12}{5}\right) = 67.4^0$

$\therefore arg(z_3) = \theta_3 = -(180^0 - 67.4^0) = -112.6^0$

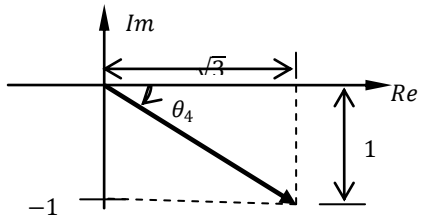
$\therefore |z_3| = 13\ units, arg(z_3) = -112.6^0$

iv. $z_4 = \sqrt{3} - i$, From $|z| = \sqrt{x^2 + y^2}$

$\therefore |z_4| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2\ units$

$arg(z_4) = \theta_4 = -tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^0 = -\frac{\pi}{6}$

$\therefore |z_2| = 2\ units, arg(z_2) = 30^0 = -\frac{\pi}{6}$



2. Given that $z_1 = 1 + i, z_2 = -1 - i, z_3 = 2 + 3i$. *Find the modulus and argument of:*

i) $z_1.z_2$ *ii)* $\frac{z_2.z_3}{z_1}$ *iii)* $\frac{{z_1}^2}{z_2.z_3}$ *iv)* $\frac{{z_2}^3}{z_1.z_3}$ *v)* ${z_1}^2$

Solution

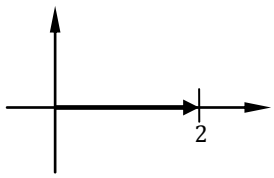
i. $z_1.z_2 = (1+i).(1+i) = 1 - i + i - i^2$

$= 2 + 0i$

$\therefore |z_1.z_2| = \sqrt{(2)^2 + 0^2} = 2\ units$

$arg(z_1.z_2) = tan^{-1}\left(\frac{0}{2}\right) = 0^0$

$\therefore |z_1.z_2| = 2\ units\ ;\ arg(z_1.z_2) = 0^0$



ii. $\frac{z_2.z_3}{z_1} = \frac{(1-i).(2+3i)}{(1+i)} = \frac{2+3i-2i-i^2.3}{1+i}$

$= \frac{2+3+i}{1+i} = \frac{5+i}{1+i}$, rationalizing denominator

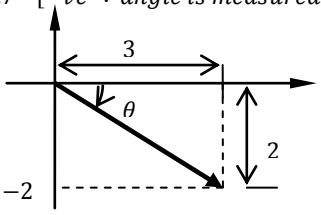
$\frac{(5+i).(1-i)}{(1+i).(1-i)} = \frac{5-5i+i-i^2}{1+i-i-i^2} = \frac{5+1-4i}{1+1} = \frac{6-4i}{2} = 3 - 2i$

$\therefore \frac{z_1.z_3}{z_1} = 3 - 2i$

$\Rightarrow \left|\frac{z_1.z_3}{z_1}\right| = \sqrt{3^2 + 2^2} - \sqrt{13}\ units$

From the figure, $tan\theta = \frac{2}{3}\ \therefore \theta = 33.7^0$

$\therefore arg\left(\frac{z_1.z_3}{z_1}\right) = -33.7^0$ $[-ve\ \because angle\ is\ measured\ in\ clockwise\ direction]$



iii. $\frac{{z_1}^2}{z_2.z_3} = \frac{(1+i)^2}{(1-i).(2+3i)} = \frac{1+i+i-i^2}{2+3i-2i-i^2.3} = \frac{2i}{5+i}$

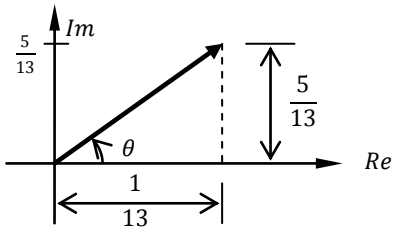
$\frac{2i}{5+i} = \frac{2i.(5-i)}{(5+i).(5-i)} = \frac{10i-2i^2}{25-5i+5i-i^2} = \frac{2+10i}{25+1} = \frac{2+10i}{26}$

$\therefore \frac{{z_1}^2}{z_2.z_3} = \frac{1}{13} + \frac{5}{13}i$

$\Rightarrow \left|\frac{{z_1}^2}{z_2.z_3}\right| = \sqrt{\left(\frac{1}{13}\right)^2 + \left(\frac{5}{13}\right)^2} = \sqrt{\left(\frac{1}{13}\right)^2[1+25]} = \frac{\sqrt{26}}{13} units$

$arg\left(\frac{{z_1}^2}{z_2.z_3}\right) = tan^{-1}\left(\frac{5/13}{1/13}\right) = tan^{-1}(5) = 78.7^0$

$\therefore \left|\frac{{z_1}^2}{z_2.z_3}\right| = \frac{\sqrt{26}}{13} units\,;\ arg\left(\frac{{z_1}^2}{z_2.z_3}\right) = 78.7^0$



iv. $\frac{{z_2}^3}{z_1.z_3} = \frac{(1-i)^3}{(1+i).(2+3i)} = \frac{(1-i)^2.(1-i)}{2+3i+2i+3i^2} = \frac{(1-2i+i^2).(1-i)}{2-3+5i}$

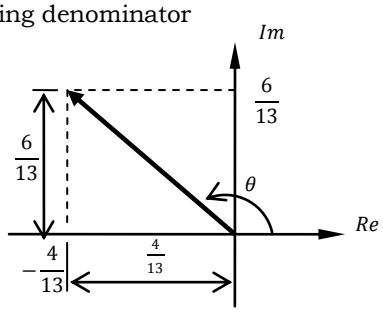
$= \frac{(-2i).(1-i)}{-1+5i} = \frac{(-2i+2i^2)}{(-1+5i)} = \frac{(-2-2i)}{(-1+5i)}$; rationalizing denominator

$\frac{(-2-2i)}{(-1+5i)} = \frac{(-2-2i).(-1-5i)}{(-1+5i).(-1-5i)} = \frac{2+10i+2i+10i^2}{1+5i-5i-25i^2}$

$\therefore \frac{{z_2}^3}{z_1.z_3} = \frac{2-10+12i}{1+25} = \frac{-8+12i}{26} = -\frac{4}{13} + \frac{6}{13}i$

$\therefore \left|\frac{{z_2}^3}{z_1.z_3}\right| = \sqrt{\left(\frac{4}{13}\right)^2 + \left(\frac{6}{13}\right)^2} = \sqrt{\left(\frac{1}{13}\right)^2[16+36]}$

$= \frac{\sqrt{52}}{13} = \frac{\sqrt{4\times13}}{13} = \frac{2}{\sqrt{13}} units$



$arg\left(\frac{{z_2}^3}{z_1.z_3}\right) = 180^0 - tan^{-1}\left(\frac{\frac{6}{13}}{\frac{4}{13}}\right)$

$= 180^0 - tan^{-1}\left(\frac{6}{4}\right) = 180^0 - 56.3^0 = 123.7^0$

$\therefore \left|\frac{{z_2}^3}{z_1.z_3}\right| = \frac{2}{\sqrt{13}} units\,;\ arg\left(\frac{{z_2}^3}{z_1.z_3}\right) = 123.7^0$

STANDARD FORM OF A COMPLEX NUMBER

A complex number of the form $z = x + iy\ where\ x, y \epsilon \mathbb{R}$ is said to be in standard form.

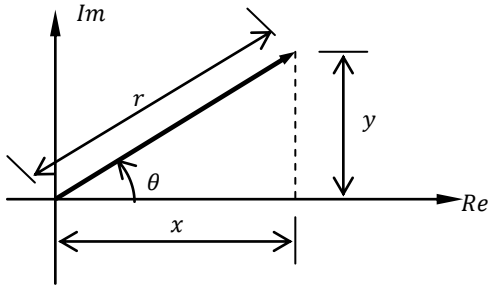
$.e.g. z = 2 + 3i\,, z = 4 - 3i$

POLAR/MODULUS-ARGUMENT FORM OF A COMPLEX NUMBER

This originates from the standard form. A complex number in the form $z = r(cos\theta + isin\theta)$ is in polar form.

Derivation

Consider $z = x + iy$



$|z| = \sqrt{x^2 + y^2}$

$arg(z) = tan^{-1}\left(\frac{y}{x}\right)$

Now $cos\theta = \frac{x}{r}\ \therefore x = rcos\theta$

$sin\theta = \frac{y}{r}\ \therefore y = rsin\theta$

From $z = x + iy$

$\Rightarrow z = rcos\theta + rsin\theta$

$\therefore z = r(cos\theta + isin\theta)$

Note:

Expressing a complex number in polar form requires the knowledge of finding modulus and argument of a complex number.

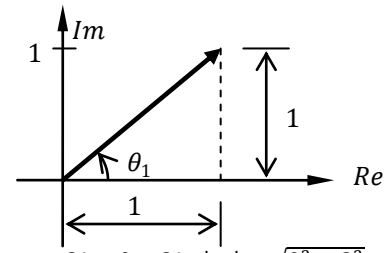
Examples

1) *Given that* $z_1 = 1 + i$, $z_2 = -3i$, $z_3 = -\sqrt{3} - i$ *and* $z_4 = \sqrt{3} - i$. *Express the complex numbers in polar form.*

Solution

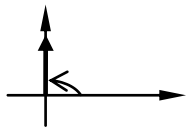
$z_1 = 1 + i$; $|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}\ units$; $arg(z_1) = tan^{-1}(1) = 45^0$

$\therefore z_1 = \sqrt{2}\big(cos(45^0) + isin(45^0)\big)$



$z_2 = -3i = 0 + 3i$; $|z_2| = \sqrt{0^2 + 3^2} = 3\ units$; $arg(z_2) = 90^0$

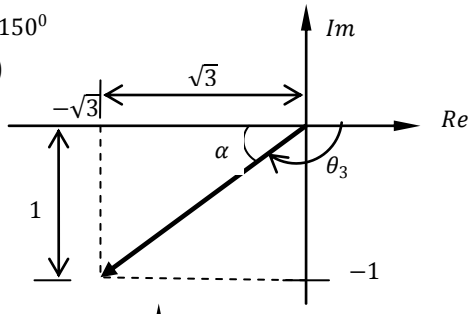
$\therefore z_2 = 3\big(cos(90^0) + isin(90^0)\big)$



$z_3 = -\sqrt{3} - i$; $|z_1| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2\ units$;

$arg(z_3) = -\left[180^0 - tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] = -150^0$

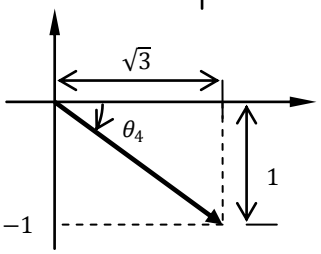
$\therefore\ z_3 = 2\big(cos(-150^0) + isin(-150^0)\big)$



$z_4 = \sqrt{3} - i$; $|z_3| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2\ units$;

$arg(z_3) = -tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -30^0$

$\therefore z_4 = 2\big(cos(-30^0) + isin(-30^0)\big)$



2) *Given that* $z_1 = 2 + 3i$, $z_2 = 1 + i$, *and* $z_3 = 1 - i$. *Express the complex numbers in modulus-argument form*

i) $\frac{z_1.z_2}{z_3}$ *ii)* $\frac{z_2.z_3}{z_1}$ *iii)* $z_2 + \frac{z_1.z_2}{z_3}$ *iv)* $\frac{z_1+z_2}{z_2.z_3}$ *v)* $\frac{z_1+z_2}{z_2-z_3}$

Solution

i. $\frac{z_1.z_2}{z_3} = \frac{(2+3i).(1+i)}{1-i} = \frac{2+2i+3i+3i^2}{1-i} = \frac{2-3+5i}{1-i}$

$\frac{z_1.z_2}{z_3} = \frac{-1+5i}{1-i}$; rationalizing denominator

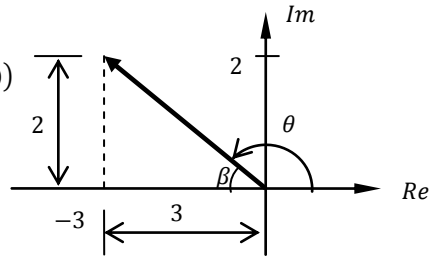
$\frac{z_1.z_2}{z_3} = \frac{(-1+5i)(1+i)}{(1-i)(1+i)} = \frac{-1-i+5i+5i^2}{(1)^2-(i)^2}$ $\left[\because (a+b)(a-b) = a^2-b^2 \text{ in the denominator}\right]$

$= \frac{-1-5+6i}{1+1} = \frac{-6+4i}{2} = -3+2i$

$\therefore \left|\frac{z_1.z_2}{z_3}\right| = \sqrt{3^2+2^2} = \sqrt{13}$

$arg\left(\frac{z_1.z_2}{z_3}\right) = 180^0 - tan^{-1}\left(\frac{2}{3}\right) = 146.3^0$

$\therefore \frac{z_1.z_2}{z_3} = \sqrt{13}\left(cos(146.3^0) + isin(146.3^0)\right)$



ii. $\frac{z_2.z_3}{z_1} = \frac{(1+i).(1-i)}{2+3i} = \frac{1^2-i^2}{2+3i} = \frac{1+1}{2+3i} = \frac{2}{(2+3i)}$

$\frac{z_2.z_3}{z_1} = \frac{2}{(2+3i)}$; rationalizing denominator

$\frac{z_2.z_3}{z_1} = \frac{2(2-3i)}{(2+3i)(2-3i)} = \frac{4-6i}{(2)^2-(3i)^2}$ $\left[\because (a+b)(a-b) = a^2-b^2 \text{ in the denominator}\right]$

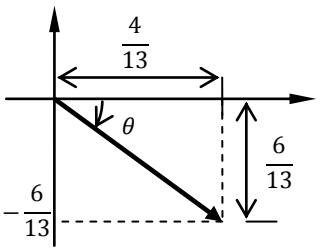
$= \frac{4-6i}{4+9} = \frac{4}{13} - \frac{6}{13}i$

$\therefore \left|\frac{z_2.z_3}{z_1}\right| = \sqrt{\left(\frac{4}{13}\right)^2 + \left(\frac{6}{13}\right)^2} = \frac{2}{\sqrt{13}}$ *units*

$arg\left(\frac{z_2.z_3}{z_1}\right) = -tan^{-1}\left(\frac{6/13}{4/13}\right) = -56.3^0$

$\therefore \frac{z_2.z_3}{z_1} = \frac{2}{\sqrt{13}}\left(cos(-56.3^0) + isin(-56.3^0)\right)$

$= \frac{2}{\sqrt{13}}\left(cos(56.3^0) - isin(56.3^0)\right)$



iii. $z_2 + \frac{z_1.z_2}{z_3} = (1+i) + \frac{(2+3i).(1+i)}{1-i} = 1+i+\frac{2+2i+3i+3i^2}{1-i}$

$= 1+i+\frac{2-3+5i}{1-i} = 1+i+\frac{-1+5i}{1-i}$

$= 1+i+\frac{(-1+5i)(1+i)}{(1-i)(1+i)} = 1+i+\frac{-1-i+5i+5i^2}{(1)^2-(i)^2}$

$= 1+i+\frac{-1-5+6i}{1+1} = 1+i+\frac{-6+4i}{2}$

$= 1+i+-3+2i$

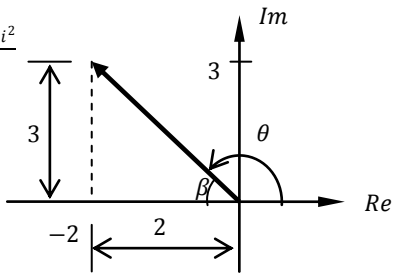
$\therefore z_2 + \frac{z_1.z_2}{z_3} = -2+3i$

$\left|z_2 + \frac{z_1.z_2}{z_3}\right| = \sqrt{2^2+3^2} = \sqrt{13}$ *units*

$arg\left(z_2 + \frac{z_1.z_2}{z_3}\right) = 180^0 - tan^{-1}\left(\frac{3}{2}\right)$

$= 123.7^0$

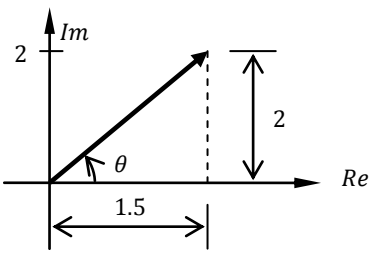
$\therefore z_2 + \frac{z_1.z_2}{z_3} = \sqrt{13}\left(cos(123.7^0) + isin(123.7^0)\right)$



iv. $\frac{z_1+z_2}{z_2.z_3} = \frac{2+3i+1+i}{(1+i).(1-i)} = \frac{3+4i}{(1)^2-(i)^2} = \frac{3+4i}{2} = \frac{3}{2}+2i$

$\left|\frac{z_1+z_2}{z_2.z_3}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2} = \frac{5}{2}$ *units*

$arg\left(\frac{z_1+z_2}{z_2.z_3}\right) = tan^{-1}\left(\frac{2}{1.5}\right) = 53.1^0$



$\therefore \frac{z_1+z_2}{z_2.z_3} = \frac{5}{2}\left(cos(53.1^0) + isin(53.1^0)\right)$

v. $\frac{z_1+z_2}{z_2-z_3} = \frac{2+3i+1+i}{(1+i)-(1-i)} = \frac{3+4i}{2i} = \frac{3+4i}{0+2i} = \frac{(3+4i)(-2i)}{(2i)(-2i)}$

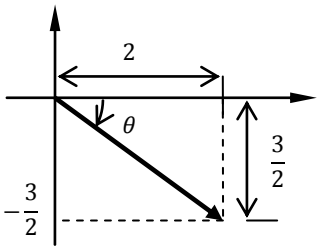
$= \frac{-6i-8i^2}{-4i^2} = \frac{8-6i}{4} = 2 - \frac{3}{2}i$

$\therefore \left|\frac{z_1+z_2}{z_2-z_3}\right| = \sqrt{2^2 + \left(\frac{3}{2}\right)^2} = \frac{5}{2}\ units$

$arg\left(\frac{z_1+z_2}{z_2-z_3}\right) = -tan^{-1}\left(\frac{1.5}{2}\right) = -36.9^0$

$\therefore \frac{z_1+z_2}{z_2-z_3} = \frac{5}{2}\left(cos(-36.9^0) + isin(-36.9^0)\right)$

$= \frac{5}{2}\left(cos(36.9^0) - isin(36.9^0)\right)$



Trial questions

Given that $z_1 = 4 + 3i$, $z_2 = -4 - 3i$, *and* $z_3 = 1 - i$. *Express the complex numbers in polar form.*

i) $z_1.z_2$ *ii)* $z_1.z_2.z_3$ *iii)* $z_3 + \frac{z_1.z_2}{z_3}$ *iv)* $\frac{z_1.{z_2}^3}{{z_3}^2}$ *v)* $\frac{z_1.z_3}{z_2} + \frac{z_2}{z_2.z_3}$

$\boldsymbol{ANSWER: i)\ r = 25\ , \theta = -106.3^0\ \ ii)\ r = 35.4\ , \theta = -151.3^0\ \ iii)\ r = 19\ , \theta = -60.1^0}$
$\boldsymbol{iv)\ r = 312.5\ , \theta = 57.5^0\ \ v)\ r = 1.581\ , \theta = 108.4^0}$

MULTIPLICATION OF COMPLEX NUMBERS IN POLAR FORM

Consider two complex numbers $z_1 = x_1 + iy_1\ and\ z_2 = x_2 + iy_2$ which can be expressed in the polar form.$i.e.$

$z_1 = r_1(cos\theta_1 + isin\theta_1)\ and\ z_2 = r_2(cos\theta_2 + isin\theta_2)$ respectively.

To find $z_1.z_2$, proceed as follows;

$z_1.z_2 = [r_1(cos\theta_1 + isin\theta_1)].[r_2(cos\theta_2 + isin\theta_2)]$

$= r_1.r_2[(cos\theta_1 + isin\theta_1).(cos\theta_2 + isin\theta_2)]$

$= r_1.r_2[cos\theta_1 cos\theta_2 + icos\theta_1 sin\theta_2 + isin\theta_1 cos\theta_2 + i^2 sin\theta_1 sin\theta_2]$

$= r_1.r_2[(cos\theta_1 cos\theta_2 + i^2 sin\theta_1 sin\theta_2) + i(cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2)]$

$= r_1.r_2[(cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2) + i(cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2)]$

$= r_1.r_2[cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)]$

In general when multiplying complex numbers in polar form, first multiply their moduli, add their arguments and write the result in polar form.$i.e.$

If $z_1 = \ r_1(cos\ \theta_1 + \ isin\ \theta_1)$

$z_n = \ r_n(cos\ \theta_n + isin\ \theta_n)$

$\Rightarrow z_1.z_2\ .....\ z_n = r_1.r_2\ .....\ r_n[cos(\theta_1 + \theta_2 + \cdots + \theta_n) + isin(\theta_1 + \theta_2 + \cdots + \theta_n)]$

Deduction

When multiplying complex numbers in polar form,

i. $arg(z_1.z_2\ .....\ z_n) = arg(z_1) + arg(z_2) + ... + arg(z_n)$

ii. $|z_1.z_2\ .....\ z_n| = |z_1||z_2|\ \ldots\ldots\ldots\ldots\ |z_n|$

Note

The sense of measurement of the argument *must* be the same. *i.e.* clockwise/anticlockwise or just consider the *principal argument* of complex numbers.

Example

1. *Given that* $z_1 = 1 + i\sqrt{3}, z_2 = \sqrt{3} - i,\ z_3 = -1 - i$ *and* $z_4 = -1 + i$ *. Express the complex numbers in polar form.* $z_1.z_2,\ z_3.z_4$ *and* $z_1.z_2.z_3$

Solution

$z_1 = 1 + i\sqrt{3}\ ;\ |z_1| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2\ units$

$arg(z_1) = tan^{-1}\left(\sqrt{3}\right) = 60^0$

$\therefore z_1 = 2(cos60^0 + isin60^0)$

$z_2 = \sqrt{3} - i\ ;\ |z_2| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2\ units$

$arg(z_2) = -tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -30^0$ [***principal argument***]

$\therefore z_2 = 2\left(cos(-30^0) + isin(-30^0)\right)$

$z_3 = -1 - i\ ;\ |z_3| = \sqrt{1^2 + 1^2} = \sqrt{2}\ units$

$arg(z_3) = -\left[180^0 - tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] = -135^0$ [***principal argument***]

$\therefore z_3 = \sqrt{2}\left(cos(-135^0) + isin(-135^0)\right)$

$z_4 = -1 + i\ ;\ |z_3| = \sqrt{1^2 + 1^2} = \sqrt{2}\ units$

$arg(z_3) = 180^0 - tan^{-1}(1) = 135^0$ [***principal argument***]

$\therefore z_4 = \sqrt{2}\left(cos(135^0) + isin(135^0)\right)$

$z_1.z_2 = [2(cos60^0 + isin60^0)].\left[2\left(cos(-30^0) + isin(-30^0)\right)\right]$

$= 4[cos(60^0 + -30^0) + isin(60^0 + -30^0)]$

$= 4\left(cos(30^0) + isin(30^0)\right)$

$z_3.z_4 = \left[\sqrt{2}\left(cos(-135^0) + isin(-135^0)\right)\right].\left[\sqrt{2}\left(cos(135^0) + isin(135^0)\right)\right]$

$= 2[cos(-135^0 + 135^0) + isin(-135^0 + 135^0)]$

$= 2[cos0^0 + isin0^0]$

$z_1.z_2.z_3 = [2(cos60^0 + isin60^0)].\left[2\left(cos(-30^0) + isin(-30^0)\right)\right].\left[\sqrt{2}\left(cos(-135^0) + isin(-135^0)\right)\right]$

$= 4\sqrt{2}[cos(60^0 + -30^0 + -135^0) + isin(60^0 + -30^0 + -135^0)]$

$= 4\sqrt{2}\left[\left(cos(-105^0) + isin(-105^0)\right)\right]$

2. *Given that* $z_1 = 2 + i, z_2 = -2 + i$ *and* $z_3 = 2 - 3i$ *. Express the complex numbers in polar form.* $z_1.z_2,\ z_3.z_4$ *and* $z_1.z_2.z_3$

Solution

$|z_1| = \sqrt{2^2 + 1^2} = \sqrt{5}\ ; arg(z_1) = 26.6^0$

$|z_2| = \sqrt{2^2 + 1^2} = \sqrt{5}\ ; arg(z_2) = 153.4^0$

$|z_3| = \sqrt{2^2 + 3^2} = \sqrt{13}\ ; arg(z_3) = -56.3^0$

$\therefore z_1.z_2 = \left[\sqrt{5}\left[\left(cos(26.6^0) + isin(26.6^0)\right)\right]\right].\sqrt{5}\left[\left(cos(153.4^0) + isin(153.4^0)\right)\right]$

$= 5[cos(26.6^0 + 153.4^0) + isin(26.6^0 + 153.4^0)]$

$= 5\left(cos(180^0) + isin(180^0)\right)$

$z_3.z_4 = \left[\sqrt{5}\left[\left(cos(153.4^0) + isin(153.4^0)\right)\right]\right].\sqrt{13}\left[\left(cos(-56.3^0) + isin(-56.3^0)\right)\right]$

$= \sqrt{65}[cos(-56.3^0 + 153.4^0) + isin(-56.3^0 + 153.4^0)]$

$= \sqrt{65}\left(cos(97.1^0) + isin(97.1^0)\right)$

$z_1.z_2.z_3 =$

$\left[\sqrt{5}\left[\left(cos(26.6^0) + isin(26.6^0)\right)\right]\right].\sqrt{5}\left[\left(cos(153.4^0) + isin(153.4^0)\right)\right].\sqrt{13}\left[\left(cos(153.4^0) + isin(153.4^0)\right)\right]$

$= 5\sqrt{13}[cos(26.6^0 + 153.4^0 - 56.3^0) + isin(26.6^0 + 153.4^0 - 56.3^0)]$

$= 5\sqrt{13}\left[\left(cos(123.7^0) + isin(123.7^0)\right)\right]$

Trial questions

Given that $z_1 = 2, z_2 = -1 + i$ *and* $z_3 = 2i$.

Express z_1, z_2, z_3, $z_1.z_2$, $z_2.z_3$ *and* $z_1.z_2.z_3$ *in polar form.*

ANSWER: $\boldsymbol{r = 2, \theta = 0; r = \sqrt{2}, \theta = 135^0; r = 2, \theta = -90^0; r = 2\sqrt{2}, \theta = 135^0; r = 4\sqrt{2}, \theta = 45^0}$

DIVIDING COMPLEX NUMBERS IN POLAR FORM

Consider two complex numbers $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$ which can be expressed in the polar form.$i.e.$

$z_1 = r_1(cos\theta_1 + isin\theta_1)$ *and* $z_2 = r_2(cos\theta_2 + isin\theta_2)$ respectively.

To find $\frac{z_1}{z_2}$, proceed as follows;

$\frac{z_1}{z_2} = \frac{r_1(cos\,\theta_1 + isin\,\theta_1)}{r_2(cos\,\theta_2 + isin\,\theta_2)} = \frac{r_1}{r_2}.\frac{(cos\,\theta_1 + isin\,\theta_1)}{(cos\,\theta_2 + isin\,\theta_2)}$; rationalizing the denominator

$\frac{z_1}{z_2} = \frac{r_1}{r_2}.\frac{(cos\,\theta_1 + isin\,\theta_1).(cos\,\theta_2 - isin\,\theta_2)}{(cos\,\theta_2 + isin\,\theta_2).(cos\,\theta_2 - isin\,\theta_2)}$

$= \frac{r_1}{r_2}.\frac{cos\,\theta_1 cos\,\theta_2 - icos\,\theta_1 sin\,\theta_2 + isin\,\theta_1 cos\,\theta_2 - i^2 sin\,\theta_1 sin\,\theta_2}{(cos\,\theta_2)^2 - (isin\,\theta_2)^2}$ $\left[\because (a+b)(a-b) = a^2 - b^2 \text{ in the denominator}\right]$

$= \frac{r_1}{r_2}.\frac{(cos\,\theta_1 cos\,\theta_2 + sin\,\theta_1 sin\,\theta_2) + i(sin\,\theta_1 cos\,\theta_2 - cos\,\theta_1 sin\,\theta_2)}{cos^2\theta_2 + sin^2\theta_2}$

$= \frac{r_1}{r_2}.\frac{cos\,(\theta_1 - \theta_2) + isin\,(\theta_1 - \theta_2)}{1}$

$\therefore \frac{z_1}{z_2} = \frac{r_1}{r_2}.[cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)]$

In general when dividing complex numbers in polar form, first divide their moduli, subtract their arguments and write the result in polar form.

Deduction

When dividing complex numbers in polar form

i. $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

ii. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Note

The sense of measurement of the argument *must* be the same.$i.e.$ clockwise/anticlockwise or just consider the *principal argument* of complex numbers.

Example

1. *Given that* $z_1 = 1 + i, z_2 = -1 + i, z_3 = -i$ *and* $z_4 = 2$. *Express in polar form;*

   i) $\frac{z_1}{z_2}$ *ii)* $\frac{z_2}{z_3}$ *iii)* $\frac{z_4}{z_2}$

Solution

$z_1 = 1 + i = \sqrt{2}\left(cos(45^0) + isin(45^0)\right)$

$z_2 = -1 + i = \sqrt{2}\left(cos(135^0) + isin(135^0)\right)$

$z_3 = -i = 0 - i = 1\left(cos(-90^0) + isin(-90^0)\right)$

$z_4 = 2 = 2 + 0i = 2\big(cos(0^0) + isin(0^0)\big)$

$[Expressing\ complex\ numbers\ in\ polar\ form\ is\ left\ as\ an\ exercise]$

i. $\frac{z_1}{z_2} = \frac{\sqrt{2}(cos\,(45^0)+isin\,(45^0))}{\sqrt{2}(cos\,(135^0)+isin\,(135^0))} = cos(45^0 - 135^0) + isin(45^0 - 135^0)$

$\therefore \frac{z_1}{z_2} = [cos(-90^0) + isin(-90^0)]$

ii. $\frac{z_2}{z_3} = \frac{\sqrt{2}(cos\,(135^0)+isin\,(135^0))}{1(cos\,(-90^0)+isin\,(-90^0))} = \sqrt{2}[cos(135^0 - -90^0) + isin(135^0 - -90^0)]$

$= \sqrt{2}[cos(225^0) + isin(225^0)]$

Expressing angles trigonometric ratios in terms *principal arguments*

Angle 225^0 lies in the 3rd *quadrant*. Thus the angle nearest to $+ve\ real\ axis\ is\ 135^0$. Hence principal argument is-135^0

$\therefore \frac{z_2}{z_3} = \sqrt{2}[cos(-135^0) + isin(-135^0)]$

iii. $\frac{z_4}{z_2} = \frac{2(cos\,(0^0)+isin\,(0^0))}{\sqrt{2}(cos\,(135^0)+isin\,(135^0))} = \frac{2}{\sqrt{2}}[cos(0^0 - 135) + isin(0^0 - 135)]$

$\therefore \frac{z_4}{z_2} = \frac{2}{\sqrt{2}}[cos(-135^0) + isin(-135^0)]$

Summary of rules in complex numbers

Complex number	**Modulus rules**	**Argument rules**
$z_1.z_2$	$\|z_1\|.\|z_2\|$	$arg(z_1.z_2) = arg(z_1) + arg(z_2)$
$\frac{z_1}{z_2}$	$\left\|\frac{z_1}{z_2}\right\| = \frac{\|z_1\|}{\|z_2\|}$	$arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

Note:

i. $arg(z_1) \times arg(z_2) \neq arg(z_1 + z_2) \neq arg(z_1) + arg(z_2)$

ii. $\frac{arg\,(z_1)}{arg\,(z_2)} \neq arg(z_1) - arg(z_2)$

2. *Given that* $z_1 = 2 + 3i\,, z_2 = 1 - i\,, z_3 = -\sqrt{3} + i$ *and* $z_4 = -1 - i\sqrt{3}$. *Express the complex numbers in polar form. Hence express the following in polar form:*

a) $\frac{z_1.z_2}{z_3}$

b) $\frac{z_1}{z_2.z_3}$ c) $\frac{z_1.z_2}{z_3.z_4}$

Solution

$z_1 = 2 + 3i = \sqrt{13}[cos(56.3^0) + isin(56.3^0)]$

$z_2 = 1 - i = \sqrt{2}[cos(-45^0) + isin(-45^0)]$

$z_3 = -\sqrt{3} + i = 2[cos(150^0) + isin(150^0)]$

$z_4 = -1 - i\sqrt{3} = 2[cos(-120^0) + isin(-120^0)]$

$[To\ express\ the\ complex\ numbers\ in\ polar\ form\ see\ notes\ on\ polar\ form\ of\ a\ complex\ number]$

a) $\frac{z_1.z_2}{z_3} = r(cos\theta + isin\theta)$

$r = \left|\frac{z_1.z_2}{z_3}\right| = \frac{|z_1.z_2|}{z_3} = \frac{|z_1|.|z_2|}{|z_3|} = \frac{\sqrt{13}\times\sqrt{2}}{2} = \frac{\sqrt{26}}{2}$

$\theta = arg\left(\frac{z_1.z_2}{z_3}\right) = arg(z_1.z_2) - arg(z_3) = arg(z_1) + arg(z_2) - arg(z_3)$

$= 56.3^0 + -45^0 - 150^0 = -138.7^0$

$\therefore \frac{z_1.z_2}{z_3} = \frac{\sqrt{26}}{2}[cos(-138.7^0) + isin(-138.7^0)]$

b) $\frac{z_1}{z_2.z_3} = r(cos\theta + isin\theta)$

$r = \left|\frac{z_1}{z_2.z_3}\right| = \frac{|z_1|}{|z_2|.|z_3|} = \frac{\sqrt{13}}{\sqrt{2}\times 2} = \frac{1}{2}\sqrt{\left(\frac{13}{2}\right)}$

$\theta = arg\left(\frac{z_1}{z_2.z_3}\right) = arg(z_1) - arg(z_2.z_3) = arg(z_1) - [arg(z_2) + arg(z_3)]$

$= 56.3^0 - [-45^0 + 150^0] = -48.7^0$

$\therefore \frac{z_1}{z_2.z_3} = \frac{1}{2}\sqrt{\left(\frac{13}{2}\right)}[cos(-48.7^0) + isin(-48.7^0)]$

c) $\frac{z_1.z_2}{z_3.z_4} = r(cos\theta + isin\theta)$

$r = \left|\frac{z_1.z_2}{z_3.z_4}\right| = \frac{|z_1|.|z_2|}{|z_3|.|z_4|} = \frac{\sqrt{13}\times\sqrt{2}}{2\times 2} = \frac{\sqrt{26}}{4}$

$\theta = arg\left(\frac{z_1.z_2}{z_3.z_4}\right) = arg(z_1) + arg(z_2) - [arg(z_3) + arg(z_4)]$

$= 56.3^0 + -45^0 - [150^0 + -120^0] = -18.7^0$

$\therefore \frac{z_1.z_2}{z_3.z_4} = \frac{\sqrt{26}}{4}[cos(-18.7^0) + isin(-18.7^0)]$

3. *Given that $z_1 = 1 + i$, $z_2 = 1 - i$, $z_3 = 2 + 3i$. Express the complex numbers in polar form. Hence express the following in polar form:*

i) $\frac{z_2.z_3}{z_1}$ *ii)* $\frac{z_1.z_2}{z_3}$ *iii)* $z_1.z_2 + \frac{1}{z_3}$ *iv)* $\frac{z_1}{z_2} + \frac{z_2.z_3}{z_1}$

Solution

$z_1 = 1 + i = \sqrt{2}[cos(45^0) + isin(45^0)]$

$z_2 = 1 - i = \sqrt{2}[cos(-45^0) + isin(-45^0)]$

$z_3 = 2 + 3i = \sqrt{13}[cos(56.3^0) + isin(56.3^0)]$

i. $\frac{z_2.z_3}{z_1} = r(cos\theta + isin\theta)$

$r = \left|\frac{z_2.z_3}{z_1}\right| = \frac{|z_2|.|z_3|}{|z_1|} = \frac{\sqrt{2}\times\sqrt{13}}{\sqrt{2}} = \sqrt{13}$

$\theta = arg\left(\frac{z_2.z_3}{z_1}\right) = arg(z_2) + arg(z_3) - arg(z_1)$

$= -45^0 + 56.3^0 - 45^0 = -33.7^0$

$\therefore \frac{z_1.z_2}{z_3.z_4} = \sqrt{13}[cos(-33.7^0) + isin(-33.7^0)]$

ii. $\frac{z_1.z_2}{z_3} = r(cos\theta + isin\theta)$

$r = \left|\frac{z_1.z_2}{z_3}\right| = \frac{|z_1|.|z_2|}{|z_3|} = \frac{\sqrt{2}\times\sqrt{2}}{\sqrt{13}} = \frac{2}{\sqrt{13}}$

$\theta = arg\left(\frac{z_1.z_2}{z_3}\right) = arg(z_1) + arg(z_2) - arg(z_3)$

$= 45^0 + -45^0 - 56.3^0 = -56.3^0$

$\therefore \frac{z_1.z_2}{z_3} = \frac{2}{\sqrt{13}}[cos(-56.3^0) + isin(-56.3^0)]$

iii. $z_1.z_2 + \frac{1}{z_3} = r(cos\theta + isin\theta)$

Dealing with $z_1.z_2$

$\Rightarrow z_1.z_2 = (1+i)(1-i) = 1^2 - i^2 = 2$

$\Rightarrow \frac{1}{z_3} = \frac{1}{2+3i} = \frac{1.(2-3i)}{(2+3i).(2-3i)} = \frac{(2-3i)}{(2)^2-(3i)^2} = \frac{2}{13} - \frac{3}{13}i$

$\therefore z_1.z_2 + \frac{1}{z_3} = 2 + \frac{2}{13} - \frac{3}{13}i = \frac{28}{13} - \frac{3}{13}i$

$r = \left|z_1.z_2 + \frac{1}{z_3}\right| = \sqrt{\left(\frac{28}{13}\right)^2 + \left(\frac{3}{13}\right)^2} = \frac{\sqrt{793}}{13}$

$\theta = arg\left(z_1.z_2 + \frac{1}{z_3}\right) = -tan^{-1}\left(\frac{3/13}{2/13}\right) = -6.1^0$

$\therefore z_1.z_2 + \frac{1}{z_3} = \frac{\sqrt{793}}{13}[cos(-6.1^0) + isin(-6.1^0)]$

iv. $\frac{z_1}{z_2} + \frac{z_2.z_3}{z_1} = r(cos\theta + isin\theta)$

$\frac{z_1}{z_2} = \frac{(1+i)}{(1-i)} = \frac{(1+i).(1+i)}{(1-i).(1+i)} = \frac{1+2i+i^2}{1^2-i^2} = \frac{2i}{2} = i$

$\frac{z_2.z_3}{z_1} = \frac{(1-i).(2+3i)}{(1+i)} = \frac{2+3i-2i-3i^2}{(1+i)} = \frac{2+i+3}{(1+i)} = \frac{(5+i)}{(1+i)}$

$\frac{(5+i)}{(1+i)} = \frac{(5+i).(1-i)}{(1+i).(1-i)} = \frac{5-5i+i-i^2}{1^2-i^2} = \frac{5-4i+1}{2} = \frac{6-4i}{2}$

$\therefore \frac{z_2.z_3}{z_1} = 3 - 2i$

$\therefore \frac{z_1}{z_2} + \frac{z_2.z_3}{z_1} = i + 3 - 2i = 3 - i$

$\Rightarrow r = \left|\frac{z_1}{z_2} + \frac{z_2.z_3}{z_1}\right| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$\Rightarrow \theta = arg\left(\frac{z_1}{z_2} + \frac{z_2.z_3}{z_1}\right) = -tan^{-1}\left(\frac{1}{3}\right) = -18.4^0$

$\therefore \frac{z_1}{z_2} + \frac{z_2.z_3}{z_1} = \sqrt{10}[cos(-18.4^0) + isin(-18.4^0)]$

Task

Given that $z_1 = 1 + i$, $z_2 = 2i$, $z_3 = -1 + i$. Express the complex numbers in polar form. Hence express the following in polar form:

i) $\frac{1}{z_1}$ *ii)* $\frac{1}{z_2}$ *iii)* $\frac{1}{z_3}$ *iv)* $\frac{z_1}{z_2} + z_3$ *iv)* $\frac{z_1.z_2}{z_3} + z_3$ *v)* $\frac{z_1.z_2}{z_3} + \frac{z_3}{z_1}$

Answer: $\boldsymbol{i)\ r = \frac{1}{\sqrt{2}}, \theta = -45^0\ \ ii)\ r = \frac{1}{2}, \theta = -90^0\ \ iii)\ r = \frac{1}{\sqrt{2}}, \theta = -135^0}$

$\boldsymbol{iv)\ r = \frac{1}{\sqrt{2}}, \theta = -135^0\ \ v)\ r = \sqrt{2}, \theta = 45^0\ \ vi)\ r = 2.236, \theta = 26.5^0}$

DEMOIVRE'S THEOREM

This is derived from multiplication of complex numbers in polar form.

Consider $z_1 = r_1(cos\theta_1 + isin\theta_1)$

$z_2 = r_2(cos\theta_2 + isin\theta_2)$

$z_3 = r_3(cos\theta_3 + isin\theta_3)$

$\Rightarrow z_1.z_2.z_3 = r_1.r_2.r_2[cos(\theta_1 + \theta_2 + \theta_3) + isin(\theta_1 + \theta_2 + \theta_3)]$

If $z = z_1 = z_2 = z_3 = r(cos\theta + isin\theta)$

$\Rightarrow z.z.z = r.r.r[cos(\theta + \theta + \theta) + isin(\theta + \theta + \theta)]$

$\therefore z^3 = r^3[cos3\theta + isin3\theta]$

$$\therefore z^3 = r^3[cos3\theta + i\ sin3\theta]$$

$$\therefore z^n = r^n[cosn\theta + i\ sinn\theta]$$

$\Rightarrow [r(cos\theta + \imath sin\theta)]^n = r^n[cosn\theta + \imath\ sinn\theta]$

$\Rightarrow r^n(cos\theta + isin\theta)^n = r^n[cosn\theta + i\ sinn\theta]$

$$\boxed{\therefore (\mathbf{cos\theta + isin\theta})^{\mathbf{n}} = \mathbf{cos(n)\theta + i\ sin(n)\theta}}$$

This is Demoivre's theorem.

Proof

The proof is based on induction.

$(cos\theta + isin\theta)^n = cos(n)\theta + i\, sin(n)\theta$

For $n = 1$

$\boldsymbol{L.H.S} = (cos\theta + isin\theta)^1 = cos\theta + i\, sin\theta$

$\boldsymbol{R.H.S} = cos(1)\theta + i\, sin(1)\theta = cos\theta + i\, sin\theta$

Since $\boldsymbol{L.H.S} = \boldsymbol{R.H.S}$, the proof holds for $n = 1$

For $n = k$

$\Rightarrow (cos\theta + isin\theta)^k = cos(k)\theta + i\, sin(k)\theta$ ..... (1)

Let the proof hold for $n = k$

Now for $n = k + 1$

$\boldsymbol{L.H.S} = (cos\theta + isin\theta)^{k+1} = (cos\theta + isin\theta)^k.(cos\theta + i\, sin\theta)^1$ ..... (2)

$eqn.(1) in\ to\ eqn.(2)$

$\boldsymbol{L.H.S} = (cos\theta + isin\theta)^{k+1} = [cos(k)\theta + i\, sin(k)\theta]\,.(cos\theta + i\, sin\theta)^1$

$= [cos(k)\theta + i\, sin(k)\theta].(cos\theta + i\, sin\theta)$

$= cos(k)\theta cos\theta + icos(k)\theta sin\theta + i\, sin(k)\theta cos\theta + i^2 sin(k)\theta sin\theta$

$= [cos(k)\theta cos\theta - sin(k)\theta sin\theta] + i[cos(k)\theta sin\theta + sin(k)\theta cos\theta]$

$= cos[k\theta + \theta] + i\, sin[k\theta + \theta]$

$= cos(k + 1)\theta + i\, sin(k + 1)\theta$

$\boldsymbol{R.H.S} = cos(k + 1)\theta + i\, sin(k + 1)\theta$

Since $\boldsymbol{L.H.S} = \boldsymbol{R.H.S}$, the proof holds for $n = k + 1$

Since the proof holds for $n = 1, 2, 3 \ldots\ldots\ldots k, k + 1$, then the proof holds for all values of n.

Deduction from Demoivre's theorem

If $z = r(cos\theta + isin\theta)$, then $z^n = r^n(cos\theta + isin\theta)^n = r^n[cosn\theta + i\, sinn\theta]$

i. $Arg(z^n) = nArg(z)$

ii. $|z^n| = |z|^n$

Properties of Argument

$Arg(\lambda z) = Arg(z)$, where $\lambda \geq 0\ and\ \lambda \epsilon \mathbb{R}$

Application of Demoivre's Theorem

- Simplifying powers of complex numbers in polar form
- Proof of trigonometric identities
- Simplifying complex number expressions
- Expansions

SIMPLIFYING EXPRESSIONS

1) *Simplify*

a) $\frac{cos\,3\theta + isin\,3\theta}{cos\theta + isin\theta}$

b) $\frac{\cos\frac{3}{2}\theta + isin\frac{3}{2}\theta}{\cos\frac{1}{2}\theta + isin\frac{1}{2}\theta}$

c) $\frac{cos\,2\theta + isin\,2\theta}{cos\theta - isin\theta}$

d) $\frac{cos\,3\theta - isin\,3\theta}{cosn\theta + isinn\theta}$

Solution

a) $\frac{cos\,3\theta + isin\,3\theta}{cos\theta + isin\theta} = \frac{(cos\theta + isin\theta)^3}{(cos\theta + isin\theta)^1} = (cos\theta + isin\theta)^2 = cos2\theta + isin2\theta$

b) $\frac{\cos\frac{3}{2}\theta+i\sin\frac{3}{2}\theta}{\cos\frac{1}{2}\theta+i\sin\frac{1}{2}\theta}=\frac{(cos\theta+isin\theta)^{\frac{3}{2}}}{(cos\theta+isin\theta)^{\frac{1}{2}}}=(cos\theta+isin\theta)^{\frac{3}{2}-\frac{1}{2}}=(cos\theta+isin\theta)^{1}=cos\theta+isin\theta$

c) $\frac{cos\,2\theta+isin\,2\theta}{cos\theta-isin\theta}=\frac{cos\,2\theta+isin\,2\theta}{cos\,(-1)\theta+isin\,(-1)\theta}=\frac{(cos\theta+isin\theta)^{2}}{(cos\theta+isin\theta)^{-1}}=(cos\theta+isin\theta)^{3}=cos3\theta+isin3\theta$

d) $\frac{cos\,3\theta-isin\,3\theta}{cosn\theta+isinn\theta}=\frac{cos\,(-3)\theta+isin\,(-3)\theta}{cos\,(n)\theta+isin\,(n)\theta}=\frac{(cos\theta+isin\theta)^{-3}}{(cos\theta+isin\theta)^{n}}=(cos\theta+isin\theta)^{-3-n}$

$=cos(3+n)\theta-isin(3+n)\theta$

2) *Express in the form* $a+ib$

a) $\sqrt{\left(\frac{cos\theta+isin\theta}{cos\,2\theta-isin\,2\theta}\right)}$

b) $\frac{\left(\cos\frac{\pi}{4}+isin\frac{\pi}{4}\right)^{2}}{\left(\cos\frac{\pi}{6}+isin\frac{\pi}{6}\right)^{3}(\cos\,\pi-isin\pi)^{\frac{1}{3}}}$

c) $\frac{\left(\cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3}-isin\frac{\pi}{3}\right)^{2}}{(\cos\,4\pi-isin\,4\pi)}$

Solution

a) $\sqrt{\left(\frac{cos\theta+isin\theta}{cos\,2\theta-isin\,2\theta}\right)}=\left[\frac{cos\theta+isin\theta}{cos\,2\theta-isin\,2\theta}\right]^{\frac{1}{2}}=\frac{(cos\theta+isin\theta)^{\frac{1}{2}}}{[(cos\theta+isin\theta)^{-2}]^{\frac{1}{2}}}=\frac{(cos\theta+isin\theta)^{\frac{1}{2}}}{(cos\theta+isin\theta)^{-1}}$

$=(cos\theta+isin\theta)^{\frac{1}{2}--1}=\cos\frac{3}{2}\theta+isin\frac{3}{2}\theta$

b) $\frac{\left(\cos\frac{\pi}{4}+isin\frac{\pi}{4}\right)^{2}}{\left(\cos\frac{\pi}{6}+isin\frac{\pi}{6}\right)^{3}(\cos\,\pi-isin\pi)^{\frac{1}{3}}}=\frac{\left[(cos\pi+isin\pi)^{\frac{1}{4}}\right]^{2}}{\left[(cos\pi+isin\pi)^{\frac{1}{6}}\right]^{3}[(cos\pi+isin\pi)^{-1}]^{\frac{1}{3}}}$

$=\frac{(cos\pi+isin\pi)^{\frac{1}{2}}}{(cos\pi+isin\pi)^{\frac{1}{2}}.(cos\pi+isin\pi)^{-\frac{1}{3}}}=(cos\pi+isin\pi)^{\frac{1}{3}}=\cos\frac{\pi}{3}+isin\frac{\pi}{3}$

c) $\frac{\left(\cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3}-isin\frac{\pi}{3}\right)^{2}}{(\cos\,4\pi-isin\,4\pi)}=\frac{\left(\cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)\left(\cos\left(-\frac{\pi}{3}\right)+isin\left(-\frac{\pi}{3}\right)\right)^{2}}{(\cos\,(-4)\pi+isin\,(-4)\pi)}=\frac{(\cos\,\pi+isin\pi)^{\frac{1}{3}}(\cos\,\pi+isin\pi)^{-\frac{2}{3}}}{(\cos\,\pi+isin\pi)^{-4}}$

$(\cos\pi+isin\pi)^{\frac{1}{3}-\frac{2}{3}--4}=(\cos\pi+isin\pi)^{\frac{11}{3}}=\cos\frac{11}{3}\theta+isin\frac{11}{3}\,\theta$

Task

Prove that

a) $\frac{(cos\,3\theta+i\,sin3\theta)^{4}(cos\,4\theta-i\,sin\,4\theta)^{5}}{(cos\,4\theta+i\,sin\,4\theta)^{3}(cos\,5\theta+i\,sin\,5\theta)^{-4}}=1$

b) $\frac{(cos\,5\theta-i\,sin\,5\theta)^{2}(cos\,7\theta+i\,sin\,7\theta)^{-3}}{(cos\,4\theta-i\,sin\,4\theta)^{9}(cos\theta+i\,sin\theta)^{5}}=1$

c) $\frac{(cos\alpha+i\,sin\alpha)^{2}}{(sin\beta+i\,cos\beta)^{5}}=sin(4\alpha+5\beta)-i\,cos(4\alpha+5\beta)$

d) $\left(\frac{cos\theta+i\,sin\theta}{sin\theta+i\,cos\theta}\right)^{4}=cos8\theta+i\,sin8\theta$

EXPANSION OF *cos nθ and sin nθ*

From Demoivre's theorem $(cos\theta+isin\theta)^{n}=cosn\theta+i\,sinn\theta$, to expand ***cos nθ and sin nθ*** the **L.H.S** $(cos\theta+isin\theta)^{n}$ is expanded.

Examples

1. *Express* $cos3\theta\,,sin3\theta$ *in terms of* $cos\theta$ *and* $sin\theta$ *respectively. Hence find* $tan3\theta$

Solution

For $n=3$ in the Demoivre's theorem

$\Rightarrow\ (cos\theta+isin\theta)^{3}=cos3\theta+i\,sin3\theta$

$(cos\theta + isin\theta)^3 = \mathbf{1}(cos\theta)^3(isin\theta)^0 + \mathbf{3}(cos\theta)^2(isin\theta)^1 + \mathbf{3}(cos\theta)^1(isin\theta)^2 + (isin\theta)^3$

$= cos^3\theta + i3cos^2\theta sin\theta - 3cos\theta sin^2\theta - isin^3\theta$

$= (cos^3\theta - 3cos\theta sin^2\theta) + i(3cos^2\theta sin\theta - sin^3\theta)$

$\therefore cos3\theta = cos^3\theta - 3cos\theta sin^2\theta = cos^3\theta - 3cos\theta(1 - cos^2\theta)$

$= cos^3\theta - 3cos\theta + 3cos^3\theta = 4cos^3\theta - 3cos\theta$

$\therefore cos3\theta = 4cos^3\theta - 3cos\theta$

$\therefore sin3\theta = 3cos^2\theta sin\theta - sin^3\theta = 3(1 - sin^2\theta)sin\theta - sin^3\theta$

$= 3sin\theta - 3sin^3\theta - sin^3\theta = 3sin\theta - 4sin^3\theta$

$\therefore sin3\theta = 3sin\theta - 4sin^3\theta$

$\therefore tan3\theta = \frac{sin\,3\theta}{cos\,3\theta} = \frac{3cos^2\theta sin\theta - sin^3\theta}{cos^3\theta - 3cos\theta\, sin^2\theta}$, dividing Numerator and Denominator by $cos^3\theta$

$\Rightarrow tan3\theta = \frac{\frac{3cos^2\theta sin\theta}{cos^3\theta} - \frac{sin^3\theta}{cos^3\theta}}{\frac{cos^3\theta}{cos^3\theta} - \frac{3cos\theta\, sin^2\theta}{cos^3\theta}} = \frac{3tan\theta - tan^3\theta}{1 - 3tan^2\theta}$

$\therefore tan3\theta = \frac{3tan\theta - 4tan^3\theta}{1 - 3tan^2\theta}$

2. *Show that;*

   a) $tan4\theta = \frac{4tan\theta - 4tan^3\theta}{1 - 6tan^2\theta + tan^4\theta}$

   b) $tan5\theta = \frac{5tan\theta - 10tan^3\theta + tan^5\theta}{1 - 6tan^2\theta + 5tan^4\theta}$

Solution

a) $tan4\theta = \frac{4tan\theta - 4tan^3\theta}{1 - 6tan^2\theta + tan^4\theta}$

$\boldsymbol{L.H.S}: tan4\theta = \frac{sin\,4\theta}{cos\,4\theta}$

For $n = 4$ in the Demoivre's theorem

$\Rightarrow (cos\theta + isin\theta)^4 = cos4\theta + i\,sin4\theta$

$(cos\theta + isin\theta)^3 = \mathbf{1}(cos\theta)^4(isin\theta)^0 + \mathbf{4}(cos\theta)^3(isin\theta)^1 + \mathbf{6}(cos\theta)^2(isin\theta)^2 + \mathbf{4}(cos\theta)^1(isin\theta)^3 + \mathbf{1}(isin\theta)^4$

$= cos^4\theta + i4cos^3\theta sin\theta - 6cos^2\theta sin^2\theta - i4cos\theta sin^3\theta + sin^4\theta$

$= (cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta) + i(4cos^3\theta sin\theta - 4cos\theta sin^3\theta)$

$\therefore cos4\theta = cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta$

$\therefore sin4\theta = 4cos^3\theta sin\theta - 4cos\theta sin^3\theta$

$\therefore tan4\theta = \frac{sin\,4\theta}{cos\,4\theta} = \frac{4cos^3\theta sin\theta - 4cos\theta\, sin^3\theta}{cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta}$, dividing Numerator and Denominator by $cos^4\theta$

$\Rightarrow tan4\theta = \frac{\frac{4cos^3\theta sin\theta}{cos^4\theta} - \frac{4cos\theta\, sin^3\theta}{cos^4\theta}}{\frac{cos^4\theta}{cos^4\theta} - 6\frac{cos^2 sin^2\theta}{cos^4\theta} + \frac{sin^4\theta}{cos^4\theta}} = \frac{4tan\theta - 4tan^3\theta}{1 - 6tan^2\theta + tan^4\theta}$

$\therefore tan4\theta = \frac{4tan\theta - 4tan^3\theta}{1 - 6tan^2\theta + tan^4\theta}$

b) *Left as an exercise for the students*

3. *Prove that;*

   a) $cos4\theta = \frac{tan^4\theta - 6tan^2\theta + 1}{tan^4\theta + 2tan^2\theta + 1}$

   b) $sin4\theta = \frac{4tan\theta - 4tan^3\theta}{tan^4\theta + 2tan^2\theta + 1}$

Solution

From example (2) above,

$cos4\theta = cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta$

$sin4\theta = 4cos^3\theta sin\theta - 4cos\theta sin^3\theta$

a) $cos4\theta = cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta$

$= \frac{cos^4\theta - 6cos^2 sin^2\theta + sin^4\theta}{1}$

Dividing Numerator and Denominator by $cos^4\theta$

$\Rightarrow cos4\theta = \frac{\frac{cos^4\theta}{cos^4\theta} - \frac{6cos^2 sin^2\theta}{cos^4\theta} + \frac{sin^4\theta}{cos^4\theta}}{\frac{1}{cos^4\theta}} = \frac{1 - 6tan^2\theta + tan^4\theta}{sec^4\theta}$

$= \frac{1 - 6tan^2\theta + tan^4\theta}{(sec^2\theta)^2} = \frac{1 - 6tan^2\theta + tan^4\theta}{(1 + tan^2\theta)^2}$

$\therefore cos4\theta = \frac{tan^4\theta - 6tan^2\theta + 1}{tan^4\theta + 2tan^2\theta + 1}$ *As required*

b) $sin4\theta = 4cos^3\theta sin\theta - 4cos\theta sin^3\theta$

$= \frac{sin4\theta = 4cos^3\theta sin\theta - 4cos\theta\, sin^3\theta}{1}$

Dividing Numerator and Denominator by $cos^4\theta$

$\Rightarrow sin4\theta = \frac{\frac{4cos^3\theta sin\theta}{cos^4\theta} - \frac{4cos\theta\, sin^3\theta}{cos^4\theta}}{\frac{1}{cos^4\theta}} = \frac{4tan\theta - 4tan^3\theta}{sec^4\theta}$

$= \frac{4tan\theta - 4tan^3\theta}{(sec^2\theta)^2} = \frac{4tan\theta - 4tan^3\theta}{(1 + tan^2\theta)^2}$

$\therefore sin4\theta = \frac{4tan\theta - 4tan^3\theta}{tan^4\theta + 2tan^2\theta + 1}$

Task

Use Demoivre's theorem to show that;

i. $cos2\theta = \frac{1 - t^2}{1 + t^2}\ ; t = tan\theta$

ii. $sin2\theta = \frac{2t}{1 + t^2}\ ; t = tan\theta$

EXPANSION OF $cos^n\theta$ and $sin^n\theta$

This is based on $z^n + \frac{1}{z^n}$ *and* $z^n - \frac{1}{z^n}$ results.

Consider $z = cos\theta + i\, sin\theta$

$\Rightarrow z^n = cos\, n\theta + i\, sin\, n\theta$ ..... (1)

$z^{-n} = \frac{1}{z^n} = cos(-n)\,\theta + i\, sin(-n)\,\theta = cos\, n\,\theta - i\, sin\, n\,\theta$

$\Rightarrow \frac{1}{z^n} = cos\, n\,\theta - i\, sin\, n\,\theta$ ..... (2)

Adding the eqns. (1)*and* (2)

$z^n + \frac{1}{z^n} = 2\, cos\, n\theta$ ..... (3)

Subtracting the eqns. (1)*and* (2)

$z^n - \frac{1}{z^n} = i2\, sin\, n\theta$ ..... (4)

From eqns. (3) and (4), when $n = 1$

$\Rightarrow z + \frac{1}{z} = 2cos\theta$ ..... (5)

$\Rightarrow z - \frac{1}{z} = i2sin\theta$ ..... (6)

The above equations are important in this kind of expansions.

Example

1. *Express*
   a) $cos^3\theta$ in terms of $cos\theta$ and $cos3\theta$
   b) $sin^3\theta$ in terms of $sin\theta$ and $sin3\theta$
   c) $cos^4\theta$ in terms of $cos4\theta$ and $cos2\theta$
   d) $sin^4\theta$ in terms of $sin4\theta$ and $sin2\theta$

Solution

a) To expand $cos^3\theta, z+\frac{1}{z}=2cos\theta$ is used.

Cubing the equation to create $cos^3\theta$

$\Rightarrow \left(z+\frac{1}{z}\right)^3=(2cos\theta)^3=8cos^3\theta$

$z^3+3.z^2.\frac{1}{z}+3z.\frac{1}{z^2}+\frac{1}{z^3}=8cos^3\theta$

$\left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right)=8cos^3\theta$

But from $z^n+\frac{1}{z^n}=2\cos n\theta, for\ n=3,$

$z^3+\frac{1}{z^3}=2cos3\theta$

$\therefore 8cos^3\theta=2cos3\theta+3(2cos\theta)$

$=2cos3\theta+6cos\theta$

$\therefore cos^3\theta=\frac{1}{4}cos3\theta+\frac{3}{4}cos\theta$

b) To expand $sin^3\theta, z-\frac{1}{z}=i2sin\theta$ is used.

Cubing the equation to create $cos^3\theta$

$\Rightarrow \left(z-\frac{1}{z}\right)^3=(i2sin\theta)^3=-i8sin^3\theta$

$z^3+3.z^2.\left(-\frac{1}{z}\right)+3z.\left(-\frac{1}{z}\right)^2+\left(-\frac{1}{z}\right)^3=-i8sin^3\theta$

$\left(z^3-\frac{1}{z^3}\right)-3\left(z-\frac{1}{z}\right)=-i8sin^3\theta$

But from $z^n-\frac{1}{z^n}=i2\sin n\theta, for\ n=3,$

$z^3-\frac{1}{z^3}=i2sin3\theta$

$\therefore -i8sin^3\theta=i2sin3\theta-3(i2sin\theta)$

$=i2sin3\theta-i6sin\theta$, dividing through by $-i8$

$\therefore sin^3\theta=-\frac{1}{4}sin3\theta+\frac{3}{4}sin\theta$

c) To expand $cos^4\theta, z+\frac{1}{z}=2cos\theta$ is used.

Quarting both sides

$\Rightarrow \left(z+\frac{1}{z}\right)^4=(2cos\theta)^4=16cos^4\theta$

$z^4+4.z^3.\frac{1}{z}+6z^2.\frac{1}{z^2}+4z.\frac{1}{z^3}+\frac{1}{z^4}=16cos^4\theta$

$\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6=16cos^4\theta$

But from $z^n+\frac{1}{z^n}=2\cos n\theta, for\ n=4\ and\ 2,$

$z^4+\frac{1}{z^4}=2cos4\theta\ ;\ z^2+\frac{1}{z^2}=2cos2\theta$

$\therefore 16cos^4\theta=2cos4\theta+4(2cos2\theta)+6$

$=2cos4\theta+8cos2\theta+6$, dividing through by16

$\therefore cos^4\theta=\frac{1}{8}cos3\theta+\frac{1}{2}cos2\theta+\frac{3}{8}$

d) $Left\ as\ an\ exercise\ for\ students$

Task

a) Prove that $sin^5\theta=\frac{1}{16}(sin5\theta-5sin3\theta+10sin\theta)$

b) Prove that $cos^6\theta=\frac{1}{32}(cos6\theta+6cos\theta+15cos\theta+10)$

FINDING THE N^{TH} ROOT OF A COMPLEX NUMBER

Consider $z = r(cos\theta + i\,sin\theta)$. To find $\sqrt[n]{z} = z^{\frac{1}{n}}$, proceed as follows.

$z^{\frac{1}{n}} = [r(cos\theta + i\,sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}(cos\theta + i\,sin\theta)^{\frac{1}{n}}$

Since the result must also be in polar form, then let

$r^{\frac{1}{n}}(cos\theta + i\,sin\theta)^{\frac{1}{n}} = R(cos\varphi + i\,sin\varphi)$

Recall: From trigonometry,

$cos(\theta) = cos(\theta + 360^0 k)$

$sin(\theta) = sin(\theta + 360^0 k)$, where $k = 0,1,2,3 \ldots\ldots$

$\therefore r^{\frac{1}{n}}(cos\theta + i\,sin\theta)^{\frac{1}{n}} = R(cos\varphi + i\,sin\varphi)$, becomes;

$r^{\frac{1}{n}}\big(cos(\theta + 360^0 k) + i\,sin(\theta + 360^0 k)\big)^{\frac{1}{n}} = R(cos\varphi + i\,sin\varphi)$

$\Rightarrow r^{\frac{1}{n}}\left[cos\left(\frac{\theta+360^0 k}{n}\right) + i\,sin\left(\frac{\theta+360^0 k}{n}\right)\right] = R(cos\varphi + i\,sin\varphi)$

Comparing

$\Rightarrow R = r^{\frac{1}{n}}$ …… (1)

$\Rightarrow \varphi = \frac{\theta+360^0 k}{n}$ …… (2)

$where\ k = 0,1,2,3 \ldots\ldots\ldots\ldots (n-1)$

Taking values of k in to equation (2) yields the argument of different roots to $\sqrt[n]{z}$.

Example

1. *Given that* $z_1 = 1 + \sqrt{3}i$, $z_2 = 2 + i$, $z_3 = 1 - i$, *find;*
   a) $\sqrt{z_1}$ d) $\sqrt[3]{z_1}$
   b) $\sqrt{z_2}$ e) $\sqrt[3]{z_3}$
   c) $\sqrt{z_3}$ f) $\sqrt[4]{z_1}$

Solution

Expressing complex numbers in polar form first,

a) $z_1 = 1 + \sqrt{3}i = 2\big(cos(60^0) + i\,sin(60^0)\big)$

$\sqrt{z_1} = \big[2\big(cos(60^0) + i\,sin(60^0)\big)\big]^{\frac{1}{2}}$

$= \sqrt{2}\big[\big(cos(60^0) + i\,sin(60^0)\big)\big]^{\frac{1}{2}}$

$= \sqrt{2}\left[cos\left(\frac{60^0+360^0 k}{2}\right) + i\,sin\left(\frac{60^0+360^0 k}{2}\right)\right]$

Now $k = 0,1$ [$\because n = 2\ \therefore k\ takes\ on\ 0\ and\ 1$]

For $k = 0$

$Root\ 1 = \sqrt{2}\left[cos\left(\frac{60^0+360^0\times 0}{2}\right) + i\,sin\left(\frac{60^0+360^0\times 0}{2}\right)\right]$

$= 1.225 + 0.707i$

For $k = 1$

$Root\ 2 = \sqrt{2}\left[cos\left(\frac{60^0+360^0\times 1}{2}\right) + i\,sin\left(\frac{60^0+360^0\times 1}{2}\right)\right]$

$= -1.225 - 0.707i$

$\therefore \sqrt{z_1} = \pm(1.225 + 0.707i)$

b) $z_2 = 2 + i = \sqrt{5}\big(cos(26.6^0) + i\,sin(26.6^0)\big)$

$\sqrt{z_2} = \big[\sqrt{5}\big(cos(26.6^0) + i\,sin(26.6^0)\big)\big]^{\frac{1}{2}}$

$= \sqrt[4]{5}\left[cos\left(\frac{26.6^0+360^0 k}{2}\right) + i\,sin\left(\frac{26.6^0+360^0 k}{2}\right)\right]$

Now $k = 0,1$ [$\because n = 2\ \therefore k\ takes\ on\ 0\ and\ 1$]

For $k = 0$

$Root\ 1 = \sqrt[4]{5}\left[cos\left(\frac{26.6^0+360^0\times0}{2}\right) + i\ sin\left(\frac{26.6^0+360^0\times0}{2}\right)\right]$

$= 1.455 + 0.344i$

For $k = 1$

$Root\ 2 = \sqrt[4]{5}\left[cos\left(\frac{26.6^0+360^0\times1}{2}\right) + i\ sin\left(\frac{26.6^0+360^0\times1}{2}\right)\right]$

$= -1.455 - 0.344i$

$\therefore \sqrt{z_1} = \pm(1.455 + 0.344i)$

c) $z_3 = 1 - i = \sqrt{2}\left(cos(-45^0) + i\ sin(-45^0)\right)$

$\sqrt{z_3} = z_3{}^{\frac{1}{2}} = \left[\sqrt{2}\left(cos(-45^0) + i\ sin(-45^0)\right)\right]^{\frac{1}{2}}$

$= \sqrt[4]{2}\left[cos\left(\frac{-45^0+360^0k}{2}\right) + i\ sin\left(\frac{-45^0+360^0k}{2}\right)\right]$

Now $k = 0,1$ [$\because n = 2\ \ \therefore k\ takes\ on\ 0\ and\ 1$]

For $k = 0$

$Root\ 1 = \sqrt[4]{2}\left[cos\left(\frac{-45^0+360^0\times0}{2}\right) + i\ sin\left(\frac{-45^0+360^0\times0}{2}\right)\right]$

$= 1.099 - 0.455i$

For $k = 1$

$Root\ 2 = \sqrt[4]{2}\left[cos\left(\frac{-45^0+360^0\times1}{2}\right) + i\ sin\left(\frac{-45^0+360^0\times1}{2}\right)\right]$

$= -1.099 + 0.455i$

$\therefore \sqrt{z_3} = \pm(1.099 - 0.455i)$

d) $\sqrt[3]{z_1} = z_1{}^{\frac{1}{3}} = \left[2\left(cos(60^0) + i\ sin(60^0)\right)\right]^{\frac{1}{3}}$

$= \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0k}{3}\right) + i\ sin\left(\frac{60^0+360^0k}{3}\right)\right]$

Now $k = 0,1,2$ [$\because n = 3\ \ \therefore k\ takes\ on\ 0\ ,1\ and\ 2$]

For $k = 0$

$Root\ 1 = \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0\times0}{3}\right) + i\ sin\left(\frac{60^0+360^0\times0}{3}\right)\right]$

$= 1.184 + 0.431i$

For $k = 1$

$Root\ 2 = \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0\times1}{3}\right) + i\ sin\left(\frac{60^0+360^0\times1}{3}\right)\right]$

$= -0.965 + 0.810i$

For $k = 2$

$Root\ 3 = \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0\times2}{3}\right) + i\ sin\left(\frac{60^0+360^0\times2}{3}\right)\right]$

$= -0.219 - 1.241i$

$\therefore \sqrt[3]{z_1} = 1.184 + 0.431i\ ; -0.965 + 0.810i\ ; -0.219 - 1.241i$

e) $\sqrt[3]{z_3} = z_3{}^{\frac{1}{3}} = \left[\sqrt{2}\left(cos(-45^0) + i\ sin(-45^0)\right)\right]^{\frac{1}{3}}$

$= \sqrt[6]{2}\left[cos\left(\frac{-45^0+360^0k}{3}\right) + i\ sin\left(\frac{-45^0+360^0k}{3}\right)\right]$

Now $k = 0,1,2$ [$\because n = 3\ \ \therefore k\ takes\ on\ 0\ ,1\ and\ 2$]

For $k = 0$

$Root\ 1 = \sqrt[6]{2}\left[cos\left(\frac{-45^0+360^0\times0}{3}\right) + i\ sin\left(\frac{-45^0+360^0\times0}{3}\right)\right]$

$= 1.084 - 0.291i$

For $k = 1$

$Root\ 2 = \sqrt[3]{2}\left[cos\left(\frac{-45^0+360^0\times1}{3}\right) + i\ sin\left(\frac{-45^0+360^0\times1}{3}\right)\right]$

$= -0.291 + 1.084i$

For $k = 2$

$Root\ 3 = \sqrt[3]{2}\left[cos\left(\frac{-45^0+360^0\times 2}{3}\right) + i\ sin\left(\frac{-45^0+360^0\times 2}{3}\right)\right]$

$= -0.794 - 0.794i$

$\therefore \sqrt[3]{z_1} = 1.084 - 0.291i\ ; -0.291 + 1.084i\ ; -0.794 - 0.794i$

f) $\sqrt[4]{z_1} = \sqrt[4]{z_1} = {z_1}^{\frac{1}{4}} = \left[2\left(cos(60^0) + i\ sin(60^0)\right)\right]^{\frac{1}{4}}$

$= \sqrt[4]{2}\left[cos\left(\frac{60^0+360^0 k}{4}\right) + i\ sin\left(\frac{60^0+360^0 k}{4}\right)\right]$

Now $k = 0,1,2\ ,3$ [$\because n = 4\ \therefore k\ takes\ on\ 0\ ,1\ ,2\ and\ 3$]

For $k = 0$

$Root\ 1 = \sqrt[4]{2}\left[cos\left(\frac{60^0+360^0\times 0}{4}\right) + i\ sin\left(\frac{60^0+360^0\times 0}{4}\right)\right]$

$= 1.149 + 0.308i$

For $k = 1$

$Root\ 2 = \sqrt[4]{2}\left[cos\left(\frac{60^0+360^0\times 1}{3}\right) + i\ sin\left(\frac{60^0+360^0\times 1}{3}\right)\right]$

$= -0.308 + 1.149i$

For $k = 2$

$Root\ 3 = \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0\times 2}{3}\right) + i\ sin\left(\frac{60^0+360^0\times 2}{3}\right)\right]$

$= -1.149 - 0.308i$

$Root\ 3 = \sqrt[3]{2}\left[cos\left(\frac{60^0+360^0\times 2}{3}\right) + i\ sin\left(\frac{60^0+360^0\times 2}{3}\right)\right]$

$= 0.308 - 1.149i$

$\therefore \sqrt[3]{z_1} = 1.149 + 0.308i\ ; -0.308 + 1.149i\ ;\ -1.149 - 0.308i; 0.308 - 1.149i$

Task

Find the fourth root of:

a. $5 + 12i$

b. $-4 + 3i$

c. $4 - 3i$

ANSWER:

EXPRESSING POWERED COMPLEX NMBER IN POLAR FORM

The rules of modulus and argument of complex numbers are employed.

Example

1) *Express the following in polar form:*

a. $\frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3}$

b. $\frac{(1-i)^2(3+4i)^3}{(-5+12i)^2}$

c. $\frac{1+2i}{1-(1-i)^2}$

d. $\frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}}$

Solution

a. let $z = \frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3} = r(cos\theta + i\ sin\theta)$

$r = |z| = \left|\frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3}\right| = \frac{|(1+i)^2||1+i\sqrt{3}|}{\left|(1-i\sqrt{3})^3\right|}$

$= \frac{|1+i|^2|1+i\sqrt{3}|}{|1-i\sqrt{3}|^3} = \frac{(\sqrt{1+1})^2(\sqrt{1+3})}{(\sqrt{1+3})^3} = \frac{2\times 2}{2^3} = \frac{1}{2}$

$\theta = arg(z) = arg\left(\frac{(1+i)^2(1+i\sqrt{3})}{(1-i\sqrt{3})^3}\right)$

$= arg(1+i)^2.(1+i\sqrt{3}) - arg(1-i\sqrt{3})^3$

$\left[\because arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)\right]$

$= 2arg(1+i) + arg(1+i\sqrt{3}) - 3arg(1-i\sqrt{3})$

$[\because Arg(z^n) = nArg(z)]$

$= 2\times 45^0 + 60^0 - 3\times(-60^0) = 330^0 \; or \; -30^0$ [***Principal angle***]

$\therefore z = \frac{1}{2}(cos(-30^0) + i\, sin(-30^0))$

b. let $z = \frac{(1-i)^2(3+4i)^3}{(-5+12i)^2} = r(cos\theta + i\, sin\theta)$

$r = |z| = \left|\frac{(1-i)^2(3+4i)^3}{(-5+12i)^2}\right| = \frac{|(1-i)^2||(3+4i)^3|}{|(-5+12i)^2|}$

$= \frac{|1-i|^2|3+4i|^3}{|-5+12i|^2} = \frac{\left(\sqrt{1+1}\right)^2\left(\sqrt{9+16}\right)^3}{\left(\sqrt{25+144}\right)^2} = \frac{2\times 5^3}{13^2} = \frac{250}{169}$

$\theta = arg(z) = arg\left(\frac{(1-i)^2(3+4i)^3}{(-5+12i)^2}\right)$

$= arg(1-i)^2.(3+4i)^3 - arg(-5+12i)^2$

$= 2arg(1-i) + 3arg(3+4i) - 2arg(-5+12i)$

$= 2\times -45^0 + 3\times 53.1^0 - 2\times 112.6^0 = -155.9^0$ [***Principal angle***]

$\therefore z = \frac{250}{169}(cos(-155.9^0) + i\, sin(-155.9^0))$

c. let $z = \frac{1+2i}{1-(1-i)^2} = \frac{1+2i}{1-(1-2i+i^2)} = \frac{1+2i}{1+2i} = 1 + 0i = r(cos\theta + i\, sin\theta)$

$r = |z| = |1+0i| = \sqrt{1+0} = 1$

$\theta = arg(z) = 0$

$\therefore z = 1(cos0 + i\, sin0)$

d. let $z = \frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}} = r(cos\theta + i\, sin\theta)$

$r = |z| = \left|\frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}}\right| = \frac{\left|(1+i\sqrt{3})^{12}\right|}{\left|(\sqrt{3}-i)^{11}\right|} = \frac{|1+i\sqrt{3}|^{12}}{|\sqrt{3}-i|^{11}}$

$= \frac{(\sqrt{1+3})^{12}}{(\sqrt{3+1})^{11}} = \frac{2^{12}}{2^{11}} = 2$

$\theta = arg(z) = arg\left(\frac{(1+i\sqrt{3})^{12}}{(\sqrt{3}-i)^{11}}\right)$

$= arg(1+i\sqrt{3})^{12} - arg(\sqrt{3}-i)^{11}$

$= 12arg(1+i\sqrt{3}) - 11arg(\sqrt{3}-i)$

$= 12\times 60^0 - 11\times -30^0 = 1050^0 or -30^0$ [***Principal angle***]

$\therefore z = 2(cos(-30^0) + i\, sin(-30^0))$

STATING THE PRINCIPAL ANGLE OF A COMPLEX NUMBER

The interval of a principal angle is $-\pi < \theta \leq \pi \; or \; -180^0 < \theta \leq 180^0$

The knowledge of trigonometry of angles in quadrants is required.

Definition

The *principal angle* is the angle whose complex number (number) is nearest to the *positive Real axis.*

Example

1) *Given that* $z_1 = \sqrt{3} - i$, $z_2 = -1 + i$ *and* $z_3 = 1 - i\sqrt{3}$. *Find the principal argument of;*

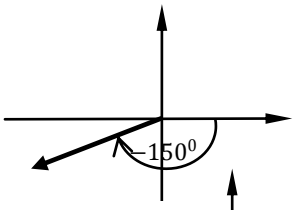
   a) $z_1{}^5$

b) $z_2{}^7$ c) $z_3{}^8$

Solution

a) $arg(z_1{}^5) = 5arg(z_1)$

$= 5 \times -30^0 = -150^0$

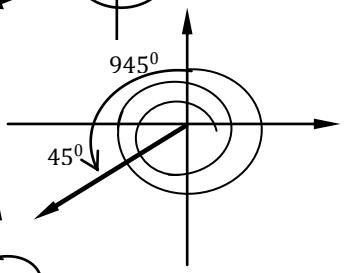


b) $arg(z_2{}^7) = 7arg(Z_2)$

$= 7 \times 135^0 = 945^0$

But $tan945^0 = -tan135^0$

$\therefore arg(z_2{}^7) = -135^0$

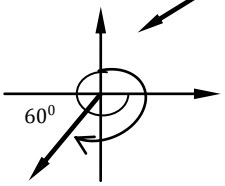


c) $arg(z_3{}^8) = 8arg(Z_3)$

But $arg(z_3) = -60^0$

$\therefore arg(z_3{}^8) = -60^0 \times 8 = -480^0$

$\therefore arg(z_3{}^8) = -120^0$



2) *Given that* $z_1 = 1 + i, z_2 = -1 + i$ *and* $z_3 = -1 - i$. *Find the principal argument of;*

c) $z_1{}^5$

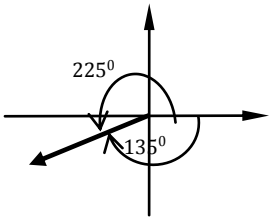
d) $z_2{}^8$ c) $z_3{}^6$

Solution

a) $arg(z_1{}^5) = 5arg(z_1)$

$= 5 \times 45^0 = 225^0$

$\therefore arg(z_1{}^5) = -135^0$

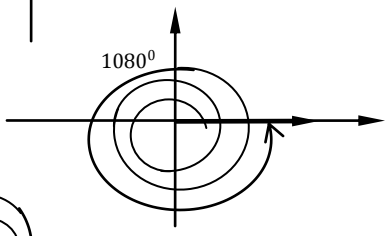


b) $arg(z_2{}^8) = 8arg(Z_3)$

But $arg(z_3) = 135^0$

$\therefore arg(z_3{}^8) = 135^0 \times 8 = 1080^0$

$\therefore arg(z_3{}^8) = 0^0$

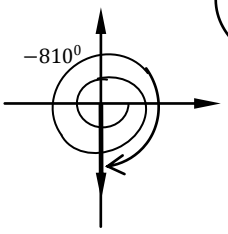


c) $arg(z_3{}^6) = 6arg(Z_3)$

But $arg(z_3) = -135^0$

$\therefore arg(z_3{}^8) = -135^0 \times 6 = -810^0$

$\therefore arg(z_3{}^8) = -90^0$



Task

Given that $z_1 = 5 + 12i, z_2 = -2 + i$ *and* $z_3 = 8 - i4$. *Find the principal argument of;*

a) $z_1{}^3$

b) $z_2{}^5$ c) $z_3{}^8$

Answer: $(\boldsymbol{a})\ \mathbf{-157.9^0}\ (\boldsymbol{b})\ \mathbf{-47.2^0}\quad (\boldsymbol{c})\ \mathbf{-65^0}$

FURTHER EXAMPLES ON PROOFS BY DEMOIVRE'S THEOREM

The knowledge of trigonometry is applicable.

Example

1. *If* $z_1 = cos\alpha + i\ sin\alpha, z_2 = cos\beta + i\ sin\beta$, *show that* $\frac{1}{2i}\left[\frac{z_1}{z_2} - \frac{z_2}{z_1}\right] = sin(\alpha - \beta)$

Solution

$\frac{z_1}{z_2} = \frac{cos\alpha + i\, sin\alpha}{cos\beta + i\, sin\beta} = cos(\alpha - \beta) + i\, sin(\alpha - \beta)$

$\frac{z_2}{z_1} = \frac{cos\beta + i\, sin\beta}{cos\alpha + i\, sin\alpha} = cos(\beta - \alpha) + i\, sin(\beta - \alpha)$

$\therefore \frac{z_1}{z_2} - \frac{z_2}{z_1} = cos(\alpha - \beta) + i\, sin(\alpha - \beta) - [cos(\beta - \alpha) + i\, sin(\beta - \alpha)\,]$

$= cos(\alpha - \beta) + i\, sin(\alpha - \beta) - cos(\beta - \alpha) - i\, sin(\beta - \alpha)$

$= cos(\alpha - \beta) + i\, sin(\alpha - \beta) - cos[-(\alpha - \beta)] - i\, sin[-(\alpha - \beta)]$

$= cos(\alpha - \beta) + i\, sin(\alpha - \beta) - cos(\alpha - \beta) + i\, sin(\alpha - \beta)$

$\therefore \frac{z_1}{z_2} - \frac{z_2}{z_1} = 2i\, sin(\alpha - \beta)$

$\therefore \frac{1}{2i}\left[\frac{z_1}{z_2} - \frac{z_2}{z_1}\right] = \frac{1}{2i} \times 2i\, sin(\alpha - \beta) = sin(\alpha - \beta)$

2. *If* $z = cos\theta + i\, sin\theta$ *, prove that* $\frac{2}{1+z} = 1 - i\, tan\frac{\theta}{2}$ *and* $\frac{1+z}{1-z} = i\, cot\frac{\theta}{2}$

Solution

$\frac{2}{1+z} = \frac{2}{1 + cos\theta + i\, sin\theta} = \frac{2}{2cos^2\frac{\theta}{2} + i\, 2sin\frac{\theta}{2}cos\frac{\theta}{2}}$

$= \frac{1}{cos\frac{\theta}{2}\left(cos\frac{\theta}{2} + i\, sin\frac{\theta}{2}\right)} = \frac{1}{cos\frac{\theta}{2}} . \left(cos\frac{\theta}{2} + i\, sin\frac{\theta}{2}\right)^{-1}$

$= \frac{1}{cos\frac{\theta}{2}} . cos\left(-\frac{\theta}{2}\right) + i\, sin\left(-\frac{\theta}{2}\right) = \frac{1}{cos\frac{\theta}{2}} . \left[cos\frac{\theta}{2} - i\, sin\frac{\theta}{2}\right]$

$\therefore \frac{2}{1+z} = 1 - i\frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}} = 1 - i\, tan\frac{\theta}{2}$

$\frac{1+z}{1-z} = \frac{1 + cos\theta + i\, sin\theta}{1 - cos\theta - i\, sin\theta} = \frac{(1 + cos\theta) + i\, sin\theta}{(1 - cos\theta) - i\, sin\theta}$

$= \frac{2cos^2\frac{\theta}{2} + i\, 2sin\frac{\theta}{2}cos\frac{\theta}{2}}{2sin^2\frac{\theta}{2} - i\, 2sin\frac{\theta}{2}cos\frac{\theta}{2}} = \frac{2cos\frac{\theta}{2}\left[cos\frac{\theta}{2} + i\, sin\frac{\theta}{2}\right]}{2sin\frac{\theta}{2}\left[sin\frac{\theta}{2} - i\, cos\frac{\theta}{2}\right]}$

Writing $sin\frac{\theta}{2} - i\, cos\frac{\theta}{2}$ in the form $cos\varphi + i\, sin\varphi$

Recall: $cos\left(90^0 - \frac{\theta}{2}\right) = sin\frac{\theta}{2}$ *and* $sin\left(90^0 - \frac{\theta}{2}\right) = cos\frac{\theta}{2}$

$= cot\frac{\theta}{2} . \frac{\left[cos\frac{\theta}{2} + i\, sin\frac{\theta}{2}\right]}{\left[cos\left(90^0 - \frac{\theta}{2}\right) - i\, sin\left(90^0 - \frac{\theta}{2}\right)\right]}$

$= cot\frac{\theta}{2} . \frac{\left[cos\frac{\theta}{2} + i\, sin\frac{\theta}{2}\right]}{\left[cos\left\{-\left(90^0 - \frac{\theta}{2}\right)\right\} - i\, sin\left\{-\left(90^0 - \frac{\theta}{2}\right)\right\}\right]}$

$= cot\frac{\theta}{2} . \left[cos\frac{\theta}{2} + i\, sin\frac{\theta}{2}\right] . \left[cos\left(\frac{\theta}{2} - 90^0\right) + i\, sin\left(\frac{\theta}{2} - 90^0\right)\right]^{-1}$

$= cot\frac{\theta}{2} . \left[cos\left(\frac{\theta}{2}\right) + i\, sin\left(\frac{\theta}{2}\right)\right] . \left[cos\left\{-\left(\frac{\theta}{2} - 90^0\right)\right\} + i\, sin\left\{-\left(\frac{\theta}{2} - 90^0\right)\right\}\right]$

$= cot\frac{\theta}{2} . \left[cos\left(\frac{\theta}{2}\right) + i\, sin\left(\frac{\theta}{2}\right)\right] . \left[cos\left(90^0 - \frac{\theta}{2}\right) + i\, sin\left(90^0 - \frac{\theta}{2}\right)\right]$

$= cot\frac{\theta}{2} . \left[cos\left(\frac{\theta}{2} + 90^0 - \frac{\theta}{2}\right) + i\, sin\left(\frac{\theta}{2} + 90^0 - \frac{\theta}{2}\right)\right]$

$= cot\frac{\theta}{2} . [cos(90^0) + i\, sin(90^0)]$

$\therefore \frac{1+z}{1-z} = i\, cot\frac{\theta}{2}$

3. *Prove that* $(sin\theta + i\, cos\theta)^n = cos\left(\frac{n\pi}{2} - n\theta\right) + i\, sin\left(\frac{n\pi}{2} - n\theta\right)$

Solution

$(sin\theta + i\, cos\theta)^n = ?$

$sin\theta = cos\left(\frac{\pi}{2} - \theta\right)$ *and* $cos\theta = sin\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow sin\theta + i\, cos\theta = cos\left(\frac{\pi}{2} - \theta\right) + i\, sin\left(\frac{\pi}{2} - \theta\right)$

$\therefore (sin\theta + i\,cos\theta)^n = \left[cos\left(\frac{\pi}{2} - \theta\right) + i\,sin\left(\frac{\pi}{2} - \theta\right)\right]^n$

$= cos\left\{n\left(\frac{\pi}{2} - \theta\right)\right\} + i\,sin\left\{n\left(\frac{\pi}{2} - \theta\right)\right\}$

$\therefore (sin\theta + i\,cos\theta)^n = cos\left(\frac{n\pi}{2} - n\theta\right) + i\,sin\left(\frac{n\pi}{2} - n\theta\right)$

4. (a) *Find the modulus and argument of* $(a + ib)^2$ *where* a *and* b *are real numbers. Hence show that* $tan^{-1}\left(\frac{2ab}{a^2-b^2}\right) = 2\,tan^{-1}\left(\frac{b}{a}\right)$

(b) *Two complex numbers* z_1 *and* z_2 *are such that* $|z_1 - z_2| = |z_1 + z_2|$.

Show that $|\theta_1 - \theta_2| = \frac{\pi}{2}$, *where* θ_1 *and* θ_2 *are arguments of* z_1 *and* z_2 *respectively.*

Solution

a) let $z = (a + ib)^2$

$|z| = |a + ib|^2 = \left[\sqrt{a^2 + b^2}\right]^2$

$\therefore |z| = a^2 + b^2$

$arg(z) = arg(a + ib)^2 = 2arg(a + i\,b) = 2\,tan^{-1}\left(\frac{b}{a}\right)$

Also $z = (a + ib)^2 = a^2 + i2ab + i^2b^2$

$= a^2 - b^2 + i2ab$

$\Rightarrow arg(z) = tan^{-1}\left(\frac{2ab}{a^2-b^2}\right) = 2\,tan^{-1}\left(\frac{b}{a}\right)$

b) $|\theta_1 - \theta_2| = \frac{\pi}{2}$; $|z_1 - z_2| = |z_1 + z_2|$

Let $z_1 = r_1(cos\theta_1 + i\,sin\theta_1)$, $z_2 = r_2(cos\theta_2 + i\,sin\theta_2)$

$z_1 - z_2 = r_1(cos\theta_1 + i\,sin\theta_1) - r_2(cos\theta_2 + i\,sin\theta_2)$

$= r_1cos\theta_1 + i\,r_1sin\theta_1 - r_2cos\theta_2 - r_2i\,sin\theta_2$

$= (r_1cos\theta_1 - r_2cos\theta_2) + i\,(r_1sin\theta_1 - r_2sin\theta_2)$

$z_1 + z_2 = r_1(cos\theta_1 + i\,sin\theta_1) + r_2(cos\theta_2 + i\,sin\theta_2)$

$= r_1cos\theta_1 + i\,r_1sin\theta_1 + r_2cos\theta_2 + r_2i\,sin\theta_2$

$= (r_1cos\theta_1 + r_2cos\theta_2) + i\,(r_1sin\theta_1 + r_2sin\theta_2)$

$|z_1 - z_2| = \sqrt{[(r_1cos\theta_1 - r_2cos\theta_2)]^2 + [(r_1sin\theta_1 - r_2sin\theta_2)]^2}$

$= \sqrt{\begin{matrix} r_1{}^2cos^2\theta_1 - r_1cos\theta_1r_2cos\theta_2 + r_2{}^2cos^2\theta_2 \\ +r_1{}^2sin^2\theta_1 - r_1sin\theta_1r_2sin\theta_2 + r_2{}^2sin^2\theta_2 \end{matrix}}$

$= \sqrt{\begin{matrix} r_1{}^2(cos^2\theta_1 + sin^2\theta_1) - 2r_1r_2(cos\theta_1cos\theta_2 + sin\theta_1sin\theta_2) \\ +r_2{}^2(cos^2\theta_2 + sin^2\theta_2) \end{matrix}}$

$\therefore |z_1 - z_2| = \sqrt{r_1{}^2 + r_2{}^2 - 2r_1r_2cos(\theta_1 - \theta_2)}$

$|z_1 + z_2| = \sqrt{[(r_1cos\theta_1 + r_2cos\theta_2)]^2 + [(r_1sin\theta_1 + r_2sin\theta_2)]^2}$

$= \sqrt{\begin{matrix} r_1{}^2cos^2\theta_1 + 2r_1cos\theta_1r_2cos\theta_2 + r_2{}^2cos^2\theta_2 \\ +r_1{}^2sin^2\theta_1 + 2r_1sin\theta_1r_2sin\theta_2 + r_2{}^2sin^2\theta_2 \end{matrix}}$

$= \sqrt{\begin{matrix} r_1{}^2(cos^2\theta_1 + sin^2\theta_1) + 2r_1r_2(cos\theta_1cos\theta_2 + sin\theta_1sin\theta_2) \\ +r_2{}^2(cos^2\theta_2 + sin^2\theta_2) \end{matrix}}$

$\therefore |z_1 + z_2| = \sqrt{r_1{}^2 + r_2{}^2 + 2r_1r_2cos(\theta_1 - \theta_2)}$

Now $|z_1 - z_2| = |z_1 + z_2|$; squaring both sides;

$r_1{}^2 + r_2{}^2 - 2r_1r_2cos(\theta_1 - \theta_2) = r_1{}^2 + r_2{}^2 + 2r_1r_2cos(\theta_1 - \theta_2)$

Simplifying

$-cos(\theta_1 - \theta_2) = cos(\theta_1 - \theta_2)$ *or* $2cos(\theta_1 - \theta_2) = 0$
$cos(\theta_1 - \theta_2) = 0 \Rightarrow \theta_1 - \theta_2 = cos^{-1}(0) = \frac{\pi}{2}$
$\therefore \theta_1 - \theta_2 = \frac{\pi}{2}$ *or* $|\theta_1 - \theta_2| = \frac{\pi}{2}$

Task

1. *If* $z = 1 + cos2\theta + i\, sin2\theta$, *where* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. *Prove that* $|z| = 2cos\theta$ *and* $arg(z) = \theta$
2. *If* $x = cos\alpha + i\, sin\alpha, y = cos\beta + i\, sin\beta, z = cos\lambda + i\, sin\lambda$ *and* $x + y + z = 0$, *prove that* $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$
3. *If* $arg\left(\frac{z_1 + z_2}{z_1 - z_2}\right) = \frac{\pi}{2}$, *show that* $|z_1| = |z_2|$

SOLVING EQUATIONS IN COMPLEX NUMBERS

a) **Equations involving equality of complex numbers**

Equality of complex numbers occurs if:

i. If $a + i\, b = 0, then\ a = 0\ ,\ b = 0$
ii. If $a + i\, b = c + i\, d, then\ a = c\ ,\ b = d$

Example

1) *Solve the simultaneous equations;*
$z_1 + z_2 = 8, 4z_1 - i3\, z_2 = 26 + 8\, i$

Solution

Let $z_1 = x_1 + iy_1\ ,\ z_2 = x_2 + iy_2$
$\Rightarrow z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = 8 + 0i$
$\therefore x_1 + x_2 = 8$ ..... (1)
$y_1 + y_2 = 0$ ..... (2)
$4z_1 - i3\, z_2 = 26 + 8\, i$
$\Rightarrow 4(x_1 + iy_1) - i3(x_2 + iy_2) = 26 + i\, 8$
$4x_1 + i4y_1 - i3x_2 - i^2 3y_2 = 26 + i\, 8$
$(4x_1 + 3y_2) + i(4y_1 - 3x_2) = 26 + i\, 8$
$4x_1 + 3y_2 = 26$ ..... (3)
$4y_1 - 3x_2 = 8$ ..... (4)
From eqn. (1)
$x_2 = 8 - x_1$ ..... (5)
$eqn.(5) into\ (4)$
$4y_1 - 3(8 - x_1) = 8 \Rightarrow 4y_1 + 3x_1 - 24 = 8$
$4y_1 + 3x_1 = 32$ ..... (6)
From eqn. (2)
$y_2 = -y_1$ ..... (7)
$eqn.(7) into\ (3)$
$4x_1 - 3y_1 = 26$ ..... (8)
$Solving\ eqn.(6)\ and\ (8)$
$x_1 = 8, y_1 = 2$
From eqn. (2) and eqn. (5)
$y_2 = -2, x_2 = 0$
$\therefore z_1 = 8 + 2\, i, z_2 = 0 - 2\, i$

Alternatively

$z_1 + z_2 = 8$ ..... (1)

$4z_1 - i3\,z_2 = 26 + 8\,i$ ..... (2)

$eqn.(1) \times 4 - eqn.(2)$

$4z_1 + 4\,z_2 = 32 + 0\,i$

$-$

$4z_1 - i3\,z_2 = 26 + 8\,i$

$(4 + 3\,i)z_2 = 6 - 8\,i$

$\therefore z_2 = \frac{6-8\,i}{4+3\,i} = \frac{(6-8\,i)(4-3\,i)}{(4+3\,i)(4-3\,i)}$

$= \frac{24-18i-32i+i^2 24}{16-(3i)^2} = \frac{0-50\,i}{16+9}$

$\therefore z_2 = 0 - 2\,i$

From eqn. (1)

$z_1 = 8 - z_2 = 8 - -2i = 8 + 2i$

$\therefore z_1 = 8 + 2\,i$

2) *Given that* $(3 + 2z^*)z = 5 + 2z$, *where* z^* *is the conjugate of z. find z.*

Solution

$3z + 2zz^* = 5 + 2z$

$\Rightarrow 2zz^* = 5 - z$

Let $z = x + i\,y \;\Rightarrow z^* = x - i\,y$

$\therefore 2(x + i\,y)(x - i\,y) = 5 - (x + i\,y)$

$\Rightarrow 2(x^2 - (i\,y)^2) = (5 - x) - i\,y$

$\Rightarrow 2x^2 + 2y^2 = (5 - x) - i\,y$

Equating Real parts and imaginary parts

$\therefore 2x^2 + 2y^2 = 5 - x$ ..... (1)

$\therefore 0 = -y \Rightarrow y = 0$

$For\ y = o, x = ?$

$\Rightarrow 2x^2 = 5 - x \;\; or\; 2x^2 + x - 5 = 0$

$\therefore x = 1.351\,, -1.851$

$\therefore z = 1.351 + 0\,i\,;\; -1.851 + 0\,i$

3) *Given that z is a complex number such that* $z = \frac{p}{2-i} + \frac{q}{1+3i}$ *, where p and q are real.*

Given that $arg(z) = \frac{\pi}{2}$ *and* $|z| = 7$, *find p and q.*

Solution

$z = \frac{p}{2-i} + \frac{q}{1+3i} = \; z = \frac{p(2+i)}{(2-i)(2+i)} + \frac{q(1-3i)}{(1+3i)(1-3i)}$

$= \frac{2p+2i}{4-i^2} + \frac{q-3qi}{1-i^2 3^2} = \frac{2p}{5} + \frac{2}{5}i + \frac{q}{10} - \frac{3q}{10}i$

$\therefore z = \left(\frac{2p}{5} + \frac{q}{10}\right) + i\left(\frac{2}{5} - \frac{3q}{10}\right)$

$arg(z) = tan^{-1}\left(\frac{\left(\frac{2}{5} - \frac{3q}{10}\right)}{\left(\frac{2p}{5} + \frac{q}{10}\right)}\right) = tan^{-1}\left(\frac{2p-3q}{4p+q}\right) = \frac{\pi}{2}$

$\therefore \frac{2p-3q}{4p+q} = \tan\frac{\pi}{2} = \infty$

$\Rightarrow 4p + q = 0$ ..... (1)

$|z| = 7 = \sqrt{\left(\frac{2p}{5} + \frac{q}{10}\right)^2 + \left(\frac{2}{5} - \frac{3q}{10}\right)^2}$

Squaring both sides

$\Rightarrow 49 = \frac{(4p+q)^2 + (2p-3q)^2}{100}$

$\therefore 4900 = (4p+q)^2 + (2p-3q)^2$ ..... (2)

From eqn. (1)

$q = -4p$ ..... (3)

eqn. (3) *in to* (2)

$\Rightarrow 4900 = (4p-4p)^2 + \big(2p-3(-4p)\big)^2$

$\Rightarrow 4900 = (14p)^2$

Finding square root on both sides

$\pm 70 = 14p \ \therefore p = \pm 5$

For $p = \pm 5$, q =?

From eqn. (1)

$q = -4p = \pm 5 \times (-4) = \pm 20$

$\therefore p = 5. q = -20; p = -5, q = 20$

4) *Given that* $u = -10 + i\,9$ *as a complex number, Find the complex number* v *which satisfies the equation* $uv = -11 + i\,28$. *Hence verify that* $|u+v| = 8\sqrt{2}$.

Solution

$uv = -11 + i\,28 \Rightarrow v = \frac{-11+i\,28}{u} = \frac{-11+i\,28}{-10+i\,9}$

$= \frac{(-11+i\,28)(-10-i\,9)}{(-10+i\,9)(-10-i\,9)}$

$= \frac{110+i\,99-i\,280-i^2 252}{(-10)^2-(i9)^2} = \frac{110-i\,181+252}{100+81}$

$= \frac{362}{181} - i\frac{181}{181} = 2 - i \ \therefore v = 2 - i$

$|u+v| = |-1 + i\,9 + 2 - i| = 8\sqrt{2}$

$= |-8 + 8i| = \sqrt{8^2+8^2} = 8\sqrt{2}$

5) *Find* z *if* $arg(z+1) = \frac{\pi}{4}$ *and* $arg(z-1) = \frac{2\pi}{3}$

Solution

Let $z = x + i\,y$, $z + 1 = x + 1 + i\,y$

$\Rightarrow arg(z+1) = tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{4}$

$\Rightarrow tan\frac{\pi}{4} = \frac{y}{x+1} = 1$

$\therefore y = x + 1$ ..... (1)

$z - 1 = x - 1 + i\,y$

$\Rightarrow arg(z-1) = tan^{-1}\left(\frac{y}{x-1}\right) = \frac{2\pi}{3}$

$\Rightarrow tan\frac{2\pi}{3} = \frac{y}{x-1} = -\sqrt{3}$

$\therefore -\sqrt{3}x + \sqrt{3} = y$ ..... (2)

eqn. (1) = *eqn.* (2)

$x + 1 = -\sqrt{3}x + \sqrt{3}$

$x\big(1+\sqrt{3}\big) + 1 - \sqrt{3} = 0 \ \therefore x = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$x = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$

$= \frac{3-2\sqrt{3}+1}{3-1} = 1 - \sqrt{3} \ \therefore x = 1 - \sqrt{3}$

From eqn. (1)

$y = x + 1 = 1 - \sqrt{3} + 1 = 2 - \sqrt{3}$

$\therefore z = \big(1-\sqrt{3}\big) + i\big(2-\sqrt{3}\big)$

6) *If* $(1+i3)z_1 = 5(1+i)$, *find* z_1

Solution

$(1+i3)z_1 = 5(1+i) \Rightarrow z_1 = \frac{5(1+i)}{(1+i3)}$

$z_1 = \frac{(5+5i)}{(1+i3)} = \frac{(5+5i)(1-i3)}{(1+i3)(1-i3)} = \frac{5-i\,15+i\,5-i^2\,15}{1^2-(i3)^2}$

$= \frac{5+15-10i}{1+9} = \frac{20-i\,10}{10} = 2-i$

$\therefore\ z_1 = 2-i$

7) *Given that x and y are real, find the values of x and y which satisfy the equation*

$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$

Solution

$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$

$\frac{2y}{2x+y} + \frac{4}{2x+y}\ i - \frac{y(x+i)}{(x-i)(x+i)} = 0$

$\frac{2y}{2x+y} + \frac{4}{2x+y}\ i - \frac{xy}{x^2+1} - \frac{y}{x^2+1}\ i = 0$

For equality of complex numbers

For real part: $\frac{2y}{2x+y} - \frac{xy}{x^2+1} = 0$ ..... (1)

For imaginary part: $\frac{4}{2x+y} - \frac{y}{x^2+1} = 0$ ..... (2)

From eqn. (2)

$\frac{4}{2x+y} = \frac{y}{x^2+1}$, substituting in eqn. (1)

$\Rightarrow \frac{2y}{2x+y} - \frac{4}{2x+y}.x = 0$

$\therefore \frac{2y-4x}{2x+y} = 0 \Rightarrow 2y-4x = 0$

$\therefore y = 2x$ ..... (3)

Eqn. (3) in to eqn. (1)

$\Rightarrow \frac{2(2x)}{2x+2x} - \frac{x(2x)}{x^2+1} = 0 \ \therefore 1 - \frac{2x^2}{x^2+1} = 0$

$\Rightarrow x^2 - 2x^2 + 1 = 0 \ or\ x^2 = 1$

$\therefore x = \pm 1$

From eqn. (3)

$y = 2(\pm 1) = \pm 2$

$\therefore x = 1, y = 2; x = -1, y = -2$

8) Find the values of x and y in the equation

$\frac{x}{2+i\,3} - \frac{y}{3-i\,2} = \frac{6+2\,i}{1+i\,8}$ $(\boldsymbol{Uneb\ 2006})$

Solution

Expressing complex numbers in the form $a+ib$

$\frac{x}{2+i\,3} - \frac{y}{3-i\,2} = \frac{6+2\,i}{1+i\,8}$

$\frac{x(2-i\,3)}{(2+i\,3)(2-i\,3)} - \frac{y(3+i\,2)}{(3-i\,2)(3+i\,2)} = \frac{(6+2\,i)(1-i\,8)}{(1+i\,8)(1-i\,8)}$

$\frac{2x-i\,3x}{4+9} - \frac{3y+i\,2y}{9+4} = \frac{6-i\,48+i\,2-i^2\,16}{1+64}$

$\frac{2x-i\,3x}{13} - \frac{3y+i\,2y}{13} = \frac{6+16-i\,46}{65}$

$\Rightarrow \frac{2x}{13} - i\frac{3x}{13} - \frac{3y}{13} - i\frac{2y}{13} = \frac{22}{65} - i\frac{46}{65}$

Equating real parts and imaginary parts

For real part: $\Rightarrow \frac{2x}{13} - \frac{3y}{13} = \frac{22}{65} \ldots\ldots\ldots\ldots\ldots\ldots \times 65$

$\therefore 10x - 15y = 22$ ..... (1)

For imaginary part: $-\frac{3x}{13} - \frac{2y}{13} = -\frac{46}{65} \ldots\ldots\ldots\ldots \times 65$

$\therefore 15x + 10y = 46$ ..... (2)

Solving eqn. (1) and eqn. (2) simultaneously

$x = 2.8\,, y = 0.4$

Alternatively

Making the equation linear

$\frac{x}{2+i\,3} - \frac{y}{3-i\,2} = \frac{6+2\,i}{1+i\,8}$

$\Rightarrow \frac{x(3-i\,2)-y(2+i\,3)}{(2+i\,3)(3-i\,2)} = \frac{6+2\,i}{1+i\,8}$

$\Rightarrow \frac{3x-i\,2x-2y-i\,3y}{6-i\,4+i\,9-i^2\,6} = \frac{6+2\,i}{1+i\,8}$

$\Rightarrow \frac{(3x-2y)+i\,(-2x-3y)}{12+i\,5} = \frac{(6+2\,i)}{(1+i\,8)} \ldots\ldots\ldots\ldots \times (12 + i\,5)$

$\Rightarrow (3x - 2y) + i\,(-2x - 3y) = \frac{(6+2\,i)(12+i\,5)}{(1+i\,8)}$

$= \frac{72+i\,30+i\,24+i^2\,10}{1+i8}$

$\frac{72+i\,30+i\,24+i^2\,10}{1+i8} = \frac{62+i\,54}{1+8\,i} = \frac{(62+i\,54)(1-i\,8)}{(1+8\,i)(1-i\,8)}$

$= \frac{62+i\,54-i\,496-i^2\,432}{1+64}$

$= \frac{494}{65} - i\frac{442}{65} = \frac{38}{5} - i\frac{34}{5}$

$\therefore (3x - 2y) = \frac{38}{5}$ *or* $15x - 10y = 38$ ..... (1)

$(-2x - 3y) = -\frac{38}{5}$ *or* $10x + 15y = 34$ ..... (2)

Solving eqn. (1) and eqn. (2) simultaneously

$\therefore x = 2.8\,, y = 0.4$

Task

Given that x and y are real, find the values of x and y which satisfy the equation

a) $\frac{x+4\,i}{2x+y} - \frac{x+i}{y} = 0$

b) $\frac{3z}{1-i} + \frac{3z}{i} = \frac{4}{3-i}$

***Answer*: $(b)\ x = 0.27\,, y = 0.53$**

ROOTS OF A GIVEN EQUATION

It can be proved that if $A + iB = x$ is a root of a quadratic equation $ax^2 + bx + c = 0$, then its conjugate is also a root.

Formulation of a quadratic equation with two given roots

The general formula used is:

$x^2 - (sum\ of\ the\ roots)x + product\ of\ roots = 0$

Example

1) *Find the equation whose one root is;*

(i) $2 - 3i$ (ii) $1 + i$ (iii) $3 + 4i$

Solution

i) Since $x = 2 - 3i$ is a root, then the other root $\bar{x} = 2 + 3i$ is also a root.

Sum of the roots: $2-3i+2+3i=4$
Product of roots: $(2-3i)(2+3i)=(2)^2-(i3)^2=4+9=13$
$\therefore x^2-4x+13=0$

ii) $x=1+i \Rightarrow x=1-i$ is also a root.
Sum of the roots: $1+i+1-i=2$
Product of roots: $(1+i)(1-i)=(1)^2-(i)^2=1+1=2$
$\therefore x^2-2x+2=0$

iii) $x=3+4i \Rightarrow x=3-4i$ is also a root.
Sum of the roots: $3+4i+3-4i=6$
Product of roots: $(3+4i)(3-4i)=(3)^2-(i4)^2=9+16=25$
$\therefore x^2-6x+25=0$

2) *Show that* $z=1$ *is the root of the equation* $z^3-5z^2+7z-5=0$. *Hence solve the equation.*(***Uneb* 2003**)

Solution

Let $f(z)=z^3-5z^2+7z-5=0$
For $z=1$
$\Rightarrow f(1)=(1)^3-5(1)^2+7(1)-5=0$. Hence $z=1$ is a root. Thus $(z-1)$ is a root of $f(z)$. Factorizing $f(z)$

$$
\begin{array}{r|l}
z-1 & z^2-4z+5 \\
\hline
 & z^3-5z^2+7z-5 \\
 & z^3-z^2 \\
\hline
 & -4z^2+9z-5 \\
 & -4z^2+4z \\
\hline
 & 5z-5 \\
 & 5z-5 \\
\hline
\end{array}
$$

$\therefore f(z)=(z-1)(z^2-4z+5)=0$
$z^2-4z+5=0$
$z=\frac{4\pm\sqrt{16-4(5)}}{2}=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm2i}{2}=2\pm i$
$\therefore z=2\pm i$

3) *Show that* $z=2+3i$ *is a root of the equation* $z^4-5z^3+18z^2-17z+13=0$

Solution

$z=2+3i$
$z^2=(2+3i)^2=4+12i+i^2 9=-5+12i$
$z^3=(-5+12i)(2+3i)=-10-15i+24i+i^2 36$
$\quad=-10+9i-36=-46+9i$
$z^4=(-5+12i)^2=25-120i+i^2 144=-119-120i$
Let $f(z)=z^4-5z^3+18z^2-17z+13=0$
$f(z)=-119-120i-5(-46+9i)+18(-5+12i)-17(2+3i)+13$
$\quad=(-119+230-90-34+13)+i(-120-45+216-51)=0$
Hence $z=2+3i$ is a root of the equation.
Since $2+3i$ is a root of the equation, also $2-3i$ is a root.
Quadratic factor in $f(z)=z^4-5z^3+18z^2-17z+13$:
Sum of roots: $2+3i+2-3i=4$

Product of roots: $(2+3i)(2-3i) = (2)^2 - (3i)^2 = 4+9 = 13$
$\therefore z^2 - 4z + 13$ is a quadratic factor of $f(z)$
Factorizing $f(z)$

$$\begin{array}{r|l} z^2 - 4z - 13 & z^2 - z + 1 \\ & z^4 - 5z^3 + 18z^2 - 17z + 13 \\ & z^4 - 4z^3 + 13z^2 \\ \hline & -z^3 + 5z^2 - 17z + 13 \\ & -z^3 + 4z^2 - 3z \\ \hline & z^2 - 4z + 13 \\ & z^2 - 4z + 13 \\ \hline \end{array}$$

$\therefore f(z) = (z^2 - 4z + 13)(z^2 - z + 1) = 0$
$z^2 - z + 1 = 0$
$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2} \quad \therefore z = \frac{1 \pm i\sqrt{3}}{2}$

4) *Show that* $z - 1 - i$ *is a factor of the expression* $z^4 - 6z^3 + 23z^2 - 34z + 26$. *Hence solve the equation* $z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$

Solution

If $z - 1 - i$ is a factor then the root of the equation $f(z) = 0$ must be $z - 1 - i = 0$ *or*
$z = 1 + i$. Thus showing that $z = 1 + i$ is a root of $f(z) = 0$.
Let $f(z) = z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$
$z = 1 + i$
$z^2 = (1+i)^2 = 1 + 2i + i^2 = 2i$
$z^3 = (2i)(1+i) = 2i + i^2 2 = -2 + 2i$
$z^4 = (2i)^2 = -4$
From $f(z) = z^4 - 6z^3 + 23z^2 - 34z + 26$
$f(z) = -4 - 6(-2 + 2i) + 23(2i) - 34(1+i) + 26$
$= (-4 + 12 - 34 + 26) + i(-12 + 46 - 34) = 0$
Hence $z = 1 + i$ is a root of the equation hence $z - 1 - i$ is a factor of the expression
$z^4 - 6z^3 + 23z^2 - 34z + 26$
Since $1 + i$ is a root of the equation, also $1 - i$ is a root.
Quadratic factor in $f(z) = z^4 - 6z^3 + 23z^2 - 34z + 26$:
Sum of roots: $1 + i + 1 - i = 2$
Product of roots: $(1+i)(1-i) = (1)^2 - (i)^2 = 1 + 1 = 2$
$\therefore z^2 - 2z + 2$ is a quadratic factor of $f(z)$
Factorizing $f(z)$

$$\begin{array}{r|l} z^2 - 2z + 2 & z^2 - 4z + 13 \\ & z^4 - 6z^3 + 23z^2 - 34z + 26 \\ & z^4 - 2z^3 + 2z^2 \\ \hline & -4z^3 + 21z^2 - 34z + 26 \\ & -4z^3 + 8z^2 - 8z \\ \hline & 13z^2 - 26z + 26 \\ & 13z^2 - 26z + 26 \\ \hline \end{array}$$

$\therefore f(z) = (z^2 - 2z + 2)(z^2 - 4z + 13) = 0$

$z^2 - 4z + 13 = 0$

$z = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm i6}{2} = 2 \pm 3i$

$\therefore z = 2 \pm 3i$

5) *Solve the equation* $z^4 - 6z^2 + 25 = 0$

Solution

$z^4 - 6z^2 + 25 = 0$ is an equation reducable to quadratic of the form $\boldsymbol{au^2 + bu + c = 0}$

Let $z^2 = u$

$\Rightarrow u^2 - 6u + 25 = 0$

$\therefore u = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$

But $z^2 = u \ \therefore z = \sqrt{u}$

For $u = 3 \pm 4i$; let $z = a + bi$

$\Rightarrow (a + bi)^2 = 3 + 4i$

$\Rightarrow a^2 + i2ab - b^2 = 3 \pm 4i$

$\therefore a^2 - b^2 = 3$ ..... (1)

$\Rightarrow 2ab = \pm 4 \ or \ ab = \pm 2$

$\therefore a = \pm \frac{2}{b}$ ..... (2)

$eqn.(2) \ in \ to \ eqn.(1)$

$\Rightarrow \left(\frac{2}{b}\right)^2 - b^2 = 3 \Rightarrow \frac{4}{b^2} - b^2 = 3$

$Let \ m = b^2$

$\Rightarrow \frac{4}{m} - m = 3 \ \therefore m^2 + 3m - 4 = 0$

Solving the equation yields $m = -4, 1$

$\Rightarrow b = \sqrt{m}$. Now for $m = -4, b = \sqrt{-4} = \pm 2i$

For $m = 1, b = \sqrt{1} = \pm 1$

From eqn. (3)

$a = \pm \frac{2}{b} \Rightarrow a = \frac{2}{b}$. For $b = \pm 2i \Rightarrow a = \pm \frac{2}{\pm 2i} = \pm i$

Also for $b = \pm 1 \Rightarrow a = \pm \frac{2}{\pm 1} = \pm 2$

$\therefore b = \pm 1, a = \pm 2 \ ; b = \pm 2i, a = \pm i$

$\Rightarrow z = \pm(2 + i)$

Task

1. *Show that* $z = 1$ *is a root of the equation* $z^4 - 3z^3 + 4z^2 - 3z + 1 = 0$. *Hence find the other roots.* ***Answer***: $\mathbf{1}, \mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{1} \pm \boldsymbol{i}\sqrt{\mathbf{3}}\right)$
2. *Given that that the complex number z and its conjugate* $\bar{z}$ *satisfy the equation:* $z\bar{z} + 3\bar{z} = 34 - 12i$. *Find the values of z.* (***Uneb* 2010**)
3. *Given the complex number such that* $z = \frac{p}{2-i} + \frac{q}{1+3i}$ *where p and q are real. If* $arg(z) = \frac{\pi}{2}$ *and* $|z| = 7$, *find the values of p and q*
4. *Find the real values of a and b such that* $(a + ib)^2 = i$, *hence or otherwise solve the equation* $z^2 + 2z + 1 - i = 0$, *where* $z = a + ib$.
5. *Solve for z in the form* $a + ib$ if $\frac{2z-3}{1-4i} = \frac{3}{1+i}$

Equations involving n^{th} root of a complex number

Example

Solve the following equations

i) $z^3 + 1 = 0$

ii) $z^3 - 8 = 0$

iii) $z^4 - 8 - i\,8\sqrt{3} = 0$

iv) $z^3 - 5 + i\,12 = 0$

v) $z^3 - \left(\frac{2-i}{2+i}\right) = 0$

Solution

i) $z^3 + 1 = 0 \Rightarrow z^3 = -1 + 0\,i \;\therefore z = \sqrt[3]{-1 + 0\,i}$

Expressing z in polar form

$\Rightarrow z = 1\left(cos(-180^0) + i\,sin(-180^0)\right)$

$\sqrt[3]{z} = [cos(-180^0) + i\,sin(-180^0)]^{\frac{1}{3}}$

$= cos\left(\frac{-180^0+360^0k}{3}\right) + i\,sin\left(\frac{-180^0+360^0k}{3}\right)$, where $k = 0, 1\ and\ 2$

For $k = 0$, $\sqrt[3]{z} = cos\left(\frac{-180^0+360^0\times0}{3}\right) + i\,sin\left(\frac{-180^0+360^0\times0}{3}\right)$

$= \frac{1}{2} - \frac{i\sqrt{3}}{2}$

For $k = 1$, $\sqrt[3]{z} = cos\left(\frac{-180^0+360^0\times1}{3}\right) + i\,sin\left(\frac{-180^0+360^0\times1}{3}\right)$

$= \frac{1}{2} + \frac{i\sqrt{3}}{2}$

For $k = 2$, $\sqrt[3]{z} = cos\left(\frac{-180^0+360^0\times2}{3}\right) + i\,sin\left(\frac{-180^0+360^0\times2}{3}\right)$

$= -1 + 0\,i$

$\therefore \sqrt[3]{z} = \frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}, -1$

ii) $z^3 - 8 = 0 \Rightarrow z^3 = 8 + 0\,i \;\therefore z = \sqrt[3]{8 + 0\,i}$

Expressing z in polar form

$\Rightarrow z = 8\left(cos(0^0) + i\,sin(0^0)\right)$

$\sqrt[3]{z} = \left[8\left(cos(0^0) + i\,sin(0^0)\right)\right]^{\frac{1}{3}}$

$= 2\left[cos\left(\frac{0^0+360^0k}{3}\right) + i\,sin\left(\frac{0^0+360^0k}{3}\right)\right]$, where $k = 0, 1\ and\ 2$

For $k = 0$, $\sqrt[3]{z} = 2\left[cos\left(\frac{0^0+360^0\times0}{3}\right) + i\,sin\left(\frac{0^0+360^0\times0}{3}\right)\right]$

$= 2 + 0\,i$

For $k = 1$, $\sqrt[3]{z} = 2\left[cos\left(\frac{0^0+360^0\times1}{3}\right) + i\,sin\left(\frac{0^0+360^0\times1}{3}\right)\right]$

$= -1 + i\sqrt{3}$

For $k = 2$, $\sqrt[3]{z} = 2\left[cos\left(\frac{0^0+360^0\times2}{3}\right) + i\,sin\left(\frac{0^0+360^0\times2}{3}\right)\right]$

$= -1 - i\sqrt{3}$

$\therefore \sqrt[3]{z} = 2 + 0\,i,\ , -1 \pm i\sqrt{3}$

iii) $z^4 - 8 - i\,8\sqrt{3} = 0 \Rightarrow z^4 = -8 + i\,8\sqrt{3} = z_1\ (say)$

$\therefore z = \sqrt[4]{z_1}$

Expressing z_1 in polar form

$|z_1| = \sqrt{(8)^2 + \left(8\sqrt{3}\right)^2} = 8\sqrt{1+3} = 16$

$arg(z_1) = arg\left[8\left(-1 + i\sqrt{3}\right)\right]$

$= arg(-1 + i\sqrt{3})$ $[\because arg(\lambda z) = arg(z)\ for\ \lambda > 0]$

$\therefore arg(z_1) = 120^0$

$\Rightarrow z_1 = 16(cos(120^0) + i\ sin(120^0))$

$\Rightarrow z = \sqrt[4]{z_1} = \sqrt[4]{[16(cos(120^0) + i\ sin(120^0))]}$

$= 2[cos(120^0) + i\ sin(120^0)]^{\frac{1}{4}}$

$= 2\left[cos\left(\frac{120^0+360^0k}{4}\right) + i\ sin\left(\frac{120^0+360^0k}{4}\right)\right]$, where $k = 0, 1, 2\ and\ 3$

For $k = 0$, $\sqrt[4]{z_1} = 2\left[cos\left(\frac{120^0+360^0\times0}{4}\right) + i\ sin\left(\frac{120^0+360^0\times0}{4}\right)\right]$

$= \sqrt{3} + i$

For $k = 1$, $\sqrt[3]{z} = 2\left[cos\left(\frac{120^0+360^0\times1}{4}\right) + i\ sin\left(\frac{120^0+360^0\times1}{4}\right)\right]$

$= -1 + i\sqrt{3}$

For $k = 2$, $\sqrt[3]{z} = 2\left[cos\left(\frac{120^0+360^0\times2}{4}\right) + i\ sin\left(\frac{120^0+360^0\times2}{4}\right)\right]$

$= -\sqrt{3} - i$

For $k = 3$, $2\left[cos\left(\frac{120^0+360^0\times3}{4}\right) + i\ sin\left(\frac{120^0+360^0\times3}{4}\right)\right]$

$= 1 - i\sqrt{3}$

$\therefore z = \pm(\sqrt{3} + i), \pm(1 - i\sqrt{3})$

iv) Left as an exercise ***Answer***: $\mathbf{1\left|102^0 18'\right.}$, $\mathbf{1\left|222^0 18'\right.}$, $\mathbf{1\left|342^0 18'\right.}$, $\mathbf{0.953 - i\ 0.304}$

LOCUS PROBLEMS IN COMPLEX NUMBERS

Consider a complex number $z = x + i\ y$. If z varies when subjected to some given condition, the corresponding set of points in the Argand diagram is called the *locus* of a point $P(x, y)$ representing z.

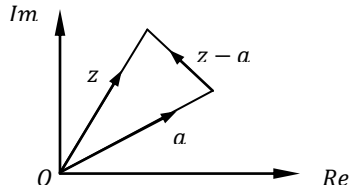
Loci are defined by:

- Distances
- Arguments/ angles

In general, equations of loci in Argand diagram involve moduli and arguments of complex variables.

Locus from Distances:

Consider the figure below.



$z = x + i\ y, P(x, y)$

$a = a_1 + i\ a_2$

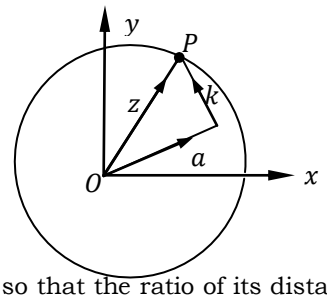
$z - a = \overrightarrow{AP}$

Finding magnitude of $\overrightarrow{AP}$

$\Rightarrow \left|\overrightarrow{AP}\right| = |z - a| = k\ (+ve\ constant)$

Thus Distance of P from A is a constant. Hence

$\|z - a\| = k$ is the equation of a circle with centre a at the point A and radius k



NOTE

If a point $P(x, y)$ move so that the ratio of its distance from two fixed points A and B is constant, then the locus is a circle. This locus is referred to as *Apollonius* circle and is represented in the Argand diagram by the equation of the form

$$|z-a| = k|z-b| \text{ or } \left|\frac{z-a}{z-b}\right| = k \text{ for } k \neq 1$$

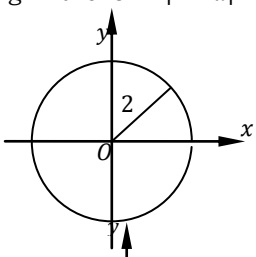
Example

1) *Sketch the loci defined by the equations:*
   i) $|z| = 2$
   ii) $|z-3+2i| = 5$
   iii) $|z-4+3i| = 4$
   iv) $|z+i| = 3$

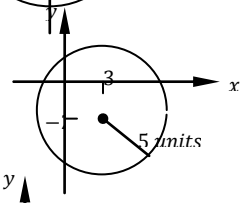
Solution

These are straight forward equations. Re-writing in the form $|z-a| = k$

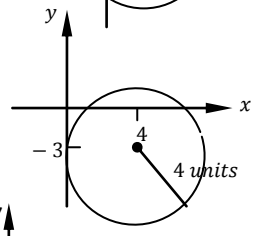
i) $|z| = 2 \Rightarrow |z-(0+0i)| = 2$
$\therefore Centre\ of\ circle: (0, 0)\ ; Radius: 2$
$|x+iy| = 2 \Rightarrow x^2 + y^2 = 2^2$



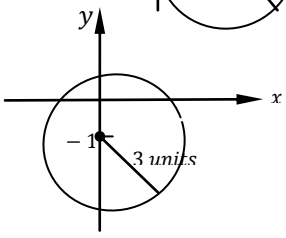
ii) $|z-3+2i| = 5 \Rightarrow |z-(3-2i)| = 5$
Centre $(3, -2): Radius: 5$
$|(x-3)+i(y+2)| = 5 \Rightarrow (x-3)^2 + (y+2)^2 = 5^2$



iii) $|z-4+3i| = 4 \Rightarrow |z-(4-3i)| = 4$
Centre $(4, -3): Radius: 4$
$|(x-4)+i(y+3)| = 5 \Rightarrow (x-4)^2 + (y+3)^2 = 4^2$



iv) $|z+i| = 3 \Rightarrow |z-(0-i)| = 3$
Centre $(0, -1): Radius: 3$
$|(x-0)+i(y+1)| = 3 \Rightarrow x^2 + (y+2)^2 = 3^2$



2) *A point P represents a complex number $z = x + iy$ on an Argand diagram. Describe the locus of P if;*
   a) $2|z+1| = |z-2|$

b) $|z+4i| = 3|z-4|$

c) $\left|\frac{z+i}{z-5-2i}\right| = 2$

d) $\left|\frac{z-1}{z+1-i}\right| = \frac{2}{3}$

e) $\left|\frac{z}{z+6}\right| = 5$

Solution

a) $2|z+1| = |z-2| \Rightarrow \frac{|z-2|}{|z+1|} = 2$

Let $z = x + yi$

$\frac{|z-2|}{|z+1|} = \frac{|(x-2)+iy|}{|(x+1)+iy|} = 2$

$\Rightarrow \frac{\sqrt{(x-2)^2+(y)^2}}{\sqrt{(x+1)^2+(y)^2}} = 2$, squaring both sides

$\Rightarrow \frac{(x-2)^2+(y)^2}{(x+1)^2+(y)^2} = 4 \ \therefore \frac{x^2-4x+4+y^2}{x^2+2x+1+y^2} = 4$

$\Rightarrow x^2 - 4x + 4 + y^2 = 4[x^2 + 2x + 1 + y^2]$

$\Rightarrow 3x^2 + 12x + 3y^2 = 0 \quad \therefore x^2 + y^2 + 4x = 0$

Circle with centre $(-2\,, 0)$, Radius= $\sqrt{4+0-0} = 2\ units$

b) $|z+4i| = 3|z-4| \Rightarrow \frac{|z+4i|}{|z-4|} = 3$

$\frac{|z+4i|}{|z-4|} = \frac{|x+i(y+4)|}{|(x-4)+iy|} = 3$

$\Rightarrow \frac{\sqrt{x^2+(y+4)^2}}{\sqrt{(x-4)^2+y^2}} = 3$, squaring both sides

$\Rightarrow \frac{x^2+(y+4)^2}{(x-4)^2+y^2} = 9 \therefore \frac{x^2+y^2+8y+16}{x^2-8x+16+y^2} = 9$

$\Rightarrow x^2 + y^2 + 8y + 16 = 9[x^2 - 8x + 16 + y^2]$

$\Rightarrow 8x^2 - 72x - 8y + 128 + 8y^2 = 0 \ \therefore x^2 + y^2 - 9x - y + 16 = 0$

Circle with centre $\left(\frac{9}{2}\,, \frac{1}{2}\right)$, Radius= $\sqrt{\frac{81}{4} + \frac{1}{4} - 16} = \frac{\sqrt{21}}{2}\ units$

c) $\left|\frac{z+i}{z-5-2i}\right| = 2 \Rightarrow \frac{|z+i|}{|z-5-2i|} = 2$

$\frac{|z+i|}{|z-5-2i|} = \frac{|x+i(y+1)|}{|(x-5)+i(y-2)|} = 2$

$\Rightarrow \frac{\sqrt{x^2+(y+1)^2}}{\sqrt{(x-5)^2+(y-2)^2}} = 2$, squaring both sides

$\Rightarrow \frac{x^2+(y+1)^2}{(x-5)^2+(y-2)^2} = 4 \therefore \frac{x^2+y^2+2y+1}{x^2-10x+25+y^2-4y+4} = 4$

$\Rightarrow x^2 + y^2 + 2y + 1 = 4[x^2 + y^2 - 10x - 4y + 29]$

$\Rightarrow 3x^2 + 3y^2 - 40x - 18y + 115 = 0 \quad \therefore x^2 + y^2 - \frac{40}{3}x - 6y + \frac{115}{3} = 0$

Circle with centre $\left(\frac{20}{3}\,, 3\right)$, Radius= $\sqrt{\frac{400}{9} + 9 - \frac{115}{3}} = \frac{\sqrt{136}}{3} = \frac{2}{3}\sqrt{34}\ units$

d) $\left|\frac{z-1}{z+1-i}\right| = \frac{2}{3} \Rightarrow \frac{|z-1|}{|z+1-i|} = \frac{2}{3}$

$\frac{|z-1|}{|z+1-i|} = \frac{|(x-1)+iy|}{|(x+1)+i(y-1)|} = \frac{2}{3}$

$\Rightarrow \frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+(y-1)^2}} = \frac{2}{3}$, squaring both sides

$\Rightarrow \frac{(x-1)^2+y^2}{(x+1)^2+(y-1)^2} = \frac{4}{9} \ \therefore \frac{x^2+y^2-2x+1}{x^2+2x+1+y^2-2y+1} = \frac{4}{9}$

$\Rightarrow 9[x^2 + y^2 - 2x + 1] = 4[x^2 + 2x + 1 + y^2 - 2y + 1]$

$\Rightarrow 9x^2 + 9y^2 - 18x + 9 = 4x^2 + 8x + 8 + 4y^2 - 8y$

$\therefore 5x^2 + 5y^2 - 26x + 8y + 1 = 0$

Circle with centre $\left(\frac{13}{5}, -\frac{4}{5}\right)$, Radius= $\sqrt{\frac{169}{25}+\frac{16}{25}-1}=\frac{\sqrt{184}}{5}=\frac{2}{5}\sqrt{46}\ units$

Task

If $z = x + iy$, determine the Cartesian equation of loci of the point z which moves in the Argand diagram so that;

i) $|z+2i|^2+|z-2i|^2=40$

ii) $|z+2i|^2-|z-2i|^2=24$

iii) $|z+ki|^2+|z-ki|^2=30k^2$

ANSWER: $(i)\ x^2+y^2=16\ (ii)\ y=3\ (iii)\ x^2+y^2=4k^2$

NOTE:

For $\left|\frac{z-a}{z-b}\right| = k$ *and* $k = 1$, this is a case where locus is a *straight line*

Example

Sketch the locus of the point $P(x, y)$ representing the complex number $z = x + iy$ given that

i) $|z-1|=|z+i|$

ii) $\left|\frac{z+i}{z-5-2i}\right|=1$

Write also the Cartesian equation of the locus.

Solution

i) $|z-1|=|z+i|\ ; z=x+iy$

$|(x-1)+iy|=|x+i(y+1)|$

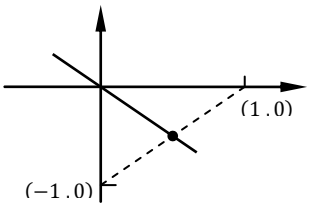
$\Rightarrow\sqrt{(x-1)^2+y^2}=\sqrt{x^2+(y+1)^2}$;

squaring both sides

$\Rightarrow(x-1)^2+y^2=x^2+(y+1)^2$

$\Rightarrow x^2+y^2-2x+1=x^2+y^2+2y+1$

$\Rightarrow -2x=2y\ \therefore y=-x\ (Line\ with\ slope=-1)$



ii) $\left|\frac{z+i}{z-5-2i}\right|=1\Rightarrow|z+i|=|z-5-2i|$

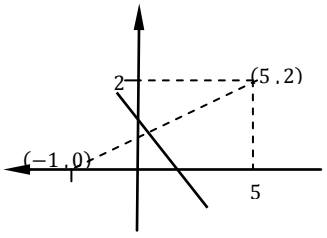
$|x+i(y+1)|=|(x-5)+i(y-2)|$

$\Rightarrow\sqrt{x^2+(y+1)^2}=\sqrt{(x-5)^2+(y-2)^2}$; squaring both sides

$\Rightarrow x^2+(y+1)^2=(x-5)^2+(y-2)^2$

$\Rightarrow x^2+y^2+2y+1=x^2+y^2-10x-4y+25+4$

$\Rightarrow 6y=-10x+28\ \therefore y=-\frac{5}{3}x+\frac{14}{3}\ \left(Line\ with\ slope=-\frac{5}{3}\right)$



LOCUS PROBLEMS FROM EQUALITY OF COMPLEX NUMBERS

1. *Given that* $Re\left(\frac{z-2i}{z+4}\right)=0$, *prove that in the Argand diagram, the locus of z is a circle. Find the centre and write as a complex number. Show that the radius is* $\sqrt{5}$

Solution

$\frac{z-2i}{z+4}=\frac{x+iy-2i}{x+iy+4}=\frac{x+i(y-2)}{(x+4)+iy}$; rationalizing the denominator,

$\Rightarrow\frac{[x+i(y-2)][(x+4)-iy]}{[(x+4)+iy][(x+4)-iy]}=\frac{x^2+4x-ixy+i(x+4)(y-2)-i^2(y^2-2y)}{(x+4)^2-i^2y^2}$

$\Rightarrow\frac{z-2i}{z+4}=\frac{x^2+4x-ixy+i(x+4)(y-2)+(y^2-2y)}{(x+4)^2+y^2}$

$\therefore\frac{z-2i}{z+4}=\frac{(x^2+4x+y^2-2y)}{(x+4)^2+y^2}+i\ \frac{(x+4)(y-2)-xy}{(x+4)^2+y^2}$

For $Real\ part=0$

$\Rightarrow\frac{(x^2+4x+y^2-2y)}{(x+4)^2+y^2}=0$

$\therefore x^2+4x+y^2-2y=0$, which is the required locus of a circle.

$Centre:\left(\frac{4}{-2}\ ,\ \frac{-2}{-2}\right)=(-2\ ,1)$

$\therefore Centre:\ -2+i$

$Radius:\sqrt{2^2+1^2-0}=\sqrt{5}$

2. *If* $z=x+iy$ *where x and y are real, show that*

   i) when $\frac{z+i}{z+2}$ is purely imaginary the locus of the point $P(x\ ,y)$ is a circle of radius $\frac{1}{2}\sqrt{5}$

   ii) When $\frac{z-2i}{2z-i}$ is purely imaginary the locus of a point $P(x\ ,y)$ in the Argand diagram is a circle. Write its centre as a complex number.

   iii) When $\frac{z-i}{z+2}$ is purely real the locus of the point $P(x\ ,y)$ is a straight line.

Solution

i) For purely imaginary complex number, $Real\ part=0$

$\frac{z+i}{z+2}=\frac{x+iy+i}{x+iy+2}=\frac{x+i(y+1)}{(x+2)+iy}$; rationalizing the denominator,

$\Rightarrow\frac{[x+i(y+1)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}=\frac{x^2+2x-ixy+i(x+2)(y+1)-i^2(y^2+y)}{(x+2)^2-i^2y^2}$

$\Rightarrow\ \frac{z+i}{z+2}=\frac{x^2+2x-ixy+i(x+2)(y+1)+(y^2+y)}{(x+2)^2+y^2}$

$\therefore\frac{z+i}{z+2}=\frac{(x^2+2x+y^2+y)}{(x+2)^2+y^2}+i\ \frac{(x+2)(y+1)-xy}{(x+2)^2+y^2}$

For $Real\ part=0$

$\Rightarrow\frac{(x^2+2x+y^2+y)}{(x+2)^2+y^2}=0$

$\therefore x^2+y^2+2x+y=0$, which is the required locus of a circle.

$\therefore Centre:\left(-1\ ,\ -\frac{1}{2}\right)$

$\therefore Centre:1-\frac{1}{2}i$

$Radius:\sqrt{1^2+\left(\frac{1}{2}\right)^2-0}=\frac{1}{2}\sqrt{5}$

ii) $\frac{z-2i}{2z-i}=\frac{x+iy-2i}{2x+i2y-i}=\frac{x+i(y-2)}{2x+i(2y-1)}$; rationalizing the denominator,

$\Rightarrow\frac{z-2i}{2z-i}=\frac{[x+i(y-2)][2x-i(2y-1)]}{[2x+i(2y-1)][2x-i(2y-1)]}=\frac{2x^2-i(2xy-x)+i(2xy-4x)-i^2(y-2)(2y-1)}{(2x)^2-i^2(2y-1)^2}$

$\Rightarrow \frac{z-2i}{2z-i} = \frac{2x^2-i(2xy-x)+i(2xy-4x)+2y^2-5y+2}{4x^2+(2y-1)^2}$

$\therefore \frac{z-2i}{2z-i} = \frac{(2x^2+2y^2-5y+2)}{4x^2+(2y-1)^2} + i\ \frac{-3x}{4x^2+(2y-1)^2}$

For $Real\ part = 0$

$\Rightarrow \frac{(2x^2+2y^2-5y+2)}{4x^2+(2y-1)^2} = 0$

$\therefore 2x^2 + 2y^2 - 5y + 2 = 0\ or\ x^2 + y^2 - \frac{5}{2}y + 1 = 0$, which is the required locus of a circle.

$\therefore Centre: \left(0\ ,\ \frac{5}{4}\right)$

$\therefore Centre: 0 + \frac{5}{4}i$

iii) $\frac{z-i}{z+2}$ is purely Real if $Imaginary\ part = 0$

$\frac{z-i}{z+2} = \frac{x+iy-i}{x+iy+2} = \frac{x+i(y-1)}{(x+2)+iy}$; rationalizing the denominator,

$\Rightarrow \frac{[x+i(y-1)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]} = \frac{x^2+2x-ixy+i(x+2)(y-1)-i^2(y^2-y)}{(x+2)^2-i^2y^2}$

$\Rightarrow \frac{z+i}{z+2} = \frac{x^2+2x-ixy+i(x+2)(y-1)+(y^2-y)}{(x+2)^2+y^2}$

$\therefore \frac{z+i}{z+2} = \frac{(x^2+2x+y^2-y)}{(x+2)^2+y^2} + i\ \frac{(x+2)(y-1)-xy}{(x+2)^2+y^2}$

For $Imaginary\ part = 0$

$\Rightarrow \frac{(x+2)(y-1)-xy}{(x+2)^2+y^2} = 0$

$\therefore (x+2)(y-1) - xy = 0$

$\Rightarrow xy - x + 2y - 2 - xy = 0$

$\therefore 2y = x + 2$, which is a straight line.

Task

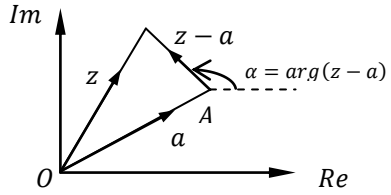
1) *If* $z = \frac{3}{3+cos\theta + i\ sin\theta}$ *, Prove that the locus of z is a circle* $x^2 + y^2 = 4x - 3$
2) *If n is a variable and* $z = 4n + i3(1-n)$*, show that the locus of z is a straight line.*

LOCUS PROBLEMS FROM ARGUMENT OF COMPLEX NUMBER

Two major cases shall be considered.

<u>Case I</u>: $\boldsymbol{arg(z-a) = \alpha}$

P is a point representing complex number $z = x + iy$. A is a complex number $a = c + id$. $\overline{AP}$ is a straight line



Deduction

$arg(z-a) = \alpha$, is the equation of a *half line* with end point A inclined at an angle α to the $+ve$ Real axis.

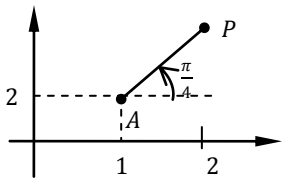
Example

1. *Sketch the loci defined by the equations*
   i) $arg(z-1-2i)=\frac{\pi}{4}$
   ii) $arg(z+2)=-\frac{2\pi}{3}$
   iii) $arg(z+1-3i)=\frac{\pi}{6}$
   iv) $arg(z+1-i)=-\frac{\pi}{4}$

Solution

i) $arg(z-1-2i)=\frac{\pi}{4}$, can be written as $arg[z-(1+2i)]=\frac{\pi}{4}\equiv\ arg(z-a)=\alpha$

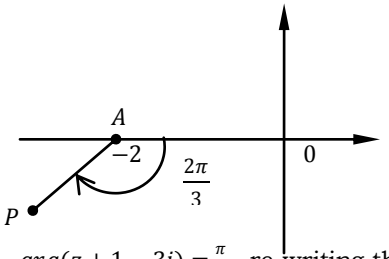
This is a locus of half line with end point $(1\,,2)$ inclined at an angle $\frac{\pi}{4}$



Note:
Direction arrow of angle measurement is in anticlockwise since
$arg(z-1-2i)=\frac{\pi}{4}=+ve$

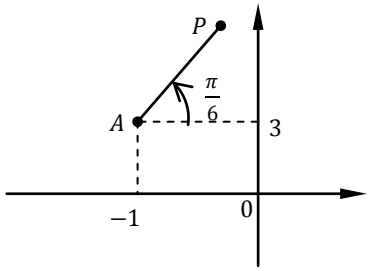
Finding Cartesian equation of line AP
Let $z=x+iy$
$\Rightarrow arg(z-1-2i)=arg\big((x-1)+(y-2)i\big)=\frac{\pi}{4}$
$\Rightarrow tan^{-1}\left(\frac{y-2}{x-1}\right)=\frac{\pi}{4}$
$\Rightarrow\frac{y-2}{x-1}=tan\frac{\pi}{4}=1$
$\therefore y-2=x-1\ \ or\ y=x+1$

ii) $arg(z+2)=-\frac{2\pi}{3}$, re-writing the equation
$\Rightarrow arg(z+2)=arg\left(z—(-2+0i)\right)=-\frac{2\pi}{3}\equiv arg(z-a)=\alpha$



Finding Cartesian equation of locus
Let $z=x+iy$
$\Rightarrow arg(z+2)=arg\big((x+2)+yi\big)=-\frac{2\pi}{3}$
$\Rightarrow tan^{-1}\left(\frac{y}{x+2}\right)=-\frac{2\pi}{3}$
$\Rightarrow\frac{y}{x+2}=tan\left(-\frac{2\pi}{3}\right)=\sqrt{3}$
$\therefore y=x\sqrt{3}+2\sqrt{3}$

iii) $arg(z+1-3i)=\frac{\pi}{6}$, re-writing the equation
$\Rightarrow arg(z+1-3i)=arg\left(z—(-1+3i)\right)=\frac{\pi}{6}\equiv arg(z-a)=\alpha$
Half line with end point $A(-1\,,\ 3)$



Finding Cartesian equation of locus
Let $z=x+iy$
$\Rightarrow arg(z+1-3i)=arg\big((x+1)+(y-3)i\big)=\frac{\pi}{6}$
$\Rightarrow tan^{-1}\left(\frac{y-3}{x+1}\right)=\frac{\pi}{6}$
$\Rightarrow\frac{y-3}{x+1}=tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$
$\Rightarrow y-3=x\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}$
$\therefore y=x\frac{1}{\sqrt{3}}+\left(\frac{1}{\sqrt{3}}+3\right)$

iv) Left as an exercise for the student

Case II: $\boldsymbol{arg\left(\frac{z-a}{z-b}\right)=\gamma}$

Let $arg(z-a)=\alpha$, $arg(z-b)=\beta$ and $arg\left(\frac{z-a}{z-b}\right)=\gamma$

where $\gamma=\alpha-\beta$ $(\pm 2\pi\ if\ necessary)$

NOTE:

γ can either be positive or negative depending on the turn of the half lines from the end points at a and b. *Anti clockwise turn* is $\overrightarrow{BP}$ to $\overrightarrow{AP}$ and γ is $+ve$ and for *clockwise turn* $.i.e\ \overrightarrow{AP}$ to $\overrightarrow{BP}$. Two cases arise here

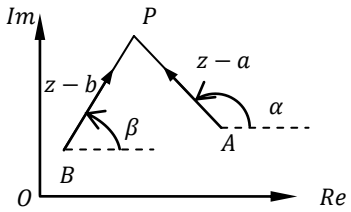


Fig $\boldsymbol{arg\left(\frac{z-a}{z-b}\right)>0}$ $.i.e.\alpha>b$

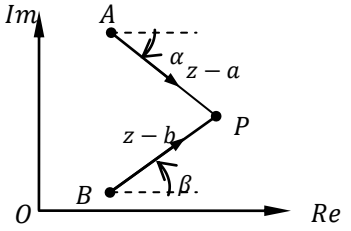


Fig $\boldsymbol{arg\left(\frac{z-a}{z-b}\right)<0}$ $.i.e.\alpha<b$

In general, the equation $\boldsymbol{arg\left(\frac{z-a}{z-b}\right)=\lambda}$, where $\lambda=constant$ represents a *circular arc* with end points A and B.

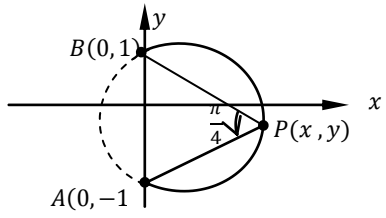
Example

1. *The point* $P(x,y)$ *represents* $z=x+iy$ *as the complex number on the Argand diagram. Describe and sketch the locus of P if;*
   a) $arg\left(\frac{z+i}{z-i}\right)=\frac{\pi}{4}$
   b) $arg\left(\frac{z-3}{z-1}\right)=\frac{\pi}{4}$
   c) $arg\left(\frac{z+2}{z-i}\right)=\frac{\pi}{3}$
   d) $arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$
   e) $arg\left(\frac{z-3-i}{z+5-3i}\right)=\frac{\pi}{3}$

Solution

a) $arg\left(\frac{z+i}{z-i}\right)=\frac{\pi}{4}$; re-writing the equation in the form $arg\left(\frac{z-a}{z-b}\right)=\lambda$

$arg\left(\frac{z+i}{z-i}\right)=\frac{\pi}{4}=arg\left[\frac{z-(0-i)}{z-(0-i)}\right]=\frac{\pi}{4}$, circular arc with end points $A(0,-1)$ and $B(0,1)$



Cartesian equation of locus:

$\frac{z+i}{z-i}=\frac{x+i(y+1)}{x+i(y-1)}=\frac{[x+i(y+1)][x-i(y-1)]}{[x+i(y-1)][x-i(y-1)]}$

$=\frac{x^2-i(xy-x)+i(xy+x)-i^2(y+1)(y-1)}{x^2-i^2(y-1)^2}$

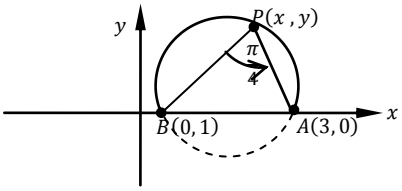
$=\frac{(x^2+y^2-1)+i(2x)}{x^2+(y-1)^2}$

$\therefore\ arg\left(\frac{z+i}{z-i}\right)=tan^{-1}\left[\frac{\frac{2x}{(x^2+(y-1)^2)}}{\frac{(x^2+y^2-1)}{(x^2+(y-1)^2)}}\right]=\frac{\pi}{4}$

$\Rightarrow \frac{2x}{x^2+y^2-1} = tan\frac{\pi}{4} = 1$

$\therefore x^2 + y^2 - 2x - 1 = 0$, centre $(1\,,0)$

b) $arg\left(\frac{z-3}{z-1}\right) = arg\left[\frac{z-(3+0i)}{z-(1+0i)}\right] = \frac{\pi}{4}$, circular arc with end points $A(3\,,0)$ and $B(0\,,1)$



Cartesian equation of locus:

$\frac{z-3}{z-1} = \frac{(x-3)+iy}{(x-1)+iy} = \frac{[(x-3)+iy][(x-1)-iy]}{[(x-1)+iy][(x-1)-iy]}$

$= \frac{(x-3)(x-1)-i(xy-3y)+i(xy-y)-i^2y^2}{(x-1)^2+(iy)^2}$

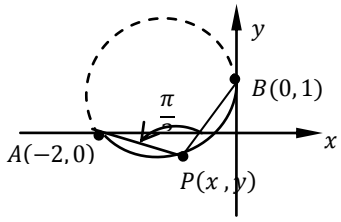
$= \frac{x^2-4x+3+i(3y-y)+y^2}{x^2-i^2(y-1)^2}$

$= \frac{(x^2+y^2-4x+3)+i(2y)}{x^2+(y-1)^2}$

$\therefore\ arg\left(\frac{z-3}{z-1}\right) = tan^{-1}\left[\frac{2y/(x^2+(y-1)^2)}{(x^2+y^2-4x+3)/(x^2+(y-1)^2)}\right] = \frac{\pi}{4}$

$\therefore x^2 + y^2 - 4x - 2y + 3 = 0$, centre $(2\,,1)$

c) $\arg\left(\frac{z+2}{z-i}\right) = \arg\left[\frac{z-(-2+0i)}{z-(0+i)}\right] = \frac{\pi}{3}$, circular arc with end points $A(-2\,,0)$ and $B(0\,,1)$



Cartesian equation of locus:

$\frac{z+2}{z-i} = \frac{(x+2)+iy}{x+i(y-1)} = \frac{[(x+2)+iy][x-i(y-1)]}{[x+i(y-1)][x-i(y-1)]}$

$\Rightarrow \frac{z+2}{z-i} = \frac{x^2+2x-i(y-1)(x+2)+ixy-i^2(y^2-y)}{x^2-i^2(y-1)^2}$

$= \frac{x^2+y^2+2x-y+i(xy+2y-x-2+xy)}{x^2+(y-1)^2}$

$= \frac{x^2+y^2+2x-y+i(-x+2y-2)}{x^2+(y-1)^2}$

$\therefore \arg\left(\frac{z+2}{z-i}\right) = \tan^{-1}\left[\frac{(-x+2y-2)/(x^2+(y-1)^2)}{(x^2+y^2+2x-y)/(x^2+(y-1)^2)}\right] = \frac{\pi}{3}$

$\Rightarrow \frac{-x+2y-2}{x^2+y^2+2x-y} = tan\frac{\pi}{3} = \sqrt{3}$

$\Rightarrow\ -x + 2y - 2 = \sqrt{3}(x^2 + y^2 + 2x - y)$

$\Rightarrow x^2 + y^2 + 2x - y = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y - \frac{2}{\sqrt{3}}$

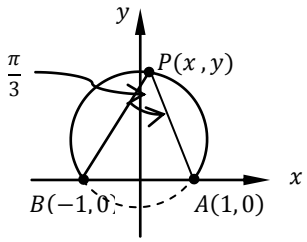
$\therefore x^2 + y^2 + \left(2 + \frac{1}{\sqrt{3}}\right)x + \left(-\frac{2}{\sqrt{3}} - 1\right)y + \frac{2}{\sqrt{3}} = 0$, centre $(-0.7\,,-0.1)$

d) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4} \Rightarrow \arg\left[\frac{z-(1+0i)}{z-(-1+0i)}\right] = \frac{\pi}{4}$, circular arc with end points $A(1\,,0)$ and $B(-1\,,0)$

Note:

Since $\overrightarrow{BP}$ is rotating through $\frac{\pi}{4}$ ($Anticlock\ wise$) to $\overrightarrow{AP}$, arrow direction is as shown.

Cartesian equation of locus:



$$\frac{z-1}{z+1}=\frac{(x-1)+iy}{(x+1)+iy}=\frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$$
$$\Rightarrow \frac{z-1}{z+1}=\frac{x^2-1-i(xy-x)+i(xy+y)-i^2y^2}{(x-1)^2-i^2y^2}$$
$$\Rightarrow \frac{z-1}{z+1}=\frac{x^2+y^2-1+i(xy+y-xy+y)}{(x-1)^2+y^2}$$
$$=\frac{(x^2+y^2-1)+i(2y)}{(x-1)^2+y^2}$$

$\therefore arg\left(\frac{z-1}{z+1}\right)=tan^{-1}\left[\frac{2y}{(x^2+y^2-1)}\right]=tan\frac{\pi}{4}=1$

$\therefore x^2+y^2-2y-1=0$, centre $(0\,,1)$

e) $\arg\left(\frac{z-3-i}{z+5-3i}\right)=\frac{\pi}{3}\Rightarrow \arg\left[\frac{z-(3+i)}{z-(-5+3i)}\right]=\frac{\pi}{3}$, circular arc with end points $A(3\,,1)$ and $B(-5\,,3)$

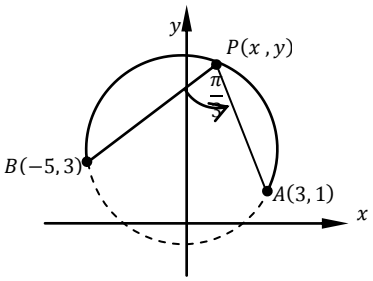
Cartesian equation of locus:

$$\frac{z-3-i}{z+5-3i}=\frac{(x-3)+i(y-1)}{(x+5)+i(y-3)}=\frac{[(x-3)+i(y-1)][(x+5)-i(y-3)]}{[(x+5)+i(y-3)][(x+5)-i(y-3)]}$$

$$=\frac{(x-3)(x+5)-i(x-3)(y-3)+i(y-1)(x+5)-i^2(y-3)(y-1)}{(x+5)^2+(y-3)^2}$$

$$=\frac{x^2+2x-15+i(xy+5y-x-5-xy+3y+3x-9)+y^2-4y+3}{(x+5)^2+(y-3)^2}$$

$$=\frac{(x^2+y^2+2x-4y-12)+i(2x+8y-14)}{(x+5)^2+(y-3)^2}$$



$\therefore arg\left(\frac{z-3-i}{z+5-3i}\right)=tan^{-1}\left[\frac{2x+8y-14}{x^2+y^2+2x-4y-12}\right]=\frac{\pi}{3}$

$\therefore \frac{2x+8y-14}{x^2+y^2+2x-4y-12}=tan\frac{\pi}{3}=\sqrt{3}$

$\therefore x^2+y^2+2x-4y-12=\frac{2}{\sqrt{3}}x+\frac{8}{\sqrt{3}}y-\frac{14}{\sqrt{3}}$

$\therefore x^2+y^2+\left(2-\frac{2}{\sqrt{3}}\right)x+\left(-4-\frac{8}{\sqrt{3}}\right)+\left(\frac{14}{\sqrt{3}}-12\right)=0$, centre $(-0.4\,,4.1)$

Task

Sketch the locus of the following

a) $arg\left(\frac{z+1}{z-1}\right)=-\frac{2\pi}{3}$

b) $arg\left(\frac{z-4}{z-2i}\right)=-\frac{\pi}{2}$

c) $arg\left(\frac{z-2i}{z-1}\right)=\frac{\pi}{3}$

2. *Represent each of the following loci on the Argand diagram*

a) $arg(z-1)=arg(z+1)$

b) $arg(z)=arg(z-1-i)$

c) $arg(z-2)=\pi+arg(z)$

d) $arg(z-1)=\pi+arg(z-i)$

e) $arg(z)=arg(z-4+2i)+\frac{\pi}{2}$

f) $arg(w-5+3i)=arg(w+2+3i)+\frac{\pi}{2}$

Solution

This example can be transformed in to forms as in example (1) above

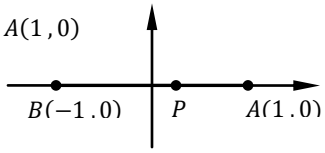
a) $arg(z-1) = arg(z+1) \Rightarrow arg(z-1) - arg(z+1) = 0$

$\therefore arg\left(\frac{z-1}{z+1}\right) = 0 \quad \left[\because arg(z_1) - arg(z_2) = arg\left(\frac{z_1}{z_2}\right)\right]$

$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$

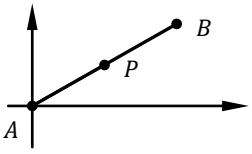
$= \frac{x^2-1+i(-xy+y+xy+y)+y^2}{(x+1)^2+y^2} = \frac{x^2+y^2-1-i2y}{(x+1)^2+y^2}$

$\Rightarrow tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = 0 \ \therefore y = 0$, is the locus of half straight line with end points $B(-1\,,0)$ and $A(1\,,0)$



b) $arg(z) = arg(z-1-i) \Rightarrow arg(z) - arg(z-1-i) = 0$

$\therefore arg\left(\frac{z}{z-1-i}\right) = arg\left[\frac{z-(0+0i)}{z-(1+i)}\right] = 0$



Since the angle $\overrightarrow{BP}$ is to turn to $\overrightarrow{AP}$ is zero, this is a straight line joining A and B (circular arc in form of a straight line).

Cartesian equation of locus:

$\frac{z}{z-1-i} = \frac{x+iy}{(x-1)+i(y-1)} = \frac{[x+iy][(x-1)-i(y-1)]}{[(x-1)+i(y-1)][(x-1)-i(y-1)]}$

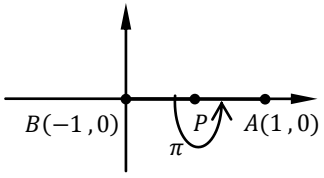
$\therefore \frac{z}{z-1-i} = \frac{x^2-x+i(-xy+x+xy-y)+y^2-y}{(x-1)^2+(y-1)^2} = \frac{x^2+y^2-x-y+i(x-y)}{(x-1)^2+(y-1)^2}$

$\therefore arg\left(\frac{z}{z-1-i}\right) = tan^{-1}\left(\frac{x-y}{x^2+y^2-x-y}\right) = 0$

$\therefore x = y$, is the locus of half straight line with end points $B(1\,,1)$ and $A(0\,,0)$

c) $arg(z-2) = \pi + arg(z) \Rightarrow arg(z-2) - arg(z) = \pi$

$\therefore arg\left(\frac{z-2}{z}\right) = arg\left[\frac{z-(2+0i)}{z-(0+0i)}\right] = \pi$, circular arc in form of a straight line with end points $B(0\,,0)$ and $A(2\,,0)$



Cartesian equation of locus:

$\frac{z-2}{z} = \frac{(x-2)+iy}{(x+iy)} = \frac{[(x-2)+iy][x-iy]}{[x+iy][x-iy]}$

$= \frac{x^2-2x+i(-xy-2y+xy)+y^2}{x^2+y^2}$

$= \frac{x^2+y^2-2x+i2y}{x^2+y^2}$

$\therefore arg\left(\frac{z-2}{z}\right) = tan^{-1}\left(\frac{2y}{x^2+y^2-2x}\right) = \pi$

$\Rightarrow \frac{2y}{x^2+y^2-2x} = tan\pi = 0 \ \therefore y = 0$

d) Left as an exercise for the students.

e) $arg(z) = arg(z-4+2i) + \frac{\pi}{2} \Rightarrow arg(z) - arg(z-4+2i) = \frac{\pi}{2}$

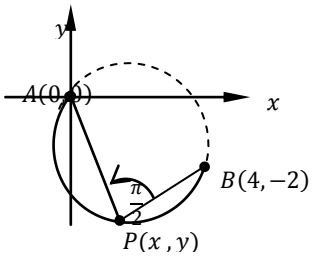
$\therefore arg\left(\frac{z}{z-4+2i}\right) = arg\left[\frac{z-(0+0i)}{z-(4-2i)}\right] = \frac{\pi}{2}$, circular arc with end points $B(4\,,-2)$ and $A(0\,,0)$

Cartesian equation of locus:

$\frac{z}{z-4+2i} = \frac{(x^2+y^2-4x+2y)+i(-2x-4y)}{(x-4)^2+(y+2)^2}$ $\left[\begin{matrix}show\ this\ as\ a\\ student\end{matrix}\right]$

$\therefore arg\left(\frac{z}{z-4+2i}\right) = tan^{-1}\left(\frac{-2x-4y}{x^2+y^2-4x+2y}\right) = \frac{\pi}{2}$

$\therefore x^2+y^2-4x+2y = 0$, since no value for $tan\frac{\pi}{2}$



Further examples on locus involving arguments

1) *Given that* $z = x + iy$, *find and sketch the locus* $arg(z) + arg(\lambda z) = \frac{\pi}{2}$ *if;*

   a) $\lambda > 0$

   b) $\lambda < 0$, where λ is a real number.

Solution

It can be shown that $\lambda = \lambda + 0i$

a) $arg(\lambda z) = arg(\lambda) + arg(z)$

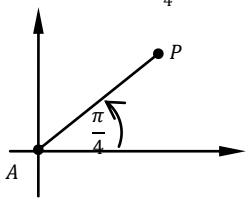
$\therefore arg(\lambda z) = 0 + arg(z) = arg(z)$

$\Rightarrow arg(\lambda z) = arg(z)$

$\therefore arg(z) + arg(z) = \frac{\pi}{2} \Rightarrow 2\,arg(z) = \frac{\pi}{2}$

$\therefore arg(z) = \frac{\pi}{4}$

$\therefore arg(z) = arg[z-(0+0i)] = \frac{\pi}{4}$, which is half line with one end point as $(0\,,0)$



b) $arg(\lambda z) = arg(\lambda) + arg(z)$, for $\lambda < 0. i.e. \lambda\ is - ve$

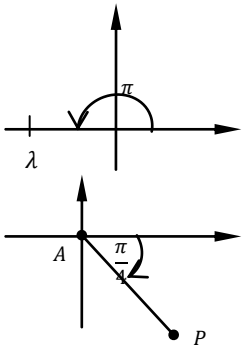
$arg(\lambda) = \pi$ $(The\ principal\ argument)$

$\Rightarrow arg(\lambda z) = \pi + arg(z)$

$\therefore arg(z) + arg(\lambda z) = arg(z) + \pi + arg(z) = \frac{\pi}{2}$

$\Rightarrow 2\,arg(z) = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$

$\therefore arg(z) = -\frac{\pi}{4}$



2) *If* $arg(2z) + \frac{\pi}{4} = arg(z+i)$, *show that the locus of* z *is a circle.*

Solution

$arg(2z) + \frac{\pi}{4} = arg(z+i) \Rightarrow arg(z+i) - arg(2z) = \frac{\pi}{4}$

$\Rightarrow arg(z+i) - arg(z) = \frac{\pi}{4}$

$\Rightarrow arg\left(\frac{z+i}{z}\right) = \frac{\pi}{4}$

$\frac{z+i}{z} = arg\left[\frac{z-(0-i)}{z-(0+0i)}\right] = \frac{\pi}{4}$, circular arc with end points $B(0\,,0)$ and $A(0\,,-1)$

Cartesian equation of locus:

$\frac{z+i}{z} = \frac{x+i(y+1)}{(x+iy)} = \frac{[x+i(y+1)][x-iy]}{[x+iy][x-iy]}$

$\therefore \frac{z+i}{z} = \frac{x^2-ixy+i(xy+x)+y^2+y}{x^2+y^2} = \frac{x^2+y^2+y+ix}{x^2+y^2}$

$\therefore arg\left(\frac{z+i}{z}\right) = tan^{-1}\left(\frac{x}{x^2+y^2+y}\right) = \frac{\pi}{4}$

$\therefore x^2 + y^2 - x + y = 0$, which is a circle of centre $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{2}}{2}$ *units*

Locus problems involving inequalities

This involves finding the equation of the locus, representing it on the Argand diagram and shading the unwanted region.

The knowledge portrayed in the previous examples shall be applied here.

Locus problems may involve arguments, modulus of inequality complex number.

Example

1. *Shade on separate Argand diagrams the regions in which;*
   - i) $|z - 4 + 4i| < 4$
   - ii) $|z - 3| > |z - 5|$
   - iii) $\left|\frac{z+i}{z-5-2i}\right| \geq 2$
   - iv) $\left|\frac{z+2}{z-i}\right| > 3$
   - v) $3 < |z| < 5$
   - vi) $4 < |z + a| \leq 5$, where $a = 1 + i$

Solution

i) *Recall:* $|z - 4 + 4i| = 4$ represents a circle of centre $(4, -4)$ and radius 4 units.

Let $z = x + iy$

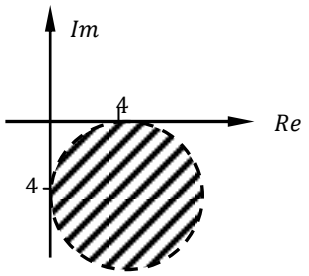
$|z - 4 + 4i| \Rightarrow |(x - 4) + i(y + 4)| < 4$

$\therefore (x - 4)^2 + (y + 4)^2 < 4^2$

Using the centre coordinate to find the wanted region to be shaded,

$\therefore (4 - 4)^2 + (-4 + 4)^2 < 4^2$, which is true.

Hence shading the outside region of a circle,



ii) $|z - 3| > |z - 5|$

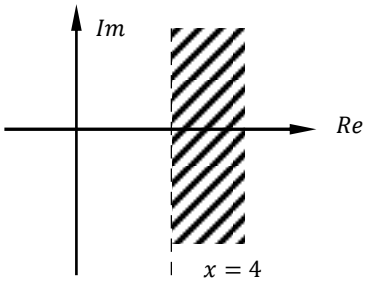
Finding the Cartesian inequality of locus

$|(x - 3) + iy| > |(x - 5) + iy|$, squaring both sides

$\Rightarrow (x - 3)^2 + y^2 > (x - 5)^2 + y^2$

$\Rightarrow x^2 - 6x + 9 > x^2 - 10x + 25$

$\Rightarrow 4x > 16 \ \therefore x > 4$



iii) $\left|\frac{z+i}{z-5-2i}\right| \geq 2$

Finding the Cartesian inequality of locus

$\Rightarrow |z + i| \geq 2|z - 5 - 2i|$

$\therefore |x + i(y + 1)| \geq 2|(x - 5) + i(y - 2)|$, squaring both sides

$\Rightarrow x^2 + (y + 1)^2 \geq 4[(x - 5)^2 + (y - 2)^2]$

$\Rightarrow x^2 + y^2 + 2y + 1 \geq 4x^2 - 40x + 100 + 4y^2 - 16y + 16$

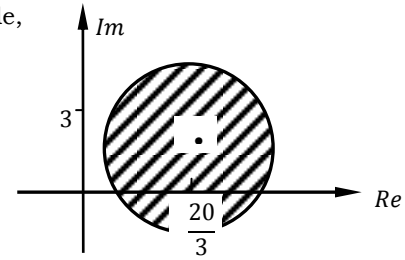
$\Rightarrow 3x^2 + 3y^2 - 40x - 18y + 115 \leq 0$

$\therefore x^2 + y^2 - \frac{40}{3}x - 6y + \frac{115}{3} \leq 0$, which is a circle of centre $\left(\frac{20}{3}, 3\right)$ and radius $\sqrt{\left(\frac{20}{3}\right)^2 + 3^2 - \frac{115}{3}} = \frac{2}{3}\sqrt{34}$

Using the centre coordinate to find the wanted region to be shaded,

$\therefore \left(\frac{20}{3}\right)^2 + (3)^2 - \frac{40}{3}\left(\frac{20}{3}\right) - 6(3) + \frac{115}{3} = -\frac{274}{9} \leq 0$, which satisfies the inequality.

Shading the inside of the circle,



iv) $\left|\frac{z+2}{z-i}\right| > 3$

Finding the Cartesian inequality of locus

$\Rightarrow |z+2| > 3|z-i|$

$\therefore |(x+2)+iy| > 3|x+i(y-1)|$, squaring both sides

$\Rightarrow (x+2)^2 + y^2 > 9[x^2 + (y-1)^2]$

$\Rightarrow x^2 + 4x + 4 + y^2 > 9x^2 + 9y^2 - 18y + 9$

$\Rightarrow 8x^2 + 8y^2 - 4x - 18y + 5 > 0$

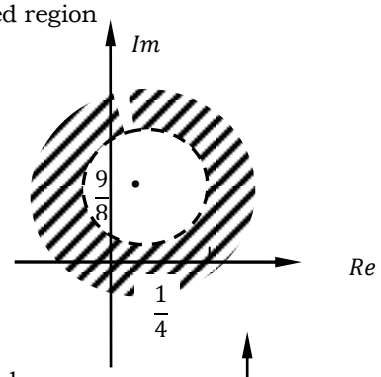
$\therefore x^2 + y^2 - \frac{1}{2}x - \frac{9}{4}y + \frac{5}{8} > 0$, which is a circle of centre $\left(\frac{1}{4}, \frac{9}{8}\right)$ and radius $\sqrt{\left(-\frac{1}{4}\right)^2 + \left(-\frac{9}{8}\right)^2 - \frac{5}{8}} = \frac{3}{8}\sqrt{5}$

Using the centre coordinate to find the wanted region to be shaded,

$\therefore \left(\frac{1}{4}\right)^2 + \left(\frac{9}{8}\right)^2 - \frac{1}{8} - \frac{81}{32} + \frac{5}{8} = -\frac{45}{64} \leq 0$,

, which does not satisfy the inequality.

Shading the outside of the circle,



v) $3 < |z| < 5$, for $3 < |z|$, $3^2 < x^2 + y^2$

For $|z| < 5$, $x^2 + y^2 \leq 5^2$

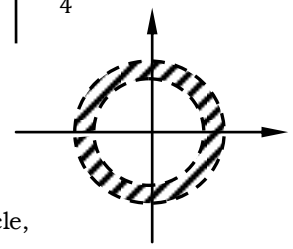
Using the centre coordinate to find the wanted region to be shaded,

For $9 < x^2 + y^2 \Rightarrow 3^2 < 0^2 + 0^2$, this is not true.

Shading the outside of circle,

Now for $|z| < 5$, $x^2 + y^2 < 5^2$

$\Rightarrow 0^2 + 0^2 < 5^2$, this is true. Shading the inside of circle,



vi) $4 < |z+a| \leq 5$, $a = 1+i$

$\Rightarrow 4^2 < (x+1)^2 + (y+1)^2 \leq 5^2$

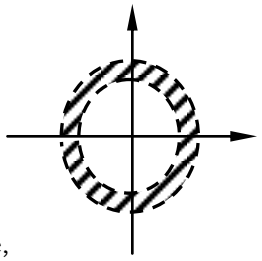
Using the centre $(-1, -1)$ to find the wanted region to be shaded,

For $4^2 < (x+1)^2 + (y+1)^2 \Rightarrow 4^2 < 0^2 + 0^2$

, this is not true shading the outside of circle,

Now for $(x+1)^2 + (y+1)^2 \leq 5^2$

$\Rightarrow 0^2 + 0^2 \geq 5^2$, this is true. Shading the inside of circle,

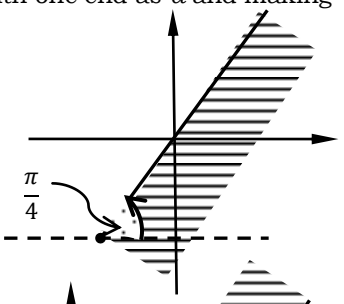


NOTE:
When testing region that is to be shaded, always use the centre to represent the inner region of the locus.

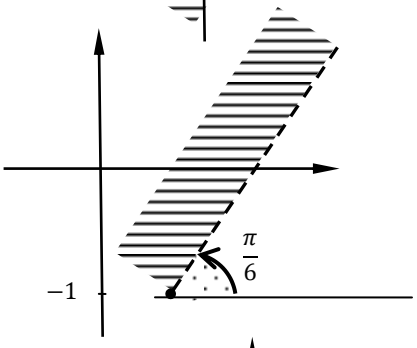
2. *Shade the region represented by;*
   a) $arg(z+a) \leq \frac{\pi}{4}$, $a = 1+i$
   b) $arg(z-1+i) > \frac{\pi}{6}$
   c) $arg(z+2-i) < -\frac{2\pi}{3}$
   d) $arg(z-i) \geq -\frac{\pi}{3}$
   e) $-\frac{\pi}{2} \leq arg(z+i) \leq \frac{2\pi}{3}$
   f) $0 < arg(z-a) \leq \frac{\pi}{2}$, where $a = 1+i$
   g) $-\frac{\pi}{6} < arg(z+a) < \frac{\pi}{4}$; $a = 2+i$

Solution

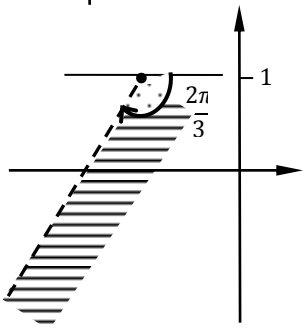
a) $arg(z+a) \leq \frac{\pi}{4}$, $a = 1+i$
Recall: $arg(z-a) = \alpha$, represents a half a line with one end as a and making an angle of α with $+ve$ real axis.
Cartesian inequality:
$arg(z+1+i) \leq \frac{\pi}{4}$
$arg[(x+1)+i(y+1)] \leq \frac{\pi}{4}$
$\Rightarrow tan^{-1}\left(\frac{y+1}{x+1}\right) \leq \frac{\pi}{4} \Rightarrow y+1 \leq (x+1)tan\frac{\pi}{4}$
$\therefore y+1 \leq x+1$ *or* $y \leq x$, shading below the line,



b) $arg(z-1+i) > \frac{\pi}{6} \Rightarrow arg[z-(1-i)] > \frac{\pi}{6}$
Cartesian inequality:
$arg[(x-1)+i(y+1)] > \frac{\pi}{6}$
$\Rightarrow tan^{-1}\left(\frac{y+1}{x-1}\right) > \frac{\pi}{6} \Rightarrow y+1 > (x-1)tan\frac{\pi}{6}$
$\therefore y+1 > (x-1)\frac{1}{\sqrt{3}}$ *or* $y > \frac{1}{\sqrt{3}}x + \left(-1-\frac{1}{\sqrt{3}}\right)$
, shading the above region of the line,



c) $arg(z+2-i) < -\frac{2\pi}{3} \Rightarrow arg[z-(-2+i)] < -\frac{2\pi}{3}$
Cartesian inequality:
$arg[(x+2)+i(y-1)] < -\frac{2\pi}{3}$
$\Rightarrow tan^{-1}\left(\frac{y-1}{x+2}\right) < -\frac{2\pi}{3}$
$\Rightarrow y-1 < (x+2)tan\left(-\frac{2\pi}{3}\right)$
$\therefore y-1 < (x+2)\sqrt{3}$ *or* $y < \sqrt{3}x + \left(1+2\sqrt{3}\right)$
, shading the below region of the line,



d) *left as an exercise*

e) $-\frac{\pi}{2} \leq arg(z+i) \leq \frac{2\pi}{3}$

$\Rightarrow -\frac{\pi}{2} \leq arg(z+i)\ ;\ arg(z+i) \leq \frac{2\pi}{3}$

Representing these on the Argand diagram,

For $-\frac{\pi}{2} \leq arg(z+i) \Rightarrow -\frac{\pi}{2} \leq arg[z-(0-i)]$

Cartesian inequality:

$-\frac{\pi}{2} \leq arg[x+i(y+1)]$

$\Rightarrow -\frac{\pi}{2} \leq tan^{-1}\left(\frac{y+1}{x}\right)\ \ or\ \ tan\left(-\frac{\pi}{2}\right) \leq \frac{y+1}{x}$

$\therefore 0 \leq x\ \ or\ x \geq 0$, shading the **R.H.S** of the line,

For $arg(z+i) \leq \frac{2\pi}{3} \Rightarrow arg[z-(0-i)] \leq \frac{2\pi}{3}$

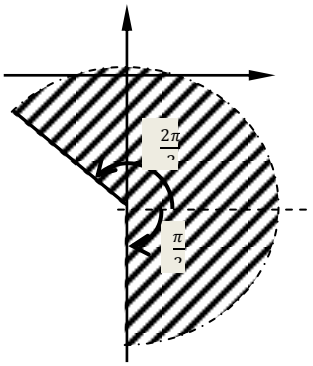
Cartesian inequality:

$arg[x+i(y+1)] \leq \frac{2\pi}{3}$

$\Rightarrow tan^{-1}\left(\frac{y+1}{x}\right) \leq \frac{2\pi}{3}\ \ or\ \ \frac{y+1}{x} \leq tan\left(\frac{2\pi}{3}\right)$

$\therefore \frac{y+1}{x} \leq -\sqrt{3}\ \ or\ y+1 \leq -\sqrt{3}x$

$\therefore y \leq -\sqrt{3}x - 1$, shading the below region of the line,



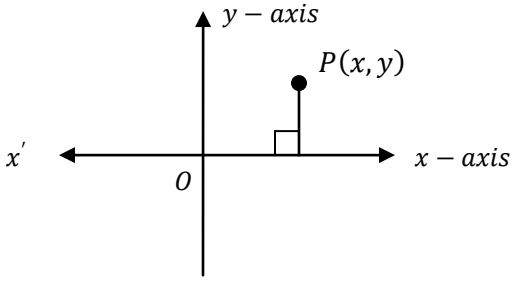
f) Left as an exercise

g) Left as an exercise

COORDINATE GEOMETRY
THE STRAIGHT LINE

1) **Position of a point in a plane**

Consider the figure below

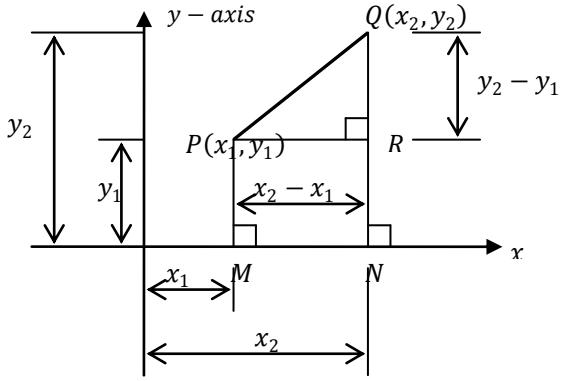


To locate a point $P(x,y)$distances OM *and* $MP$are used.
$OM = x$, $MP = y$. $\overline{MP}$ *and* $\overline{OM}$ are perpendicular lines,
to x *and* y *axes* respectively.
Distances $\overline{MP}$ *and* $\overline{OM}$ are *RECTANGULAR COORDINATES*
Ox *and* Oy are $x-axis$ *and* $y-axis$ respectively.

The $x-coordinate$ of P is called *ABSCISSA* and the $y-coordinate$ of P is called *ORDINATE. O is the origin.*

2) **To find the distance between two given coordinates.**

Let $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ be two given points in $x-y$ *plane*.
From the figure, distances are indicated $w.r.t$ coordinates.



From ΔPQR ,
$\overline{PQ}^2 = \overline{PR}^2 + \overline{QR}^2$

$$\therefore \overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

NOTE

If $P(0,0)$ is the origin, then

$$\overline{PQ} = \sqrt{x^2 + y^2}$$

Example

1) *Find the distance between the points*
   a) $A(-6, 5)$ *and* $B(-1, -7)$
   b) $A(4, 4)$ *and* $B(1, -1)$
   c) $C(-1, -3)$ *and* $D(4, -7)$

Solution

a) $A(-6, 5)$, $B(-1, -7) \Rightarrow (x_1, y_1) = (-6, 5)$ *and* $(x_2, y_2) = (-1, -7)$
$\overline{AB} = \sqrt{(-1 - -6)^2 + (-7 - 5)^2} = \sqrt{25 + 144} = 13\ units$

b) $A(4, 4)$, $B(1, -1) \Rightarrow (x_1, y_1) = (4, 4)$ *and* $(x_2, y_2) = (-1, -7)$
$\overline{AB} = \sqrt{(1 - 4)^2 + (-1 - 4)^2} = \sqrt{9 + 25} = \sqrt{34}\ units$

c) $C(-1, -3)$, $D(4, -7) \Rightarrow (x_1, y_1) = (-1, -3)$ *and* $(x_2, y_2) = (4, -7)$
$\overline{CD} = \sqrt{(4 - -1)^2 + (-7 - -3)^2} = \sqrt{25 + 16} = \sqrt{41}\ units$

2) *Given that* $A(a\,,3)\,, B(1\,,-9)$ *and* $\overline{AB} = 13$ *units, find the possible values of a.*

Solution

$\overline{AB} = 13 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, but $A(a\,,3) \equiv A(x_1\,,\ y_1)\,, B(1\,,-9) \equiv B(x_2\,,\ y_2)$

$\Rightarrow 13 = \sqrt{(1-a)^2 + (-9-3)^2}$, *squaring both sides*

$169 = (1-a)^2 + (-9-3)^2 \Rightarrow 25 = (1-a)^2$, finding the square root on both sides gives

$\Rightarrow \pm 5 = 1 - a \ \ \therefore a = 1 \pm 5$ *or* $a = 6\,, -4$

NOTE

Modulus /magnitude also represent length/distance between two given points.

Task

Given that $B(-4\,, 2b)$ *and* $D(-1\,, 1)$ *are end points of a line segment, find the possible values of b if*

a) $\overline{BD} = 5$ **Ans:** $2.5\,, -1.5$

b) $\overline{BD} = \sqrt{58}$ **Ans:** $4\,, -3$

3) **Ratio formula (Internal and External division)**

To find the coordinates of the point that divides the line joining the given points in the given ratio, say m: n

Let $A(x_1\,,\ y_1)$ *and* $B(x_2\,,\ y_2)$ be the two given points and $P(x\,, y)$, the point which divides AB in the ratio $m{:}\,n$.

The ratio of division is $\boxed{\overline{AP}{:}\,\overline{PB} = m{:}\,n}$

$\Rightarrow \frac{\overline{AP}}{\overline{PB}} = \frac{m}{n}$

By similar Δs $\ .i.e.\ \Delta PCB$

$\Rightarrow \frac{\overline{AP}}{\overline{PB}} = \frac{\overline{AE}}{\overline{PC}} = \frac{\overline{PE}}{\overline{BC}} = \frac{m}{n}$

$\therefore \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$ ..... (1)

Since x *and* y are unknowns in (1) then

$\frac{m}{n} = \frac{x - x_1}{x_2 - x} \Rightarrow mx_2 - mx = nx - nx_1$

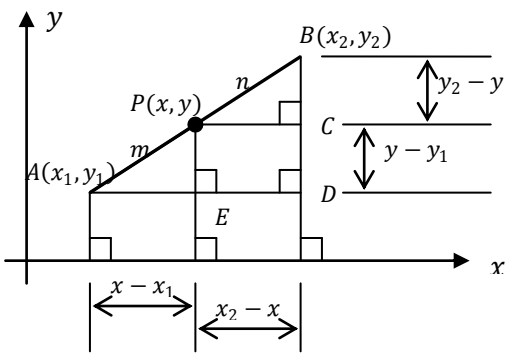
$\Rightarrow mx_2 + nx_1 = mx + nx$

$\therefore x(m+n) = mx_2 + nx_1 \Rightarrow x = \frac{mx_2 + nx_1}{m+n}$

Also $\frac{m}{n} = \frac{y - y_1}{y_2 - y} \Rightarrow my_2 - my = ny - ny_1$

$\Rightarrow my_2 + ny_1 = my + ny$

$\therefore y(m+n) = my_2 + ny_1 \Rightarrow y = \frac{my_2 + ny_1}{m+n}$



$$\therefore \boldsymbol{P(x\,, y) = P\left[\frac{mx_2 + nx_1}{m+n}\,,\ \frac{my_2 + ny_1}{m+n}\right]}$$

, which is sometimes referred to as *SECTIONAL FORMULA*

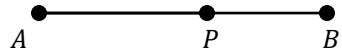
NOTE

1) The ratio of division can be stated in two ways.

a) *Internal division*

The ratio is $\overline{AP}:\overline{PB} = m:n$ $(i.e.\ both\ m\ and\ n\ are + ve\ values)$.

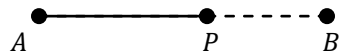
Representation



b) *External division*

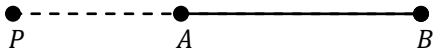
The ratio is $\overline{AP}:\overline{PB} = m:-n$ $(i.e.\ both\ m\ and\ n\ are\ of\ opposite\ signs)$.

Representation



$\overline{AP}:\overline{PB} = m:-n$

or



$\overline{AP}:\overline{PB} = -m:n$

Ratio formula of external division becomes

$$\therefore \boldsymbol{P(x\,,y) = P\left[\frac{mx_2-nx_1}{m-n}\,,\frac{my_2-ny_1}{m-n}\right]}$$

2) If P is the *midpoint of AB,* then $m = n$. Hence $\boldsymbol{P\left[\frac{x_1+x_2}{2}\,,\frac{y_1+y_2}{2}\right]}$

3) Sometimes the ratio of division *i.e.* $\overline{AP}:\overline{PB} = m:n$ can be expressed in the form $\overline{AP}:\overline{PB} = k:1$

Thus the ratio formula for the coordinates of a point P dividing $\overline{AB}$ in the ratio $k:1$ *is* $\boldsymbol{P\left[\frac{kx_2+x_1}{k+1}\,,\frac{ky_2+y_1}{k+1}\right]}$

4) If the coordinates of the dividing point are given and the ratio is required, then take the ratio as $\overline{AP}:\overline{PB} = k:1$, for simplicity.

Examples

1) *Find the coordinates of the point that divides the line* $A(-7\,,2)$ *and* $B(4\,,-4)$ *in the ratio*

a) $-1:3$ b) $1:2$ c) $4:-3$ d) $4:1$

Solution

Given: $A(-7\,,2) \equiv A(x_1\,,y_1)\,;B(4\,,-4) \equiv B(x_2\,,y_2)$

Let $P(x\,,y)$ be the point of division

a) Ratio $\overline{AP}:\overline{PB} = -1:3 \equiv m:n$ $[External\ division]$

Using the general ratio formula

$$P\left[\frac{mx_2+nx_1}{m+n}\,,\frac{my_2+ny_1}{m+n}\right] \equiv P\left[\frac{-1\times4+3\times-7}{-1+3}\,,\frac{-1\times-4+3\times2}{-1+3}\right] = P\left(-\frac{25}{2}\,,5\right)$$

b) Ratio $\overline{AP}:\overline{PB} = 1:2 \equiv m:n$ $[Internal\ division]$

$$\therefore P\left[\frac{1\times4+2\times-7}{1+2}\,,\frac{1\times-4+2\times2}{1+2}\right] = P\left(\frac{5}{3}\,,0\right)$$

c) $\overline{AP}:\overline{PB} = 4:-3 \equiv m:n$ $[External\ division]$

$$\therefore\ P\left[\frac{4\times4\pm3\times-7}{4+-3}\,,\frac{1\times-4\pm3\times2}{4+-3}\right] = P(37\,,-10)$$

d) Ratio $\overline{AP}:\overline{PB} = 4:1 \equiv m:n$

$$\therefore P\left[\frac{4\times4+1\times-7}{4+1}\,,\frac{4\times-4+1\times2}{4+1}\right] = P\left(\frac{9}{5}\,,\frac{14}{5}\right)$$

2) *Find the coordinates of the point which divides the line joining points* $(2\,,-4)$ *and* $(7\,.1)$

a) Internally in the ratio $2:3$

b) Externally in the ratio $1:3$

Solution

Let $A(2\,,-4)\equiv A(x_1\,,y_1)$ *and* $B(7\,.1)\equiv B(x_2\,,y_2)$

Let $P(x\,,y)$ be the required point

From the ratio formula $P\left[\frac{mx_2+nx_1}{m+n}\,,\frac{my_2+ny_1}{m+n}\right]$

a) For internal division, $\overline{AP}:\overline{PB}=2:3$ $[Both\ ratio\ values\ are+ve]$

$\therefore P\left[\frac{2\times7+3\times2}{2+3}\,,\frac{2\times1+3\times-4}{2+3}\right]=P(4\,,-2)$

b) For external division, $\overline{AP}:\overline{PB}=1:-3$ $[One\ of\ the\ ratio\ value\ is-ve]$

$\therefore P\left[\frac{1\times7+-3\times2}{1+-3}\,,\frac{1\times1+-3\times-4}{1+-3}\right]=P\left(-\frac{1}{2}\,,-\frac{13}{2}\right)$

3) *Find the ratio in which the given points divide the given lines*

i) Point $(1\,,3)\,,line\ through\ (0\,,1)\ and\ (-4\,,-7)$

ii) Point $(1\,,-4)\,,line\ through\ (0\,,-5)\ and\ (4\,,-1)$

iii) Point $(1\,,-1)\,,line\ through\ (-1\,,-7)\ and\ (-2\,,-10)$

Solution

i) Let the required ratio be $\overline{AP}:\overline{PB}=k:1$

$\therefore\ P\left[\frac{kx_2+x_1}{k+1}\,,\frac{ky_2+y_1}{k+1}\right]=P(1\,,3)$, using the $x-coordinate$,

$\Rightarrow\frac{kx_2+x_1}{k+1}=1\Rightarrow kx_2+x_1=k+1$,

Now $A(0\,,1)\equiv A(x_1\,,y_1)\ and\ B\ (-4\,,-7)\equiv B(x_2\,,y_2)\Rightarrow x_1=0\,,x_2=-4$

$\Rightarrow-4k+0=k+1\ \therefore k=-\frac{1}{5}$

Ratio is $k:1=-\frac{1}{5}:1\ or-1:5$ $[External\ division]$

Note: If the $y-coordinate$ was used in finding $k-value$, $k=-\frac{1}{5}\ i.e.$

$\frac{ky_2+y_1}{k+1}=3\Rightarrow\frac{-7k+1}{k+1}=3\ \therefore k=-\frac{1}{5}$

ii) $\overline{AP}:\overline{PB}=k:1,\ A(0\,,-5)\equiv A(x_1\,,y_1),B\ (4\,,-1)\equiv B(x_2\,,y_2)$

$\therefore\ P\left[\frac{kx_2+x_1}{k+1}\,,\frac{ky_2+y_1}{k+1}\right]=P(1\,,-4)\ \Rightarrow\frac{kx_2+x_1}{k+1}=1\ or\ ,\frac{ky_2+y_1}{k+1}=-4$

Using the $x-coordinate$, $\frac{k(4)+0}{k+1}=1\Rightarrow4k=k+1\ \therefore k=\frac{1}{3}$

Ratio is $k:1=\frac{1}{3}:1\ or\ 1:3$ $[Internal\ division]$

iii) $\overline{AP}:\overline{PB}=k:1,\ A(-1\,,-7)\equiv A(x_1\,,y_1),B\ (-2\,,-10)\equiv B(x_2\,,y_2)$

$\therefore\ P\left[\frac{kx_2+x_1}{k+1}\,,\frac{ky_2+y_1}{k+1}\right]=P(1\,,-1)\ \Rightarrow\frac{kx_2+x_1}{k+1}=1\ or\ ,\frac{ky_2+y_1}{k+1}=-1$

Using the $x-coordinate$, $\frac{k(-2)-1}{k+1}=1\Rightarrow-2k-1=k+1\ \therefore k=-\frac{2}{3}$

Ratio is $k:1=-\frac{2}{3}:1\ or-2:3$ $[External\ division]$

4) *Given that the three vertices of a rectangle are* $A(2\,,-2)\,,B(8\,,4)\,,C(5\,,7)$. *Find the fourth vertex.*

Solution

Let $D(x\,,y)$, diagonals $\overline{AC}\ and\ \overline{BD}$ meet at

$M = midpoint\ of\ diagonals$

$\therefore midpoint\ of\ \overline{AC} = M\left(\frac{2+5}{2}\,,\frac{7+-2}{2}\right) = M\left(\frac{7}{2}\,,\frac{5}{2}\right)$

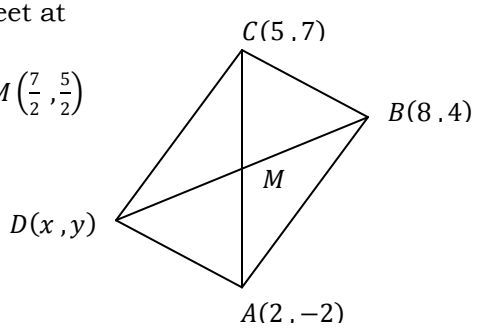
$midpoint\ of\ \overline{BD} = M\left(\frac{x+8}{2}\,,\frac{y+4}{2}\right)$

Equating $x\ nad\ y\ coordinaes\ of\ M$

$\Rightarrow \frac{7}{2} = \frac{x+8}{2}\ \therefore x = -1$

$\Rightarrow \frac{5}{2} = \frac{y+4}{2}\ \therefore y = 1$

$\therefore (-1\,,1)$, is the required vertex.



5) *If the three vertices of the triangle are points* $A(2\,,3), B(-1\,,-3)\ and\ C(3\,,2)$. *Find the coordinates of the fourth vertex if ABCD are vertices of a parallelogram.*

Solution

Diagonals $\overline{BC}\ and\ \overline{AD}$ meet at $M = mid\ point$

For diagonal $\overline{BC}$:

$M\left(\frac{-1+3}{2}\,,\frac{-3+2}{2}\right) = M\left(\frac{2}{2}\,,-\frac{1}{2}\right)$

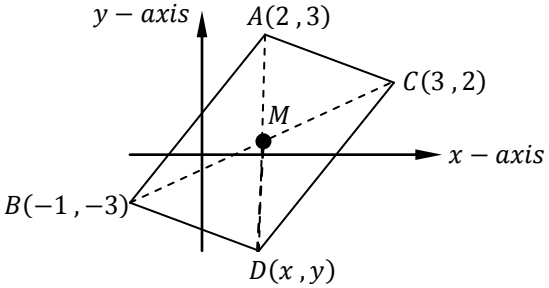
For diagonal $\overline{BC}$:

$M\left(\frac{2+x}{2}\,,\frac{3+y}{2}\right) = M\left(\frac{2}{2}\,,-\frac{1}{2}\right)$

But $\frac{2}{2} = \frac{2+x}{2} \Rightarrow x = 0$

Also $-\frac{1}{2} = \frac{3+y}{2} \Rightarrow y = -4$

$\therefore D(0\,,-4)$



NOTE:

Numbers of this kind can equally be done using distances between two points.$i.e.\ \overline{AB} = \overline{CD}\ and\ \overline{BD} = \overline{AC}$

$\Rightarrow \sqrt{(-1-2)^2 + (-3-3)^2} = \sqrt{(x-3)^2 + (y-2)^2}$

$squaring, 9 + 36 = (x-3)^2 + (y-2)^2$

$\Rightarrow 45 = x^2 - 6x + 9 + y^2 - 4y + 4\ \therefore x^2 + y^2 - 6x - 4y = 32$ ..... (1)

$\Rightarrow \sqrt{(x+1)^2 + (y+3)^2} = \sqrt{(3-2)^2 + (2-3)^2}$

$squaring, 1 + 1 = (x+1)^2 + (y+3)^2\ \therefore x^2 + y^2 + 2x + 6y = -8$ ..... (2)

Solving (1) and (2) simultaneously, (1) − (2)

$\Rightarrow -8x - 10y = 40\ or\ 4x + 5y = -20\ \therefore x = \frac{-20-5y}{4}$ ..... (3)

$eqn.(3) in\ to\ (2)$

$\Rightarrow \left[-\left(\frac{20+5y}{4}\right)\right]^2 + y^2 + 2\left[\frac{-20-5y}{4}\right] + 6y + 8 = 0$, Simplifying and collecting like terms yields

$41y^2 + 256y + 368 = 0$, solving the quadratic equation by quadratic formula gives

$\therefore y = -\frac{92}{41}\,, -4$

Now for $y=-4, x=?$

From $eqn.(3), x=\frac{-20+20}{4}=0 \ \therefore (0,-4)\ is\ the\ 4^{th}\ vertex.$

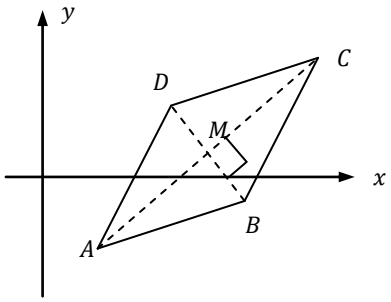
Also for $y=-\frac{92}{41}$, $x=\frac{-20+5\times 92/41}{4}=-\frac{90}{41} \Rightarrow \left(-\frac{90}{41},-\frac{92}{41}\right)$ is also a vertex. But plotting the coordinates of a parallelogram $\left(-\frac{90}{41},-\frac{92}{41}\right)$ does not give a shape of parallelogram. Hence $(0,-4)$ is the required vertex.

6) *ABCD is quadrilateral with* $A(2,-2), B(5,-1), C(6,2)$ *and* $D(3,1)$. *show that the quadrilateral is a rhombus.*

Solution

For a rhombus

- All sides are equal
- Diagonals meet at 90^0
- Opposite angles are $\neq 90^0$ but obtuse



$\overline{AB}=\sqrt{(5-2)^2+(-1--2)^2}=\sqrt{10}$

$\overline{BC}=\sqrt{(6-5)^2+(2--1)^2}=\sqrt{10}$

$\overline{CD}=\sqrt{(3-6)^2+(1-2)^2}=\sqrt{10}$

$\overline{AD}=\sqrt{(2-3)^2+(-2-1)^2}=\sqrt{10}$

Hence all sides are equal.

Now finding coordinates of M, the point of intersection of diagonals.

For $\overline{AC}: M\left[\frac{2+6}{2},\frac{-2+2}{2}\right]=M(4,0)$

Now considering ΔAMB, applying Pythagoras theorem

$\Rightarrow \overline{AM}^2+\overline{MB}^2=\overline{AB}^2 \Rightarrow \left[\sqrt{(4-2)^2+(0+2)^2}\right]^2+\left[\sqrt{(4-5)^2+(0+1)^2}\right]^2=$

$\left[\sqrt{10}\right]^2=\overline{AB}^2$. Hence this is a rhombus.

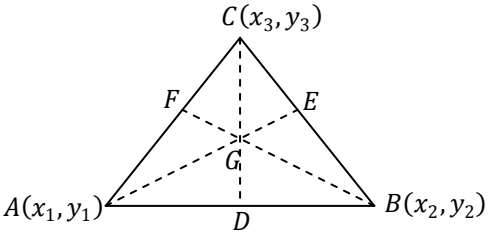
NOTE:

This question can still be handled by vectors.

COORDINATES OF THE CENTROID OF A TRIANGLE WITH GIVEN VERTICES.

Consider ΔABC $D, E\ and\ F$ are midpoints of lines $\overline{AB}, \overline{BC}\ and\ \overline{AC}$ respectively.

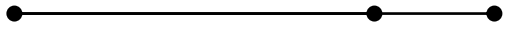
Lines joining vertices to opposite midpoints are *meridians*, and their intersection is the *CENTROID.*(G)



$.i.e.$ Meridians are concurrent at G.

Taking line $\overline{CD}$, G divides CD in the ratio $2:1$ $.i.e\ \overline{CG}:\overline{GD}=2:1$

$\left[This\ is\ a\ standard\ ratio, and\ is\ stated\ from\ the\ vertex\ to\ the\ base\ containing\ the\ midpoint\right]$



From the ratio theorem

$$G\left[\frac{2\times\left(\frac{x_1+x_2}{2}\right)+1\times x_3}{2+1}, \frac{2\times\left(\frac{y_1+y_2}{2}\right)+1\times y_3}{2+1}\right] = G\left[\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right]$$

$$\therefore \textbf{Centroid is } \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Example

1. *Find the centroid of the triangles below*
   a) $A(2, 1), B(1, 2)$ *and* $C(3, 6)$
   b) $A(0, 1), B(3, 2)$ *and* $C(-3, -9)$

Solution

Let $G(x, y)$ be the required centroid

But $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$t

a) $G\left(\frac{2+1+3}{3}, \frac{1+2+6}{3}\right) = G(2, 3)$

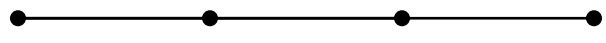
b) $G\left(\frac{0+3-3}{3}, \frac{1+2-9}{3}\right) = G(0, -2)$

Task

1) Find the coordinates of one vertex of triangle having two of its vertices as $(5, 4), (6, 9)$ and its centroid as $(5, 6)$ **Ans:** $(4, 5)$

2) Find the ratio in which the join of
   a) $(5, 8)$ *and* $(-9, -2)$ is divided by the point $(-2, 3)$ **Ans:** $1:1$
   b) $(1, 4)$ *and* $(3, -2)$ is divided by the point $(-3, 16)$
   Ans: $\mathbf{2:3}$ $(\boldsymbol{externally})$
   c) $(2, 3)$ *and* $(4, 1)$ is divided by the x-axis. **Ans:** $\mathbf{3:1}$ $(\boldsymbol{externally})$
   [**HINT**: *Find the point of division by solving line joining points and* $x-$ *axis*]

3) The line joining points $(1, -2)$ *and* $(-3, 4)$ is trisected. Find the coordinates of point of trisection.
[*HINT*: *Trisection means dividing line in to* $3-$ *equal parts*]. *i. e.*



[P and Q are needed. considering P; Ratio of division is $1:2 = \overline{AP}:\overline{PB}$
Now consdering Q; Ratio of division is $\overline{AQ}:\overline{QB} = 2:1$]

Answer: $\left(-\frac{1}{3}, \mathbf{0}\right), \left(-\frac{5}{3}, \mathbf{2}\right)$

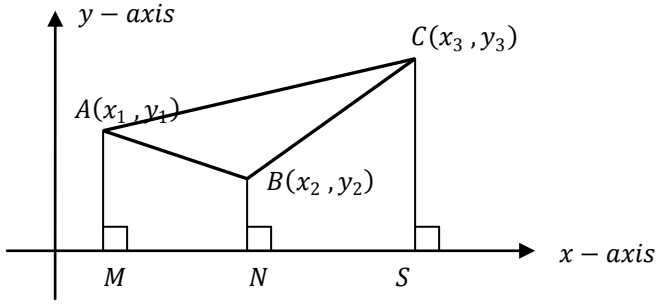
4) The three angular points of a parallelogram are;
$(-2, -1), (1, 0)$ *and* $(4, 3)$. Find the coordinates of the fourth vertex.
Ans: $(\mathbf{1}, \mathbf{2})$

5) The points $A(-2,3), B(1,-2), C(8,1)$ are vertices of a parallelogram. Find the coordinates of the fourth vertex. ***Answer***: $(\mathbf{5},\mathbf{0})$

6) $ABCD$ *with* $A(2,-2), B(5,-1), C(6,2)$ *and* $D(x,y)$is a rhombus. Find D.
Ans: $(\mathbf{3},\mathbf{1})$

AREA OF A TRIANGLE
To find area of a triangle given its three vertices

Let $A(x_1,y_1), B(x_2,y_2)$ *and* $C(x_3,y_3)$ be three vertices of ΔABC.



$Area\ of\ \Delta ABC = Area\ of\ ACMS - (Area\ of\ AMBN + Area\ of\ NSBC)$

$Area\ of\ trapezium\ ACMS = \frac{1}{2}(x_3 - x_1)(y_1 + y_3)$

$Area\ of\ trapezium\ AMNB = \frac{1}{2}(x_2 - x_1)(y_1 + y_2)$

$Area\ of\ trapezium\ NBCS = \frac{1}{2}(x_3 - x_2)(y_2 + y_3)$

$$Area\ of\ AMBN + Area\ of\ NSBC = \frac{1}{2}(x_2 - x_1)(y_1 + y_2) + \frac{1}{2}(x_3 - x_2)(y_2 + y_3)$$
$$= \frac{1}{2}[x_2(y_1 + y_2) - x_1(y_1 + y_2) + x_3(y_2 + y_3) - x_2(y_2 + y_3)]$$
$$= \frac{1}{2}[-x_1(y_1 + y_2) + x_2(y_1 - y_3) + x_3(y_2 + y_3)]$$

$$\therefore Area\ of\ \Delta ABC = \frac{1}{2}(x_3 - x_1)(y_1 + y_3) - \frac{1}{2}[-x_1(y_1 + y_2) + x_2(y_1 - y_3) + x_3(y_2 + y_3)]$$
$$= \frac{1}{2}[x_3(y_1 + y_3) - x_1(y_1 + y_3) + x_1(y_1 + y_2) - x_2(y_1 - y_3) - x_3(y_2 + y_3)]$$
$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$
$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \times determinant\ of \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

$$\therefore \textbf{Area of } \Delta\textbf{ABC} = \frac{1}{2}\begin{vmatrix} \boldsymbol{x_1} & \boldsymbol{y_1} & \mathbf{1} \\ \boldsymbol{x_2} & \boldsymbol{y_2} & \mathbf{1} \\ \boldsymbol{x_3} & \boldsymbol{y_3} & \mathbf{1} \end{vmatrix}$$

Note

During the derivation of the formula, the brackets containing the $y's$ have been maintained through out the derivation, while those of $x's$ are expanded.

Examples

Find the area of the triangle whose vertices are

a) $(4,6),(0,4)\ and\ (6,2)$

b) $(-1,2),(5,6)\ and\ (5,-4)$

c) $(4,0),(-6,0)\ and\ (2,6)$

Solution

$$\therefore\ Area\ of\ \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

a) $$Area = \frac{1}{2}\begin{vmatrix} 4 & 6 & 1 \\ 0 & 4 & 1 \\ 6 & 2 & 1 \end{vmatrix} = \frac{1}{2}\left[4\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} + 6\begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix}\right]$$

$$= \frac{1}{2}[4(4-2) - 0 + 6(6-4)] = \frac{1}{2}(8+12) = 10\ sq.units$$

b) $$Area = \frac{1}{2}\begin{vmatrix} -1 & 2 & 1 \\ 5 & 6 & 1 \\ 5 & -4 & 1 \end{vmatrix} = \frac{1}{2}\left[-1\begin{vmatrix} 6 & 1 \\ -4 & 1 \end{vmatrix} - 5\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 6 & 1 \end{vmatrix}\right]$$

$$= \frac{1}{2}[-6(6+4) - 5(2+4) + 5(2-6)] = \frac{1}{2}\times(-60) = 30\ sq.units\ (numerically)$$

c) $$Area = \frac{1}{2}\begin{vmatrix} 4 & 0 & 1 \\ -6 & 0 & 1 \\ 2 & 6 & 1 \end{vmatrix} = \frac{1}{2}\left[4\begin{vmatrix} 0 & 1 \\ 6 & 1 \end{vmatrix} - 0\begin{vmatrix} -6 & 1 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} -6 & 1 \\ 2 & 1 \end{vmatrix}\right]$$

$$= \frac{1}{2}[4(0-6) + 6(0-6) + 2(0-0)] = \frac{1}{2}\times(-60) = 30\ sq.units\ (numerically)$$

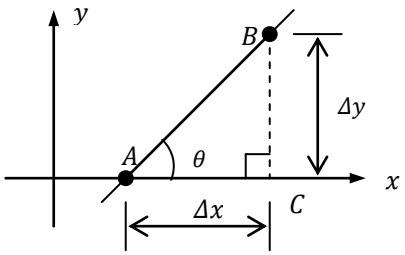
GRADIENT/SLOPE OF A LINE

Consider the figure below

Slope, $m = \frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x}$

But $tan\theta = \frac{\overline{BC}}{\overline{AC}} = \frac{\Delta y}{\Delta x}$

$$\therefore \boldsymbol{m = \tan\theta}$$



Thus slope of a line is the *tangent* of the angle which the line makes with the $+ve\ x-axis$.

NOTE

1) The measurement of the angle can either be $+ve.i.e.$ anticlockwise direction measurement or $-ve.i.e.$ clockwise direction measurement $w.r.t. +ve\ x-axis$.
2) If $0^0 < \theta < 90^0$, the slope of line is $+ve$ and if $90^0 < \theta < 180^0$, the slope of line is $-ve$.
   If $\theta = 0^0$, $m = tan0 = 0$, line is parallel to $x-axis$

If $\theta = 180^0$, $m = tan180 = 0$, line is oppositely directed to the $+ve\ sense\ of\ x-axis$

If $\theta = 90^0$, $m = tan90 = \infty$, line is parallel to the $y-axis. i.e\ vertical$

Example

State the slope of the following lines

a) $y = x + 1$

b) $y = -\sqrt{3}x + 1$

c) $2y + 3x - 1 = 0$

d) $4y + 3x + 1 = 0$

e) *Hence state the angle the lines make with the* $x-axis$

Solution

a) $y = x + 1, slope, m = 1 \ \therefore tan\theta = 1 \Rightarrow \theta = 45^0$

b) $y = -\sqrt{3}x + 1$, $slope, m = -\sqrt{3} \ \therefore tan\theta = -\sqrt{3} \Rightarrow \theta = -60^0\ or\ 120^0$

c) $2y + 3x - 1 = 0 \Rightarrow y = -\frac{3}{2}x + \frac{1}{2}$, $m = -\frac{3}{2} \therefore tan\theta = -\frac{3}{2} \Rightarrow \theta = -56.3^0\ or\ 123.7^0$

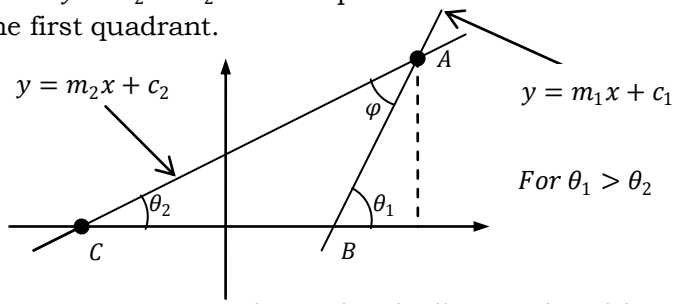
d) $Left\ as\ an\ exercise$. **Ans:** $\mathbf{-36.9^0}$ or $\mathbf{143.1^0}$

NOTE

1. The knowledge of trigonometry of finding tangents of angles is applicable
2. Before stating the gradient of a line, make sure that it is expressed in the form $Y = Mx + C$

ANGLE BETWEEN TWO LINES

Let $y = m_1x + c_1$ and $y = m_2x + c_2$ be the equations of two lines intersecting in the first quadrant.



Let $\varphi = angle\ between\ lines, \theta_1\ and\ \theta_2$ be angles the lines make with $x-axis$

From the figure

$\varphi + \theta_2 = \theta_1 \ \therefore \varphi = \theta_1 - \theta_2$

Introducing tangent on both sides

$tan\varphi = tan(\theta_1 - \theta_2) = \frac{tan\,\theta_1 - tan\,\theta_2}{1 + tan\,\theta_1 . tan\,\theta_2}$, $but\ m_1 = tan\theta_1$, $m_2 = tan\theta_2$

$\therefore \tan\varphi = \frac{m_1 - m_2}{1 + m_1 . m_2}$

$\therefore \boldsymbol{\varphi = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 . m_2}\right)}$, which gives angle between two lines.

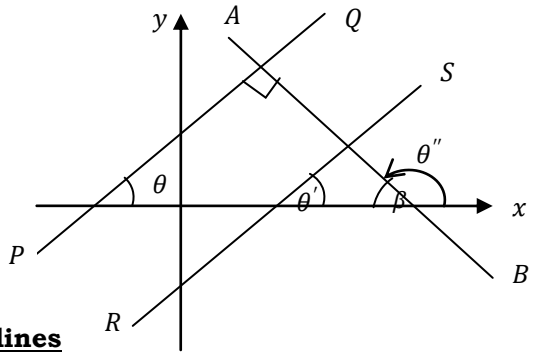
NOTE

1. From the figure above, θ_1 *and* θ_2 both are *acute* and $\theta_1 > \theta_2$ so that $tan\theta_1 > tan\theta_2$ $\therefore \varphi$ *is acute*. however if $m_2 > m_1$, then φ will be *obtuse.*$(90^0 < \varphi 180^0)$
2. *If the lines are parallel* , then $\varphi = 0$ *or* 180^0
   $\therefore \tan 0 = \frac{m_1 - m_2}{1 + m_1 . m_2} = 0 \Rightarrow m_1 - m_2 = 0$

   $\therefore$ **For parallel lines,** $\boldsymbol{m_1 = m_2}$
3. If the lines are *perpendicular*, then $\varphi = 90^0$
   $\therefore \tan 90 = \frac{m_1 - m_2}{1 + m_1 . m_2} = \infty \Rightarrow m_1 m_2 + 1 = 0$

   $\therefore$ **For perpendicular lines,** $\boldsymbol{m_1 m_2 = -1}$

Alternative derivation of condition for parallelism and perpendicularity of lines $\boldsymbol{y = m_1 x + c_1}$ **and** $\boldsymbol{y = m_2 x + c_2}$**.**
Consider the figure below



Parallel lines

If PQ and RS are parallel, then
$\theta = \theta' \; or \; \tan\theta = \tan\theta' \Rightarrow m_1 = m_2$
$\therefore$ If PQ is the line $y = m_1 x + c_1$ and RS is
$y = m_2 x + c_2$, then the lines will be parallel if $\boldsymbol{m_1 = m_2}$

Perpendicular line

If PQ is perpendicular to AB, then $\beta + \theta'' = 180^0$
Also $\theta + \beta = 90^0 \Rightarrow \beta = 90 - \theta$
$\therefore 90 - \theta + \theta'' = 180 \Rightarrow \theta'' = 90 + \theta$
From trigonometry, $tan\theta'' = tan(90 + \theta) = -cot\theta$
$\therefore \tan\theta'' = -\frac{1}{\tan\theta} \Rightarrow m_2 = -\frac{1}{m_1}$

$\therefore \boldsymbol{m_1 m_2 = -1}$

, which is the condition for perpendicular line

Examples

Find the angle between two straight lines whose equations are

a) $y = 2x + 7 \; and \; y = \frac{1}{3}x - 4$

b) $2y + 3x + 1 = 0 \; and \; y = x + 3$

c) $2x - 3y + 1 = 0 \; and \; 2x - 3y + 2 = 0$

d) $2x + 3y - 2 = 0 \; and - 3x + 2y + 3 = 0$

Solution

a) $y = 2x + 7;\; m_1 = 2 \; and \; for \; y = \frac{1}{3}x - 4;\; m_2 = \frac{1}{3}$

Using $\therefore tan\varphi = \frac{m_1 - m_2}{1 + m_1.m_2}\;,\; m_1 > m_2$

$\Rightarrow \tan\varphi = \frac{2 - \frac{1}{3}}{1 + \frac{2}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1 \;\therefore \varphi = tan^{-1}(1) = 45^0$

b) $2y + 3x + 1 = 0 \Rightarrow y = -\frac{3}{2}x - \frac{1}{2}; m_1 = -\frac{3}{2} \; and \; for \; y = x + 3; m_2 = 1$

$m_2 > m_1 \therefore tan\varphi = \frac{m_2 - m_1}{1 + m_1.m_2} \Rightarrow \varphi = tan^{-1}\left(\frac{1 - -\frac{3}{2}}{1 + -\frac{3}{2}}\right) = -78.69^0\;, 101.31^0$

c) $2x + 3y - 2 \Rightarrow y = \frac{2}{3}x + \frac{1}{3};\; m_1 = \frac{2}{3} \; and$

$-3x + 2y + 3 = 0 \;\Rightarrow y = \frac{2}{3}x + \frac{2}{3};\; m_2 = \frac{2}{3}$

$\therefore \varphi = tan^{-1}\left(\frac{\frac{2}{3} - \frac{2}{3}}{1 + \frac{2}{3} \times \frac{2}{3}}\right) = 0$. Hence lines are parallel

d) $2x + 3y - 2 \Rightarrow y = -\frac{2}{3}x + \frac{2}{3};\; m_2 = -\frac{2}{3} \; and$

$-3x + 2y + 3 = 0 \Rightarrow y = \frac{3}{2}x - \frac{3}{2};\; m_1 = \frac{3}{2}$

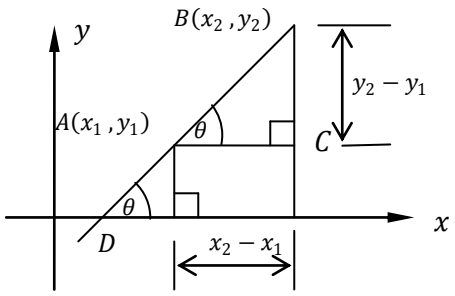
$\therefore tan\varphi = \frac{m_1 - m_2}{1 + m_1.m_2} = \frac{\frac{3}{2} - -\frac{2}{3}}{1 + -\frac{2}{3} \times \frac{3}{2}} = +\infty \therefore \varphi = 90^0$. Hence lines are perpendicular.

SLOPE OF THE LINE JOINING POINTS $A(x_1\,, y_1)$ AND $B(x_2\,, y_2)$

Consider the figure below

$< B\hat{D}X =< B\hat{A}C = \theta$

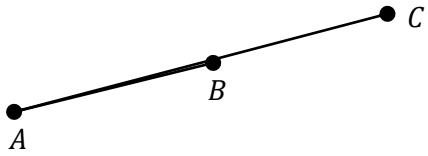
Slope of $\overline{AB} = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$



$$\therefore \boldsymbol{Slope = m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{Diff.in\ Ordinates}{Diff.in\ Absiscea}}$$

COLLINEARITY OF THREE POINTS

Consider points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ lying on the same line.



$\text{Grad}\,\overline{AB} = \text{Grad}\,\overline{BC} = \text{Grad}\,\overline{AC}$

$$\Rightarrow \frac{y_2-y_1}{x_2-x_1} = \frac{y_3-y_2}{x_3-x_2} = \frac{y_3-y_1}{x_3-x_1}$$

This is the condition for three collinear points.

NOTE

Also three points will be collinear if the area of triangle formed by joining the 3 points is ***ZERO***. *i.e.*$\therefore$ $\boldsymbol{Area\ of\ \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0}$

Examples

1) *Show that the points are collinear*
   a. $(2, 5), (3, 7)\ and\ (-1, -1)$
   b. $(2, -3), (4, 1)\ and\ (-2, -7)$
   c. $(0, -4), (2, 2)\ and\ (-1, -7)$

Solution

a. Let $A(2, 5), B(3, 7)\ and\ C(-1, -1)$

$Grad\,\overline{AB} = Grad\,\overline{BC} \Rightarrow Grad\,\overline{AB} = \frac{7-5}{3-2} = 2\,; Grad\,\overline{BC} = \frac{-1-7}{-1-3} = 2$

Since $Grad\,\overline{AB} = Grad\,\overline{BC} = 2$, points lie on a straight line.

Alternatively

For points to be on a straight line, then $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 2 & 5 & 1 \\ 3 & 7 & 1 \\ -1 & -1 & 1 \end{vmatrix} = \frac{1}{2}\left[2\begin{vmatrix} 7 & 1 \\ -1 & 1 \end{vmatrix} - 3\begin{vmatrix} 5 & 1 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 7 & 1 \end{vmatrix}\right]$$

$= \frac{1}{2}[2(7+1) - 3(5+1) - (5-7)] = 0$. Hence points are collinear

b. Let $A(2, -3), B(4, 1)\ and\ C(-2, -7)$

For A, B and C to be collinear, $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 2 & -3 & 1 \\ 4 & 1 & 1 \\ -2 & -7 & 1 \end{vmatrix} = \frac{1}{2}\left[2\begin{vmatrix} 1 & 1 \\ -7 & 1 \end{vmatrix} - 4\begin{vmatrix} -3 & 1 \\ -7 & 1 \end{vmatrix} - 2\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}\right]$$

$= \frac{1}{2}[2(8) - 4(4) + 8] = 0$ $\therefore$ Points are collinear.

NOTE

3 –Collinear points form 3 –parallel lines. Hence the condition for parallel lines is the same as that of collinear points.

c. Left as an exercise for the reader.

VARIOUS FORMS OF EQUATION OF A STRAIGHT LINE

A. **To find equation of a line given any two coordinate points on it.**
***i.e.*Two point form of equation of a line**

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points.

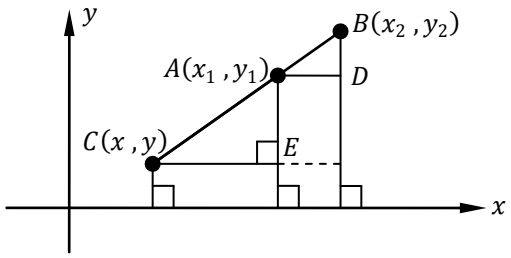
Let $C(x, y)$ be a general point on line.

By similar $\Delta s, \Delta ACE \text{ and } \Delta BAD$are similar.

$\therefore \frac{\overline{CE}}{\overline{AD}} = \frac{\overline{AE}}{\overline{BD}}$

$\Rightarrow \frac{x_1 - x}{x_2 - x_1} = \frac{y_1 - y}{y_2 - y_1} \Rightarrow \frac{x_1 - x}{y_1 - y} = \frac{x_2 - x_1}{y_2 - y_1}$

$\Rightarrow \frac{y_1 - y}{x_1 - x} = \frac{y_2 - y_1}{x_2 - x_1}$ *or* $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$



Sometimes the equation is written as

$\frac{y - y_1}{x - x_1} = m \text{ or } y - y_1 = m(x - x_1) \text{ where } m = \frac{y_2 - y_1}{x_2 - x_1}$

$\boxed{\therefore \boldsymbol{y - y_1 = m(x - x_1) \text{ or } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m}}$, which is the equation of a line.

From this equation other forms can be deduced.

B. **To find the equation of a line passing through the given point with a given slope , m**

Let $P(x_1, y_1)$ be the given point.

But $m = (known)$

$\boxed{\Rightarrow \boldsymbol{y - y_1 = m(x - x_1)}}$, which is the equation of the line.

NOTE

1) If the line cuts the $y - axis$at a distance c from the origin $i.e.$ given point is $(0, c)$, then the equation will be

$y - c = m(x - 0) \Rightarrow$ $\boxed{\boldsymbol{y = mx + c}}$

2) If the line passes through the origin , $(0, 0)$ its equation will be given by

$y - 0 = m(x - 0) \Rightarrow$ $\boxed{\boldsymbol{y = mx}}$

3) If the line passes through the point (x_1, y_1) with a *zero* gradient $i.e. m = 0$, its equation will be given by

$y - y_1 = 0(x - x_1) \Rightarrow$ $\boxed{\boldsymbol{y = y_1 \ (Horizantal)}}$

4) If the line passes through the point $(x_1\,, y_1)$ with an *infinite* gradient $i.e. m = \infty$, its equation will be given by

$$\boxed{x = x_1\ (\boldsymbol{Verticlal})}$$

Example

1) *Find the equation of a line passing through point(s)*
   a. $(2\,, 3)$ *and* $(-3\,, -5)$
   b. $(-2\,, 1)$ and having the gradient -3
   c. $(-4\,, 1)$ and making an angle of 135^0 *with* $x - axis$
   d. $(-4, -5)$ *and* $(-5\,, 1)$

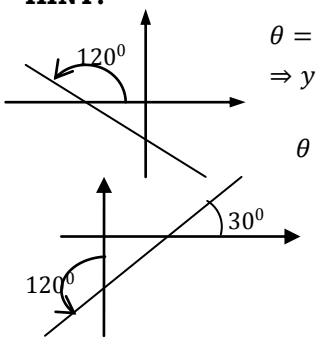
Solution

a. Equation of line passing through $2 - points$ is $y - y_1 = m(x - x_1)$, $m = \frac{y_2 - y_1}{x_2 - x_1}$
b. Let $(2\,, 3) \equiv (x_1\,, y_1) and\ (-3\,, -5) \equiv (x_2\,, y_2)$
c. $\Rightarrow m = \frac{-5-3}{-3-2} = \frac{8}{5}\ \therefore y - 3 = \frac{8}{5}(x - 2)\ or\ 8x - 5y - 1 = 0$
d. Equation of a line passing through a known point and having a known gradient is
   $y - y_1 = m(x - x_1)\,; m = -3\,, (x_1\,, y_1) \equiv (-2\,, 1)$
   $y - 1 = -3(x - -2) \therefore 3x + y + 5 = 0$
c. Given: $(-4\,, 1)$ lies on line, angle made by line and $x - axis\ is\ \theta = 135^0$
   $\therefore m = tan\theta = tan135^0 = -1$
   $\therefore y - y_1 = m(x - x_1)$, must be used.
   $\Rightarrow y - 1 = -1(x + 4)\ \therefore x + y + 5 = 0$
d. Left as an exercise***Answer***: $\boldsymbol{6x + y + 29 = 0}$

Task

Find the equation of a line passing through $(0\,, -3)$ *and making an angle of*
i) $120^0\ with\ x - axis$
ii) $120^0\ with\ y - axis$

HINT:



$\theta = +120^0; hence\ slope\ is - ve. i.e\ m = tan120^0 = -\sqrt{3}$
$\Rightarrow y + 3 = -\sqrt{3}(x - 0)\ \therefore y + \sqrt{3}x + 3 = 0$

$\theta = +120^0; measured\ w.r.t\ y - axis.$
$slope\ is + ve. i.e\ m = tan30^0 = \frac{1}{\sqrt{3}}$
$\Rightarrow y + 3 = \frac{1}{\sqrt{3}}(x - 0)\ \therefore y - \frac{1}{\sqrt{3}}x + 3 = 0$

C. **To find of a line passing through a given point $(x_1\ ,\ y_1)$ and parallel to a given line $ax + by + c = 0$**

From $ax + by + c = 0\ or\ y = -\frac{a}{b}x - \frac{c}{b} \Rightarrow slope\ , m = -\frac{a}{b}$

Required line:

$\Rightarrow y - y_1 = m'(x - x_1)$, but for parallel lines$m = m'\ (slopes\ are\ equal)$

$\Rightarrow y - y_1 = -\frac{a}{b}(x - x_1) \Rightarrow by - by_1 = -ax + ax_1$

$\therefore ax + by = ax_1 + by_1$, but $ax_1 + by_1$ can be evaluated to give a constant, say d

$\Rightarrow\ ax + by = d$, which is the required line.

In general, the equation of a line passing through a given point $(x_1\ ,\ y_1)$ and parallel to the given line $ax + by + c = 0$ is

$$\boxed{ax + by + k = 0}$$

NOTE

The two lines always differ in only constants.

Example

Find the equation of a line passing through

a. *Point* $(4\ ,1)$ *and parallel to* $2x + y + 1 = 0$
b. *Point*$(-1\ ,2)$ *and parallel to* $y = 2x - 5$
c. *Point* $(4\ ,7)$ *and parallel t*$2y = 4x - 50$

Solution

a. The line parallel to $2x + y + 1 = 0$ is $2x + y + k = 0$
b. But $(4\ ,1)$ lies on this required line
c. $\Rightarrow 2(4) + 1 + k = 0\ \therefore k = 9$
d. $\therefore 2x + y - 9 = 0$, is the line.
e. The line parallel to the line $y = 2x - 5\ or\ 2x - y - 5 = 0$ is $2x - y + k = 0$
   This line passes through the point $(-1\ ,2)$
   $\Rightarrow 2(-1) - 2 + k = 0\ \therefore k = 4$
   $\therefore\ 2x - y + 4 = 0$

   Note

   The **R.H.S** of the given parallel line to required must be equal to zero.
f. Line parallel to $2y = 4x - 5\ or\ 4x - 2y - 5 = 0$ is $4x - 2y + k = 0$
   This line passes through $(4\ ,7)$
   $\Rightarrow 4(4) - 2(7) + k = 0\ \therefore k = -2$
   $\therefore 4x - 2y - 2 = 0$

D. **To find equation of a line passing through a point $(x_1\ ,y_1)$ and perpendicular to the given line $ax + by + c = 0$**

From $ax + by + c = 0\ or\ y = -\frac{a}{b}x - \frac{c}{b}$, slope of this line is $m_1 = -\frac{a}{b}$

But for perpendicular lines, $m_1 \times m_2 = -1$; m_2 is the slope of the required line $\ \therefore m_2 = \frac{b}{a}$

Required line: $y - y_1 = m_2(x - x_1)$

$\Rightarrow y - y_1 = \frac{b}{a}(x - x_1)\ or\ ay - ay_1 = bx - bx_1 \Rightarrow bx - ay = bx_1 - ay_1$

$\Rightarrow bx - ay = d$

Generally the line parallel to $ax + by + c = 0$ is $\boxed{\boldsymbol{bx - ay + k = 0}}$

Example

Find the equation of a line passing through

a) $(2, 3)$ and perpendicular to $x + 2y + 5 = 0$ (***Uneb* 2011**)

b) $(-3, 1)$ and perpendicular to $2x + 3y - 4 = 0$

c) $(4, 5)$ and perpendicular to $2y = 4x - 5$

d) $(4, 6)$ and perpendicular to $x + y - 5 = 0$

Solution

a) $x + 2y + 5 = 0\ \ or\ y = -\frac{1}{2}x - \frac{5}{2}\ \therefore m_1 = -\frac{1}{2}$

Let m_2 be slope of the required line

From $m_1 \times m_2 = -1;\ m_2 = 2$ for perpendicular lines

Required line:

$y - y_1 = m_2(x - x_1) \Rightarrow y - 3 = 2(x - 2) = 2x - 4$

$\therefore 2x - y - 1 = 0$

b) $2x + 3y - 4 = 0\ \ or\ y = -\frac{2}{3}x + \frac{4}{3}\ \therefore m_1 = -\frac{2}{3}$

For perpendicular lines $m_1 \times m_2 = -1;\ m_2 = \frac{3}{2}$

Required line:

$y - y_1 = m_2(x - x_1) \Rightarrow y - 1 = \frac{3}{2}(x - -3) \Rightarrow 2y - 2 = 3x + 9$

$\therefore 3x - 2y + 11 = 0$

c) $2y = 4x - 5\ \ or\ y = 2x - \frac{5}{2}\ \therefore m_1 = 2$

From $m_1 \times m_2 = -1;\ m_2 = -\frac{1}{2}$

Required line:

$y - y_1 = m_2(x - x_1) \Rightarrow y - 5 = -\frac{1}{2}(x - 4) \Rightarrow 2y - 10 = -x + 4$

$\therefore x + 2y - 14 = 0$

d) Left as an exercise to the reader **Ans:** $y = x + 2$

E. **To find the equation of a line in terms of intercepts which it makes on the axes. (*Intercept form)***

Consider the figure below.

Let the line AB cut the $x - axis$ at A and $y - axis$ at B.

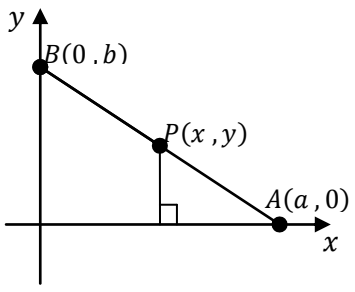
Let $\overline{OA} = a\ units\ and\ \overline{OB} = b\ units$

$\overline{OA}$ and $\overline{OB}$ are intercepts on $x - axis\ and\ y - axis$ respectively.

Let $P(x, y)$ be any point on the line

By similar triangles,

$\Delta PAC\ and\ \Delta OAB$ are similar



$\Rightarrow \frac{\overline{PC}}{\overline{OB}} = \frac{\overline{AC}}{\overline{OA}} \Rightarrow \frac{y}{b} = \frac{a-x}{a} = 1 - \frac{x}{a}$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1 \text{ or } \frac{x}{x-intercept} + \frac{y}{y-intercept} = 1$$

Example

1) *Find the equation of a line that cuts off intercepts of +2 and +3 on the +ve x and y − axis respectively*

Solution

Required line: $\frac{x}{a} + \frac{y}{b} = 1$, $a = x - intercept, b = y - intercept$

$\therefore \frac{x}{2} + \frac{y}{3} = 1$

2) *Find the equation of a line that cuts off intercepts of −3 and −4 on x and y − axis respectively.*

Solution

Required line: $\frac{x}{x-intercept} + \frac{y}{y-intercept} = 1$

$\therefore \frac{x}{-3} + \frac{y}{-4} = 1 \; or \frac{x}{3} + \frac{y}{4} = -1$

3) *Find the equation of the line cutting off an intercept of a on x − axis and making an angle of θ with x − axis in the first quadrant.Take* $\theta < 90^0$

Solution

Given $\overline{OA} = a \; and \; < O\hat{A}B = \theta$

Required $B(0\,, b)$ and the line

From trigonometry

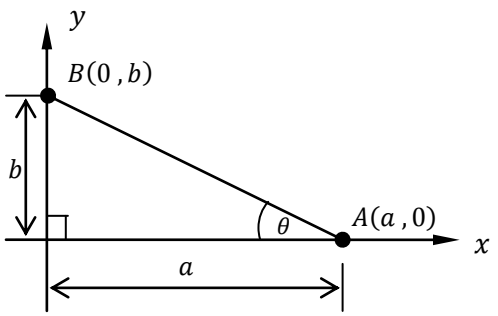
$\tan\theta = \frac{b}{a} \; \therefore b = atan\theta$

$\therefore B(0\,, atan\theta)$

Required line:

$\frac{x}{x-intercept} + \frac{y}{y-intercept} = 1$

$\Rightarrow \frac{x}{a} + \frac{y}{atan\theta} = 1 \;\; or \; x + ycot\theta = a$



4) *Find the equation of a line cutting off +ve intercepts on x-and y − axes, making an angle of* $\theta < 90^0$ *with x − axis and distance between intercepts is 13 units*

Solution

Now expressing intercepts in terms of 13 and θ

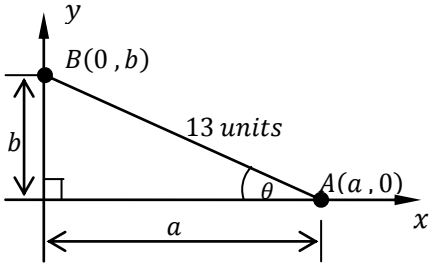
$\cos\theta = \frac{a}{13} \; \therefore a = 13\cos\theta$

$\sin\theta = \frac{b}{13} \; \therefore b = 13\sin\theta$

$\therefore A(13\cos\theta \;, 0) \; and \; B(0\,, 13\sin\theta)$

Required line:

$\frac{x}{x-intercept} + \frac{y}{y-intercept} = 1$



$\Rightarrow \frac{x}{13cos\theta} + \frac{y}{13sin\theta} = 1 \; or \;\; xsec\theta + ycosec\theta = 13$

Task

The points A and B lie on the positive sides of x and $y-axis$ respectively. If the length of AB is 15 units and $< OAB = \theta$, where O is the origin, find the equation of line AB (Leave θ in your answer) (***Uneb* 2010**)

F. **To find the equation of a line in terms of length of perpendicular, d upon it from origin and the angle θ the perpendicular makes with the $x-axis$**

Consider the figure below

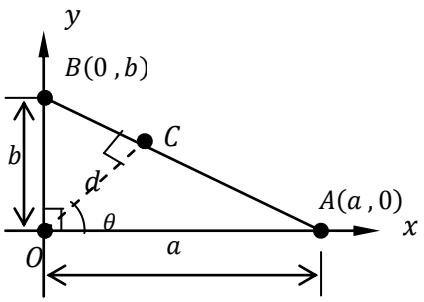
From the figure, $< O\hat{B}A = \alpha, < B\hat{O}C = 90 - \theta$

And $< A\hat{O}C = \theta$

From $\Delta OAC, cos\theta = \frac{d}{a} \; \therefore a = \frac{d}{cos\theta}$

From $\Delta OBC, sin\theta = \frac{d}{b} \; \therefore b = \frac{d}{sin\theta}$

From intercept equation $\frac{x}{\frac{d}{cos\theta}} + \frac{y}{\frac{d}{sin\theta}} = 1$

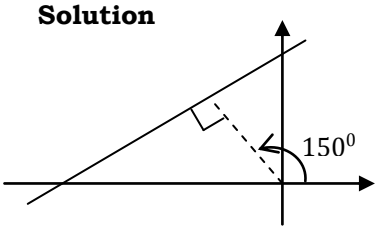


Required line is $\boxed{\boldsymbol{xcos\theta + ysin\theta - d = 0}}$

Examples

1. *Find the equation of a line whose perpendicular distance from the origin is 8 units and angle made by the perpendicular with $x-axis$ is 150^0*

Solution



Using $xcos\theta + ysin\theta - d = 0$

$\theta = 150^0 \, , d = 8$

$\Rightarrow xcos150^0 + ysin150^0 - 8 = 0$

$\Rightarrow x[-cos30^0] + y[sin30^0] = 8$

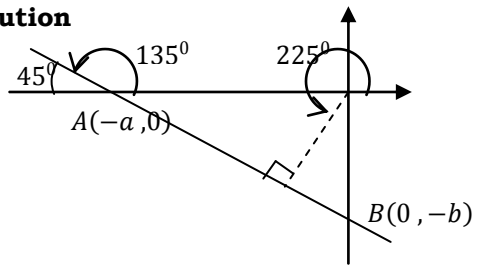
$\Rightarrow -x\sqrt{3} + \frac{y}{2} = 8 \; or \; 2\sqrt{3}x - y + 16 = 0$

Note

Numbers involving angles and distances need a clear sketch for illustration.

2. *Find the equation of a line cutting off negative intercepts on x and $y-axis$ and its perpendicular distance from the origin is 16 units if the line makes an angle of 135^0 with $x-axis$*

Solution



From interpretation of the diagram, angle made by the perpendicular with $+ve \; x-axis$ is 225^0

Required line:

$xcos\theta + ysin\theta - d = 0$

$\Rightarrow\ xcos225^0 + ysin225^0 - 16 = 0$
$\Rightarrow x[-cos45^0] + y[-sin45^0] - 16 = 0$
$\therefore -\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} = 16\ or\ x + y + 16\sqrt{2} = 0$

3. *Find the equation of a line which passes through* $(4, \sqrt{3})$ *and whose perpendicular distance from the origin on to the line makes an angle of* 60^0 *with the* $x-axis$

Solution

Required line:

$xcos60^0 + ysin60^0 - d = 0$ This line passes through $(4, \sqrt{3})$

$\Rightarrow \frac{4}{2} + \sqrt{3}.\frac{\sqrt{3}}{2} = d\ \ \therefore d = \frac{7}{2}$

$\therefore \frac{x}{2} + y\frac{\sqrt{3}}{2} = \frac{7}{2}\ or\ x + \sqrt{3}y - 7 = 0$

G. **To find equation of a line passing through the point of intersection of the two given lines and satisfying any other given condition**

Let the given lines be

$U = a_1x + b_1y + c_1 = 0\ .i.e.U = 0$
$V = a_2x + b_2y + c_2 = 0\ .i.e.V = 0$

Required line is $\boxed{(\boldsymbol{a_1x + b_1y + c_1}) + \boldsymbol{\lambda}(\boldsymbol{a_2x + b_2y + c_2}) = \boldsymbol{0}}$

Where $\lambda = arbitrary\ constant$, and can be obtained from any other given condition.

Example

1. *Find the equation of a straight line passing through the point of intersection of the pair of lines*

a) $3x - 4y + 5 = 0$ *and* $5x + 2y - 6 = 0$ *and is parallel to the line* $2x - y + 5 = 0$
b) $x - 2y + 3 = 0$ *and* $2x + 4y - 9 = 0$ *and containing the point* $(1, 1)$
c) $2x - 5y + 4 = 0$ *and* $5x - 6y - 3 = 0$ *and passing through the point* $(-5, -2)$
d) $5x + 2y - 4 = 0$ *and* $3x + y - 1 = 0$ *and perpendicular to the line* $4x - 15y + 6 = 0$

e) Solution

a) Equation of a line passing through point of intersection of the other two lines is $(3x - 4y + 5) + \lambda(5x + 2y - 6) = 0$

$\Rightarrow (3 + 5\lambda)x + (2\lambda - 4)y + (5 - 6\lambda) = 0$ ..... (1)

$eqn.(1)$ is parallel to $2x - y + 5 = 0$ when both have the same gradient

$\Rightarrow -\frac{(3+5\lambda)}{(2\lambda-4)} = 2\ or\ 3 + 5\lambda = -2(2\lambda - 4) = 8 - 4\lambda$

$\therefore \lambda = \frac{5}{9}$ ..... (2)

$eqn.(2) in\ to\ eqn.(1)$

$\Rightarrow \left(3 + \frac{25}{9}\right)x + \left(\frac{10}{9} - 4\right)y + \left(5 - \frac{30}{9}\right) = 0$ .....$(\times 9)$

$\therefore 52x - 26y + 15 = 0$ is the line required.

Alternatively

Solving the equations simultaneously for point of intersection

$3x - 4y + 5 = 0$ .....$(\times 1) \Rightarrow 3x - 4y + 5 = 0$

$5x + 2y - 6 = 0$ .....$(\times 2) \Rightarrow 10x + 4y - 12 = 0$

Adding the equations to eliminate y

$13x = 7 \Rightarrow x = \frac{7}{13}$

From $3x - 4y + 5 = 0$, and $x = \frac{7}{13} \Rightarrow 4y = 3 \times \frac{7}{13} + 5 \ \therefore y = \frac{43}{26}$

$\therefore \left(\frac{7}{13}, \frac{43}{26}\right)$, is the point of intersection

Now finding equation of a line passing through $\left(\frac{7}{13}, \frac{43}{26}\right)$ and parallel to

$2x - y + 5 = 0$ is $2x - y + k = 0$; $\left(\frac{7}{13}, \frac{43}{26}\right)$ lies on this line

$\Rightarrow \frac{14}{13} - \frac{43}{26} + k = 0 \ \therefore k = \frac{15}{26}$

$\therefore 2x - y + \frac{15}{26} = 0 \Rightarrow 52x - 26y + 15 = 0$

b) Required line:

$(x - 2y + 3) + \lambda(2x + 4y - 9) = 0$, $(1, 1)$ lies on this line

$\Rightarrow (1 - 2 + 3) + \lambda(2 + 4 - 9) = 0 \therefore \lambda = \frac{2}{3}$

$\Rightarrow (x - 2y + 3) + \frac{2}{3}(2x + 4y - 9) = 0$

$\Rightarrow 3x - 6y + 9 + 4x + 8y - 18 = 0$

$\therefore 7x + 2y - 9 = 0$ is the required line

c) Left as an exercise ***Answer***: $\boldsymbol{x - 2y + 1 = 0}$

d) Required line:

$(5x + 2y - 4) + \lambda(3x + y - 1) = 0$

$\Rightarrow (5 + 3\lambda)x + (2 + \lambda)y + (-4 - \lambda) = 0$ ..... (1)

$slope\ m_1 = -\frac{(5+3\lambda)}{(2+\lambda)}$

$eqn.(1)$ is perpendicular to $4x - 15y + 6 = 0$

Slope $m_2 = \frac{4}{15}$

But for perpendicular lines $m_1 \times m_2 = -1$

$\Rightarrow -\frac{(5+3\lambda)}{(2+\lambda)} \times \frac{4}{15} = -1 \Rightarrow 20 + 12\lambda = 30 + 13\lambda \ \therefore \lambda = -\frac{10}{3}$ ..... (2)

$eqn.(2) in\ to\ eqn.(1)$

$\Rightarrow \left(5 - \frac{30}{3}\right)x + \left(2 - \frac{10}{3}\right)y + \left(-4 + \frac{10}{3}\right) = 0 \Rightarrow -5x - \frac{4}{3}y - \frac{2}{3} = 0$ ..... $(\times -3)$

$15x + 4y + 2 = 0$

Task

Do (b) , (c) and (d) using the 2nd *alternative. i. e.* 1st *get point of intersection , and apply the suitable form of equation of line*

Note

When finding point of intersection of two lines, solve the equations simultaneously.

PERPENDICULAR DISTANCE OF A GIVEN POINT FROM A GIVEN LINE

Let the given line be $ax + by + k = 0$, which can be expressed as $y = mx + c \; or \; mx - y + c = 0$, and the point $A(p\,,q)$

Consider the figures below

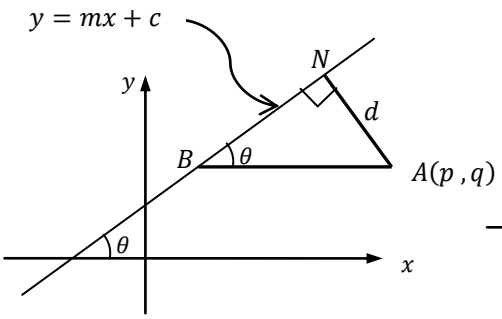


Fig. I. A and Origin on same side

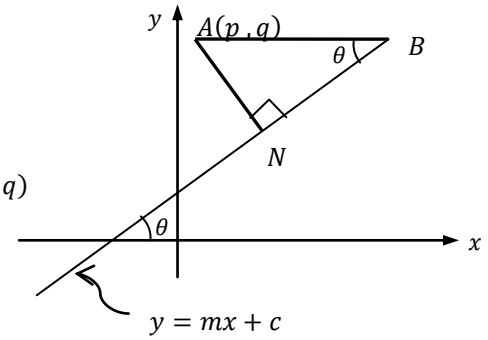


Fig. I. A and Origin on opposite side

$A(p\,,q)$ is at a distance , d from the line

$\overline{AB}$ is horizontal $.i.e. parallel\ to\ x-axis$

Finding coordinates

At B, $y - coordinate = q \;\; \therefore from\ y = mx + c\,, x = \frac{q-c}{m}$

$\therefore B\left(\frac{q-c}{m}\,, q\right)$

$From\ fig. I;\ \overline{AB} = p - \left(\frac{q-c}{m}\right) = \frac{mp-q+c}{m}$

$From\ fig. I;\ \overline{AB} = \frac{q-c}{m} - p = -\left[p - \left(\frac{q-c}{m}\right)\right] = -\left[\frac{mp-q+c}{m}\right]$

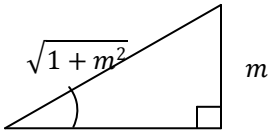
$\Rightarrow For\ any\ point\ A,$

$\overline{AB} = \pm\left[\frac{mp-q+c}{m}\right]$

From trigonometry,

$d = \overline{AB} sin\theta$, but $m = tan\theta$

$\therefore sin\theta = \frac{m}{\sqrt{1+m^2}}$



$\therefore perpendicular\ distance, d = \overline{AB} sin\theta$

$$= \pm\left[\frac{mp-q+c}{m}\right] \cdot \frac{m}{\sqrt{1+m^2}}$$

$$= \pm\left[\frac{mp-q+c}{\sqrt{1+m^2}}\right]$$

$\therefore$ Perpendicular distance of a point $(p\,,q)$ from line $y = mx + c \;\; or \;\; mx - y + c = 0$ is $\pm\left[\frac{\boldsymbol{mp-q+c}}{\sqrt{\boldsymbol{1+m^2}}}\right]$

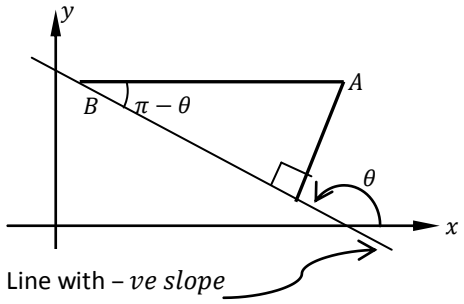
Line is $ax + by + k = 0 \Rightarrow y = -\frac{a}{b}x - \frac{k}{b}$, *where* $m = -\frac{a}{b}$, $c = -\frac{k}{b}$ in the equation $y = mx + c$, where $m = -\frac{a}{b}$, $c = -\frac{k}{b}$ in the equation $y = mx + c$

$\therefore$ Perpendicular distance $= \pm\left(\frac{-\frac{a}{b}p - q - \frac{k}{b}}{\sqrt{\left(-\frac{a}{b}\right)^2 + 1}}\right) = \pm\left[\frac{\boldsymbol{ap+bq+k}}{\sqrt{\boldsymbol{a^2+b^2}}}\right]$

$\therefore$ Perpendicular Distance of $(\boldsymbol{p}, \boldsymbol{q})$ from the line $\boldsymbol{ax + by + k = 0}$ is $\pm\left[\frac{\boldsymbol{ap+bq+k}}{\sqrt{\boldsymbol{a^2+b^2}}}\right]$

Note

1) When finding perpendicular distance of a point (p, q) from the line $ax + by + k = 0$ $(\mathbf{R.H.S = 0})$
   "Substitute the point in the line when $\boldsymbol{R.H.S = 0}$ *and divide the result by the square root of the sum of squares of coefficient of x and y in the equation"*
2) The above result is also valid for lines with negative gradient since $\overline{AN} = d = \overline{AB}sin(\pi - \theta) = \overline{AB}sin\theta$



3) The sign $(\pm)$ before the expression is adjusted with the sign of k or with sign of b if $k = 0$ in the equation $ax + by + k = 0$, so the perpendicular distance, d is $+ve$ when the *point* and the *origin* are on *opposite sides* of the line, and $-ve$ if they are on the *same side* of line.

Example

1. *Find the perpendicular distance of the given point from the stated lines in each of the following*

   a) Point $(-5, -6)$, $line: 3x + 4y + 12 = 0$
   b) Point $(2, -5)$, $line: 12x - 5y + 7 = 0$
   c) Point $(-5, 6)$, $line: 12x + 5y - 9 = 0$

Hence state whether the point and the origin are on the same side of the line.

Solution

a) $(-5, -6) \equiv (p, q)$; $3x + 4y + 12 = 0 \equiv ax + by + k = 0$

$\therefore perpendicular\ distance = \pm\left[\frac{ap+bq+k}{\sqrt{a^2+b^2}}\right]$

$= \pm\left[\frac{3\times(-5)+4\times(-6)+12}{\sqrt{3^2+4^2}}\right]$

Adjusting the sign on $k = 12$ before the expression

$\therefore perpendicular\ distance = -\left[\frac{3\times(-5)+4\times(-6)+12}{\sqrt{3^2+4^2}}\right] = \frac{27}{5} units$

Since perpendicular distance is $+ve$, then the point and the origin are on the opposite sides of the line.

b) $(2, -5) \equiv (p, q)\ ;\ 12x - 5y + 7 = 0 \equiv\ ax + by + k = 0$

$\therefore perpendicular\ distance\ = \pm\left[\frac{12\times(3)+(-5)\times(-5)+7}{\sqrt{12^2+(-5)^2}}\right]$

$= -\left[\frac{36+25+7}{13}\right]\quad \left\{\begin{matrix} sign\ on \\ k = 7\ is + ve, hence\ taking - ve\ sign\ before\ expression\end{matrix}\right\}$

$= -\frac{68}{13}, hence\ distance\ is \frac{68}{13} units\ (taking\ numerical\ value)$

Since the perpendicular distance is negative, then the origin and the point are on the same side of the line.

c) $(-5, 6) \equiv (p, q)\ ;\ 12x + 5y - 9 = 0 \equiv\ ax + by + k = 0$

$\therefore perpendicular\ distance\ = \pm\left[\frac{12\times(-5)+(5)\times(6)-9}{\sqrt{12^2+5^2}}\right]$

$= +\left[\frac{-60+30-9}{13}\right]$

$\{+ve\ sign\ before\ expression\ is\ taken\ \because k = -7\}$

$\therefore perpendicular\ distance = 3units\ (taking\ numerical\ value)$

Since perpendicular distance is $-ve$, the origin and the point are on the same side of the line.

2. *If p and q be lengths of the perpendiculars from the origin upon the straight lines* $4x + 3y - 5cos\alpha = 0$ *and* $3x - 4y - \frac{5}{2}sin\alpha = 0$ *respectively, prove that* $p^2 + 4q^2 = 1$

Solution

$p = \pm\left[\frac{4\times0+3\times0-5cos\alpha}{\sqrt{4^2+3^2}}\right] = \pm cos\alpha\ \Rightarrow p = -5cos\alpha$ ..... (1)

$q = \pm\left[\frac{3\times0-4\times0-\frac{5}{2}sin\alpha}{\sqrt{3^2+4^2}}\right] = \pm\frac{sin\alpha}{2}\ \Rightarrow q = -\frac{sin\alpha}{2}$ ..... (2)

$eqn.(1)^2 + (2)^2$

$\Rightarrow p^2 + 4q^2 = sin^2\alpha + cos^2\alpha = 1\ \therefore p^2 + 4q^2 = 1$

3. *Given that U and V are lengths of perpendicular from the point* $(1, -1)$ *upon the straight lines* $4x + 3y - 5cos\theta = 0$ *and* $3x - 4y - 5sin\theta = 0$. *Show that the equation connecting perpendicular distances U and V is an equation of a circle. Hence state its centre and radius*

Solution

$U = \pm\left[\frac{4\times1+3\times(-1)-5cos\theta}{\sqrt{4^2+3^2}}\right] = \pm\frac{1-5cos\theta}{5}$

$\therefore U = -\left(\frac{1-5cos\theta}{5}\right)\ or\ U + \frac{1}{5} = cos\theta$ ..... (1)

$V = \pm\left[\frac{3\times1-4\times(-1)-5sin\theta}{\sqrt{3^2+4^2}}\right] = \pm\frac{7-5sin\theta}{5}$

$\therefore V = -\left(\frac{7}{5} - sin\theta\right)\ or\ V + \frac{7}{5} = sin\theta$ .....(2)

$From\ cos^2\theta + sin^2\theta = 1$

$\Rightarrow \left(U + \frac{1}{5}\right)^2 + \left(V + \frac{7}{5}\right)^2 = 1$, which is the equation of a circle of centre $\left(-\frac{1}{5}, -\frac{7}{5}\right)$ and radius 1 $unit$

NOTE

This example illustrates the relevance of perpendicular distance of a point from a line in determining the path traced by the point $w.r.t.$ the line $i.e.$ locus of a point.

EQUATIONS OF BISECTORS OF THE ANGLES BETWEEN TWO STARIGHT LINES

Let AB and CD be two given lines whose equations are:

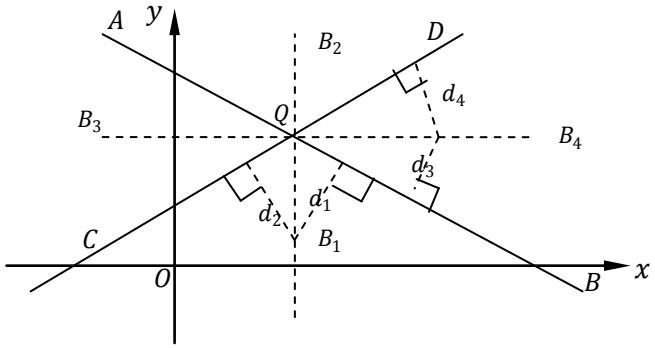
$a_1x + b_1y + c_1 = 0$, and

$a_2x + b_2y + c_2 = 0$ respectively

Let Q be the point of intersection of the lines.

Consider the figure below

$\overline{B_1B_2}$, $\overline{B_3B_4}$ $are\ angle\ bisectors$



From the figure, $B_1$is the point on the angle bisector having coordinates$B_1(x_1, y_1)$ and $B_2(x_2, y_2)$

$d_1\ and\ d_2$ are perpendicular distances of B_1 from lines AB and CD respectively, and $d_3\ and\ d_4$ are perpendicular distances of B_4 from AB and CD respectively.

$\Rightarrow d_1 = \pm\frac{a_1x_1+b_1y_1+c_1}{\sqrt{{a_1}^2+{b_1}^2}}\ and\ d_2 = \pm\frac{a_2x_1+b_2y_1+c_2}{\sqrt{{a_2}^2+{b_2}^2}}$

$\Rightarrow d_3 = \pm\frac{a_1x_2+b_1y_2+c_1}{\sqrt{{a_1}^2+{b_1}^2}}\ and\ d_4 = \pm\frac{a_2x_2+b_2y_2+c_2}{\sqrt{{a_2}^2+{b_2}^2}}$

Assuming $c_1\ and\ c_2$ are of the same sign, then $d_1 = d_2\ and\ d_3 = d_4$

NOTE:

From the figure B_1 and the origin are on the same side of the lines, and B_4 and origin are on the same side of CD but opposite side of AB

$\therefore d_1 = -ve, and\ d_2 = -ve$

$$\Rightarrow -\left(\frac{a_1x_1+b_1y_1+c_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) = -\left(\frac{a_2x_1+b_2y_1+c_2}{\sqrt{{a_2}^2+{b_2}^2}}\right)$$

$\therefore d_3 = +ve\ and\ d_4 = -ve$

$$+\left(\frac{a_1x_2+b_1y_2+c_1}{\sqrt{{a_1}^2+{b_1}^2}}\right) = -\left(\frac{a_2x_2+b_2y_2+c_2}{\sqrt{{a_2}^2+{b_2}^2}}\right)$$

$\Rightarrow$ The point $B_1(x_1, y_1)$ lies on the line whose equation is

$$\frac{a_1x+b_1y+c_1}{\sqrt{{a_1}^2+{b_1}^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{{a_2}^2+{b_2}^2}}$$

$\Rightarrow$ The point $B_4(x_2, y_2)$ lies on the line whose equation is

$$+\frac{a_1x+b_1y+c_1}{\sqrt{{a_1}^2+{b_1}^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{{a_2}^2+{b_2}^2}}$$

Combining the equations

Angle bisector equations of angles made by the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by

$$\boxed{\frac{a_1x+b_1y+c_1}{\sqrt{{a_1}^2+{b_1}^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{{a_2}^2+{b_2}^2}}}$$

Note

The upper sign $(+ve)$ gives the equation of the bisector of the angle containing the origin when $c_1\ and\ c_2$ are of the same sign. $i.e.$ Always re-arrange the equations to make $c_1\ and\ c_2$ of the same sign before using them in the formula

Example

1) *Find the equations of angle bisectors between the following lines*
   a) $7x - 24y + 55 = 0\ and\ 4x - 3y - 65 = 0$
   b) $3x + 4y - 5 = 0\ and\ 6x - 8y + 7 = 0$
   c) $\sqrt{3}x + y - 7 = 0\ and\ x - \sqrt{3}y + 1 = 0$
   d) $3x + 4y - 2 = 0\ and\ 5x - 12y + 2 = 0$

Solution

a) $7x - 24y + 55 = 0\ and\ 4x - 3y - 65 = 0$ since the constants are not of the same sign, re-arranging the equations for same constants

$7x - 24y + 55 = 0\ and - 4x + 3y + 65 = 0\ (Re - arranged)$

$\Rightarrow$ Equation of the bisector containing the origin is

$$-\left[\frac{7x-24y+55}{-\sqrt{7^2+24^2}}\right] = -\left[\frac{-4x+3y+65}{-\sqrt{4^2+3^2}}\right] \Rightarrow \frac{7x-24y+55}{25} = \frac{-4x+3y+65}{5}$$

$\therefore 9x - 13y - 90 = 0$ is the required line

And equation of bisector of the angle which does not contain the origin is

$+\left[\frac{7x-24y+55}{-\sqrt{7^2+24^2}}\right] = -\left[\frac{-4x+3y+65}{-\sqrt{4^2+3^2}}\right] \Rightarrow \frac{7x-24y+55}{-25} = \frac{-4x+3y+65}{5}$

$\therefore 13x + 9y - 380 = 0$ is the required line

b) $3x + 4y - 5 = 0\ and\ 6x - 8y + 7 = 0$ since the constants are not of the same sign,

Re-arranging the equations for same constants

$3x + 4y - 5 = 0\ and - 6x + 8y - 7 = 0\ (Re - arranged)$

Equation of bisector containing the origin is

$-\left[\frac{3x+4y-5}{+\sqrt{3^2+4^2}}\right] = -\left[\frac{-6x+8y-7}{+\sqrt{6^2+8^2}}\right] \Rightarrow \frac{3x+4y-5}{5} = \frac{-6x+8y-7}{10}$

$\therefore 16y - 17 = 0$

c) $\sqrt{3}x + y - 7 = 0\ and\ x - \sqrt{3}y + 1 = 0$ re-arranging the equations for same constants

$\sqrt{3}x + y - 7 = 0\ and - x + \sqrt{3}y - 1 = 0$

Equation of bisector containing the origin is

$-\left[\frac{\sqrt{3}x+y-7}{+\sqrt{3+1}}\right] = -\left[\frac{-x+\sqrt{3}y-1}{+\sqrt{3+1}}\right] \Rightarrow \frac{\sqrt{3}x+y-7}{2} = \frac{x-\sqrt{3}y+1}{2}$

$\therefore \left(\sqrt{3}-1\right)x + \left(1+\sqrt{3}\right)y - 8 = 0$

d) $3x + 4y - 2 = 0\ and\ 5x - 12y + 2 = 0\ or - 5x + 12y - 2 = 0$

Equation of bisector containing the origin is

$-\left(\frac{3x+4y-2}{+\sqrt{3^2+4^2}}\right) = -\left(\frac{-5x+12y-2}{+\sqrt{5^2+12^2}}\right) \Rightarrow \frac{3x+4y-2}{5} = \frac{5x+12y-2}{13}$

$\therefore 64x - 8y - 16 = 0\ or\ 8x - y - 2 = 0$

Equation of bisector of angle not containing origin

$+\left(\frac{3x+4y-2}{+\sqrt{3^2+4^2}}\right) = -\left(\frac{-5x+12y-2}{+\sqrt{5^2+12^2}}\right) \Rightarrow \frac{3x+4y-2}{5} = \frac{5x-12y+2}{13}$

$\therefore 14x + 112y - 36 = 0\ or\ 7x + 56y - 18 = 0$

NOTE

Referring to the last example, when finding equation of bisector containing the origin, the distances must be $-ve$, hence $-ve$ sign before the bracketed terms, and the inside sign in the bracket must be opposite to that of the constant in the equation, and when finding the equation of bisector not containing the origin the equated distances must be opposite in sign. $i.e. +ve$ and then $-ve$ to the second.

TRIGONOMETRY

Introduction:

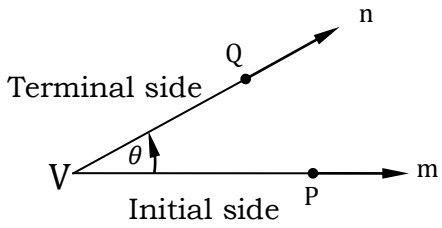
This deals with the study of relationship between the sides and angles of a triangle.

Angles and their measure

Angles

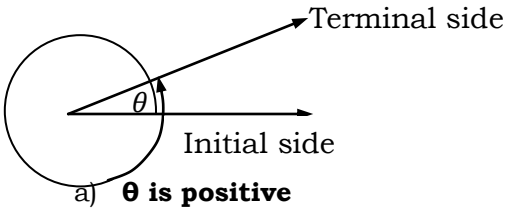
The **angle** is formed by rotating (in planes) a ray *m*, called the *initial side* of the angle, around the end point until it coincides with a ray *n*, called the *terminal side* of the angle.

The common end point *V* of *m* and *n* is called the *vertex.*

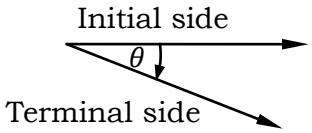


A counter clockwise produces *a positive angle* and a clockwise rotation produces a *negative angle.*

Consider the fig. below



a) **θ is positive**



b**) θ is negative**

Co terminal angles:

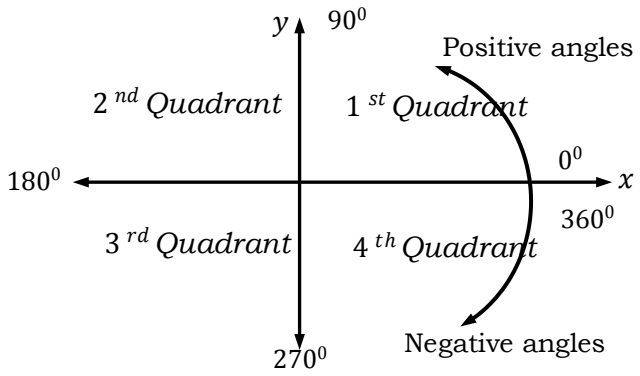
Any angles having the same terminal and initial sides are said to be *co terminal*.

The six ratios are called *trigonometrical ratios, and each gas a name.*

Angle measured with respect to x- axis

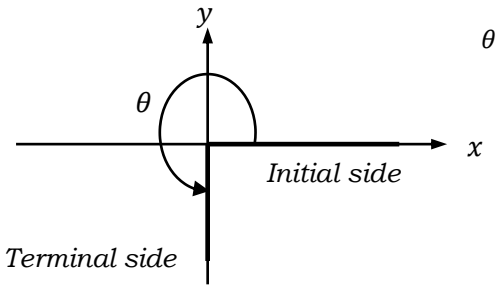
Any angle in a rectangular coordinate system (x – y axes) is said to be in *standard position* If its vertex is at the *origin* and the initial side is along the positive x – axis.

Consider the fig. below



Quadrantal angle

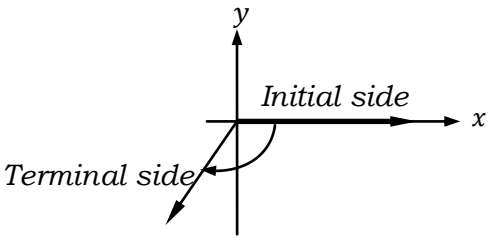
If the terminal side of an angle is in standard position lies along the coordinate axes, the angle is said to be a *Quadrantal angle*



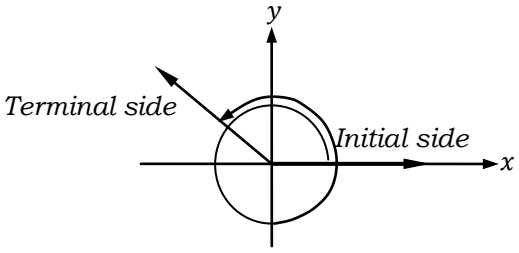
θ is a quadrantal angle

Angle in a quadrant

If the terminal side does not along a coordinate axis, the angle is referred to in terms of the quadrant in which the terminal side lies.



a) θ is in the 3rd quadrant

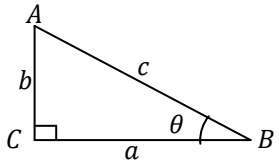


b) θ is in 2nd quadrant

The Trigonometrical functions of acute angles:

In many applications of trigonometry, you are given the measure of *two sides* of *right angled triangle* or the measure of an *acute angle* and *one side,* and you are asked to find the measure of the remaining sides and acute angles, this is called *solving a triangle.*

Solving a triangle is accomplished by using special *trigonometrical ratios* associated with each acute angle θ consider the fig. below,



There are altogether *six* ratios between the sides of the Δ ABC viz:

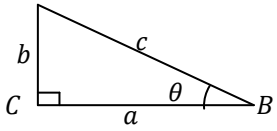
$$\frac{b}{c}, \frac{c}{b}, \frac{a}{b}, \frac{b}{a}, \frac{a}{c}, \frac{c}{a}$$

The six ratios are called *trigonometric ratios* and each has a name

. i. e. *Sine* (sin). *Cosine* (cos), *tangent* (tan), *Cosecant* $(cosec)$, *Secant* (sec) *and cotangent* (cot).

Definitions of the ratios:

Consider the right angled Δ ABC



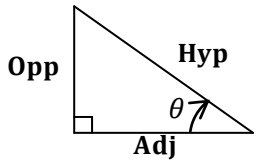
$$\cos\theta = \frac{a}{c} \qquad \text{cosec}\,\theta = \frac{c}{b}$$

$$\sin\theta = \frac{b}{c} \qquad \sec\theta = \frac{c}{a}$$

$$\tan\theta = \frac{b}{a} \qquad \cot\theta = \frac{a}{b}$$

Side a = *side adjacent* to angle θ
Side b = *side opposite to the angle* θ
Side c = *opposite side to the right angle or hypotenuse*

Using these definitions, then the alternative form;



$$\cos\theta = \frac{Adj}{Hyp} \qquad \text{cosec}\,\theta = \frac{Hpy}{Opp}$$

$$\sin\theta = \frac{Opp}{Hyp} \qquad \sec\theta = \frac{Hyp}{Adj}$$

$$\tan\theta = \frac{Opp}{Adj} \qquad \cot\theta = \frac{Adj}{Opp}$$

Relation between the six ratios

From the above definitions, three of these ratios are the reciprocals of the other viz:

$$\text{cosec}\,\theta = \frac{1}{\sin\theta}\ ,\ \sec\theta = \frac{1}{\cos\theta} \text{ and } \cot\theta = \frac{1}{\tan\theta}$$

NOTE:

$\tan\theta = \frac{b}{a} = \left(\frac{b}{c}\right) / \left(\frac{a}{c}\right) = \frac{\sin\theta}{\cos\theta}$ and hence $cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{tan\theta}$$

NOTE:

1. When using reciprocal relations many students tend to associate *cosecant* with *cosine* and *secant* with *sine,* just the opposite is the case. Cosecant goes with sine and secant with cosine.
2. In remembrance of reciprocal relations, in each of the three pairs of reciprocals, there is one ratio with the prefix "*CO"* thus the reciprocal of sin is cosec, of sec is cos, of tan is cot.

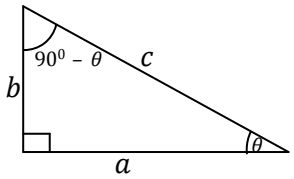
Application of reciprocal trig Ratios:

- Proof of identities
- Solution to trig equations
- Elimination of parameters
- Simplifying trig. Expressions
- Integration

Related examples shall be tackled later.

Complementary Angles and Co-functions;

Consider a right angled triangle below.



$\sin\theta = \frac{b}{c}$, $\cos(90 - \theta)$, $\tan\theta = \frac{b}{a} = \frac{1}{a/b} = \frac{1}{\tan(90-\theta)}$

$\therefore \tan\theta = \cot(90 - \theta)$

$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{a/c} = \frac{1}{\sin(90-\theta)} \therefore \sec\theta = \text{cosec}(90 - \theta)$

$\therefore \sin\theta = \cos(90 - \theta), \sec\theta = \text{cosec}\,\theta\ (90 - \theta)$, and $\tan\theta = \cot(90 - \theta)$

The trigonometrical ratios cosine, cotangent and cosecant contain the prefix "CO". so some times are referred to as the *co functions* of sine, tangent and secant respectively.

Application of complementary Angles and co functions:

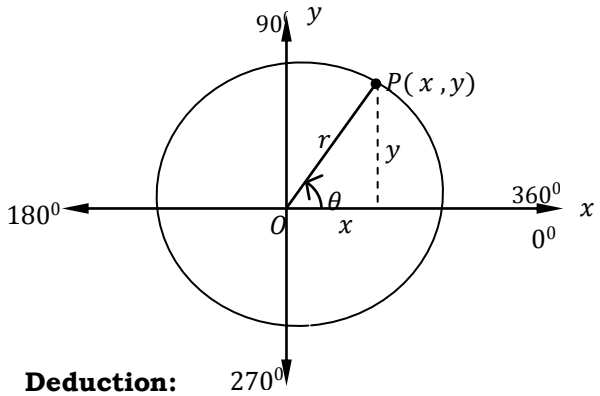
- Proof of identities
- Solution to trig equations
- Elimination of parameters
- Simplifying trig. Expressions
- Integration

Related examples shall be tackled later.

General angle (angles of any magnitude)

Consider an angle θ as an arbitrary in standard position in rectangular coordinate system and *P(x, y* a point *r units)* from the origin on the terminal side of θ. Six ratios involving *r* and coordinates of P can be derived, taking $\overline{OP}$ = r (always +ve),

ANGLES IN 1st QUADRANT, $0^0 \leq \theta \leq 90^0$ (*Acute angle case)*

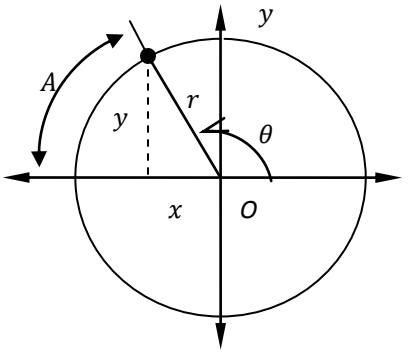


$$\sin\theta = \frac{+y}{+r} = +\frac{y}{r}$$
$$\cos\theta = \frac{+x}{+r} = +\frac{x}{r}$$
$$\tan\theta = \frac{+y}{+x} = +\frac{y}{x}$$
$$\cot\theta = \frac{+x}{+y} = +\frac{x}{y}$$
$$\sec\theta = \frac{+r}{+x} = +\frac{r}{x}$$
$$\operatorname{cosec}\theta = \frac{+r}{+y} = +\frac{r}{y}$$

Deduction:

All trig. Ratios of angles in 1st quadrant are positive

ANGLES IN 2nd OUADRANT, $90^0 \leq \theta \leq 180^0$ (*Obtuse angle case)*

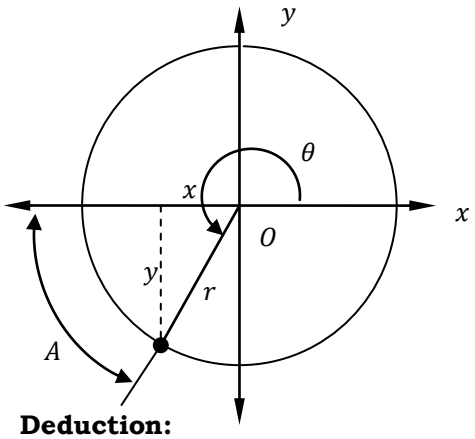


$$A = 180 - \theta$$
$$\sin\theta = \frac{+y}{+r} = +\frac{y}{r} \qquad \cos\theta = \frac{-x}{+r} = -\frac{x}{r}$$
$$\tan\theta = \frac{+y}{-x} = -\frac{y}{x} \qquad \cot\theta = \frac{-x}{+y} = -\frac{x}{y}$$
$$\sec\theta = \frac{+r}{-x} = -\frac{r}{x} \qquad \operatorname{cosec}\theta = \frac{+r}{+y} = +\frac{r}{y}$$

Deduction:

Only sines and cosecants of angles in 2nd quadrant are positive, the other trig. Ratios are negative.

ANGLES IN 3rd QUADRANT, $180^0 \leq \theta \leq 270^0$



$A = \theta - 180^0$

$\sin\theta = \frac{-y}{+r} = -\frac{y}{r}$

$\cos\theta = \frac{-x}{+r} = -\frac{x}{r}$

$\cot\theta = \frac{-x}{-y} = +\frac{x}{y}$

$\operatorname{cosec}\theta = \frac{+r}{-y} = -\frac{r}{y}$

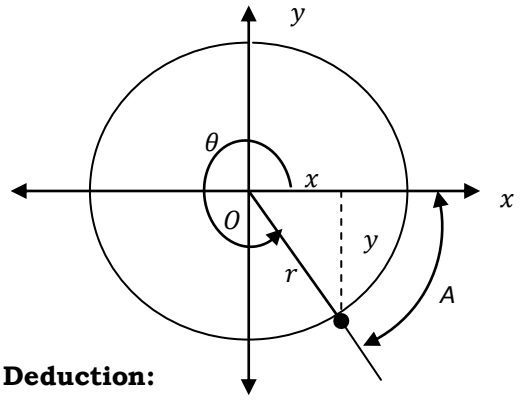
$\sec\theta = \frac{+r}{-x} = -\frac{r}{x}$

$\tan\theta = \frac{-y}{-x} = +\frac{y}{x}$

Deduction:

Only tangents and cotangent of angles in 3rd quadrant are positive, the other trig ratios are negative.

ANGLES IN 4th QUADRANT, $270^0 \leq \theta \leq 360^0$ (*Reflex angle case)*



$A = 360^0 - \theta$

$\sin\theta = \frac{-y}{+r} = -\frac{y}{r}$

$\cos\theta = \frac{+x}{+r} = +\frac{x}{r}$

$\cot\theta = \frac{+x}{-y} = -\frac{x}{y}$

$\sec\theta = \frac{+r}{+x} = +\frac{r}{x}$

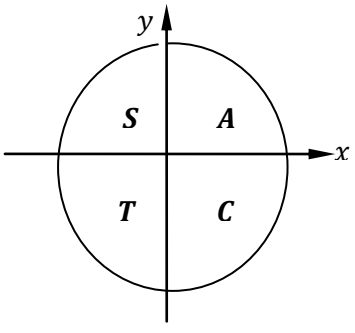
$\operatorname{cosec}\theta = \frac{+r}{-y} = -\frac{r}{y}$

$\tan\theta = \frac{-y}{+x} = -\frac{y}{x}$

Deduction:

Only cosines and secants of angles in 4th quadrant are positive, the other trig. Ratios are negative

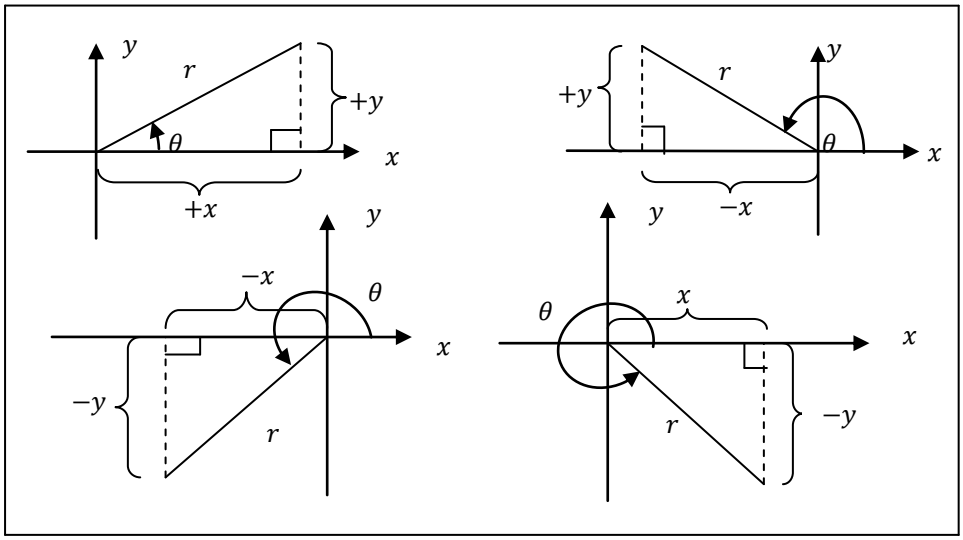
Summary of the results:



The abbreviated letters S, T and C denote theSines , Tangents and Cosines of angles being positive. Even their reciprocal ratios are positive. A denotes ALL trig ratios of angles in that quadrant are positive; the rest undefined letters are negative for the ratios.

NOTE:

1. The positive/negative sign on x and y are assumed as the sign conventions of rectangular coordinates axes.
2. The triangle being formed with its base on the horizontal axis (x- axis) is called the *reference triangle.*
3. The acute angle (always taken positive) between the terminal side of θ and the horizontal axis (x – axis) is called the *reference angle*.
4. r = $\sqrt{x^2 + y^2} > 0$.
5. A student should b very careful to verify each item on the reference triangle below in each quadrant.



Application of the above results:

- Evaluating trig. functions
- Expressing trig ratios of angles in terms of acute angled trig. Ratios
- Proof of identities
- Evaluation of inverse of trig functions in terms of acute angles.

Evaluation of trigonometric functions:

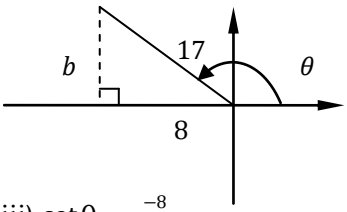
1. *Given that* $\cos\theta = -\frac{8}{17}$ *and* θ *is obtuse,*

Find(i) $\sin\theta$ (ii) $\tan\theta$ (iii) $\cot\theta$ (iv) $\operatorname{cosec}\theta$ (v) $\sec\theta$.

Solution

Given $\cos\theta = -\frac{8}{17}$, $90^0 < \theta < 180^0$,

$\cos\theta = \frac{Adj}{Hyp} = \frac{-8}{17}$



By Pythagoras theorem, $b^2 + 8^2 = 17^2$

$\therefore b = 15$

i) $\sin\theta = +\frac{15}{17}$ ii) $\tan\theta = \frac{15}{-8}$ iii) $\cot\theta = \frac{-8}{15}$

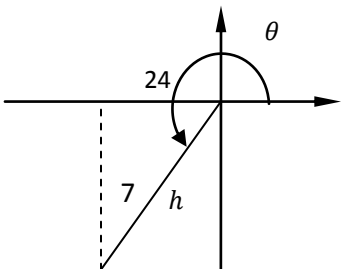
iv) $\sec\theta = \frac{+17}{15}$ v) $\sec\theta = \frac{-17}{8}$

2. *If* $\tan\theta = \frac{7}{24}$ *and* θ *is reflex, find without using tables/calculator the values;*

i) $\sec\theta$ *ii)* $\sin\theta$ iii) $\cot\theta$ iv) $\operatorname{cosec}\theta$ v) $\cos\theta$

Solution

Given θ *is reflex, then* $180^0 < \theta < 270^0$



Since $\tan\theta = \frac{Opp}{Adj} = \frac{7}{24}$

By Pythagoras theorem,

$7^2 + 24^2 = h^2,\ h = 25$

i) $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-24/25} = -\frac{25}{24}$

ii) $\sin\theta = \frac{-7}{25}$

iii) $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{7/24} = \frac{24}{7}$

iv) $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{-7/25} = -\frac{25}{7}$ v) $\cos\theta = -\frac{24}{25}$

Task:

If $\sin\theta = \frac{-4}{5}$ *for* $270^0 < \theta < 360^0$, *find*

i) $\operatorname{cosec}\theta$ ii) $\sec\theta$ iii) $\cot\theta$ iv) $\cos\theta$

v) $\tan\theta$ vi) $3\cos\theta - 5\tan\theta$ vi) $4\tan\theta - 3\operatorname{cosec}\theta$.

Evaluating inverse of trig Functions in terms of acute angles:

Find the exact vales of two angles less 360^0.

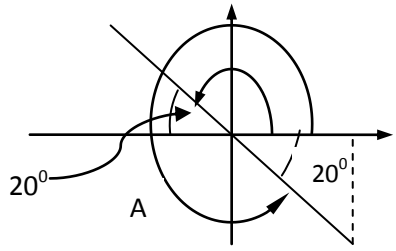
i) $\tan^{-1}(-0.36397)$ iv) $\operatorname{cosec}^{-1}(-2.7290)$

ii) $\cot^{-1}(1.73205)$ v) $\sec^{-1}(-1.0670)$

iii) $\sin^{-1}(0.866)$

Solution

i) let $A = \tan^{-1}(-0.36397) \Rightarrow \tan A = -0.36397\ (-ve)$



Now finding $\tan^{-1}(-0.36397) = 20^0$ (*Reference angle)*

$\therefore A = 180 - 20 = 160^0$

$A = 360 - 20 = 340^0$

$\therefore \tan^{-1}(-0.36397) = 160^0, 340^0$

ii) Let $B = \cot^{-1}(1.73205)$

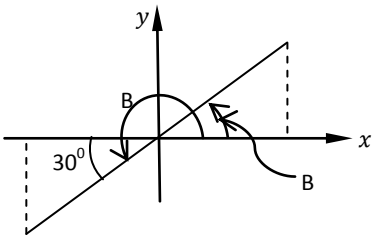
$\cot B = 1.73205$, but $\cot B = \frac{1}{\tan\theta}$

$\therefore \tan\theta = \frac{1}{1.73205} = 0.5774\ (+ve)$

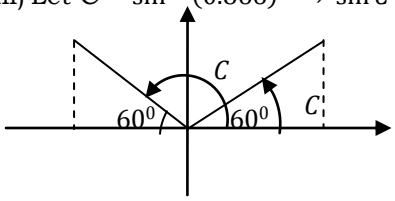
$\tan^{-1} 0.5774 = 30^0$ (*Reference angle)*

$\therefore B = 30^0, 180 + 30^0 = 210^0$

$\therefore \cot^{-1}(1.73205) = 30^0, 210^0$



iii) Let C = $\sin^{-1}(0.866) \Rightarrow \sin C = 0.866$ (+ve)



$\sin^{-1}(0.866) = 60^0 (reference\ angle)$

$\therefore C = 180 - 60^0 = 120^0, 60^0$

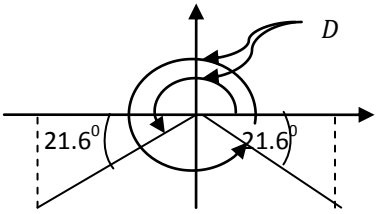
$\therefore \sin^{-1}(0.866) = 60^0, 120^0$

iv) Let D = $\text{cosec}^{-1}(-2.7290)$

$\Rightarrow$ cosecD $= -2{,}7290$

, but cosec D = $\frac{1}{\sin D} = -2.7290$

$\therefore \sin D = -0.3664\ (-ve)$



$\sin^{-1}(+0.3664) = 21.5^0\ (reference\ angle)$

$\therefore D = 180 + 21.5 = 201.5^0\ or\ 360 - 21.5 = 338.5^0$

$\therefore \text{cosec}^{-1}(-2.7290) = 201.6^0, 338.5^0$

Task:

Find two values of φ less than 360⁰ which satisfy the following trig functions.

i) $\cot\varphi = -0.06315$ $[\cot^{-1}(-0.06315)]$ $(\boldsymbol{Ans}: \mathbf{93.6^0, 273.6^0})$

ii) $\sec\varphi = -1.0670$ $[\sec^{-1}(-1.0670)]$ $(\boldsymbol{Ans}: \mathbf{159.6^0, 200.4^0})$

iii) $\text{cosec}\,\varphi = -1.4142$ $[\text{cosec}^{-1}(-1.4142)]$ $(\boldsymbol{Ans}: \mathbf{225^0, 315^0})$

Expressing trig ratios in terms of trig ratios of acute angles:

Express in terms of trig ratios of acute angles,

$\sin 170^0$ *iv)* $\sec(-172^0)$

i) $\tan 200^0$ v) $\text{cosec}(-590^0)$

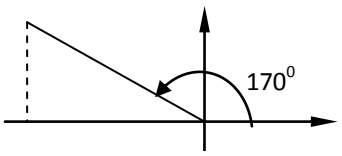
ii) $\cot 200^0$ vi) $\cot(-860^0)$

Solution:

i) $\sin 170^0$, $+170^0$ means counter clockwise direction

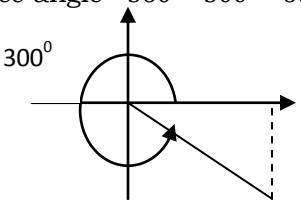
Reference angle = $180 - 170 = 10^0$

$\therefore \sin 170^0 = +\sin 10^0$



ii) $\tan 300^0$

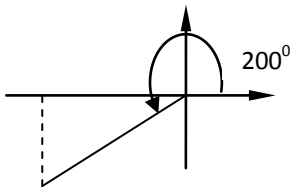
Reference angle = $360 - 300 = 60^0$



$\tan 300^0 = -\tan 60^0$

(*−ve sign means quadrant in which angle lies*)

iii) $\cot 200^0 = \frac{1}{\tan\ 200}$

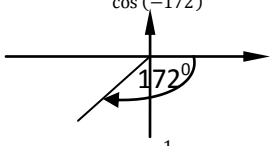


Reference angle $= 200 - 180 = 20^0$

$\tan 200^0 = +\tan 20^0$

$\therefore \cot 200^0 = \frac{1}{+\tan\ 20} = +\cot 20^0$

iv) $\sec(-172^0) = \frac{1}{\cos(-172)}$, -172^0 means clockwise measurement



Reference angle$=180 - 172 = 8^0$

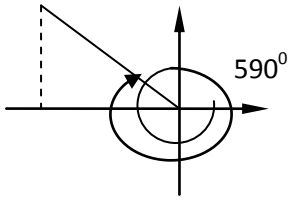
$\therefore \cos(-172^0) = -\cos 8^0$

$\therefore \sec(-172^0) = \frac{1}{-\cos\ 8^0} = -\sec 8^0$

v) $\text{cosec}(-590^0) = \frac{1}{\sin(-590^0)}$

but $\sin(-590^0) = -\sin 50^0$

$\text{cosec}(-590^0) = \frac{1}{-\sin\ 50^0} = -\text{cosec}\, 50^0$

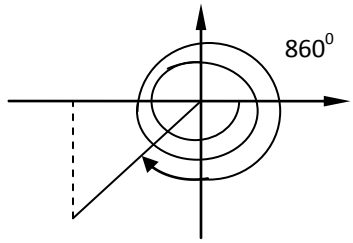


Note:

$590^0 = 360^0 + 230^0, 230^0 = 180^0 + 50^0$

The final reference angle is 50^0.

vi) $\cot(-860^0) = \frac{1}{\tan(-860^0)}$



$860 = 2 \times 360 + 140, 140 = 180 - 40$

$\therefore Reference\ angle = 40^0$

$\tan(-860^0) = +\tan 40^0$

$\cot(-860^0) = \frac{1}{\tan(-860^0)} = \frac{1}{+tan\ 40^0} = +\cot 40^0$

Task:

Express in terms of trigonometric ratios of acute angles

a) $\tan(-500^0)$ c) $\text{cosec}(+950^0)$

b) $\cot(-773^0)$

Ans: a) $+\tan 40^0$ b) $-\cot 55^0$ c) $-\text{cosec}\, 50^0$

Exact values for trigonometrical ratios of special angles

Special angles take on *30^0 and 45^0 or $\frac{\pi}{6}$ and $\frac{\pi}{4}$ radians* multiples.

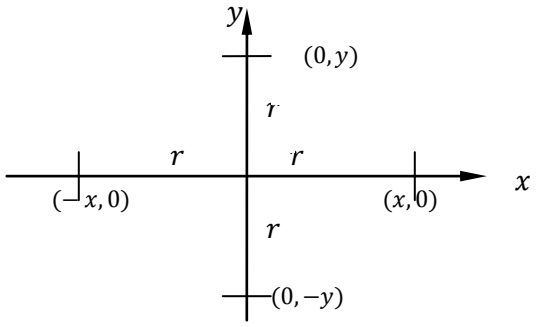
The trig functions of these angles can easily be evaluated exactly with the use of tables or calculator.

Quadrantal angles:

Are angles with their terminal side lying along the coordinate axes.

These angles are multiples of 90^0 or $\frac{\pi}{2}$*radians*. They are the easiest angles to work with.

Consider a point *(x, y)* on the coordinate axes *r-units* from the *origin.*



$\sqrt{x^2 + y^2}$;
$x = y = r$

a) **The function of zero degrees.** $(\boldsymbol{y = 0, x = 0})$

$\sin 0^0 = \frac{y}{r} = \frac{0}{r} = 0$, $\tan 0^0 = \frac{y}{x} = \frac{0}{r} = 0$
$\cos 0^0 = \frac{+x}{r} = \frac{r}{r} = +1$ And $\sec 0^0 = \frac{r}{+x} = \frac{r}{r} = +1$
Cotangent and cosecant of 0^0 do not exist since a calculation of them involves a division by a zero

b) **The function of 90^0,** $(\boldsymbol{x = 0, y = r})$

$\sin 90^0 = \frac{+y}{r} = \frac{+r}{+r} =$, $\cot 90^0 = \frac{+x}{+y} = \frac{0}{r} = 0$
$\cos 90^0 = \frac{+x}{r} = \frac{0}{r} = 0$ And $\operatorname{cosec} 90^0 = \frac{r}{+y} = \frac{r}{r} = +1$
Tangent and secant of 90^0 do not exist since a calculation of them involves a division by a

c) **The function of 180^0,** $(\boldsymbol{x = -r, y = 0})$

$\sin 180^0 = \frac{y}{r} = \frac{0}{+r} = 0$, $\tan 180^0 = \frac{y}{-x} = \frac{0}{-r} = 0$
$\cos 180^0 = \frac{-x}{r} = \frac{-}{r} = -1$ And $\sec 180^0 = \frac{r}{-x} = \frac{r}{-r} = -1$
Cotangent and cosecant of 180^0 do not exist since a calculation of them involves a division by a zero

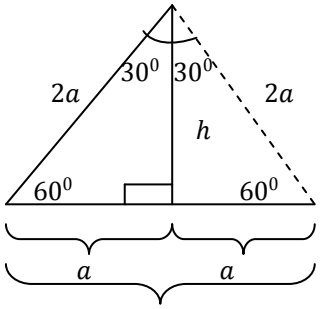
d) **The function of 270⁰ and 360⁰ ,** ($x = -r,\ y = 0$)

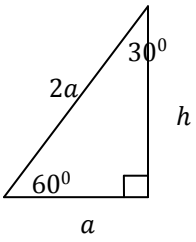
It is left for the student to verify the following.

$\sin 270^0 = -1$	$\sin 360^0 = 0$
$\cos 270^0 = 0$	$\cos 360^0 = 1$
$\tan 270^0$ *does not exist*	$\tan 360^0 = 0$
$\cot 270^0 = 0$	
$\cot 360^0$ *doesnot exist*	
$\sec 270^0$ *does not exist*	$\sec 360^0 = 1$
$\text{cosec} 270^0 = -1$	$\text{cosec}\ 360^0$ *does ot exist*

e) **Evaluation for Angles with 30^0 , 60^0 *or* 45^0**

Consider the equilateral triangle of side *2a.*



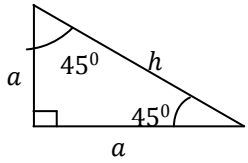


By Pythagoras theorem

$h^2 = (2a)^2 - a^2$

$h = \sqrt{3a^2} = a\sqrt{3}$

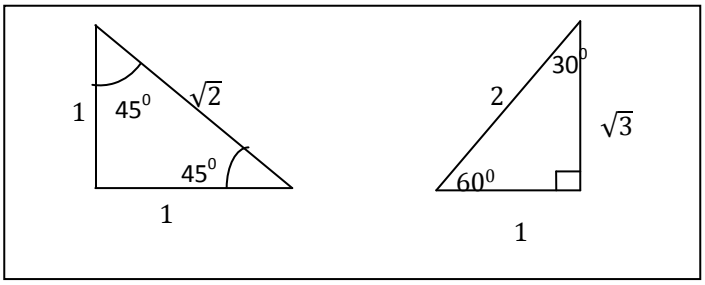
Consider also an isosceles right angled triangle of side a.



$h^2 = a^2 + a^2$

$\therefore h = \sqrt{2a^2} = a\sqrt{2}$

Taking $a = 1$ for both reference triangles, then



Application of the special angles reference triangle

- Evaluation of exact angle trigonometric ratios. e.g. $\sin 30^0, \sec(-240)^0\ etc.$
- Solving trigonometric equations.

Evaluation of exact angle trig ratios

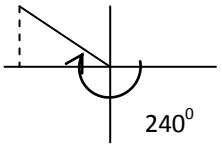
Find the exact value of;

i) $\sec(-240^0)$ iii) $\cot(315^0)$

ii) $\text{cosec}(600^0)$ iv) $\tan(-765^0)$

Solution

i) $\sec(-240^0) = \frac{1}{cos\,(-240^0)}$

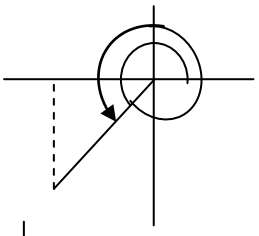


Reference angle = $240 - 180 = 60^0$

$\cos(-240^0) = -cos 60^0 = -1/2$

$\therefore\ \sec(-240^0) = \frac{1}{-1/2} = -2$

ii) $\text{cosec}(600^0) = \frac{1}{\sin(600^0)}$

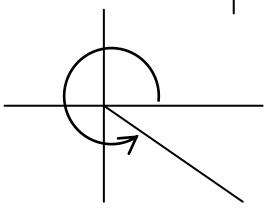


Reference angle = $(360 + 240) - 540^0$

$= (360 + 180 + 60) - 540 = 60^0$

$\text{cosec}(600^0) = \frac{1}{-sin\,60^0} = \frac{1}{-\sqrt{3}/2} = \frac{-2}{\sqrt{3}}$

iii) $\cot(315^0) = \frac{1}{\tan(315^0)}$



Reference angle= $360 - 315 = 45^0$

$\tan(315^0) = -\tan(45^0) = \frac{-1}{\sqrt{2}}$

$\therefore\ cot(315^0) = \frac{1}{-1/\sqrt{2}} = -\sqrt{2}$

iv) Left as an exercise to the student. **Ans**: $-\mathbf{1}$

Solving trigonometric equations

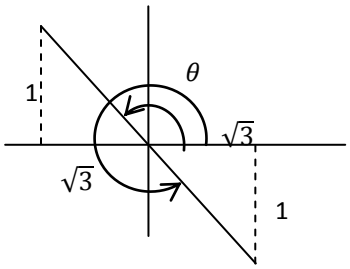
Solve for $0^0 < \theta < 360^0$ *the equations;*

i) $\cot\theta = -\sqrt{3}$ *iii)* $\sin\theta = \frac{\sqrt{3}}{2}$

ii) $\text{cosec}\theta = -2$

Solution

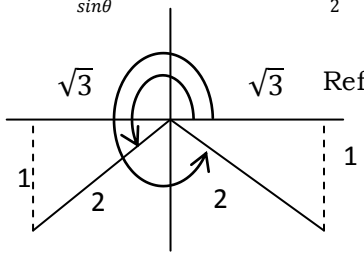
i) $\cot\theta = -\sqrt{3} \Rightarrow \frac{1}{tan\theta} = -\sqrt{3}\ \therefore \tan\theta = \frac{-1}{\sqrt{3}} = \frac{Opp}{Adj}$



$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^0 = reference\ angle$

$\therefore\ \theta = 150^0, 300^0$

ii) $\text{cosec}\,\theta = -2 \Rightarrow \frac{1}{sin\theta} = -2 \;\; \therefore \; \sin\theta = \frac{-1}{2} = \frac{Opp}{Hyp}$



Reference angle = $\sin^{-1}\left(\frac{1}{2}\right) = 30^0$

$\therefore \; \theta = 210^0, 330^0$

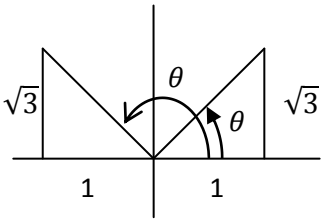
iii) $\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = 180 - Reference\ angle,$

θ is the reference angle

Reference angle = $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^0$

$\theta = 120^0, 60^0$



TRIGONOMETERIC IDENTIES.

Solving trigonometric equations using algebraic approach

Suggested approach for solving Trig equations algebraically.

1. Regard one particular trig function as function and solve for it,
   a) Consider using algebraic manipulation such as factoring, combining or separating functions
   b) Consider using identities
2. After solving for trig function solve for it.

Example:

Exact solutions using factoring

Solve for $-180^0 < \theta < +180^0$ *the equations;*

i) $2\cos^2\theta - \cos\theta = 0$

ii) $\tan^2\theta + \tan\theta = 0$

iii) $4\cos^3\theta = \cos\theta$

iv) $2\cos^2\theta + 3\cos\theta + 1 = 0$

v) $4\sin^2\theta - \sin\theta - 1 = 0$

Solution:

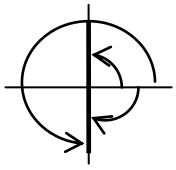
Step 1: Solve for $cosx$: $2cos^2\theta - cos\theta = 0$, factor out $cosx$

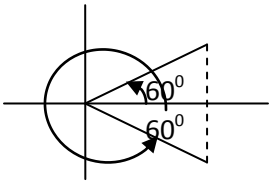
$\cos x\,(\cos x - 1) = 0 \qquad [\text{a.b} = 0 \text{ if a} = 0, \text{b} = 0]$

$\therefore \cos x = 0, 2\cos x - 1 = 0 \quad or \quad \cos x = 1/2$

Step 2: Solve each equation for the required interval.

$\cos x = 0, \;\; x = \cos^{-1}(0) = 90^0, 270^0, -90^0, -270^0$





For $\cos x = 1/2$,

$\cos^{-1}(1/2) = 60^0$

$\therefore x = \pm 60^0, \pm 90^0$

ii) $\tan^2\theta + \tan\theta = 0 \quad [\because \tan\theta = x, x^2 + x - 0]$

$\tan\theta\,(\tan\theta + 1) = 0\ [x(x+1) = 0]$

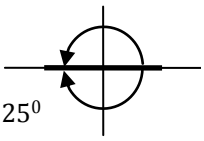
$\therefore \tan\theta = 0, \tan\theta + 1 = 0$ or $\tan\theta = -1$

For $\tan\theta = 0$

$\theta = 0^0, \pm 180^0$

$\theta = 135^0, 315^0, -45^0, -225^0$

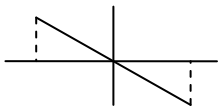
$\therefore \theta = 0^0, \pm 180^0, 135^0$



For $\tan\theta = -1$

$\Rightarrow \tan^{-1}(1) = 45^0$

$\theta = 135^0, 315^0, -45^0, -225^0$

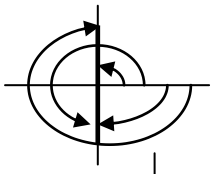


iii) $4\cos^3\theta = \cos\theta \Rightarrow 4\cos^3\theta - \cos\theta = 0$

$\cos\theta\,[4\cos^2\theta - 1] = 0 \Rightarrow \cos\theta = 0, 4\cos^2\theta - 1 = 0$ or $\cos\theta = \pm\frac{1}{2}$

From $\cos\theta = 0$

$\theta = \pm 90^0, \pm 270^0$

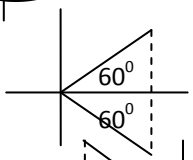


From $\cos\theta = \pm\frac{1}{2}$

Considering

$\cos\theta = +\frac{1}{2}$, $\cos^{-1}(1/2) = 60^0$

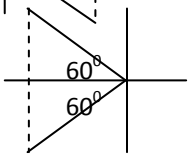
$\therefore\ \theta = \pm 60^0, \pm 300^0$



Considering $cos\theta = -\frac{1}{2}$, $\cos^{-1}\left(\frac{1}{2}\right) = 60^0$

$\therefore\ \theta = \pm 120^0, \pm 240^0$

$\therefore\ \theta = \pm 60^0, \pm 120^0$c



iv) $2cos^2\theta + 3cos\theta + 1 = 0 \quad [let \cos\theta = x,\ 2x^2 + 3x + 1 = 0]$

$2\cos^2\theta + 2\cos\theta + \cos\theta + 1 = 0 \quad [2x^2 + 2x + x + 1 = 0]$

$2\cos\theta(\cos\theta + 1) + 1(\cos\theta + 1) = 0 \quad [2x(x+1) + 1(x+1) = 0]$

$(\cos\theta + 1)(2\cos\theta + 1) = 0 \quad [(2x+1)(x+1) = 0]$

$\therefore \cos\theta + 1 = 0$ or $\cos\theta = -1$

$2\cos\theta + 1 = 0$ or $\cos\theta = -1/2$

Considering $\cos\theta = -1$

$\cos^{-1}(1) = 180^0$

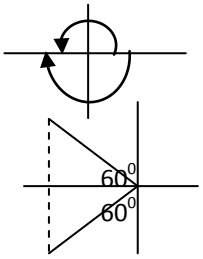
$\therefore\ \theta = \pm 180^0$

Considering $cos\theta = -1/2$

$\cos^{-1}(1/2) = 60^0$

$\therefore\ \theta = \pm 120^0, \pm 240^0$

$\therefore\ \theta = \pm 120^0, \pm 180^0$



a) $4\sin^2\theta = 1 \quad [\text{let } x = \sin\theta,\ 4x^2 = 1]$

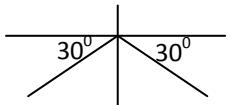
$4\sin^2\theta - 1 = 0 \quad [4x^2 - 1 = 0]$

$(2\sin\theta + 1)(2\sin\theta - 1) = 0 \quad [(2x+1)(2x-1) = 0]$

$\therefore 2\sin\theta + 1 = 0$ or $\sin\theta = -1/2$

$2\sin\theta - 1 = 0$ or $\sin\theta = 1/2$

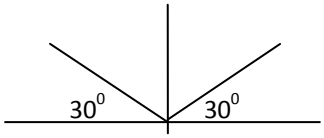
Dealing with $\sin\theta = -1/2$



$\sin^{-1}(1/2) = 30^0$

$\theta = 210^0, 330^0, -30^0, -150^0$

Dealing with $\sin\theta = 1/2$



$\sin^{-1}(1/2) = 30^0$

$\theta = 30^0, 150^0$

$\therefore \theta = \pm 30^0, \pm 150^0$

v) $2\sin^2\theta - \sin\theta - 1 = 0$ [let $x = \sin\theta, 2x^2 - x - 1 = 0$]

$2\sin^2\theta - 2\sin\theta - 1 = 0$ $[2x^2 - 2x + x - 1 = 0]$

$2\sin\theta(\sin\theta - 1) + 1(\sin\theta - 1) = 0$ $[2x(x-1) + 1(x-1) = 0]$

$(\sin\theta - 1)(2\sin\theta + 1) = 0$ $[(x-1)(2x+1) = 0]$

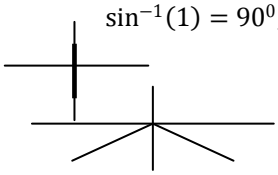
$\therefore \sin\theta - 1 = 0$ or $\sin\theta = 1$

$2\sin\theta + 1 = 0$ or $\sin\theta = -1/2$

Dealing with $\sin\theta = 1$

$+90^0, -270^0$

$\sin^{-1}(1) = 90^0, \therefore \theta =$



Dealing with $\sin\theta = -1/2$

$\sin^{-1}(1/2) = 30^0$

$\therefore \theta = 210^0, 330^0, -30^0, -150^0$

$\Rightarrow \therefore \theta = -30^0, -150^0, +90^0$

Example

Approximate solutions using substitution

Solve for $0^0 < \theta < 360^0$ *the equations*

i) $5\sin(2\theta - 5) = -3.045$

ii) $\sin(2\theta + 30^0) = 0.8$

iii) $\tan(3\theta - 45^0) = \frac{1}{2}$

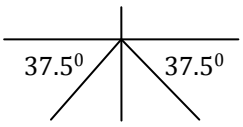
iv) $\sec(2\theta + 30^0) = 3$

v) $\cot\left(\frac{1}{2}\theta + 45^0\right) = \sqrt{3}$

Solution

i) $5\sin(2\theta - 5) = -3.045$, let $u = 2\theta - 5 \Rightarrow 5\sin u = -3.045$

$\Rightarrow \sin u = \frac{-3.045}{5} = -0.609$



$\sin^{-1}(0.609) = 37.5^0$

$u = 217.5^0, 322.5^0 + 360^0 k, k = 0,1,2,3 \ldots..$

[$\because 360k$ is added to get other angles giving the same result]

$\Rightarrow u = 2\theta - 5 = 217.5^0, 322.5^0, 577.5^0, 682.5^0$

$2\theta = (217.5^0, 322.5^0, 577.5^0, 682.5^0) + 5^0$

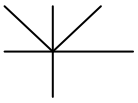
$= 222.5^0, 327.5^0, 582.5^0, 687.5^0$

$\therefore \theta = 111.25^0, 163.75^0, 291.25^0, 343.75^0$

NOTE:

If $\sin\theta = a$, then $\theta = \sin^{-1}(a) \pm 360^0 k; k = 0,1,2,3,\ldots.$, and $\sin^{-1}(a) < 90^0$

ii) $\sin(2\theta + 30^0) = 0.8$, let $u = 2\theta + 30^0 \Rightarrow \sin u = 0.8$



$\sin^{-1}(0.8) = 53.1^0$

$u = 53.1^0, 126.7^0 + 360^0 k$

$\therefore 2\theta + 30^0 = 53.1^0, 126.7^0, 413.1^0, 486.7^0 773.1^0$

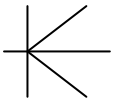
$2\theta = 23.1^0, 96.7^0, 383.1^0, 456.7^0, 743.1^0$(Ignored)

$\theta = 11.55^0, 48.36^0, 191.55^0, 228.35^0.$

iii) left as an exercise **Ans:** $\theta = 35.79^0, 81.86^0, 215.79^0, 203.86^0, 263.86^0, 323.86^0$

iv) $sec(2\theta + 30^0) = 3 \Rightarrow \frac{1}{cos(2\theta + 30^0)} = 3$, $\therefore cos(2\theta + 30^0) = 1/3$

let $u = 2\theta + 30^0 \Rightarrow \cos u = 1/3$



$\cos^{-1}(1/3) = 70.5^0$

$u = 70.5^0, 289.5^0 + 360^0 k$

$\therefore 2\theta + 30^0 = 70.5^0, 289.5^0, 430.5^0, 649.5^0$

$2\theta = 40.5^0, 259.5^0, 400.5^0, 619.5^0$

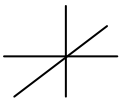
$\therefore \theta = 20.25^0, 129.75^0, 200.25^0, 309.75^0$

NOTE:

If $\cos\theta = b, \theta = \cos^{-1}(b) \pm 360^0 k$ and $\cos^{-1}(b) < 90^0$

v) $\cot\left(\frac{1}{2}\theta + 45^0\right) = \sqrt{3} \Rightarrow \frac{1}{\tan\left(\frac{1}{2}\theta + 45^0\right)} = \sqrt{3}$

$\Rightarrow \tan\left(\frac{1}{2}\theta + 45^0\right) = \frac{1}{\sqrt{3}}$, let $u = \frac{1}{2}\theta + 45^0$



$\Rightarrow \tan u = 1/\sqrt{3}$

$\tan^{-1}\left(1/\sqrt{3}\right) = 30^0$

$\therefore u = 30^0, 210^0 + 180^0 k; k = 0,1,2,3, \ldots\ldots\ldots\ldots$

$\therefore \frac{1}{2}\theta + 45^0 = 30^0, 210^0, 390^0, 570^0, 750^0$

$\frac{1}{2}\theta = -15^0, 165^0, 345^0, 525^0, 705^0 \therefore \theta = (-15^0, 165^0, 345^0, 525^0, 705^0) \times 2$

$\therefore \theta = 330^0$

NOTE:

If $\tan\theta = \alpha$, then $\theta = \tan^{-1}(\alpha) \pm 180^0 k$ .i.e cumulate the angle $\tan^{-1}(\alpha)$ with $\pm 180^0$

Approximate solutions to trig equations using a quadratic formula

Solve for $-180^0 \leq \theta \leq +180^0$ *the equation*

i) $2\cos^2\theta - 4\cos\theta + 1 = 0$

ii) $3\tan^2\theta + 5\tan\theta - 1 = 0$

iii) $3\text{cosec}^2\theta + \text{cosec}\theta - 8 = 0$

iv) $2\sec^2\theta - \sec\theta - 2 = 0$

Solution

i) $2\cos^2\theta - 4\cos\theta + 1 = 0$ [let $u = \cos\theta, 2u^2 - 4u + 1 = 0$]

$\cos\theta = u = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)}$

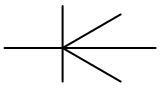
$= \frac{4\pm\sqrt{16-8}}{4} \Rightarrow \cos\theta = \frac{4+\sqrt{8}}{4}$ or $\frac{4-\sqrt{8}}{8}$

$\therefore \cos\theta = 1.707107, cos\theta = 0.292893$

Dealing with $\cos\theta = 1.707107$

No values of θ since $-1 \le \theta \le +1$

Dealing with $cos\theta = 0.292893$, $\cos^{-1}(0.292893) = 72.93^0$



$\therefore \theta = \pm 72.97^0, 287.03^0$

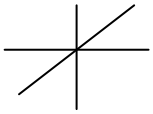
$\therefore \theta = \pm 72.97^0$

ii) $3\tan^2\theta + 5\tan\theta - 1 = 0$ [let $u = \tan\theta, 3u^2 + 5u - 1 = 0$]

$$\tan\theta = \frac{-5 \pm \sqrt{(5)^2 - 4(3)(-1)}}{2(3)} = \frac{-5 \pm \sqrt{37}}{6}$$

$\therefore \tan\theta = \frac{-5+\sqrt{37}}{6} = 0.18046$ *or* $\frac{-5-\sqrt{37}}{6} = -1.84713$

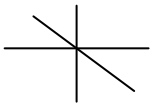
Dealing with $\tan\theta = 0.18046$



$\tan^{-1}(0.18046) = 10.2^0$

$\therefore \theta = 10.2^0, 190.2^0, -169.8^0$

Dealing with $\tan\theta = -1.84713$



$\tan^{-1}(1.84713) = 61.6^0$

$\Rightarrow \theta = -61.6^0, -241.6^0, 118.4^0, 298.4^0$

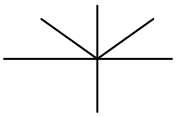
$\therefore \theta = 10.2^0, 118.4^0, -61.6^0, -169.8^0$

iii) $3\text{cosec}^2\theta + \text{cosec}\theta - 8 = 0$ [let $u = \text{cosec}\theta, 3u^2 + u - 8 = 0$]

$\text{cosec}\theta = \frac{-1\pm\sqrt{(1)^2-4(3)(-8)}}{2(3)} = \frac{-1\pm\sqrt{97}}{6}$

$\therefore \text{cosec}\theta = \frac{-1+\sqrt{97}}{6}$ or $\frac{-1-\sqrt{97}}{6} \Rightarrow \text{cosec}\theta = 1.4748$ *or* -1.80814

But $cosec\theta = \frac{1}{\sin\theta} \Rightarrow \frac{1}{\sin\theta} = 1.4748 \Rightarrow \sin\theta = 0.6781$ also



$\sin\theta = \frac{1}{-1.80814} = -0.5531$

Dealing with $\sin\theta = -0.6781$, $\sin^{-1}(0.6781) = 42.7^0$

$\therefore \theta = 42.7^0, 137.3^0, (no\ need\ for\ negative\ angles\ sine\ are\ out\ of\ range)$

Dealing with $\sin\theta = -0.5531$, $\sin^{-1}(0.5531) = 33.6^0$

$\therefore \theta = -33.6^0, -146.4^0, (no\ need\ for\ positive\ angles\ since\ are\ out\ of\ range)$

$\therefore \theta = 42.7^0, 137.3^0, -33.6^0, -146.4^0$

iv) Left as an exercise. ***Ans***: $\pm$ **38.9^0**

Solve $8cos^4x - 10cos^2x + 3 = 0$ *for x in the range*

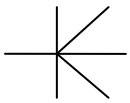
$0^0 \le x \le 180^0$ (**Uneb2012**)

Solution

Let $u = \cos^2 x \Rightarrow 8u^2 - 10u + 3 = 0$ Solving $u = \frac{3}{4}, \frac{1}{2} \Rightarrow \cos^2 x = \frac{3}{4}, \frac{1}{2}$

$\therefore \cos x = \pm\frac{\sqrt{3}}{2}$ *or* $\cos x = \pm\frac{1}{\sqrt{2}}$

For $\cos x = +\frac{\sqrt{3}}{2}$



$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^0$

$For \cos x = -\frac{\sqrt{3}}{2}$ $\Rightarrow x = 30^0, 330^0$

$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^0$

$\Rightarrow x = 150^0, 210^0$

For $\cos x = +\frac{1}{\sqrt{2}}$ $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^0$

$\Rightarrow x = 45^0, 315^0$

For $\cos x = -\frac{1}{\sqrt{2}}$ $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^0$

$\Rightarrow x = 135^0, 225^0$

$\therefore x = 30^0, 45^0, 135^0, 150^0$ $Reject\ x = 210^0, 225^0, 330^0$

Equations involving creation of tangent ratio

Solve for $0^0 < \theta < 360^0$ *the equations*

a) $\cos\theta + \sin\theta = 0$

b) $2\sin\theta - 4\cos\theta = 0$

c) $4\cos\theta - 3\sin\theta = 0$

d) $3\sin^2\theta - \sin\theta\cos\theta - 2\cos^2\theta = 0$

e) $2\cos^2\theta - 8\sin\theta\cos\theta + 3\sin^2\theta = 0$

Solution

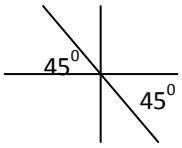
Equation of the form $\boxed{\mathbf{a\cos(m\theta) + b\sin(m\theta) = 0}}$ where *a, m, and b* are constants require creation of *tangent*, dividing through by $\boldsymbol{cos(m\theta)}$.

a) $\cos\theta + \sin\theta = 0$ $(dividing\ by\ cos\theta)$

$\frac{cos\theta}{cos\theta} + \frac{\sin\theta}{cos\theta} = \frac{0}{cos\theta} \Rightarrow 1 + \tan\theta = 0 \ \therefore \tan\theta = -1$

$\tan^{-1}(1) = 45^0$

$\therefore \theta = 135^0, 315^0$

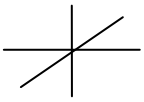


b) $2sin\theta - 4cos\theta = 0$ $(dividing\ by \cos\theta)$

$\frac{2\sin\theta}{\cos\theta} - \frac{4\cos\theta}{\cos\theta} = \frac{0}{\cos\theta} \Rightarrow 2\tan\theta - 4 = 0$ or $\tan\theta = 2$

$\tan^{-1}(2) = 63.4^0$

$\therefore \theta = 63.4^0, 243.4^0$



c) Left as an exercise **Ans**: $\mathbf{53.1^0, 233.1^0}$

d) $3\sin^2\theta - \sin\theta\cos\theta - 2\cos^2\theta = 0$ is of the form

$$\boldsymbol{a\sin^2(m\theta) + b\sin(m\theta)\cos(m\theta) + c\cos^2\theta = d}$$

Or $\boldsymbol{a\sin^2\theta + b\sin(2m\theta) + c\cos^2(m\theta) = d}$ is solved by division by $\mathbf{\cos^2(m\theta)}$

Dividing through by $\cos^2\theta$ yields $\frac{3sin^2\theta}{cos^2\theta} - \frac{\sin\theta cos\theta}{cos^2\theta} - \frac{2cos^2\theta}{cos^2\theta} = 0$

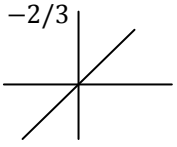
$\Rightarrow 3\tan^2\theta - \tan\theta - 2 = 0 \Rightarrow \tan\theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-2)}}{2(3)}$

$\therefore \tan\theta = \frac{1 \pm \sqrt{25}}{6} = \frac{1 \pm 5}{6} \Rightarrow \tan\theta = \frac{1+5}{6} = 1\ of\ \frac{1-5}{6} = -2/3$

Dealing with $\tan\theta = 1$

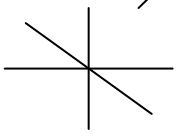
$\tan^{-1}(1) = 45^0$

$\therefore \theta = 45^0, 225^0$



Dealing with $\tan\theta = -2/3$

$\tan^{-1}(2/3) = 23.69^0$



$\Rightarrow \theta = 146.31^0, 326.31^0$

$\therefore \theta = 45^0, 225^0, 146.31^0, 326.31^0$

e) $2\cos^2\theta - 8\sin\theta\cos\theta + 3\sin^2\theta = 0$, dividing through by $\cos^2\theta$ yields

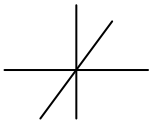
$2 - 8\tan\theta + 3\tan^2\theta = 0$

Solving the equation yields $\tan\theta = 2.3874, 0.2792\ (CAL)$

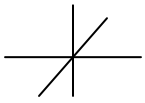
Dealing with $\tan\theta = 2.3874$

$\tan^{-1}(2.3874) = 67.3^0$

$\therefore \theta = 67.3^0, 247.3^0$



Dealing with $\tan\theta = 0.2792$



$\tan^{-1}(0.2792) = 15.6^0$

$\therefore \theta = 15.6^0, 195.6^0$

$\therefore \theta = 67.3^0, 247.3^0, 15.6^0, 195.6^0$

Task:

Solve for $0^0 < \theta < 360^0$ *the equations*

i) $4\cos 2\theta - 3\sin 2\theta = 0$

ii) $3\sin^2\theta - \sin 2\theta\cos 2\theta - 4\cos^2 2\theta = 0$

iii) $10\sin^2 x + 10\sin x\cos x - \cos^2 x = 0$

$\boldsymbol{Ans}: (\boldsymbol{i})\ \mathbf{26.6^0, -153.4^0}\ (\boldsymbol{ii})\ \mathbf{26.6^0, 116.6^0, -63.5^0}\ (\boldsymbol{iii})\ \mathbf{14^0, -166^0, -56.3^0, 123.7^0}$

Evaluating trig equations involving reciprocal of trig ratios, and solving using factorizing 0r quadratic formula

Solve for $0^0 < \theta < 360^0$ *the equations*

a) $3cos\theta + 2sec\theta + 7 = 0$

b) $5\sin\theta + 6\text{cosec}\theta = 17$

c) $\cot\theta = 5cos\theta$

d) $\sec\theta = 2\cos\theta$

e) $\tan\theta + 2\cot\theta = 3$

Solution

a) $3\cos\theta + 2\sec\theta + 7 = 0 \Rightarrow 3\cos\theta + \frac{2}{\cos\theta} + 7 = 0$ (multiplying through by$cos\theta$)

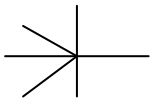
$3(\cos\theta)^2 + 7\cos\theta + 2 = 0 \quad [let\ u = \cos\theta, 3u^2 + 7u + 2 = 0]$

$3u^2 + 7u + 2 = 0 \Rightarrow u = -\frac{1}{3}\ \ or -2\ (CAL)$

$\therefore \cos\theta = -\frac{1}{3}, \cos\theta = -2$

For $\cos\theta = -2$, *No values since* $-1 \le \theta \le +1$

For $\cos\theta = -\frac{1}{3}$



$\cos^{-1}\left(\frac{1}{3}\right) = 70.5^0$

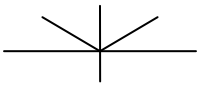
$\therefore \theta = \pm 109.5^0$

b) $5\sin\theta + 6\text{cosec}\theta = 17 \Rightarrow 5\sin\theta + \frac{6}{\sin\theta} = 17$

$5(\sin\theta)^2 - 17\sin\theta + 6 = 0 \quad [let\ u = \sin\theta, 5u^2 - 17u + 6 = 0]$

Solving $u = 3, 0.4 \Rightarrow \sin\theta = 3$ and $\sin\theta = 0.4$

For $\sin\theta = 3$, *no values of* θ since $-1 \le \theta \le +1$

For $\sin\theta = 0.4$  $\sin^{-1}(0.4) = 23.6^0$

$\therefore \theta = 23.6^0, 156.4^0$

c) $\cot\theta = 5\cos\theta \Rightarrow \frac{\cos\theta}{\sin\theta} = 5\cos\theta \Rightarrow \cos\theta = 5\cos\theta\sin\theta$

$\cos\theta - 5\cos\theta\sin\theta = 0 \Rightarrow \cos\theta(1 - \sin\theta) = 0$

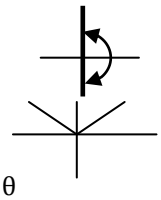
$\therefore \cos\theta = 0$ or $1 - \sin\theta = 0, \sin\theta = 1$

For $\cos\theta = 0$, $\cos^{-1}(0) = 90^0$

$\therefore \theta = \pm 90^0$

For $\sin\theta = \frac{1}{5}$, $\sin^{-1}(1/5) = 11.5^0$

$\therefore \theta = 11.5^0, \pm 90^0, 168.5^0$



d) $\sec\theta = 2\cos\theta \Rightarrow \frac{1}{\cos\theta} = 2\cos\theta$ or $1 = 2\cos^2\theta$

$\Rightarrow 2\cos^2\theta - 1 = 0$, solving yoelds $\cos\theta = \pm 1/\sqrt{2}$

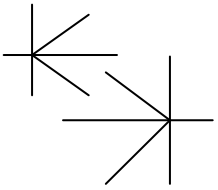
For $\cos\theta = +1/\sqrt{2}$, $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^0$

$\therefore \theta = \pm 45^0$

For $\cos\theta = -1/\sqrt{2}$, $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^0$

$\therefore \theta = \pm 135^0$

$\therefore \theta = \pm 45^0, \pm 135^0$



e) $\tan\theta + 2\cot\theta = 3 \Rightarrow \tan\theta + \frac{2}{\tan\theta} = 0 \Rightarrow \tan^2\theta - 3\tan\theta + 2 = 0$

Solving the quadratic equation yields $\tan\theta = 2, 1$

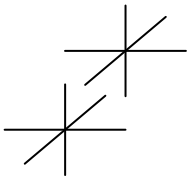
For $\tan\theta = 2$, $\tan^{-1}(2) = 63.4^0$

$\therefore \theta = 63.4^0, -116.5^0$

For $\tan\theta = 1$, $\tan^{-1}(1) = 45^0$

$\therefore \theta = 45^0, -135^0$

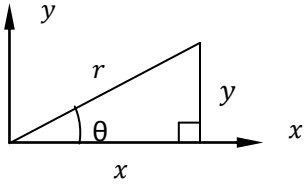
$\therefore \theta = 63.4^0, -116.5^0, 45^0, -135^0$



TRIGONOMETRIC IDENTITIES:

The fundamental identity:

This is derived basing on the Pythagoras theorem. Consider the figure below.



By Pythagoras theorem

$x^2 + y^2 = r^2$ $(dividing\ by\ r^2)$

$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$, but $cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$

$\Rightarrow (cos\theta)^2 + (\sin\theta)^2 = 1$ $\boxed{\therefore \cos^2\theta + \sin^2\theta = 1}$

In gener $\boxed{\cos^2(m\theta) + \sin^2(m\theta) = 1}$ this is the fundamental identity.

Dividing he fundamental identity by $cos^2\theta\ or sin^2\theta$,

$\frac{cos^2\theta}{cos^2\theta} + \frac{sin^2\theta}{cos^2\theta} = \frac{1}{cos^2\theta} \Rightarrow$ $\boxed{1 + \tan^2\theta = \sec^2\theta}$

Now dividing by $\sin^2\theta$, $\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$

$\Rightarrow 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Also in general, $1 + \tan^2(m\theta) = \sec^2(m\theta)$
$1 + \cot^2(m\theta) = \operatorname{cosec}^2(m\theta)$

Application of the identities:

- Solving trigonometric equations
- Proof of identities
- Simplifying trigonometric expressions
- Elimination of parameters.

Solution to Trigonometrical Equations

Solve for $0^0 < \theta < 360^0$ *the equations.*

i) $4cot^2\theta + 39 = 24cosec\theta$
ii) $5sec\theta - 2sec^2\theta = tan^2\theta + 1$
iii) $5cos\theta = 2(1 + 2sin^2\theta)$
iv) $4sec^2\theta = 3tan\theta + 5$
v) $(2tanx - 1)^2 = 3(sec^2x - 2)$
vi) $4cosx + 5 = 6sin^2x$

Solution

i) $4cot^2\theta + 39 = 24cosec\theta$ ..... (1)

Identity relating $cosec\theta$ *and* $cot^2\theta$ is $1 + cot^2\theta = cosec^2\theta$

$\Rightarrow \cot^2\theta = \operatorname{cosec}^2\theta - 1$ ..... (2)

Put (2) in to (1) gives

$4(\operatorname{cosec}^2\theta - 1) + 39 = 24\operatorname{cosec}\theta$

$4\operatorname{cosec}^2\theta - 4 + 39 = 24\operatorname{cosec}\theta$ or $4\operatorname{cosec}^2\theta - 24\operatorname{cosec}\theta + 35 = 0$

Let $u = \operatorname{cosec}\theta \Rightarrow 4u^2 - 24u + 35 = 0 \ \therefore u = 3.5, 2.5$ (cal)

$\Rightarrow \operatorname{cosec}\theta = 3.5$ or $\frac{1}{\sin\theta} = 3.5 \ \therefore \sin\theta = \frac{2}{7}$ and $\operatorname{cosec}\theta = 2.5$ or $\frac{1}{\sin\theta} = 2.5 \ \therefore \sin\theta = \frac{2}{5}$

Dealing with $\sin\theta = \frac{2}{7}$

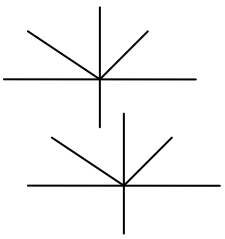
$\sin^{-1}(2/7) = 16.6^0$

$\Rightarrow \theta = 16.6^0, 163.4^0$

Dealing with $\sin\theta = \frac{2}{5}$

$\sin^{-1}(2/5) = 23.6^0$

$\Rightarrow \theta = 23.6^0, 156.4^0$



$\therefore \theta = 16.6^0, 163.4^0, 23.6^0, 156.4^0$

ii) $5sec\theta - 2sec^2\theta = tan^2\theta + 1$ ..... (1)

Identity relating $sec\theta$ *and* $tan\theta$ is $1 + tan^2\theta = sec^2\theta$ ..... (2)

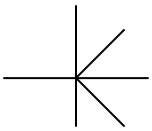
Put (2) in to (1) gives

$5\sec\theta - 2\sec^2\theta = \sec^2\theta \Rightarrow 5\sec\theta - 3\sec^2\theta = 0$

$sec\theta(5 - 3esc\theta) = 0 \ \Rightarrow sec\theta = 0 \ or \ (5 - 3esc\theta) = 0 \Rightarrow sec\theta = \frac{5}{3}$

Dealing with $\sec\theta = \frac{1}{cos\theta} = 0$, *hence no values of the equation*

Dealing with $\sec\theta = \frac{1}{cos\theta} = \frac{5}{3} \Rightarrow \cos\theta = \frac{3}{5}$



$\cos^{-1}(3/5) = 53.1^0$

$\therefore \theta = 53.1^0, 306.9^0$

iii) $5cos\theta = 2(1 + 2sin^2\theta)$..... (1)

Identity relating $cos\theta$ *and* $\sin\theta$ is $cos^2\theta + sin^2\theta = 1$

$\Rightarrow sin^2\theta = 1 - cos^2\theta$..... (2)

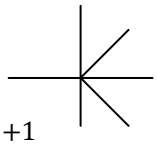
Put (2) in to (1) gives

$5cos\theta = 2\big(1 + 2(1 - cos^2\theta)\big)$ or $5cos\theta = 2 + 4 - 4cos^2\theta$

$\Rightarrow 4\cos^2\theta + 5\cos\theta - 6 = 0$

Let $u = cos\theta$ then $4u^2 + 5u - 6 = 0 \Rightarrow u = 0.75, -2$

From $cos\theta = 0.75$



$\cos^{-1}(0.75) = 41.4^0$

$\therefore \theta = 41.4^0, 318.6^0$

From $\cos\theta = -2$, *no values of* θ *since* $-1 \leq cos\theta \leq +1$

iv) Left as an exercise.

v) $(2\tan x - 1)^2 = 3(\sec^2 x - 2)$

Expanding gives $4\tan^2 x - 4\tan x + 1 = 3\sec^2 x - 6$..... (1)

Identity relating $\tan x$ *and* $\sec x$ *is* $1 + \tan^2 x = \sec^2 x$

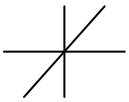
Put (2) in to (1) gives

$4\tan^2 x - 4\tan x + 1 = 3(1 + \tan^2 x) - 6$ or

$4\tan^2 x - 4\tan x + 1 = 3 + 3\tan^2 x - 6$

$\Rightarrow \tan^2 x - 4\tan x + 4 = 0$ *or* $(\tan x - 2)^2 = 0$

Dealing with $\tan x = 2$



$\tan^{-1}(2) = 63.4^0$

$\therefore x = 63.4^0, 243.4^0$

vi) Left as an exercise. ***Ans***: $\mathbf{78.8^0, 281.2^0, 149.4^0, 210.6^0}$

Solve $cot^2\theta = 5(cosec\theta + 1)$ *for* $0^0 \leq \theta 360^0$ **(Uneb 2000)**

Solution

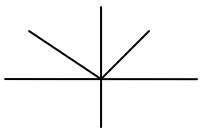
$cot^2\theta = 5(cosec\theta + 1)$, using $1 + cot^2\theta = cosec^2\theta \Rightarrow cot^2\theta = cosec^2\theta - 1$

$\Rightarrow cosec^2\theta - 1 = 5cosec\theta + 5$ *or* $cosec^2\theta - 5cosec\theta - 6 = 0$

Let $u = cosec\theta \Rightarrow u^2 - 5u - 6 = 0$ solving $u = 6, -1$

$\therefore \text{cosec}\theta = 6$ *or* $\frac{1}{\sin\theta} = 6 \Rightarrow \sin\theta = \frac{1}{6}$

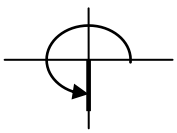
And $\text{cosec}\theta = -1$ *or* $\sin\theta = -1$



Dealing with $\sin\theta = \frac{1}{6}$, $\sin^{-1}(1/6) = 9.6^0$

$\Rightarrow \theta = 9.6^0, 170.6^0$

Dealing with $\sin\theta = -1$, $\sin^{-1}(1) = 90^0$



$\Rightarrow \theta = 270^0$

$\therefore \theta = 9.6^0, 170.6^0, 270^0$

Trigonometric equations and identities involving algebraic surds

Example

1. a) If $\sec\theta + \tan\theta = 4$, *find* $\sin\theta, \cos\theta, \tan\theta$ *and* $\sec\theta$

   b) $\tan\theta = 2 - \sqrt{3}$, *If find* $\sin\theta, \cos\theta$ *and* $\sec\theta$

Solution

a) $\sec\theta + \tan\theta = 4, \Rightarrow \tan\theta = 4 - \sec\theta$

Squaring both sides

$\tan^2\theta = 16 - 8\sec\theta + \sec^2\theta$

But $1 + \tan^2\theta = \sec^2\theta$ then $\sec^2\theta - 1 = 16 - 8sec\theta + sec^2\theta$

$\Rightarrow -1 = 16 - 8\sec\theta$ *or* $\sec\theta = \frac{17}{8}$

But $\sec\theta = \frac{1}{cos\theta} \Rightarrow \cos\theta = \frac{8}{17}$

Now from $\cos^2\theta + \sin^2\theta = 1$ then $\sin\theta = \sqrt{1 - cos^2\theta} = \sqrt{(1 - (8/17)^2)}$

$\therefore \sin\theta = \pm\frac{15}{17}$

Now $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\pm 15/17}{8/17} = \pm\frac{15}{8}$

Alternatively

$sec\theta + tan\theta = 4$…… (1)

Creating $\sec\theta - \tan\theta$

$\sec\theta - \tan\theta = (\sec\theta - \tan\theta)\frac{(\sec\theta - \tan\theta)}{(\sec\theta - \tan\theta)} = \frac{\sec^2\theta - \tan^2\theta}{\sec\theta + \tan\theta}$

$= \frac{1}{sec\theta + tan\theta} = \frac{1}{4}$ $\quad \left[\begin{array}{l}\sec^2\theta - \tan^2\theta = 1 \\ \because \sec^2\theta = 1 + \tan^2\theta\end{array}\right]$

$\therefore \sec\theta - \tan\theta = \frac{1}{4}$…… (2)

Eliminating $\tan\theta$, (1) + (2) $\Rightarrow 2\sec\theta = 4 + \frac{1}{4} = \frac{17}{4}$ $\therefore \sec\theta = \frac{17}{4}$,

But $\cos\theta = \frac{1}{\sec\theta} \Rightarrow \cos\theta = \frac{8}{17}$

Eliminating $sec\theta$, (1) − (2) $\Rightarrow 2tan\theta = 4 - \frac{1}{4} = \frac{15}{4}$ $\therefore tan\theta = \frac{15}{8}$

Using$cos^2\theta + sin^2\theta = 1$

$\sin\theta = \sqrt{1 - cos^2\theta} = \sqrt{(1 - (8/17)^2)} = \pm\frac{15}{17}$ *as before*

b) $\tan\theta = 2 - \sqrt{3}$ *a*nd using $1 + \tan^2\theta = \sec^2\theta$

$\sec^2\theta = 1 + \left(2 - \sqrt{3}\right)^2 = 1 + 4 - 4\sqrt{3} + 3 = 8 - 4\sqrt{3}$

$\therefore \frac{1}{cos^2\theta} = 8 - 4\sqrt{3} \Rightarrow cos^2\theta = \frac{1}{8-4\sqrt{3}}$ $\therefore cos\theta = \pm\frac{1}{\sqrt{4(2-\sqrt{3})}} = \pm\frac{1}{2(2-\sqrt{3})}$

Also using $cos^2\theta + sin^2\theta = 1$

$\sin^2\theta = 1 - cos^2\theta = 1 - \frac{1}{8-4\sqrt{3}} = \frac{8-4\sqrt{3}-1}{8-4\sqrt{3}} = \frac{7-4\sqrt{3}}{8-4\sqrt{3}}$,

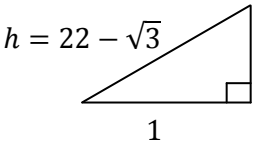
Rationalizing denominator

$= \frac{(7-4\sqrt{3})(8+4\sqrt{3})}{(8-4\sqrt{3})(8+4\sqrt{3})} = \frac{56+28\sqrt{3}-32\sqrt{3}-48}{64-16\times 3} = \frac{8-4\sqrt{3}}{16} = \frac{2-\sqrt{3}}{4}$

$\therefore \sin\theta = \pm\frac{\sqrt{2-\sqrt{3}}}{2}$

Alternatively

$\tan\theta = 2-\sqrt{3} = \frac{2-\sqrt{3}}{1} = \frac{OPp}{Adj}$



$h^2 = 1^2 + \left(2-\sqrt{3}\right)^2$

$= 1 + 4 - 4\sqrt{3} + 3 = 8 - 4\sqrt{3}$

$\therefore h = \sqrt{4\left(2-\sqrt{3}\right)}$

Note θ *lies in* 1st *and* 3rd *quadrant*

$\cos\theta = \pm\frac{1}{2(2-\sqrt{3})}$, $\sin\theta = \pm\frac{2-\sqrt{3}}{2\sqrt{2-\sqrt{3}}} = \pm\frac{\sqrt{2-\sqrt{3}}}{2}$ and

$\sec\theta = \frac{1}{cos\theta} = \pm 2\sqrt{2-\sqrt{3}}$

2. If $\text{cosec}\theta + \cot\theta = 2-\sqrt{3}, find\ \sin\theta, \tan\theta\ and \cos\theta$

Solution

$\text{cosec}\,\theta + \cot\theta = 2-\sqrt{3}$..... (1)

Creating $\text{cosec}\,\theta - \cot\theta$

$\text{cosec}\,\theta - \cot\theta = (\text{cosec}\,\theta - \cot\theta)\frac{(\text{cosec}\,\theta + \cot\theta)}{(\text{cosec}\,\theta + \cot\theta)}$

$= \frac{cosec^2\theta - cot^2\theta}{cosec\theta + cot\theta}$ $\quad \left[\because 1 + cot^2\theta = cosec^2\theta \atop cosec^2\theta - cot^2\theta = 1\right]$

$= \frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}$ $\quad Rationalising\ denominator$

$\therefore\ cosec\theta - \cot\theta = 2+\sqrt{3}$..... (2)

Adding (1) and (2)

$2\text{cosec}\,\theta = 4\ or\ \sin\theta = \frac{1}{2}$

Subtracting (1) and (2)

$2\cot\theta = -2\sqrt{3}\ \ or\ \tan\theta = -\frac{1}{\sqrt{3}}$

Since $\sin\theta = +ve$ and $\tan\theta = -ve$, then $90^0 < \theta\ 180^0$

$cos\theta = \sqrt{cos^2\theta} = \sqrt{1 - sin^2\theta}$, $sin\theta = \frac{1}{2} = \pm\sqrt{1-\frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$

$\therefore \cos\theta = -\frac{\sqrt{3}}{2}$

ELIMINATION OF PARAMETERS

Usually two parametric equations in terms of trig ratios of an angle, say θ are given. The parameter is the angle θ, to be eliminated.

Example

1. Show that the parametric equations defined by the pair of equations represent the given curves

i) $x = a\cos\theta, y = b\sin\theta$, curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

ii) $x = a\tan\theta\ , y = b\sec\theta\ $, curve is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

iii) $x = 1 + \tan\theta\ , y = cos\theta$, curve is $y^2(x-1)^2 + y^2 = 1$

iv) $x = a\text{cosec}\,\theta, y = bsec\,\theta$, curve is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$

v) $x = a\sec\theta, y = b + c\cos\theta$, curve is $x(y-b)=ac$

Solution

i) $x = a\cos\theta \Rightarrow \cos\theta = \frac{x}{a}$..... (1)

$y = b\sin\theta \Rightarrow \sin\theta = \frac{y}{b}$..... (2)

Relating $cos\theta$ *and* $\sin\theta$ using identity $cos^2\theta + sin^2\theta = 1$

$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as required

ii) $x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a}$..... (1)

$y = bsec\theta \Rightarrow sec\theta = \frac{y}{b}$..... (2)

Relating $tan\theta$ *and* $sec\theta$ using the identity $1 + tan^2\theta = sec^2\theta$

$\Rightarrow 1+\frac{x^2}{a^2}=\frac{y^2}{b^2} \quad \therefore \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ *as required*

iii) $x = 1 + \tan\theta \Rightarrow \tan\theta = x - 1$..... (1)

$y = cos\theta\ or \frac{1}{y}=\frac{1}{cos\theta}=sec\theta \Rightarrow sec\theta=\frac{1}{y}$..... (2)

Using $1 + tan^2\theta = sec^2\theta$

$\Rightarrow 1+(x-1)^2=\frac{1}{y^2} \quad \therefore y^2(x-1)^2+y^2=1$

iv) $x = acosec\theta \Rightarrow cosec\theta = \frac{x}{a}=\frac{1}{\sin\theta} \Rightarrow \sin\theta=\frac{a}{x}$ ..... (1)

$y = asec\theta \Rightarrow sec\theta=\frac{y}{b}=\frac{1}{cos\theta}\Rightarrow cos\theta=\frac{b}{y}$ ..... (2)

Using $cos^2\theta + sin^2\theta = 1 \Rightarrow \frac{a^2}{x^2}+\frac{b^2}{y^2}=1$ as required

v) $x = a\sec\theta \Rightarrow \sec\theta = \frac{x}{a}\ or \cos\theta=\frac{a}{x}$ ..... (1)

$y = b + c\cos\theta \Rightarrow \frac{y-b}{c}=cos\theta$..... (2)

Equating (1) and (2)

$\Rightarrow \frac{y-b}{c}=\frac{a}{x} \quad \therefore x(y-b)=ac$

2. a) If $\tan\theta + \sin\theta = x\ and \tan\theta - \sin\theta = b$ prove that $(x^2-y^2)^2 = 16xy$

b) If $p\sin\theta + q\cos\theta = a$ and $p\cos\theta - q\sin\theta = b$ show that $\frac{p+a}{q+b}+\frac{q-b}{p-a}=0$

c) If $\sin\theta - cos\theta = a\ and$ $\text{cosec}\theta + \sec\theta = b$ prove that

$b^2(1-a^2)^2 = 4(2-a^2)$

b) If $xcos\theta + y\sin\theta = a\ and\ x\sin\theta - ycos\theta = b$ prove that

$\tan\theta=\frac{bx+ay}{ax-by}\ and\ x^2+y^2=a^2+b^2$

Solution

a) $\tan\theta + \sin\theta = x$ ..... (1)

$\tan\theta - \sin\theta = b$ ..... (2)

Eliminating $\sin\theta$, (1) + (2)

$2\tan\theta = x + y \Rightarrow \tan\theta=\frac{x+y}{2}$ ..... (3)

Eliminating $\tan\theta$, (1) − (2)

$2\sin\theta = x - y \Rightarrow \sin\theta=\frac{x-y}{2}$

But $\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{2}{x-y}$ and from (3) $\cot\theta = \frac{1}{\tan\theta} = \frac{2}{x+y}$

Now using $1 + cot^2\theta = cosec^2\theta$

$\Rightarrow 1 + \frac{4}{(x+y)^2} = \frac{4}{(x-y)^2} \quad \therefore 1 = 4\left(\frac{1}{(x-y)^2} - \frac{1}{(x+y)^2}\right) = \frac{4[(x+y)^2-(x-y)^2]}{(x-y)^2(x+y)^2}$

$= \frac{4[(x+y+x-y)(x+y-x+y)]}{(x^2-y^2)^2}$, $Difference\ of\ squares$

$= \frac{4(2x)(2y)}{(x^2-y^2)^2}$

$\therefore\ (x^2 - y^2)^2 = 16xy$

b) $p\sin\theta + q\cos\theta = a$ ..... (1)

$p\cos\theta - q\sin\theta = b$ ..... (2)

Squaring equations and adding

$p^2sin^2\theta + q^2cos^2\theta + 2pq\sin\theta\,cos\theta = a^2$

$+\ p^2cos^2\theta + q^2sin^2\theta - 2pq\sin\theta\,cos\theta = b^2$

$p^2(sin^2\theta + cos^2\theta) + (cos^2\theta + sin^2\theta) = a^2 + b^2$

$p^2 + q^2 = a^2 + b^2 \Rightarrow p^2 - a^2 + q^2 - b^2 = 0$

$(p-a)(p+a) + (q-b)(q+b) = 0$, $dividing\ through\ by\ (p-a)(q+b)$

$\Rightarrow \frac{(p-a)(p+a)}{((p-a)(q+b))} + \frac{(q-b)(q+b)}{(p-a)(q+b)} = 0 \quad \therefore \frac{p+a}{q+b} + \frac{q-b}{p-a} = 0$

c) $\sin\theta - cos\theta = a$, $cosec\theta + sec\theta = b$ to prove $b^2(1-a^2)^2 = 4(2-a^2)$

Beginning from a complicated side L.H.S

$b^2(1-a^2)^2 = (cosec\theta + sec\theta)^2(1 - (\sin\theta - cos\theta)^2)^2$

$= (cosec^2\theta + 2cosec\theta sec\theta + sec^2\theta)\left(1 - (cos^2\theta + sin^2\theta - 2\sin\theta\,cos\theta)\right)$

$= \left(\frac{1}{sin^2\theta} + \frac{1}{cos^2\theta} + \frac{2}{\sin\theta cos\theta}\right)(1 - (1 - 2sin\theta cos\theta)^2)$

$= \left(\frac{cos^2\theta + sin^2\theta + 2\sin\theta cos\theta}{sin^2\theta cos^2\theta}\right)(2\sin\theta\,cos\theta)^2 = 4(1 + \sin\theta\,cos\theta)$

Now expressing $2\sin\theta\,cos\theta$ in terms of a

From $(\sin\theta - cos\theta) = a$, squaring gives

$\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta = a^2 \Rightarrow 1 - 2\sin\theta\cos\theta = a^2$

$\therefore 2\sin\theta\cos\theta = 1 - a^2 \Rightarrow b^2(1-a^2)^2 = 4(1 + 1 - a^2) = 4(2 - a^2)$

d) $xcos\theta + y\sin\theta = a$ ..... (1)

$x\sin\theta - ycos\theta = b$ ..... (2)

Eliminating constants a *and* b, $(1) \times b - (2) \times a$

$\Rightarrow bx\cos\theta + by\sin\theta = ab$

$ax\sin\theta - aycos\theta = ab$

$bxcos\theta - ax\sin\theta + by\sin\theta + aycos\theta = 0$

$cos\theta(bx + ay) + (by - ax)\sin\theta = 0$

Creating $\tan\theta$ by division by $\cos\theta$

$\frac{\cos\theta}{\cos\theta}(bx + ay) + (by - ax)\frac{\sin\theta}{\sin\theta} = 0 \Rightarrow (bx + ay) + (by - ax)tan\theta = 0$

$\therefore \tan\theta\,(ax - by) = bx + ay \Rightarrow \tan\theta = \frac{bx+ay}{ax-by}$ as required

To prove $x^2 + y^2 = a^2 + b^2$, see previous example in © above

Task: If $\sec\theta - \cos\theta = a\ and\ \text{cosec}\theta - \sin\theta - b, show\ that\ a^2b^2(a^2 + b^2 + 3 = 1)$

PROOF OF IDENTITIES

A trigonometric equation which is true for all permissible values of the angles involved is called a *trigonometric identity.*

Suggested steps in verifying Identities

1. Start with a more complicated side of the identity and transform it in to a simpler side
2. Try algebraic operations such as multiplying, factoring and combining fractions and splitting fractions
3. If the other steps fail, express each function in terms of sine and cosine function and then perform appropriate algebraic operations
4. At each step keep the other side of the identity in mind. This reveals what you should do to get there.
5. Try to manipulate either side of the equation until it is identical to the other side.

Examples

1. *Prove the following identities*

i) $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

ii) $\sin x = \tan x \cos x$

iii) $\frac{\text{cosec } x}{1-\text{cosec } x} = \frac{\sec x}{\tan x - \sec x}$

iv) $\text{cosec}^4 x - \cot^4 x = \text{cosec}^2 x + \cot^2 x$

v) $\frac{\cos^2 x}{\sin x} + \sin x = \text{cosec } x$

vi) $\frac{\cot x}{\cos x} + \frac{\sec x}{\cot x} = \sec^2 \text{x}.\text{cosec } x$

vii) $\frac{1+\sin x - \sin^2 x}{\cos x} = \cos x + \tan x$

viii) $\frac{\cos x + 1}{\sin^3 x} = \frac{\text{cosec } x}{1-\cos x}$

ix) $\frac{\tan x}{\sec x} + \frac{\cot x}{\text{cosec } x} = \sin x + \cos x$

Solution

i) $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

L.H.S: $\sin^4 x - \cos^4 x = (\sin^2 x)^2 - (\cos^2 x)^2$ [Difference of squares]

$= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)$

$= 1.(\sin^2 x - \cos^2 x) = (\sin^2 x - \cos^2 x) = \mathbf{R.H.S}$

ii) $\sin x = \tan x \cos x$

R.H.S: $\tan x \,.\cos x = \frac{\sin x}{\cos x}.\cos x = \sin x = \mathbf{L.H.S}$

iii) $\frac{\text{cosec } x}{1-\text{cosec } x} = \frac{\sec x}{\text{tanx} - \sec x}$, The tactic is to express both sides to the same expression. i.e. same trig ratios

$\mathbf{L.H.S}$: $\frac{\text{cosec } x}{1-\text{cosec } x} = \frac{1/\sin x}{1-1/\sin x} = \frac{1}{\sin x} \times \frac{\sin x}{\sin x - 1} = \frac{1}{\sin x - 1}$

$\mathbf{R.H.S}$: $\frac{\sec x}{\tan x - \sec x} = \frac{1/\cos x}{\frac{\sin x}{\cos x} - \frac{1}{\cos x}} = \frac{1}{\cos x} \times \frac{\cos x}{\sin x - 1} = \frac{1}{\sin x - 1}$

Consequently $\frac{\text{cosec } x}{1-\text{cosec } x} = \frac{\sec x}{\tan x - \sec x}$

iv) Left as an exercise. Hint: see (i) above

v) $\frac{\cos^2 \text{x}}{\text{sinx}} + \text{sinx} = \text{cosecx}$

$\mathbf{L.H.S}$: $\frac{\cos^2 x}{\text{sinx}} + \frac{\sin x}{1} = \frac{\cos^2 \text{x} + \sin^2 x}{\sin x} = \frac{1}{\sin x} = \text{cosecx} = \mathbf{R.H.S}$

vi) $\frac{\cot x}{\cos x} + \frac{\sec x}{\cot x} = \sec^2 x.\text{cosec} x$

Expressing L.H.S in terms of sines and cosines

$$\frac{\cot x}{\cos x}+\frac{\sec x}{\cot x}=\frac{\cos x}{\sin x.\cos x}+\frac{1}{\cos x.\frac{\cos x}{\sin x}}=\frac{1}{\sin x}+\frac{\sin x}{\cos^2 x}$$

$$=\frac{\cos^2 x+\sin^2 x}{\sin x.\cos^2 x}=\frac{1}{\sin x}.\frac{1}{\cos^2 x}=\text{cosecx.sec}^2\text{x}=\mathbf{R.H.S}$$

vii) $\frac{1+\sin x-\sin^2 x}{\cos x}=\cos x+\tan x$

$\mathbf{L.H.S}$: $\frac{1+\sin x-\sin^2 x}{\cos x}=\frac{1+\sin x-(1-\cos^2 x)}{\cos x}$ $[\because \cos^2 x+\sin^2 x=1]$

$$=\frac{\sin x+\cos^2 x}{\cos x}=\frac{\sin x}{\cos x}+\frac{\cos^2 x}{\cos x}=\tan x+\cos x=\mathbf{R.H.S}$$

viii) $\frac{\cos x+1}{\sin^3 x}=\frac{\text{cosec } x}{1-\cos x}$

The numerator of LHS gives rise to denominator of the RHS through rationalization

$\mathbf{L.H.S}$: $\frac{\cos x+1}{\sin^3 x}=\frac{(\cos x+1)(\cos x-1)}{\sin^3 x.(\cos x-1)}=\frac{\cos^2 x-1}{\sin^3 x.(\cos x-1)}$ ..... $\left[\times\left(\frac{-1}{-1}\right)\right]$

$$=\frac{1-\cos^2 x}{\sin^3 x.(\cos x-1)}=\frac{\sin^2 x}{\sin^3 x.(\cos x-1)}=\frac{1}{\sin x(1-\cos x)}=\frac{1}{\sin x}.\frac{1}{1-\cos x}=\frac{\text{cosec } x}{1-\cos x}$$

ix) $\frac{\tan x}{\sec x}+\frac{\cot x}{\text{cosec } x}=\sin x+\cos x$

Since the RHS is in terms of sinx and cosx, expressing LHS in terms of $\sin x$ and $\cos x$,

$\mathbf{L.H.S}$: $\frac{\text{tanx}}{\text{secx}}+\frac{\text{cotx}}{\text{cosecx}}=\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}+\frac{\frac{\cos x}{\sin x}}{\frac{1}{\text{sinx}}}=\frac{\sin x}{\cos x}.\cos x+\frac{\cos x}{\sin x}.\sin x$

$$=\text{sinx}+\text{cosx}=\mathbf{R.H.S}$$

2. Prove the following identities

a) $\frac{1-\cos\theta}{\sin\theta}=\frac{1}{\text{cosec }\theta+\cot\theta}$

b) $\frac{\tan\theta+\cot\theta}{\sec\theta+\text{cosec }\theta}=\frac{1}{\sin\theta+\cos\theta}$

c) $\sec^2\theta=\frac{\text{cosec }\theta}{\text{cosec }\theta-\sin\theta}$

d) $\frac{1+\sin\theta}{1-\sin\theta}=(\sec\theta+\tan\theta)^2$

e) $\sec\theta-\sin\theta=\frac{\tan^2\theta+\cos^2\theta}{\sec\theta+\sin\theta}$

f) $\frac{1-\sin\theta+\cos\theta}{1-\sin\theta}=\frac{1+\sin\theta+\cos\theta}{\text{cocs }\theta}$

g) $\frac{\sin^3\theta+\cos^3\theta}{1-2\cos^2\theta}=\frac{\sec\theta-\sin\theta}{\tan\theta-1}$

h) $\frac{1-2\sin\theta-3\sin^2\theta}{\cos^2\theta}=\frac{1-3\sin\theta}{1-\sin\theta}$

Solution

a) $\frac{1-\cos\theta}{\sin\theta}=\frac{1}{\text{cosec }\theta+\cot\theta}$

$\mathbf{L.H.S}$: $\frac{1-\cos\theta}{\sin\theta}=\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}=\text{cosec}\theta-\cot\theta$

Now $\mathbf{L.H.S}=\text{cosec}\theta-\cot\theta$

The denominator of **R.H.S** comes from $\text{cosec}\theta-\cot\theta$ through Rationalization

$$\mathbf{L.H.S}=\text{cosec}\theta-\cot\theta=(\text{cosec}\theta-\cot\theta).\frac{(\text{cosec }\theta+\cot\theta)}{\text{cosec }\theta+\cot\theta}$$

$$=\frac{\text{cosec }^2\theta-\cot^2\theta}{\text{cosec }\theta+\cot\theta}=\frac{1}{\text{cosec }\theta+\cot\theta}\quad[\because \text{from cosec}^2\theta=1+\cot^2\theta]$$

b) $\frac{\tan\theta+\cot\theta}{\sec\theta+\text{cosec }\theta}=\frac{1}{\sin\theta+\cos\theta}$

$\mathbf{L.H.S}$: Expressing in terms of $\cos\theta$ and $\sin\theta$

$\mathbf{L.H.S}=\frac{\frac{\sin\theta}{\text{COS }\theta}+\frac{\text{COS }\theta}{\sin\theta}}{\frac{1}{\text{COS }\theta}+\frac{1}{\sin\theta}}=\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)\div\left(\frac{\sin\theta+\cos\theta}{\cos\theta\sin\theta}\right)$

$$= \frac{\sin^2\theta+\cos^2\theta}{\sin\theta+\cos\theta} = \frac{1}{\sin\theta+\cos\theta} = \mathbf{R.H.S}$$

c) $\sec^2\theta = \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta-\sin\theta}$

$\mathbf{R,H,S}$: $\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta-\sin\theta} = \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta}-\sin\theta} = \frac{1}{\sin\theta}.\frac{\sin\theta}{1-\sin^2\theta} = \frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$

d) $\frac{1+\sin\theta}{1-\sin\theta} = (\sec\theta+\tan\theta)^2$

$\mathbf{R.H.S.}$: $(\sec\theta+\tan\theta)^2 = \left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)^2 = \frac{(1+\sin\theta)^2}{\cos^2\theta}$

$$= \frac{(1+\sin\theta)^2}{1-\sin^2\theta} \quad [\because \cos^2\theta+\sin^2\theta = 1]$$

$$= \frac{(1+\sin\theta)(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} \text{ [Denominator, difference of squares]}$$

$$= \frac{1+\sin\theta}{1-\sin\theta} = \mathbf{L.H.S}$$

e) $\sec\theta-\sin\theta = \frac{\tan^2\theta+\cos^2\theta}{\sec\theta+\sin\theta}$

Denominator of **R.H.S** originates from $\sec\theta-\sin\theta$ on the **L. H.S** and vis versa

$\mathbf{L.H.S}$: $\sec\theta-\sin\theta = (\sec\theta-\sin\theta)\frac{(\sec\theta+\sin\theta)}{(\sec\theta+\sin\theta)}$

$$= \frac{\sec^2\theta-\sin^2\theta}{\sec\theta-\sin\theta} = \frac{(1+\tan^2\theta)-(1-\cos^2\theta)}{\sec\theta+\sin\theta}$$

$$= \frac{\tan^2\theta+\cos^2\theta}{\sec\theta+\sin\theta} = \mathbf{R.H.S}$$

f) $\frac{1-\sin\theta+\cos\theta}{1-\sin\theta} = \frac{1+\sin\theta+\cos\theta}{\operatorname{cocs}\theta}$, Starting from any side

$\mathbf{L.H.S}$: $\frac{1-\sin\theta+\cos\theta}{1-\sin\theta} = \frac{1-\sin\theta}{1-\sin\theta}+\frac{\cos\theta}{1-\sin\theta} = 1+\frac{\cos\theta}{1-\sin\theta}$

Now $\frac{\cos\theta}{1-\sin\theta}$ can be rationalized since denominator yields

$1-\sin^2\theta = \cos^2\theta$

$\because \frac{\cos\theta}{1-\sin\theta} = \cos\theta.\frac{(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} = \cos\theta.\frac{1+\sin\theta}{1-\sin^2\theta}$

$$= \cos\theta.\frac{1+\sin\theta}{\cos^2\theta} = \frac{1+\sin\theta}{\cos\theta}$$

$\Rightarrow \mathbf{L.H.S}$: $\frac{1-\sin\theta+\cos\theta}{1-\sin\theta} = 1+\frac{1+\sin\theta}{\cos\theta} = \frac{1+\sin\theta+\cos\theta}{\operatorname{cocs}\theta} = \mathbf{R.H.S}$

g) $\frac{\sin^3\theta+\cos^3\theta}{1-2\cos^2\theta} = \frac{\sec\theta-\sin\theta}{\tan\theta-1}$

$\mathbf{L.H.S}$: $\frac{\sin^3\theta+\cos^3\theta}{1-2\cos^2\theta} = \frac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{1-2\cos^2\theta}$ $\left[\because a^3+b^3 = a^2-ab+b^2 \text{ an identity}\right]$

$$= \frac{(\sin\theta+\cos\theta)(\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta)}{1-2\cos^2\theta} = \frac{(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)}{1-2\cos^2\theta},$$

Dividing the Numerator and Denominator by $\cos^2\theta$

$$= \frac{(\sin\theta+\cos\theta)\frac{(1-\cos\theta\sin\theta)}{\cos^2\theta}}{\frac{1}{\cos^2\theta}-2} = \frac{\left(\frac{\sin\theta+\cos\theta}{\cos\theta}\right).\left(\frac{1-\cos\theta\sin\theta}{\cos\theta}\right)}{\sec^2\theta-2}$$

$$= \frac{(\tan\theta+1)(\sec\theta-\sin\theta)}{1+\tan^2\theta-2} = \frac{(\tan\theta+1)(\sec\theta-\sin\theta)}{\tan^2\theta-1} = \frac{(\tan\theta+1)}{(\tan\theta+1)}.\frac{(\sec\theta-\sin\theta)}{(\tan\theta-1)}$$

$$= \frac{\sec\theta-\sin\theta}{\tan\theta-1} = \mathbf{R.H.S}$$

h) $\frac{1-2\sin\theta-3\sin^2\theta}{\cos^2\theta}=\frac{1-3\sin\theta}{1-\sin\theta}$

$\mathbf{L.H.S};\frac{1-2\sin\theta-3\sin^2\theta}{\cos^2\theta}=\frac{1-2\sin\theta-3\sin^2}{1-\sin^2\theta}$

$=\frac{(1-3\sin\theta)(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}=\frac{1-3\sin\theta}{1-\sin\theta}$

Note: $1-2\sin\theta-3\sin^2\theta\equiv 1-2x-3x^2$, factoring gives $(1+x)(1-3x)$

Task

Prove the following identities

a) $\frac{\sec\theta-\operatorname{cosec}\theta}{\tan\theta-\cot\theta}=\frac{\tan\theta+\cot\theta}{\sec\theta+\operatorname{cosec}\theta}$

b) $\frac{\cot\alpha+\tan\beta}{\cot\beta+\tan\alpha}=\cot\alpha\tan\beta$

c) $\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}=2\operatorname{cosec}^2\theta$

d) $\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}1+\frac{2\tan\theta}{1-\tan\theta}$

e) $\frac{\cos\theta-1}{\sec\theta+\tan\theta}+\frac{\cos\theta+1}{\sec\theta-\tan\theta}=2(1+\tan\theta)$

3. Prove the following identities

a) $\frac{\sin\theta}{\cot\theta+\operatorname{cosec}\theta}=2+\frac{\sin\theta}{\cot\theta-\operatorname{cosec}\theta}$

b) $\frac{\tan\theta-\sec\theta+1}{\tan\theta+\sec\theta-1}=\frac{\cos\theta}{1+\sin\theta}$

c) $\cot\theta\cos\theta+\tan\theta\sin\theta=(\operatorname{cosec}\theta+\sec\theta)(1-\sin\theta\cos\theta)$

d) $\operatorname{cosec}^2\theta-\operatorname{cosec}\theta=\frac{\cot^2\theta}{1+\sin\theta}$

Solution

a) $\frac{\sin\theta}{\cot\theta+\operatorname{cosec}\theta}=2+\frac{\sin\theta}{\cot\theta-\operatorname{cosec}\theta}$

Since the **R.H.S** is containing an isolated constant 2, then

Re arranging identity for simplicity,

$\frac{\sin\theta}{\cot\theta+\operatorname{cosec}\theta}-\frac{\sin\theta}{\cot\theta-\operatorname{cosec}\theta}=2$, now proving from the **L.H.S**

$\mathbf{L.H.S}:\ \frac{\sin\theta}{\cot\theta+\operatorname{cosec}\theta}-\frac{\sin\theta}{\cot\theta-\operatorname{cosec}\theta}=\sin\theta.\left[\frac{1}{\cot\theta+\operatorname{cosec}\theta}-\frac{1}{\cot\theta-\operatorname{cosec}\theta}\right]$

$=\sin\theta.\left[\frac{\cot\theta-\operatorname{cosec}\theta-\cot\theta-\operatorname{cosec}\theta}{\cot^2\theta-\operatorname{cosec}^2\theta}\right]$

$=\sin\theta.\left[\frac{-2\operatorname{cosec}\theta}{-1}\right]=2=\mathbf{R.H.S}$

Alternatively

With out re-arranging the identity,

$\mathbf{L.H.S}:\ \frac{\sin\theta}{\cot\theta+\operatorname{cosec}\theta}=\frac{\sin\theta}{\cot\theta+\operatorname{cosec}\theta}-2+2$ (creating the 2 in $\mathbf{R.H.S}$)

$=2+\left(\frac{\sin\theta}{\cot\theta+\operatorname{cosec}\theta}-2\right)$

$=2+\left(\frac{\sin\theta-2(\cot\theta+\operatorname{cosec}\theta)}{\cot\theta+\operatorname{cosec}\theta}\right)=2+\frac{\left(\sin\theta-\left(\frac{\cos\theta}{\sin\theta}+\frac{1}{\sin\theta}\right)\right)}{\cot\theta+\operatorname{cosec}\theta}$

$=2+\left(\frac{\sin^2\theta-2\cos\theta-2}{\cot\theta+\operatorname{cosec}\theta}\right).\frac{1}{\sin\theta}$

$=2+\left(\frac{1-\cos^2\theta-2\cos\theta-2}{\cot\theta+\operatorname{cosec}\theta}\right).\frac{1}{\sin\theta}$

$=2+-\left(\frac{\cos^2\theta+2\cos\theta+1}{\cot\theta+\operatorname{cosec}\theta}\right).\frac{1}{\sin\theta}$

$$= 2 + -\frac{(\cos\theta+1)^2}{(\cot\theta+\operatorname{cosec}\theta).\sin\theta}$$

$$= 2 + \frac{-(\cos\theta+1)^2(\cot\theta-\operatorname{cosec}\theta)}{(\cot\theta+\operatorname{cosec}\theta)(\cot\theta-\operatorname{cosec}\theta).\sin\theta}$$

$$= 2 + \frac{-(\cos\theta+1)^2\left(\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}\right)}{\left(\frac{\cos\theta}{\sin\theta}+\frac{1}{\sin\theta}\right)(\cot\theta-\operatorname{cosec}\theta).\sin\theta} = 2 + \frac{-(\cos\theta+1)^2(\cos\theta-1)}{\sin^2\theta\left(\frac{\cos\theta+1}{\sin\theta}\right)(\cot\theta-\operatorname{cosec}\theta)}$$

$$= 2 + \frac{-(\cos\theta+1)(\cos\theta\theta-1)}{(\cot\theta-\operatorname{cosec}\theta).\sin\theta} = 2 + \frac{-(\cos^2\theta-1)}{(\cot\theta-\operatorname{cosec}\theta).\sin\theta}$$

$$= \frac{\sin^2\theta}{(\cot\theta-\operatorname{cosec}\theta).\sin\theta}$$

$$= 2 + \frac{\sin\theta}{(\cot\theta-\operatorname{cosec}\theta)} = \mathbf{R.H.S}$$

b) $\frac{\tan\theta-\sec\theta+1}{\tan\theta+\sec\theta-1} = \frac{\cos\theta}{1+\sin\theta}$

$\mathbf{L.H.S}: \frac{\tan\theta-\sec\theta+1}{\tan\theta+\sec\theta-1} = \frac{\tan\theta-\sec\theta+\left(\sec^2\theta-\tan^2\theta\right)}{\tan\theta+\sec\theta-1}$ $[\because\ \sec^2\theta = 1+\tan^2\theta]$

$$= \frac{(\tan\theta-\sec\theta)-\left(\tan^2\theta-\sec^2\theta\right)}{\tan\theta+\sec\theta-1}$$

$$= \frac{(\tan\theta-\sec\theta)-(\tan\theta-\sec\theta)(\tan\theta+\sec\theta)}{\tan\theta+\sec\theta-1} = \frac{(\tan\theta-\sec\theta)[1-(\tan\theta+\sec\theta)]}{\tan\theta+\sec\theta-1}$$

$$= \frac{-(\tan\theta-\sec\theta)(\tan\theta+\sec\theta-1)}{(\tan\theta+\sec\theta-1)} = \sec\theta-\tan\theta = \frac{1}{\cos\theta}-\frac{1}{\sin\theta}$$

$$= \frac{(1-\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)} = \frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)} = \frac{\cos^2\theta}{\cos\theta(1+\sin\theta)} = \frac{\cos\theta}{1+\sin\theta}$$

NOTE:

The same proof can be done using half angle identities. See proof under this topic.

c) $\cot\theta\cos\theta + \tan\theta\sin\theta = (\operatorname{cosec}\theta + \sec\theta)(1-\sin\theta\cos\theta)$

$\mathbf{L.H.S}: \cot\theta\cos\theta + \tan\theta\sin\theta = \frac{\cos\theta}{\sin\theta}.\cos\theta + \frac{\sin\theta}{\cos\theta}.\sin\theta$

$$= \frac{\cos^2\theta}{\sin\theta} + \frac{\sin^2\theta}{\cos\theta} = \frac{\cos^3\theta+\sin^3\theta}{\sin\theta\cos\theta} = \frac{(\cos\theta+\sin\theta)(\cos^2\theta-\sin\theta\cos\theta+\sin^2\theta)}{\sin\theta\cos\theta}$$

$$= \frac{(\cos\theta+\sin\theta)(\cos^2\theta+\sin^2\theta-\sin\theta\cos\theta)}{\sin\theta\cos\theta} = \left(\frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}\right).(1-\sin\theta\cos\theta)$$

$$= \left(\frac{\cos\theta}{\sin\theta\cos\theta}+\frac{\sin\theta}{\sin\theta\cos\theta}\right).(1-\sin\theta\cos\theta)$$

$$= (\operatorname{cosec}\theta+\sec\theta).(1-\sin\theta\cos\theta) = \mathbf{R.H.S}$$

d) $\operatorname{cosec}^2\theta - \operatorname{cosec}\theta = \frac{\cot^2\theta}{1+\sin\theta}$

$\operatorname{cosec}^2\theta - \operatorname{cosec}\theta = \frac{1}{\sin^2\theta}-\frac{1}{\sin\theta} = \frac{1-\sin\theta}{\sin^2\theta}$

$$= \frac{(1-\sin\theta)(1+\sin\theta)}{\sin^2\theta(1+\sin\theta)} = \frac{1-\sin^2\theta}{\sin^2\theta(1+\sin\theta)} = \frac{\cos^2\theta}{\sin^2\theta(1+\sin\theta)} = \frac{\cot^2\theta}{1+\sin\theta}$$

Task

Verify that

a) $\frac{\tan^2 A}{\sec A+1} = \sec A - 1$

b) $\frac{\cos^2\theta\tan\theta}{\sin\theta} + \frac{\sin^3\theta\cot\theta}{\cos^2\theta} = \sec\theta$

c) $\frac{2\tan A(\tan A-\cot A)}{\tan^2 A-1} = 2$

d) $\frac{\cot x\sin^3 x+\cos x(1+\tan^2 x)}{\tan x\cos^2 x\sin x} - 1 = \sec^2 x\operatorname{cosec}^2 x$

e) $\frac{(\text{cosec}^2 x-1)\sin^2 x+\sin x\cos^2 x}{\cos^2 x}=1+\sin x$

f) $2\text{cosec}x=\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}$

Simplifying Trigonometric expressions

Example

1. Simplify if $s=\sin\theta$ and $c=\cos\theta$

a) $\frac{1-s^2}{1-c^2}$ c) $\frac{s}{c^2-1}$ d) $\frac{c}{s}+\frac{s}{c}$ e) $\frac{c^4-s^4}{c^2-s^2}$

b) $\frac{sc}{\sqrt{1-s^2}}$

Solution

a) $\frac{1-s^2}{1-c^2}=\frac{1-\sin^2\theta}{1-\cos^2\theta}=\frac{\cos^2\theta+\sin^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta-\cos^2\theta}=\frac{\cos^2\theta}{\sin^2\theta}=\cot^2\theta$

b) $\frac{sc}{\sqrt{1-s^2}}=\frac{\sin\theta.\cos\theta}{\sqrt{1-\sin^2\theta}}=\frac{\sin\theta.\cos\theta}{\sqrt{\cos^2\theta}}=\frac{\sin\theta.\cos\theta}{\cos\theta}=\sin\theta\ [\because \cos^2\theta+\sin^2\theta=1]$

c) $\frac{s}{c^2-1}=\frac{\sin\theta}{\cos^2\theta-1}=\frac{\sin\theta}{-\sin^2\theta}=\frac{1}{-\sin\theta}=-\text{cosec}\theta$

d) left as an exercise

e) $\frac{c^4-s^4}{c^2-s^2}=\frac{(c^2)^2-(s^2)^2}{c^2-s^2}=\frac{(c^2+s^2)(c^2-s^2)}{c^2-s^2}=c^2+s^2=\cos^2\theta+\sin^2\theta=1$

2. Simplify using identities

a) $\sin x(\sin x-1)+\cos^2 x-1$ f) $\frac{\sin x\,(\sin x-\cos x\tan x\ \cot^2 x)}{\cos x\ (\tan x+1)}+\cos x$

b) $(1-\sin A)(1+\sin A)-\cos^2 A$ g) $\frac{\sin x}{1-\sin x}+\frac{\cos^2 x}{\sin x+1}-\frac{1+\sin x}{\cos^2 x}$

c) $3\sin^2 B+3\cos^2 B-2$ h) $\frac{\text{cosec}^2 x-\cot^2 x.\text{cosec}^2 x}{\cot^2 x}+\text{cosec}^2 x$

d) $\sin x+\cos^2 x-\sin x(1-\sin x)$

e) $\left(\frac{\sec\varphi}{\tan\varphi}-\sin\varphi\right).\sec\varphi$

Solution

a) $\sin x(\sin x-1)+\cos^2 x-1=\sin^2 x-\sin x+\cos^2 x-1$
$=\sin^2 x+\cos^2 x-\sin x-1=1-\sin x-1=-\sin x$

b) $(1-\sin A)(1+\sin A)-\cos^2 A=1-\sin^2 A-\cos^2 A$
$=1-(\sin^2 A+\cos^2 A)=1-1=0$

c) $3\sin^2 B+3\cos^2 B-2=3\sin^2 B+3\cos^2 B-2(1)$
$=3\sin^2 B+3\cos^2 B-2(\cos^2 B+\sin^2 B)$
$=3\sin^2 B+3\cos^2 B-2\cos^2 B-2\sin^2 B$
$=\sin^2 B+\cos^2 B=1$

d) $\sin x+\cos^2 x-\sin x(1-\sin x)=\sin x+\cos^2 x-\sin x+\sin^2 x$
$=\cos^2 x+\sin^2 x=1$

e) $\left(\frac{\sec\varphi}{\tan\varphi}-\sin\varphi\right).\sec\varphi=\left(\frac{\frac{1}{\cos\varphi}}{\frac{\sin\varphi}{\cos\varphi}}-\sin\varphi\right).\frac{1}{\cos\varphi}$

$=\left(\frac{1}{\cos\varphi}\times\frac{\cos\varphi}{\sin\varphi}-\sin\varphi\right).\frac{1}{\cos\varphi}$

$=\left(\frac{1}{\sin\varphi}-\sin\varphi\right).\frac{1}{\cos\varphi}=\frac{1-\sin^2\varphi}{\sin\varphi}.\frac{1}{\cos\varphi}=\frac{\cos^2\varphi}{\sin\varphi}.\frac{1}{\cos\varphi}=\frac{\cos\varphi}{\sin\varphi}=\cot\varphi$

f) $\frac{\sin x(\sin x-\cos x\tan x\cot^2 x)}{\cos x(\tan x+1)}+\cos x=\frac{\sin^2 x-\sin x.\cos x.\frac{\sin x}{\cos x}.\frac{\cos^2 x}{\sin^2 x}}{\cos x.\frac{\sin x}{\cos x}+\cos x}+\cos x$

$=\frac{\sin^2 x-\cos^2 x}{\sin x+\cos x}=\frac{(\sin x+\cos x)(\sin x-\cos x)}{\sin x+\cos x}=\sin x-\cos x$

g) $\frac{\sin x}{1-\sin x}+\frac{\cos^2 x}{\sin x+1}-\frac{1+\sin x}{\cos^2 x}=\left(\frac{\sin x}{1-\sin x}+\frac{\cos^2 x}{\sin x+1}\right)-\frac{(1+\sin x)}{\cos^2 x}$

$=\frac{(\sin^2 x+\cos^2 x)+\sin x-\sin x.\cos^2 x}{\cos^2 x}-\frac{(1+\sin x)}{\cos^2 x}$

$=\frac{1+\sin x-\sin x.\cos^2 x-1-\sin x}{\cos^2 x}=\frac{-\sin x.\cos^2 x}{\cos^2 x}=-\sin x$

h) $\frac{\operatorname{cosec}^2 x-\cot^2 x.\operatorname{cosec}^2 x}{\cot^2 x}+\operatorname{cosec}^2 x=\frac{\operatorname{cosec}^2 x-\cot^2 x.\operatorname{cosec}^2 x+\cot^2 x.\operatorname{cosec}^2 x}{\cot^2 x}$

$=\frac{\operatorname{cosec}^2 x}{\cot^2 x}=\operatorname{cosec}^2 x.\tan^2 x=\frac{1}{\sin^2 x}.\frac{\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$

Task

Using suitable identities simplify

a) $\frac{\sin A+\cos A}{\sin A}-\frac{\cos A-\sin A}{\cos A}$

b) $\frac{1-2\cos^2 x}{\sin x\cos x}+\cot x$

c) $\left(\frac{\sin x-\sin^3 x}{\sin x.\cos x+\cos^2 x}\right).\left(\frac{1+2\sin x\cos x}{\cos x\sin x}\right)$

d) $\left(\frac{\sin x+\sin x.\cos x}{1-\sin x}\right).\left(\frac{\cos^2 x}{1+\cos x}.\operatorname{cosec} x\right)$

NOTE:

This application of simplifying trig functions is very applicable in proof of identities. It is most suitable when dealing with trig ratios expressed I terms of cosines and sines

SKETCHING TRIGONOMERIC CURVES

The major trigonometric curves to be considered are

$y=a\cos Ax, y=b\sin Bx \quad and\ y=c\tan Cx$

Example

1. *Sketch the following curves for* $0^0\leq\theta\leq 360^0$

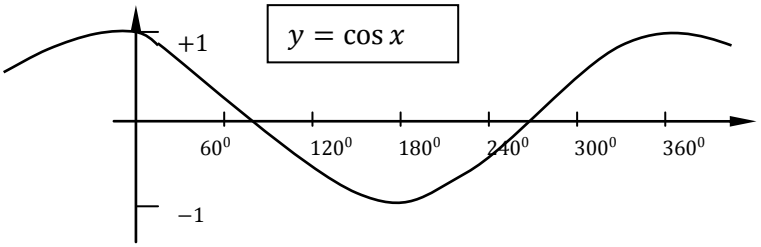
a) $y=\cos x$

b) $y=\sin x$

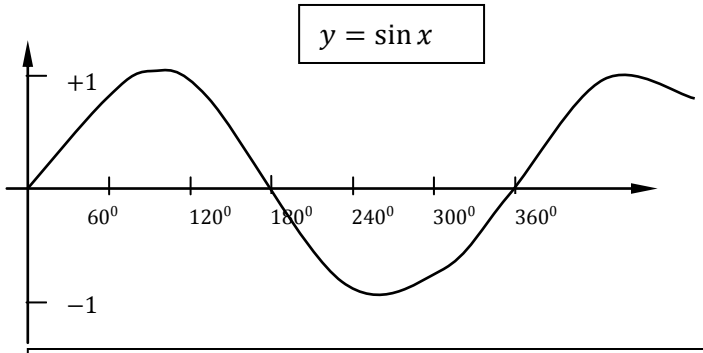
c) $y=\tan x$

Solution

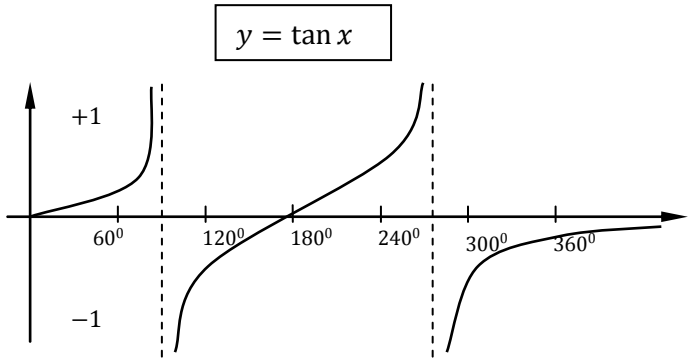
x^0	0	30	60	90	120	150	180	210	240	270	300	330	360
$y = \cos x$	1	$\frac{\sqrt{3}}{2} = 0.9$	$\frac{1}{2} = 0.5$	0	$\frac{1}{2} = 0.5$	$\frac{-\sqrt{3}}{2} = 0.9$	-1	$\frac{-\sqrt{3}}{2} = 0.9$	$-\frac{1}{2} = -0.5$	0	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} = 0.9$	1
$y = \sin x$	0	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} = 0.9$	1	$\frac{\sqrt{3}}{2} = 0.9$	$\frac{1}{2} = 0.5$	0	$-\frac{1}{2} = -0.5$	$\frac{-\sqrt{3}}{2} = 0.9$	-1	$\frac{-\sqrt{3}}{2} = 0.9$	$-\frac{1}{2} = -0.5$	0
$y = \tan x$	0	$\frac{1}{\sqrt{3}} = 0.6$	$\sqrt{3} = 1.7$	$+\infty$	$-\sqrt{3} = -1.7$	$-\frac{1}{\sqrt{3}} = -0.6$	0	$\frac{1}{\sqrt{3}} = 0.6$	$\sqrt{3} = 1.7$	$-\infty$	$-\sqrt{3} = -1.7$	$-\frac{1}{\sqrt{3}} = -0.6$	0



Deduction
Maximum value of $\cos x = 1$ and occurs at $x = 0^0$ *or* 360^0 and minimum value is -1 and occurs at $x = 180^0$
In general $-\mathbf{1} \leq \boldsymbol{cos}(\boldsymbol{Angle}) \leq +\mathbf{1}$



Deduction
Maximum value of $sinx = 1$ and occurs at $x = 90^0$ *or* 360^0 and minimum value is -1 and occurs at $x = 270^0$
In general $-\mathbf{1} \leq \mathbf{sin}(\boldsymbol{Angle}) \leq +\mathbf{1}$



Deduction

1. The curve $y = \tan x$ is un defined at $x = 90^0, 270^0$ for$0^0 \leq \theta \leq 360^0$ $i.e$ for $\tan x = \frac{sinx}{cosx}$, the curve is undefined at $\cos x = 0^0$ $i.e. at$ $90^0, 270^0, 450^0$ etc
2. The curve $y = tanx$ has no maximum and minimum values

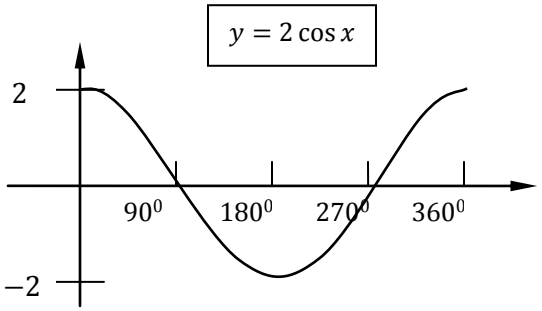
2. *Sketch the curves* $0^0 \leq \theta \leq 360^0$

a) $y = 2cosx$ b) $y = 3sinx$

Solution

Using intervals of 45^0

x^0	0	45	90	135	180	225	270	315	360
$y = 2\cos x$	2.0	1.4	0.0	−1.4	−2.0	−1.4	0.0	1.4	2.0
$y = 3\sin x$	0.0	2.1	3.0	2.12	0.0	−2.1	−3.0	−2.1	0.0

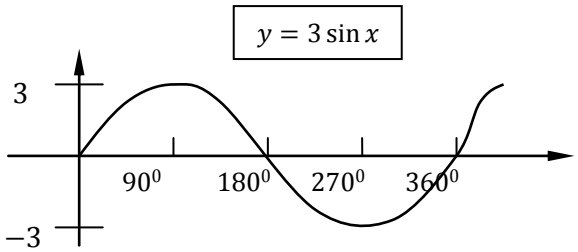


Deduction

Maximum value of $\mathbf{2cos}\,\boldsymbol{x}$ is 2 and occurs at $x = 0^0$ *or* 360^0 and minimum value is -2 and occurs at $x = 180^0$.

In general for

$\boldsymbol{y = a\cos(Angle)}$, *the maximum value is* $+\boldsymbol{a}$ *and minimum value is* $-\boldsymbol{a}$ $for\ a > 0$



Deduction
Maximum value of **3sinx** is 3 and occurs at $x = 90^0$ and minimum value is -3 and occurs at $x = 270^0$.
In general for
$\boldsymbol{y = a\sin(Angle)}$, *the maximum value is* $+\boldsymbol{a}$ *and minimum value is* $-\boldsymbol{a}$ $\quad for\ a > 0$

NOTE
The above deductions are used in deducing maximum and minimum values of trigonometric functions

STATING MAXIMUM AND MINIMUM VALUES OF $\mathbf{y = a \pm bcos(m\theta + \alpha)}$ AND $\mathbf{y = a \pm bsin(m\theta + \alpha)}$ AND THEIR RECIPRICALS AND VALUES OF θ FOR WHICH THEY OCCUR
The following rules may be used

$max(x + y) = max(x) + max(y)$
$min(x + y) = min(x) + min(y)$
$max(x - y) = max(x) - min(y)$
$min(x - y) = min(x) - max(y)$
$max(x.y) = max(x).max(y)\ \ [for\ x, y > 0]$
$min(x.y) = min(x).min(y)\quad [for\ x, y > 0]$

$max\left(\frac{x}{y}\right) = \frac{max\ (x)}{min\ (y)}$
$min\left(\frac{x}{y}\right) = \frac{min\ (x)}{max\ (y)}$

Example
Write down the maximum and minimum value of the following expressions giving the smallest positive or zero value of θ for which they occur.

i) $\frac{1}{3}cos\frac{1}{2}\theta$
ii) $\frac{1}{4}\sin 2\theta$
iii) $1 - 2\sin\theta$
iv) $3 + 2\cos 2\theta$
v) $\frac{1}{4-3\cos 2\theta}$
vi) $\frac{1}{2+3\sin\theta}$
vii) $\sec\frac{3}{2}\theta$
viii) $\text{cosec}3\theta$
ix) $-\frac{1}{2}\sin 2\theta$
x) $-\frac{1}{4}\cos 2\theta$

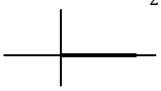
Solution

i) $max\left(\frac{1}{3}\cos\frac{1}{2}\theta\right) = \frac{1}{3}$ $\max\left(\cos\frac{1}{2}\theta\right)$ but $-1 \le \cos\frac{1}{2}\theta \le +1$

$\therefore \max\left(\cos\frac{1}{2}\theta\right) = 1 \Rightarrow \max\left(\frac{1}{3}\cos\frac{1}{2}\theta\right) = \frac{1}{3}\times 1 = \frac{1}{3}$

Finding value of θ_{max}

$\frac{1}{3}\cos\frac{1}{2}\theta = \frac{1}{3}(\max value) \Rightarrow \cos\frac{1}{2}\theta = 1$, solving the equation



$\cos^{-1}(1) = 0^0, 360^0$

$\therefore \frac{1}{2}\theta = 0^0, 360^0 \Rightarrow \theta_{max} = 0^0$

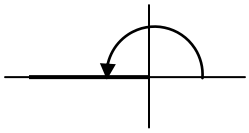
$\min\left(\frac{1}{3}\cos\frac{1}{2}\theta\right) = \frac{1}{3}\min\left(\cos\frac{1}{2}\theta\right)$ but $\min\left(\cos\frac{1}{2}\theta\right) = -1$

$\therefore \min\left(\frac{1}{3}\cos\frac{1}{2}\theta\right) = \frac{1}{3}\times -1 = -\frac{1}{3}$

Finding value of θ_{min}

$\frac{1}{3}\cos\frac{1}{2}\theta = -\frac{1}{3}(\min value) \Rightarrow \cos\frac{1}{2}\theta = -1$, solving the equation

$\frac{1}{2}\theta = 180^0 \Rightarrow \theta_{min} = 360^0$



ii) $\max\left(\frac{1}{4}\sin 2\theta\right) = \frac{1}{4}\max(\sin 2\theta)$ but $-1 \le \sin 2\theta \le +1$

$\therefore \max(\sin 2\theta) = 1, \min(\sin 2\theta) = -1$

$\Rightarrow \max\left(\frac{1}{4}\sin 2\theta\right) = \frac{1}{4}\times 1 = \frac{1}{4}$

Finding value of θ_{max}

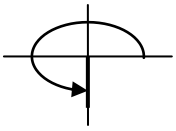
$\frac{1}{4}\sin 2\theta = \frac{1}{4}(\max value) \Rightarrow \sin 2\theta = 1$, solving the equation

$\sin^{-1}(1) = 90^0$

$\therefore 2\theta = 90^0 \Rightarrow \theta_{max} = 45^0$

$\min\left(\frac{1}{4}\sin 2\theta\right) = \frac{1}{4}\min(\sin 2\theta) = \frac{1}{4}\times -1 = -\frac{1}{4}$

Finding value of θ_{min}



$\frac{1}{4}\sin 2\theta = -\frac{1}{4}(\min value) \Rightarrow \sin 2\theta = -1$

$\therefore 2\theta = 270^0, \theta_{min} = 135^0$

iii) $\max(1 - 2\sin\theta) = \max(1) - \min(2\sin\theta)$

$= 1 - 2\min(\sin\theta) = 1 - 2(-1) = 3$

Finding value of θ_{max}

$1 - 2\sin\theta = 3(\max value) \Rightarrow \sin\theta = -1$, solving the equation

$\therefore \theta_{max} = 270^0$

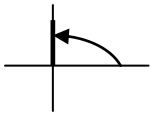
$\min(1 - 2\sin\theta) = \min(1) - \max(2\sin\theta)$

$= 1 - 2\max(\sin\theta) = 1 - 2(1) = -1$

Finding value of θ_{min}

$1 - 2\sin\theta = -1(\min value) \Rightarrow \sin\theta = 1$, *solving the equation*

$\therefore \theta_{min} = 90^0$

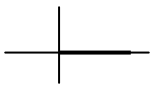


iv) $max(3 + 2\cos 2\theta) = max(3) + max(2\cos 2\theta)$

$= 3 + 2max(\cos 2\theta) = 3 + 2(1) = 5$

Finding value of θ_{max}

$3 + 2cos2\theta = 5 \Rightarrow cos2\theta = 1$, solving the equation



$\Rightarrow 2\theta = 0^0, 360^0$

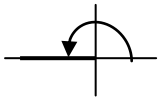
$\therefore \theta_{max} = 0^0, 180^0$

$min(3 + 2cos2\theta) = min(3) + min(2\cos 2\theta)$

$= 3 + 2min(\cos 2\theta) = 3 + 2(-1) = 1$

Now finding value of θ_{min}

$3 + 2cos2\theta = 1 \Rightarrow cos2\theta = -1$, solving the equation



$\Rightarrow 2\theta = 180^0 \quad \therefore \theta_{min} = 90^0$

v) $max\left(\frac{1}{4-3cos\,2\theta}\right) = \frac{max\,(1)}{min\,(4-3cos\,2\theta)} = \frac{1}{min\,(4)-max\,(3cos\,2\theta)}$

$= \frac{1}{4-3max\,(cos\,2\theta)} = \frac{1}{4-3(+1)} = 1$

Finding value of θ_{max}

$\frac{1}{4-3cos\,2\theta} = 1(\max value)\ or\ 4 - 3cos2\theta = 1 \Rightarrow cos2\theta = 1$

Solving the equations as before $2\theta = 90^0 \Rightarrow \theta_{max} = 45^0$

$min\left(\frac{1}{4-3cos\,2\theta}\right) = \frac{min\,(1)}{max\,(4-3cos\,2\theta)} = \frac{1}{max\,(4)-min\,(3cos\,2\theta)}$

$= \frac{1}{4-3min\,(cos\,2\theta)} = \frac{1}{4-3(-1)} = \frac{1}{7}$

Now finding value of θ_{min}

$\frac{1}{4-3cos\,2\theta} = \frac{1}{7}(\min value) \Rightarrow 4 - 3cos2\theta = 7 \Rightarrow cos2\theta = -1$

Solving the equations as before $2\theta = 180^0 \Rightarrow \theta_{min} = 90^0$

vi) $max\left(\frac{1}{2+3\sin\,\theta}\right) = \frac{max\,(1)}{min\,(2+3\sin\,\theta)} = \frac{1}{2+3min\,(\sin\,\theta)}$

$= \frac{1}{2+3(-1)} = -1$

Finding value of θ_{max}

$\frac{1}{2+3\sin\,\theta} = -1\ or\ -1 = 2 + 3\sin\theta \Rightarrow \sin\theta = -1$

Solving the equations as before $\theta_{max} = 270^0$

$min\left(\frac{1}{2+3\sin\,\theta}\right) = \frac{min\,(1)}{max\,(2+3\sin\,\theta)} = \frac{1}{2+3max\,(\sin\,\theta)}$

$= \frac{1}{2+3(+1)} = \frac{1}{5}$

Now finding value of θ_{min}

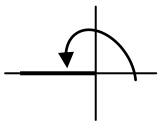
$\frac{1}{2+3\sin\,\theta} = \frac{1}{5} \Rightarrow 5 = 2 + 3\sin\theta\ \ or\ \ \sin\theta = 1$

Solving the equations as before $\theta_{min} = 90^0$

vii) $max\left(\sec\frac{3}{2}\theta\right) = max\left(\frac{1}{\cos\frac{3}{2}\theta}\right) = \frac{max\,(1)}{min\left(\cos\frac{3}{2}\theta\right)} = \frac{1}{-1} = -1$

Finding value of θ_{max}

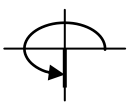
$-1 = \frac{1}{cos\frac{3}{2}\theta} \Rightarrow cos\frac{3}{2}\theta = -1$, Solving the equations as before



$\therefore \frac{3}{2}\theta = 180^0, \theta_{max} = 120^0, \min values\ is\ left\ as\ an\ exercise$

viii) $max(\text{cosec}\ 3\theta) = max\left(\frac{1}{\sin\,3\theta}\right) = \frac{1}{min\,(\sin\,3\theta)} = \frac{1}{-1} = -1$

Finding the value of θ_{max}



$\frac{1}{sin\,3\theta} = -1 \Rightarrow \sin 3\theta = -1$

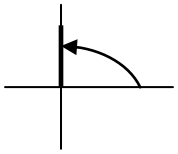
$\Rightarrow 3\theta = 270^0 \quad \therefore \theta_{max} = 90^0$

$min(\text{cosec } 3\theta) = min\left(\frac{1}{\sin 3\theta}\right) = \frac{1}{max\ (\sin 3\theta)} = \frac{1}{+1} = 1$

Finding the value of θ_{min}

$\frac{1}{\sin 3\theta} = 1 \Rightarrow \sin 3\theta = 1$

$\therefore 3\theta_{min} = 90^0 \Rightarrow \theta_{min} = 30^0$



ix) $-\frac{1}{2}sin2\theta = 0 - \frac{1}{2}sin2\theta$

$[Re-writing\ the\ expression, since\ the\ multiplying\ constant\ is\ -ve]$

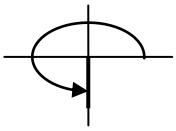
$max\left(-\frac{1}{2}\sin 2\theta\right) = max\left(0 - \frac{1}{2}\sin 2\theta\right) = max(0) - \frac{1}{2}min(\sin 2\theta)$

$= 0 - \frac{1}{2}(-1) = \frac{1}{2}$

Finding the value of θ_{max}

$\frac{1}{2} = -\frac{1}{2}\sin 2\theta\ \ or \sin 2\theta = -1$

$\Rightarrow 2\theta = 270^0 \ \ \therefore \theta_{max} = 135^0$



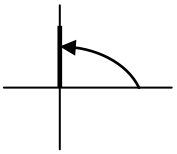
$min\left(-\frac{1}{2}sin2\theta\right) = min\left(0 - \frac{1}{2}sin2\theta\right) = min(0) - \frac{1}{2}max(sin2\theta)$

$= 0 - \frac{1}{2}(+1) = -\frac{1}{2}$

Finding the value of θ_{min}

$-\frac{1}{2} = -\frac{1}{2}\sin 2\theta\ \ or \sin 2\theta = 1$

$\Rightarrow 2\theta = 90^0 \ \ \therefore \theta_{min} = 45^0$



x) *Left as an exercise*

Answer $\frac{1}{4}, 135^0 ; -\frac{1}{4}, 45^0$

Task

Find the maximum and minimum value and values of θ for which they occur between 0^0 *and* 360^0

i) $\frac{1}{1+\text{cosec } \theta}$ $\left[\begin{array}{ll} \textbf{HINT}: max\left(\frac{1}{1+cosec\theta}\right) = \frac{1}{min\ (1+cosec\theta\ )} \Rightarrow & \boldsymbol{Ans}: \frac{1}{2}, \boldsymbol{90^0} \\ min\left(\frac{1}{1+cosec\theta}\right) = \frac{1}{max\ (1+cosec\theta\ )} \Rightarrow & \boldsymbol{Ans}: \boldsymbol{no\ min} \end{array}\right]$

ii) $\frac{\cos\theta}{\cos\theta + \sin\theta}$ $\left[\begin{array}{l} \textbf{HINT}: making\ numerator\ 1, dividing\ both\ numerator \\ and\ denominator\ by \cos\theta\ \ gives\ \ \frac{1}{1+\tan\theta}, since \tan\theta\ \ has\ no\ max \\ and \min value\ then \qquad \boldsymbol{Ans}: \boldsymbol{None} \end{array}\right]$

iii) $\frac{2}{3-2\cot 2\theta}$ $[\boldsymbol{Ans}: \boldsymbol{None}]$

TRIGONOMETRIC IDENTITES INVOLVING FUNCTIONS OF THE SUM OR DIFFERENCE OF TWO ANGLES

The formulae $\boldsymbol{sin(\varphi \pm \theta), cos(\varphi \pm \theta), tan(\varphi \pm \theta)\ and\ cot(\varphi \pm \theta)}$

The following figures shall be made of use

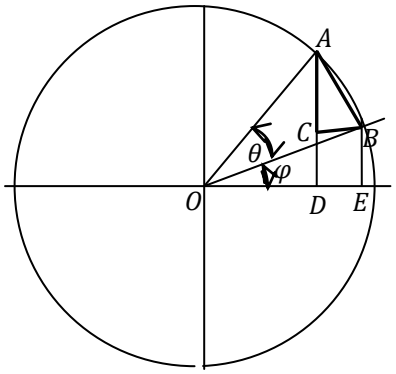


Fig.1

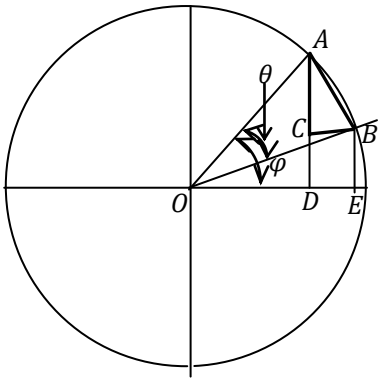


Fig.2

Functions of sum of two angles

a) $\boldsymbol{sin(\varphi + \theta)}$

Referring to fig.1, where AB ⊥OB and AD ⊥OD

From ΔOAB, $AB = OA\sin\theta \ \ < BAC = \varphi$

From $\Delta ABC, AC = AB cos\varphi$

$AC = OA\sin\theta\, cos\varphi$

From $\Delta OAB,\ OB = OA cos\theta$

From $\Delta OBE,\ BE = OB sin\varphi \Rightarrow BE = OA cos\theta sin\varphi$

But $BE = CD\ \therefore CD = OA cos\theta sin\varphi$

From $\Delta OAD, sin\ (\varphi + \theta) = \frac{AC+CD}{OA} = \frac{OA\sin\theta cos\varphi + OA cos\theta sin\varphi}{OA}$

$$\therefore \sin(\varphi + \theta) = \sin\varphi\cos\theta + \cos\varphi\sin\theta$$

NOTE: $\sin(\varphi \pm \theta) \neq \sin\varphi + \sin\theta$

b) $\boldsymbol{\cos(\varphi + \theta)}$

Again referring to fig.1

From ΔOAB, $OB = OA\cos\theta$

From ΔOBE, $OE = OB\cos\varphi \Rightarrow OE = OA\cos\theta\cos\varphi$

From $\Delta OAB,\ AB = OA\sin\theta$

From $\Delta ABC,\ CB = AB\sin\varphi$

$DE = CB = OA\sin\theta\sin\varphi$

From $\Delta OAD, \mathrm{COS}(\varphi + \theta) = \frac{OE-DE}{OA} = \frac{OA cos\varphi cos\theta\ -OA sin\varphi\ \sin\theta}{OA}$

$$\therefore \cos(\varphi + \theta) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$

NOTE: $\mathrm{COS}(\varphi + \theta) \neq cos\varphi + cos\theta$

c) $\tan(\varphi+\theta)$

$\tan(\varphi+\theta)=\frac{\sin(\varphi+\theta)}{\cos(\varphi+\theta)}$

From the above two deduced results,

$\tan(\varphi+\theta)=\frac{\sin(\varphi+\theta)}{\cos(\varphi+\theta)}=\frac{\sin\varphi\cos\theta+\cos\varphi\sin\theta}{\cos\varphi\cos\theta-\sin\varphi\sin\theta}$

Dividing Numerator and Denominator by $\cos\varphi\cos\theta$

$\tan(\varphi+\theta)=\frac{\frac{\sin\varphi}{\cos\varphi}+\frac{\sin\theta}{\cos\theta}}{1-(\sin\varphi/\cos\varphi).(\sin\theta/\cos\theta)}$

$=\frac{\tan\varphi+\tan\theta}{1-\tan\varphi\tan\theta}$

$$\therefore \tan(\varphi+\theta)=\frac{\tan\varphi+\tan\theta}{1-\tan\varphi\tan\theta}$$

NOTE: $\tan(\varphi+\theta)\neq\tan\varphi+\tan\theta$

d) $\cot(\varphi+\theta)$

$\cot(\varphi+\theta)=\frac{1}{\tan(\varphi+\theta)}=\frac{1-\tan\varphi\tan\theta}{\tan\varphi+\tan\theta}$

From $\cot\alpha=\frac{1}{\tan\alpha}\Rightarrow\tan\alpha=\frac{1}{\cot\alpha}$

$\Rightarrow\cot(\varphi+\theta)=\frac{1-\frac{1}{\cot\varphi}.\frac{1}{\cot\theta}}{\frac{1}{\cot\varphi}+\frac{1}{\cot\theta}}=\frac{\cot\varphi\cot\theta-1}{\cot\varphi+\cot\theta}$

$$\therefore \cot(\varphi+\theta)=\frac{\cot\varphi\cot\theta-1}{\cot\varphi+\cot\theta}$$

NOTE: $\cot(\varphi+\theta)\neq\cot\varphi+\cot\theta$

Functions of difference between two angles

Referring to fig.2, where OA ⊥AB and AD ⊥OD

a) $\sin(\varphi-\theta)$

From ΔOAB, $AB=OB\sin\theta$

From ΔABC, $AC=AB\cos\varphi\Rightarrow AC=OB\sin\theta\cos\varphi$

From ΔOAD, $AD=OA\sin\varphi$

From ΔOAB, $OA=OB\cos\theta$

$\Rightarrow AD=OB\sin\varphi\cos\theta$

Form ΔOBE, $\sin(\varphi-\theta)=\frac{AD-AC}{OB}=\frac{OB\sin\varphi\cos\theta-OB\cos\varphi\sin\theta}{OB}$

$$\therefore \sin(\varphi-\theta)=\sin\varphi\cos\theta-\cos\varphi\sin\theta$$

NOTE: $\sin(\varphi-\theta)\neq\sin\varphi-\sin\theta$

b) $\cos(\varphi-\theta)$

Again referring to fig.2

From ΔOAB, $OA=OB\cos\theta$

From ΔOAD, $OD=OA\cos\varphi\Rightarrow OD=OB\cos\varphi\cos\theta$

From $\Delta OAB,\ AB = OBsin\theta$

From $\Delta ABC,\ CB = ABsin\varphi \Rightarrow CB = OBsin\varphi \sin\theta$

Form $\Delta OBE,\ cos(\varphi - \theta) = \frac{OD+CB}{OB} = \frac{OBcos\varphi cos\theta\ + OBsin\varphi\ \sin\theta}{OB}$

$$\therefore \cos(\varphi - \theta) = \cos\varphi\cos\theta + \sin\varphi\sin\theta$$

NOTE: $\cos(\varphi - \theta) \neq \cos\varphi - \cos\theta$

c) $\tan(\boldsymbol{\varphi - \theta})$

$\tan(\varphi - \theta) = \frac{\sin(\varphi-\theta)}{\cos(\varphi-\theta)}$

From the above two deduced results,

$\tan(\varphi - \theta) = \frac{\sin(\varphi-\theta)}{\cos(\varphi-\theta)} = \frac{sin\varphi cos\theta\ - cos\varphi\ \sin\theta}{cos\varphi cos\theta\ + sin\varphi\ \sin\theta}$

Dividing Numerator and Denominator by $cos\varphi cos\theta$

$\tan(\varphi - \theta) = \frac{\frac{sin\varphi}{cos\varphi} - \frac{sin\theta}{cos\theta}}{1+(sin\varphi\ /cos\varphi\ ).(sin\theta\ /cos\theta\ )} = \frac{tan\varphi\ - tan\theta}{1+tan\varphi tan\theta}$

$$\therefore \tan(\varphi - \theta) = \frac{tan\varphi\ - tan\theta}{1+tan\varphi tan\theta}$$

NOTE: $\tan(\varphi - \theta) \neq tan\varphi - tan\theta$

a) **$\cot(\boldsymbol{\varphi - \theta})$**

$\cot(\varphi - \theta) = \frac{1}{tan(\varphi-\theta)} = \frac{1+tan\varphi tan\theta}{tan\varphi\ - tan\theta}$

From $cot\alpha = \frac{1}{tan\alpha} \Rightarrow\ tan\alpha = \frac{1}{cot\alpha}$

$\Rightarrow \cot(\varphi - \theta) = \frac{1+\frac{1}{cot\varphi}.\frac{1}{cot\theta}}{\frac{1}{cot\varphi} - \frac{1}{cot\theta}} = \frac{\cot\varphi\cot\theta+1}{\cot\varphi - \cot\theta}$

$$\therefore \cot(\varphi - \theta) = \frac{cot\varphi cot\theta\ +1}{cot\varphi\ - cot\theta}$$

NOTE: $cot(\varphi - \theta) \neq cot\varphi - cot\theta$

Application of the above identities

- Finding exact values of trig ratios involving the use of special angles
- Solving trig equations
- Proof of identities and more further identities
- integration

Finding exact values of trig ratios involving the use of special angles

Examples

Show that

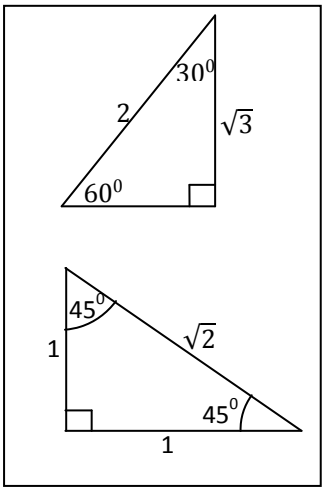
1) $\cos 15^0 = \frac{\sqrt{3}+1}{2\sqrt{2}}$

2) $\cos 75^0 = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$

3) $\sin 195^0 = \frac{\sqrt{2}-\sqrt{6}}{4}$

4) $\tan 225^0 = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

5) $\tan\ 195^0 = \frac{\sqrt{3}-1}{\sqrt{3}+1}$



Solution

1) $\cos 15^0 = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$15^0 = 60^0 - 45^0$ $[Both\ special\ angles]$

$or = 45^0 - 30^0$ $[both\ special\ angles]$

Using any of the above expressions

$\cos 15^0 = \cos(60^0 - 45^0)$

$= \cos 60^0 \cos 45^0 + \sin 60^0 \sin 45^0$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$ $As\ required$

2) $\cos 75^0 = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$

$\cos 75^0 = \cos(30^0 + 45^0)$

$= cos30^0 cos45^0 + sin30^0 sin45^0$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \ldots\ldots\ldots\ldots \left(\times \frac{\sqrt{2}}{\sqrt{2}}\right)$

$= \frac{\sqrt{2}}{4}(\sqrt{3}-1)$ $As\ required$

3) $\sin 195^0 = \frac{\sqrt{2}-\sqrt{6}}{4}$

$\sin 195^0 = sin(180^0 + 15^0)$

$= \sin 180^0 \cos 15^0 + \cos 180^0 \sin 15^0$

$= \cos 180^0 \sin 15^0$ $[\because \sin 180^0 = 0]$

$= -\sin 15^0$ $[\because \cos 180^0 = -1]$

Now $\sin 15^0 = \sin(45^0 - 30^0)$

$= \sin 45^0 \cos 30^0 - \cos 45^0 \sin 30^0$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\therefore \sin 195^0 = -\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = \frac{1-\sqrt{3}}{2\sqrt{2}} \ldots\ldots\ldots \times \left(\frac{\sqrt{2}}{\sqrt{2}}\right)$

$= \frac{\sqrt{2}-\sqrt{6}}{4}$ $As\ required$

4) $\tan 255^0 = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$\tan 255^0 = tan(180^0 + 75^0)$

$$= \frac{tan\ 180^0 + tan\ 75^0}{1 - tan\ 180^0 tan\ 75^0} = \frac{0 + tan\ 75^0}{1 - 0} = \tan 75^0$$

But $\tan 75^0 = \tan(30^0 + 45^0)$

$$= \frac{tan\ 30^0 + tan\ 45^0}{1 - tan\ 30^0 tan\ 45^0} = \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}} = \frac{1 + \sqrt{3}}{\sqrt{3} - 1}$$

$\therefore \tan 255^0 = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ *As required*

5) *Left as an exercise. Proceed as above*

Task

Show that

i) $\tan^2 225^0 = 2 + \sqrt{3}$

ii) $\sin^2 195^0 = \frac{1}{4}(2 - \sqrt{3})$

iii) $\cos^2 435^0 = \frac{1}{4}(\sqrt{3} - 2)$

[***HINT***: *First get* $tan 225^0, sin 195^0$ *and* $cos 435^0$ *then use algebra to find the required answer*]

Solving trigonometric equations

Solve for values of x *between* 0^0 *and* 360^0 *inclusive*

a. $2\sin x = \cos(x + 60^0)$

b. $\cos(x + 45^0) = \cos x$

c. $\sin(x - 30^0) = \frac{1}{2}\cos x$

d. $3\sin(x + 10^0) = 4\cos(x - 10^0)$

e. $\sin(x + 30^0) = \cos(x + 30^0)$

f. $\tan(x - 45^0) = \frac{1}{3}$

g, $\tan(45^0 + x) = 2$

Solution

NOTE: The equations (a) to (e) take the form .

$$\boldsymbol{a\sin(px + q) = b\cos(px + c)}$$

The equations are solved by creation of **tanpx.**

a. $2\sin x = \cos(x + 60^0)$

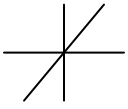
$2\sin x = \cos x \cos 60^0 - \sin x \sin 60^0 = \frac{1}{2}.\cos x - \frac{\sqrt{3}}{2}.\sin x$

$\Rightarrow 4\sin x = \cos x - \sqrt{3}.\sin x$……(*dividing through by cosx*)

$\Rightarrow 4\tan x = 1 - \sqrt{3}.\tan x \Rightarrow (4 + \sqrt{3}).\tan x = 1$

$\therefore \tan x = \frac{1}{4+\sqrt{3}} = 0.1745 \quad (\boldsymbol{Note}: \sqrt{3}\ \ is\ taken + ve)$

Solving the equation $x = 9.9^0, 189.9^0$ $\tan^{-1}(0.1745) = 9.9$

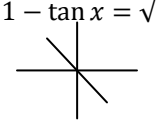


b. $\cos(x + 45^0) = \cos x$

$\cos x \cos 45^0 - \sin x \sin 45^0 = \cos x$

$\frac{1}{\sqrt{2}}.\cos x - \frac{1}{\sqrt{2}}.\sin x = \cos x\ \ or \cos x - \sin x = \sqrt{2}.\cos x$……$(\div \cos x)$

$1 - \tan x = \sqrt{2}\ or \tan xx = 1 - \sqrt{2} = -0.4142\ (for\ only + ve\sqrt{2})$



$\tan^{-1}(0.4142) = 22.5^0$

$\therefore x = 157.5^0, 337.5^0$

c. (c) and (e) are left as an exercise

Ans: $(\boldsymbol{c})\ \mathbf{49.1^0, 229.1^0}\ (\boldsymbol{e})\ \mathbf{15^0, 195^0}$

d. $3\sin(x+10^0) = 4\cos(x-10^0)$

$3(\sin x\cos 10^0 + \cos x\sin 10^0) = 4(\cos x\cos 10^0 + \sin x\sin 10^0)$

$3\sin x\cos 10^0 + 3\cos x\sin 10^0 = 4\cos x\cos 10^0 + 4\sin x\sin 10^0$

Dividing through by $\cos x$

$3\cos 10^0 . \tan x + 3\sin 10^0 = 4\cos 10^0 + 4\sin 10^0 . \tan x$

$\tan x . (3cos10 - 4\sin 10^0) = 4cos10 - 3sin10$

$\therefore \tan x = \frac{4\cos 10^0 - 3\sin 10^0}{3\cos 10^0 - 4\sin 10^0} = 1.5126,\ solving\ x = 56.5^0, 236.5^0$

e. $\tan(x-45^0) = \frac{1}{3}$

$\frac{\tan x - \tan 45}{1+\tan x tan 45} = \frac{1}{3}\ or\ 3\tan x - 3 = 1 + \tan x$

$\Rightarrow \tan x = 2,\ solving\ x = 63.4^0, 243.4^0$

Alternatively

Let $x - 45^0 = B \Rightarrow tanB = \frac{1}{3}, solving\ the\ equation\ B = 18.4^0, 198.4^0$

$\Rightarrow x - 45^0 = 18.4^0, 198.4^0$

$\therefore x = (18.4^0, 198.4^0) + 45^0\ or\ x = 63.4^0, 243.4^0$

g. Left an exercise ***Answer*: $\mathbf{18.4^0, 198.4^0}$**

Proof of identities

1. *Prove the following identities*

a) $\frac{\sin(A-B)}{\cos A\cos B} + \frac{\sin(B-C)}{\cos B\cos C} + \frac{\sin(C-A)}{\cos C\cos A} = 0$

b) $\tan A + \tan B = \frac{\sin(A+B)}{\cos A\cos B}$

c) $\sin(A+B) + \sin(A-B) = 2sinAcosB$

d) $\tan A + \tan B + \tan C = \tan A\tan B\tan C$ where A,B and C are angles of a triangle

e) $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \cos^2\beta$

Solution

a) $\frac{\sin(A-B)}{\cos A\cos B} + \frac{\sin(B-C)}{\cos B\cos C} + \frac{\sin(C-A)}{\cos C\cos A} = 0$

L.H.S: $\frac{\sin(A-B)}{\cos A\cos B} = \frac{\text{sinAcosB} - \text{cosAsinB}}{\text{cosAcosB}} = \frac{\text{sinAcosB}}{\text{cosAcosB}} - \frac{\text{cosAsinB}}{\text{cosAcosB}}$

$= \tan A - \tan B$

Similarly $\frac{\sin(B-C)}{\cos B\cos C} = \tan B - \tan C$ Also $\frac{\sin(C-A)}{\cos C\cos A} = \tan C - \tan A$

$\therefore \frac{sin(A-B)}{cosAcosB} + \frac{sin(B-C)}{cosBcosC} + \frac{sin(C-A)}{cosCcosA} = \tan A - \tan B + \tan B - \tan C - \tan C - \tan A = 0$

b) $\tan A + \tan B = \frac{\sin(\text{A+B})}{\text{cosAcosB}}$

R.H.S: $\frac{\sin(\text{A+B})}{\text{cosAcosB}} = \frac{\text{sinAcosB} + \text{cosAsinB}}{\text{cosAcosB}} = \frac{\text{sinAcosB}}{\text{cosAcosB}} + \frac{\text{cosAsinB}}{\text{cosAcosB}}$

$= \tan A + \tan B = \mathbf{L.H.S}$

c) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

L.H.S: $\sin(A+B) + \sin(A-B) = \sin A\cos B + cosA\sin B + \sin A\,cosB - \cos A\sin B$

$= 2sinAcosB = \mathbf{R.H.S}$

d) $\tan A + \tan B + \tan C = \tan A \tan B \tan C \ , A + B + C = 180^0$

$A + B = 180 - C \Rightarrow \tan(A + B) = \tan(180 - C)$

$\frac{tanA + tanB}{1 - tanA\,.tanB} = \frac{tan\,180 - tanC}{1 + tan\,180\,.tanC} = -\tan C \ \ [\because \tan 180 = 0]$

$\tan A + \tan B = -\tan C + \tan A \tan B \tan C \ \Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$

e) $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \cos^2\beta$

L.H.S: $\cos(\alpha + \beta)\cos(\alpha - \beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= (\cos\alpha\cos\beta)^2 - (\sin\alpha\sin\beta)^2 = \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta$

In the required proof **R.H.S** does not have *cosβ and sinα*

$\therefore \mathbf{L.H.S} = \cos^2\alpha[1 - \sin^2\beta] - [1 - \cos^2\alpha]\sin^2\beta$

$= \cos^2\alpha - \cos^2\alpha\sin^2\beta - \sin^2\beta + \cos^2\alpha\sin^2\beta$

$= \cos^2\alpha - \cos^2\beta = \mathbf{R.H.S}$

2. *Verify the following identities*

a) $\sin(90^0 + x) = \cos x$

b) $\tan(270^0 + \theta) = -\cot\theta$

c) $\tan(90^0 + \theta) = -\cot\theta$

d) $\tan(90^0 - \theta) = \cot\theta$

e) $\sec(90^0 + \theta) = -\text{cosec}\theta$

f) $\text{cosec}(180^0 - \theta) = \text{cosec}\theta$

g) $\frac{\cot\theta}{\cot\theta - \cot 3\theta} + \frac{\tan\theta}{\tan\theta - tan\,3\theta} = 1$

Solution

a) $\sin(90^0 + x) = \sin 90\cos x + \cos 90\sin x = \cos x \ [\because \sin 90 = 1.\cos 90 = 0]$

b) $\tan(270^0 + \theta) = \frac{\sin(270+\theta)}{\cos(270+\theta)} = \frac{\sin 270\cos\theta + \cos 270\sin\theta}{\cos 270\cos\theta - \sin 270\sin\theta} = -\frac{\cos\theta}{\sin\theta} = -\text{cot}\theta$

$[\because \sin 270^0 = -1, \ \cos 270^0 = 0]$

c) $(c), (d) and\ (g) are\ left\ as\ an\ exercise$

e) $\sec(90^0 + \theta) = \frac{1}{\cos(90+\theta)} = \frac{1}{\cos 90\cos\theta - \sin 90\sin\theta} = \frac{1}{-\sin\theta} = -\text{cosec}\theta$

f) $Left\ as\ an\ exercise$

Task

Prove the following identities

a) $\sin(\alpha + \beta)\,.\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$

b) $\sin(\alpha + \beta)\,.\sin(\alpha - \beta) = \cos^2\alpha - \cos^2\beta$

c) $If \sin(x + \alpha) = \cos(x - \beta)\,, then \tan x = \frac{\cos\beta - \sin\alpha}{\cos\alpha - \sin\beta} f$

d) $If \sin(x + \alpha) = 2\cos(x - \beta)\,, then \tan x = \frac{2 - tan\alpha}{1 - 2tan\alpha}$

e) $\frac{\sin(A-B)}{\text{sinAsinB}} + \frac{\sin(B-C)}{\text{sinBsinC}} + \frac{\sin(C-A)}{\text{sinCsinA}} = 0$

Finding exact value of $\sin(\alpha \pm \beta)$, $\cos(\alpha \pm \beta)$, $\tan(\alpha \pm \beta)$ *or* $\cot(\alpha \pm \beta)$ *given* $\sin(\alpha$ *or* $\beta)$ *or* $\cos(\alpha$ *or* $\beta)$ Or $\tan(\alpha$ *or* $\beta)$

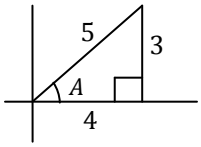
Example

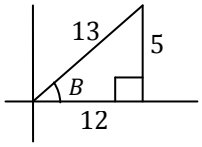
1. *If* $\sin A = \frac{3}{5}$ *and* $\sin B = \frac{5}{13}$ *where A and B are acute angles, find the value of*

a) $\sin(A+B)$ b) $\cos(A-B)$ c) $\cot(A-B)$

Solution

Given: $\sin A = \frac{3}{5}$, $\sin B = \frac{5}{13}$, $A, B < 90^0$ *i.e.* lie in 1st *quadrant*





a) $\sin(A+B) = \sin A\cos B + \cos A \sin B$

$= \frac{3}{5}\times\frac{12}{13} + \frac{4}{4}\times\frac{5}{13} = \frac{56}{65}$

b) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$= \frac{4}{5}\times\frac{12}{13} + \frac{3}{5}\times\frac{5}{13} = \frac{63}{65}$

c) $\cot(A-B) = \frac{1}{\tan(A-B)} = \frac{1+\tan A\tan B}{\tan A - \tan B} = \frac{1+\frac{3}{4}\times 5/12}{\frac{3}{4}-\frac{5}{12}} = \frac{63}{16}$

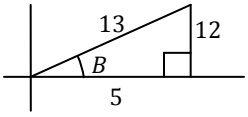
2. *If* $\sin A = \frac{4}{5}$ *and* $\cos B = \frac{12}{13}$ *where B is acute and A is obtuse, find the values of*

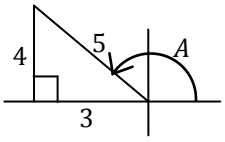
a) $\tan(A+B)$ b) $\tan(A-B)$ c) $A+B$

Solution

Given: $\sin A = \frac{4}{5}$, $90 < A < 180$ *. i.e A lies in* 2nd *quadrant* $\cos B = \frac{12}{13}$,

$0 < B < 90^0$ *. i.e. B lies in* 1st *quadrant*





a) $\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B} = \frac{-\frac{4}{3}+\frac{12}{5}}{1-\frac{4}{3}} \times \frac{12}{5}$

$\left[\because \tan A = \frac{4}{3}\left(\begin{array}{c} -ve \text{ since } A \text{ is in } 2^{nd} \\ \text{quadrant where tangents are} \\ -ve \end{array}\right)\right] = \frac{16}{63}$

b) $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B} = \frac{-\frac{4}{3}-\frac{12}{5}}{1+\left(-\frac{4}{3}\right)\times\frac{12}{5}} = \frac{56}{33}$

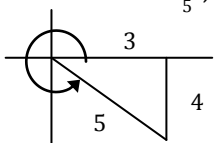
c) $A+B = ?\, From\ (a) above, A+B = \tan^{-1}\left(\frac{16}{63}\right)$

3. *If* $cosA = \frac{3}{5}$ *and* $tanB = \frac{12}{5}$ *where both A and B are reflex angles, find the values f*

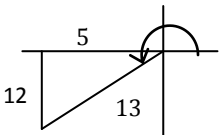
a) $sin(A-B)$ b) $tan(A-B)$ c) $cos(A+B)$

Solution

$Given: \cos A = \frac{3}{5}\ ,\ \tan B = \frac{12}{5}\ , 180^0 < (A\ or\ B) < 360^0$



$In\ 4^{th}\ quadrant\ \because cosA = +ve$



$In\ 3^{rd} quadrant\ \because \tan B = +ve$

a) $\sin(A-B) = \sin A\cos B\ - \cos A\sin B = \left(-\frac{4}{5}\right)\times\left(-\frac{5}{13}\right) - \frac{3}{5}\times\left(-\frac{12}{13}\right) = \frac{56}{65}$

b) $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B} = \frac{-\frac{4}{3}-\frac{12}{5}}{1+\left(-\frac{4}{3}\right).\left(\frac{12}{5}\right)} = \frac{56}{33}$

c) $\cos(A+B)\ = cosAcosB - sinAsinB = \frac{3}{5}\times\left(-\frac{5}{13}\right) - \left(-\frac{4}{5}\right).\left(-\frac{12}{13}\right) = -\frac{63}{65}$

Task

1. *If* $\tan A = -\frac{1}{7}$ *and* $\tan B = \frac{3}{4}$ *where A is obtuse and B is acute, find* $\tan(A-B)$ *hence find* $A-B$.

Ans: $\tan(A-B) = -1\ ; A-B = (135^0\ , 315^0) + 180^0 n$

2. *Prove that* $\frac{\tan(a-b)+\tan b}{1-\tan(a-b)\tan b} = \tan a$

MILTIPLE ANGLE IDENTITIES

a) **Double Angle Identities**

i) $\boldsymbol{\sin 2\varphi}$

Using $sin(\varphi+\theta) = sin\varphi cos\theta + cos\varphi sin\theta$

Let $\varphi = \theta\ \Rightarrow sin(\varphi+\varphi) = sin\varphi cos\varphi + cos\varphi sin\varphi$

$$\therefore \boldsymbol{\sin 2\varphi = 2\sin\varphi\ \cos\varphi}$$

In general $\quad \therefore \boldsymbol{\sin 2m\varphi = 2\sin m\varphi\ \cos m\varphi}$

ii) $\boldsymbol{\cos 2\varphi}$

Using $cos(\varphi+\theta) = cos\varphi cos\theta - sin\varphi sin\theta$

Let $\theta = \varphi \Rightarrow cos(\varphi+\varphi) = cos\varphi cos\varphi - sin\varphi sin\varphi$

$$\therefore \boldsymbol{\cos 2\varphi = cos^2\varphi - sin^2\varphi}$$

Further manipulation using $cos^2\varphi + sin^2\varphi$=1

$cos^2\varphi = 1 - sin^2\varphi\ \Rightarrow cos2\varphi = 1 - sin^2\varphi - sin^2\varphi$

$$\therefore \boldsymbol{\cos 2\varphi = 1 - 2\sin^2\varphi} \qquad \textbf{or}\ \ \boldsymbol{\sin^2\varphi = \frac{1}{2}(1 - cos2\varphi)}$$

$sin^2\varphi = 1 - cos^2\varphi\ \Rightarrow cos2\varphi = cos^2\varphi - (1 - cos^2\varphi)$

$$\therefore \boldsymbol{\cos 2\varphi = 2\cos^2\varphi - 1} \qquad \textbf{or}\ \ \boldsymbol{\cos^2\varphi = \frac{1}{2}(1 + cos2\varphi)}$$

In general

$\cos(2m)\varphi = \cos^2 m\varphi - \sin^2 m\varphi$
$= 1 - 2\sin^2 m\varphi$
$= 2\cos^2 m\varphi - 1$

$\sin^2 m\varphi = \frac{1}{2}(1 - \cos 2m\varphi)$
$\cos^2 m\varphi = \frac{1}{2}(1 + \cos 2m\varphi)$

iii) $\tan 2\varphi$

Using $\tan(\varphi + \theta) = \frac{\tan\varphi + \tan\theta}{1 - \tan\varphi\tan\theta}$

Let $\theta = \varphi \Rightarrow \tan(\varphi + \varphi) = \frac{\tan\varphi + \tan\varphi}{1 - \tan\varphi\tan\varphi} = \frac{2\tan\varphi}{1 - \tan^2\varphi}$

$\therefore \tan 2\varphi = \frac{2\tan\varphi}{1 - \tan^2\varphi}$

In general
$\therefore \tan 2m\varphi = \frac{2\tan m\varphi}{1 - \tan^2 m\varphi}$

iv) $\cot 2\varphi$

Using $\cot(\varphi + \theta) = \frac{\cot\varphi \cot\theta - 1}{\cot\varphi + \cot\theta}$

Let $\theta = \varphi \Rightarrow \cot(\varphi + \varphi) = \frac{\cot\varphi\cot\varphi - 1}{\cot\varphi + \cot\varphi} = \frac{\cot^2\varphi - 1}{2\cot\varphi}$

$\therefore \cot 2\varphi = \frac{\cot^2\varphi - 1}{2\cot\varphi}$

In general $\cot 2m\varphi = \frac{\cot^2 m\varphi - 1}{2\cot m\varphi}$

Application of double angle identities

- Solution to trig equations
- Proof of identities and derivation of other multiple angle identities
- Elimination of parameters
- Simplifying trig expressions
- integration

Solution to trig equations

1. *solve for* $0^0 < \theta < 360^0$ *the equations*

a) $3\cos 2\theta + \sin\theta = 1$
b) $3\cot 2\theta + \cot\theta = 1$
c) $4\tan\theta . tan2\theta = 1$
d) $\sin 2\theta . \cos\theta + \sin^2\theta = 1$
e) $3\tan\theta = \tan 2\theta$
f) $2\sin\theta(5\cos 2\theta + 1) = 3\sin 2\theta$

Solution

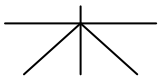
a) $3\cos 2\theta + \sin\theta = 1$

Expressing the equation in terms of only $\sin\theta$

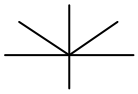
$3(1 - 2sin^2\theta) + \sin\theta = 1$

$-6sin^2\theta + \sin\theta + 2 = 0$ [Let $u = \sin\theta, -6u^2 + u + 2 = 0$]

Solving algebraically, $\sin\theta = -\frac{1}{2}, \frac{2}{3}$

Dealing with $\sin\theta = -\frac{1}{2}$  $\sin^{-1}\left(\frac{1}{2}\right) = 30^0$

$\Rightarrow \theta = 210^0, 330^0$

Dealing with $\sin\theta = \frac{2}{3}$  $\sin^{-1}\left(\frac{2}{3}\right) = 41.8^0$

$\Rightarrow \theta = 41.8^0, 138.2^0$

$\therefore \theta = 41.8^0, 138.2^0, 210^0, 330^0$

b) $3cot2\theta + cot\theta = 1$

Expressing the equation in terms of only $cot\theta$

$3\left[\frac{cot^2\theta - 1}{2cot\theta}\right] + cot\theta = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\times cot\theta)$

$3cot^2\theta - 3 + 2cot^2\theta = 2cot\theta \Rightarrow 5cot^2\theta - 2cot\theta - 3 = 0$

$\Rightarrow 5cot^2\theta - 5cot\theta + 3cot\theta - 3 = 0 \Rightarrow 5cot\theta(cot\theta - 1) + 3(cot\theta - 1) = 0$

$\therefore (5cot\theta + 3)(cot\theta - 1) = 0 \; or \; cot\theta = -\frac{3}{5}, 1$

But $cot\theta = \frac{1}{tan\theta} = -\frac{3}{5}, 1 \Rightarrow tan\theta = -\frac{5}{3}, 1$

Dealing with $tan\theta = 1$ $\tan^{-1}(1) = 45^0$

$\Rightarrow \theta = 45^0, 225^0$

Dealing with $tan\theta = -\frac{5}{3}$ $\tan^{-1}\left(\frac{5}{3}\right) = 59^0$

$\Rightarrow \theta = 121^0, 301^0$

$\therefore \theta = 45^0, 121^0, 225^0, 301^0$

c) $4\tan\theta \,.\, \tan 2\theta = 1$

Expressing in terms of $tan\theta$

$4tan\theta . \frac{2tan\theta}{1 - tan^2\theta} = 1 \Rightarrow 8tan\theta = 1 - tan^2\theta \Rightarrow 9tan^2\theta = 1 \therefore tan\theta = \pm\frac{1}{3}$

Dealing with $tan\theta = \frac{1}{3}$ $\tan^{-1}\left(\frac{1}{3}\right) = 18.4^0$

$\Rightarrow \theta = 18.4^0, 198.4^0$

Dealing with $tan\theta = -\frac{1}{3}$ $\tan^{-1}\left(\frac{1}{3}\right) = 18.4^0$

$\Rightarrow \theta = 161.6^0, 341.6^0$

$\therefore \theta = 18.4^0, 161.6^0, 198.4^0, 341.6^0$

d) $sin2\theta . cos\theta + sin^2\theta = 1$

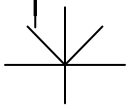
$2\sin\theta \, cos\theta . cos\theta = 1 - sin^2\theta = cos^2\theta$

$2sin\theta(cos^2\theta) - cos^2\theta = 0 \Rightarrow cos^2\theta(2sin\theta - 1) = 0$

$[Do \; not \; divide \; through \; by \; cos^2\theta, you \; limit \; the \; required \; values \; of \; \theta]$

$\Rightarrow cos^2\theta = 0 \; or \; cos\theta = 0 \;\; and \; 2\sin\theta - 1 = 0 \; or \; \sin\theta = \frac{1}{2}$

Dealing with $cos\theta = 0$ $\therefore \theta = 90^0, 270^0$

Dealing with $\sin\theta = \frac{1}{2}$  $\sin^{-1}\left(\frac{1}{2}\right) = 30^0$

$\Rightarrow \theta = 30^0, 150^0$

$\therefore \theta = 30^0, 90^0, 150^0, 270^0$

e) Left as an exercise ***Answer***: $\mathbf{0^0, 30^0, 150^0, 180^0, 210^0, 330^0}$. 360^0 rejected $\because$ it is outside the interval $0^0 < \theta < 360^0$

f) $2\sin\theta\,(5cos2\theta + 1) = 3\sin 2\theta$

$2\sin\theta\,(5\cos 2\theta + 1) - 3sin2\theta = 0$

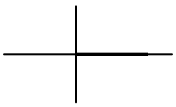
$\Rightarrow 2\sin\theta\,(5\cos 2\theta + 1) - 3.(2\sin\theta\cos\theta) = 0$

$2\sin\theta\,\{(5\cos 2\theta + 1) - 3\cos\theta\} = 0$

$\Rightarrow 2\sin\theta = 0\ or\ \sin\theta = 0$ and $5\cos 2\theta + 1 - 3\cos\theta = 0$

Dealing with $\sin\theta = 0$

$\therefore \theta = 0^0, 360^0$



Dealing with $5cos2\theta + 1 - 3cos\theta = 0$

Expressing in terms of $cos\theta$

$5(2\cos^2\theta - 1) + 1 - 3\cos\theta = 0 \Rightarrow 10\cos^2\theta - 3\cos\theta - 4 = 0$

Solving the equation gives $cos\theta = 0.8, -\frac{1}{2}$

Dealing with $\cos\theta = 0.8$ $\cos^{-1}(0.8) = 36.9^0$

$\Rightarrow \theta = 36.9^0, 323.1^0$

Dealing with $cos\theta = -\frac{1}{2}$ $\cos^{-1}\left(\frac{1}{2}\right) = 60^0$

$\Rightarrow \theta = 120^0, 300^0$

$\therefore \theta = 0^0, 36.9^0, 120^0, 300^0, 323.1^0$ $360^0 rejected\ since 0^0 < \theta < 360^0$

Task

1. *Solve for* $-180^0 \le \theta \le +180^0$ *the equations*

a) $7tan2\theta + cot2\theta = 5sec2\theta$

b) $4cos\theta - 2cos2\theta - 3 = 0$

c) $tan4\theta.tan2\theta = 1$ $\left[\text{HINT: } tan4\theta = tan2(2\theta) = \frac{2tan\,2\theta}{1-tan^2 2\theta}\right]$

2. *Solve* $sin2\theta = cos\theta\ for\ 0^0 \le \theta \le 90^0$ $(\boldsymbol{Uneb\ 2012})$

$\boldsymbol{Answers: (a) 15^0, 75^0, 9.8^0, 80.3^0, -99.8^0, -105^0\ (b) \pm 60^0}$

$\boldsymbol{(c) 15^0, 105^0, -75^0, -165^0, (2)(Uneb\ 2012)\ \theta = 30^0, 150^0}$

Proof of identities

1) *Proof the following identities*

a) $\frac{tan\left(\frac{\pi}{4}+\theta\right)-tan\left(\frac{\pi}{4}-\theta\right)}{tan\left(\frac{\pi}{4}+\theta\right)+tan\left(\frac{\pi}{4}-\theta\right)} = sin2\theta$

b) $\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha} = tan2\alpha + sec2\alpha$

c) $\cot 2\theta = \frac{1+cos\,4\theta}{sin\,4\theta}$

d) $\sec 2\theta = \frac{cosec^2\theta}{cosec^2\theta-2} = \frac{sec^2\theta}{2-sec^2\theta}$

e) $\cos 4\theta = \frac{tan^4\theta-6tan^2\theta+1}{tan^4\theta+2tan^2\theta+1}$

f) $\sin 3\theta = 3sin^3\theta - 4sin\theta$

g) $\cos 3\theta = 4cos^3\theta - 3cos3\theta$

h) $\tan 3\theta = \frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}$

i) $\tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right) = 2\tan 2\theta$ **(Uneb 2006)**

j) $\frac{\sin A}{\sin B} + \frac{\cos A}{\cos B} = \frac{2sin(A+B)}{sin\,2B}$

k) $\frac{1}{cosA+sinA} + \frac{1}{cosA-sinA} = tan2AcosecA$

l) $(\sin 2\alpha - \sin 2\beta)\tan(\alpha+\beta) = 2(\sin^2\alpha - \sin^2\beta)$

Solution

a) $\frac{tan\left(\frac{\pi}{4}+\theta\right)-tan\left(\frac{\pi}{4}-\theta\right)}{tan\left(\frac{\pi}{4}+\theta\right)+tan\left(\frac{\pi}{4}-\theta\right)} = sin2\theta$ **Note:** $1\pi radian = 180^0, \Rightarrow \frac{\pi}{4} = \frac{180^0}{4} = 45^0$

$$\textbf{L.H.S:}\ \frac{tan\left(\frac{\pi}{4}+\theta\right)-tan\left(\frac{\pi}{4}-\theta\right)}{tan\left(\frac{\pi}{4}+\theta\right)+tan\left(\frac{\pi}{4}-\theta\right)} = \frac{\frac{tan\left(\frac{\pi}{4}\right)+tan\theta}{1-tan\left(\frac{\pi}{4}\right)tan\theta} - \frac{tan\left(\frac{\pi}{4}\right)-tan\theta}{1+tan\left(\frac{\pi}{4}\right)tan\theta}}{\frac{tan\left(\frac{\pi}{4}\right)+tan\theta}{1-tan\left(\frac{\pi}{4}\right)tan\theta} + \frac{tan\left(\frac{\pi}{4}\right)-tan\theta}{1+tan\left(\frac{\pi}{4}\right)tan\theta}} = \frac{\frac{1+tan\theta}{1-tan\theta} - \frac{1-tan\theta}{1+tan\theta}}{\frac{1+tan\theta}{1-tan\theta} + \frac{1-tan\theta}{1+tan\theta}}$$

$$= \frac{\frac{(1+tan\theta)^2-(1-tan\theta)^2}{1-tan^2\theta}}{\frac{(1+tan\theta)^2+(1-tan\theta)^2}{1-tan^2\theta}} = \frac{(1+tan\theta+1-tan\theta)(1+tan\theta-1+tan\theta)}{(1+tan\theta)^2+(1-tan\theta)^2}$$

$$= \frac{2(2tan\theta)}{1+2tan\theta+tan^2\theta+1-2tan\theta+tan^2\theta} = \frac{4tan\theta}{2+2tan^2\theta} = \frac{4tan\theta}{2(1+tan^2\theta)}$$

$$= \frac{4sin\theta/cos\theta}{2sec^2\theta} = \frac{2sin\theta/cos\theta}{1/cos^2\theta} = \frac{2sin\theta}{cos\theta}.cos^2\theta = 2sin\theta cos\theta = sin2\theta$$

Alternatively

$$\frac{tan\left(\frac{\pi}{4}+\theta\right)-tan\left(\frac{\pi}{4}-\theta\right)}{tan\left(\frac{\pi}{4}+\theta\right)+tan\left(\frac{\pi}{4}-\theta\right)} = \frac{\frac{sin\left(\frac{\pi}{4}+\theta\right)}{cos\left(\frac{\pi}{4}+\theta\right)} - \frac{sin\left(\frac{\pi}{4}-\theta\right)}{cos\left(\frac{\pi}{4}-\theta\right)}}{\frac{sin\left(\frac{\pi}{4}+\theta\right)}{cos\left(\frac{\pi}{4}+\theta\right)} + \frac{sin\left(\frac{\pi}{4}-\theta\right)}{cos\left(\frac{\pi}{4}-\theta\right)}} = \frac{\frac{sin\left(\frac{\pi}{4}+\theta\right)cos\left(\frac{\pi}{4}-\theta\right)-sin\left(\frac{\pi}{4}-\theta\right)cos\left(\frac{\pi}{4}+\theta\right)}{cos\left(\frac{\pi}{4}+\theta\right)cos\left(\frac{\pi}{4}-\theta\right)}}{\frac{sin\left(\frac{\pi}{4}+\theta\right)cos\left(\frac{\pi}{4}-\theta\right)+sin\left(\frac{\pi}{4}-\theta\right)cos\left(\frac{\pi}{4}+\theta\right)}{cos\left(\frac{\pi}{4}+\theta\right)cos\left(\frac{\pi}{4}-\theta\right)}}$$

$$= \frac{sin\left(\frac{\pi}{4}+\theta\right)cos\left(\frac{\pi}{4}-\theta\right)-sin\left(\frac{\pi}{4}-\theta\right)cos\left(\frac{\pi}{4}+\theta\right)}{sin\left(\frac{\pi}{4}+\theta\right)cos\left(\frac{\pi}{4}-\theta\right)+sin\left(\frac{\pi}{4}-\theta\right)cos\left(\frac{\pi}{4}+\theta\right)} = \frac{sin\left[\left(\frac{\pi}{4}+\theta\right)-\left(\frac{\pi}{4}-\theta\right)\right]}{sin\left[\left(\frac{\pi}{4}+\theta\right)+\left(\frac{\pi}{4}-\theta\right)\right]}$$

$$= \frac{sin2\theta}{sin\left(\frac{\pi}{2}\right)}\ But\ sin\left(\frac{\pi}{2}\right) = sin90^0 = 1 \therefore \textbf{L.H.S} = \textbf{R.H.S.} = \textbf{Sin2}\boldsymbol{\theta}$$

b) $\frac{cos\alpha+sin\alpha}{cos\alpha-sin\alpha} = tan2\alpha + sec2\alpha$

$$\textbf{R.H.S:}\ tan2\alpha + sec2\alpha = \frac{sin2\alpha}{cos\alpha} + \frac{1}{cos2\alpha} = \frac{sin2\alpha+1}{cos2\alpha}$$

$$= \frac{2sin2\alpha cos\alpha + cos^2\alpha + sin^2\alpha}{cos^2\alpha - sin^2\alpha} = \frac{(cos\alpha+sin\alpha)^2}{(cocs\alpha-sin\alpha)(cocs\alpha+sin\alpha)} = \frac{(cocs\alpha+sin\alpha)}{(cocs\alpha-sin\alpha)} = \textbf{L.H.S}$$

c) $cot2\theta = \frac{1+cos4\theta}{sin4\theta}, let\ 2\theta = A\ Now\ to\ prove\ cotA = \frac{1+cos2A}{sin2A}$

$$\textbf{R.H.S:}\ \frac{1+cos2A}{sin2A} = \frac{1+(2cos^2A-1)}{2sinAcosA} = \frac{2cos^2A}{2sinAcosA} = \frac{cosA}{sinA} = cotA\ \ \therefore cot2\theta = \frac{1+cos4\theta}{sin4\theta}$$

d) $sec2\theta = \frac{cosec^2\theta}{cosec^2\theta-2} = \frac{sec^2\theta}{2-sec^2\theta}$

$$\textbf{R.H.S:}\ \frac{cosec^2\theta}{cosec^2\theta-2} = \frac{1/sin^2\theta}{\frac{1}{sin^2\theta}-1} = \frac{1}{1-2sin^2\theta} = \frac{1}{cos2\theta} = sec2\theta = \textbf{L.H.S}$$

Also $sec2\theta = \frac{sec^2\theta}{2-sec^2\theta} = \frac{1/cos^2\theta}{2-\frac{1}{cos^2\theta}} = \frac{1}{2cos^2\theta-1} = \frac{1}{cos2\theta} = sec2\theta = \textbf{L.H.S}$

e) $cos4\theta = \frac{tan^4\theta-6tan^2\theta+1}{tan^4\theta+2tan^2\theta+1}$

$$\textbf{L.H.S:}\ cos4\theta = cos2(2\theta) = cos^2 2\theta - sin^2 2\theta = \frac{cos^2 2\theta - sin^2 2\theta}{1^2}$$

$$= \frac{cos^2 2\theta - sin^2 2\theta}{(cos^2\theta+sin^2\theta)^2} = \frac{(cos^2\theta-sin^2\theta)^2-(2sin\theta cocs\theta)^2}{cos^4\theta+cos^2\theta sin^2\theta+sin^4\theta}$$

$$= \frac{cos^4\theta-2sin^2\theta cos^2\theta-4sin^2\theta cos^2\theta+sin^4\theta}{cos^4\theta+cos^2\theta sin^2\theta+sin^4\theta}$$

Dividing the numerator and denominator by $\cos^4\theta$

$\therefore cos4\theta = \frac{cos^4\theta/cos^4\theta - 6sin^2\theta cos^2\theta/cos^4\theta + sin^4\theta/cos^4\theta}{cos^4\theta/cos^4\theta + cos^2\theta sin^2\theta/cos^4\theta + sin^4\theta/cos^4\theta} = \frac{1-6tan^2\theta+tan^4\theta}{1+2tan^2\theta+tan^4\theta} = \mathbf{R.H.S}$

Task

Show that $tan4\theta = \frac{4tan\theta - 4tan^3\theta}{1-6tan^2\theta+tan^4\theta}$

f) $\sin 3\theta = 3\sin^3\theta - 4\sin\theta$

L.H.S: $sin3\theta = sin(2\theta+\theta) = sin2\theta cos\theta + cos2\theta sin\theta$

$= 2sin\theta cos\theta.cos\theta + cos2\theta sin\theta = 2sin\theta cos^2\theta + (1-2sin^2\theta).\sin\theta$

$= 2\sin\theta(1-sin^2\theta) + \sin\theta - 2sin^3\theta$

$= 2sin\theta - 2sin^3\theta + \sin\theta - 2sin^3\theta = 3sin\theta - 4sin^3\theta = \mathbf{R.H.S}$

g) $cos3\theta = 4cos^3\theta - 3cos3\theta$

L.H.S: $cos3\theta = cos(2\theta+\theta) = cos2\theta cos\theta - sin2\theta sin\theta$

$= (2cos^2\theta - 1).cos\theta - (2sin\theta cos\theta).sin\theta$

$= 2cos^3\theta - cos\theta - 2sin^2\theta cos\theta = 2cos^3\theta - cos\theta - 2(1-cos^2\theta)cos\theta$

$= 2cos^3\theta - cos\theta - 2cos\theta + 2cos^3\theta = 4cos^3\theta - 3cos\theta = \mathbf{R.H.S}$

h) $\tan 3\theta = \frac{3tan\theta - tan^3\theta}{1-3tan^2\theta}$

$\tan 3\theta = tan(2\theta+\theta) = \frac{tan2\theta+tan\theta}{1-tan2\theta tan\theta} = \frac{\frac{2tan\theta}{1-tan^2\theta}+tan\theta}{1-\frac{2tan\theta\ .tan\theta}{1-tan^2\theta}}$

$= \frac{(2tan\theta+tan\theta-tan^3\theta)/(1-tan^2\theta)}{(1-tan^2\theta-2tan^2\theta)/(1-tan^2\theta)} = \frac{3tan\theta-tan^3\theta}{1-3tan^2\theta} = \mathbf{R.H.S}$

NOTE:

The multiple identities of thrice the angles are very useful.

Note them carefully

$\mathbf{\cos 3\theta = 4\cos^3\theta - 3\cos 3\theta}$

$\mathbf{\sin 3\theta = 3\sin^3\theta - 4\sin\theta}$

$\mathbf{\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}}$

i) $left\ as\ an\ exercise.See\ (a)aove$

j) $\frac{sinA}{sinB} + \frac{cosA}{cosB} = \frac{2sin(A+B)}{sin2B}$

R.H.S: $\frac{2sin(A+B)}{sin2B} = \frac{2sinAcosB + 2cosAsinB}{2sinBcosB}$

$= \frac{2sinAcosB}{2sinBcosB} + \frac{2cosAsinB}{2sinBcosB} = \frac{sinA}{sinB} + \frac{cosA}{cosB} = \mathbf{L.H.S}$

k) $\frac{1}{cosA+sinA} + \frac{1}{cosA-sinA} = tan2AcosecA$

L.H.S: $\frac{1}{cosA+sinA} + \frac{1}{cosA-sinA} = \frac{cosA+sinA+cosA-sinA}{cos^2A-sin^2A} = \frac{2cosA}{cos2A}$

Now creating $tan2A$ by suitable manipulation

$\frac{1}{cosA+sinA} + \frac{1}{cosA-sinA} = \frac{2cosA}{cos2A}.\frac{sin2A}{sin2A} = \left(\frac{sin2A}{cos2A}\right).\frac{2cosA}{sin2A} = tan2A.\frac{2cosA}{2sinAcosA}$

$= tan2AcosecA = \mathbf{R.H.S}$

l) $(sin2\alpha - sin2\beta)tan(\alpha+\beta) = 2(sin^2\alpha - sin^2\beta)$

L.H.S: $(sin2\alpha - sin2\beta)tan(\alpha+\beta) = (sin2\alpha - sin2\beta).\frac{sin(\alpha+\beta)}{cos(\alpha+\beta)}$

$$= \frac{(2sin\alpha cos\alpha\ -2sin\beta cos\beta\ )(sin\alpha cos\beta\ +cos\alpha sin\beta\ )}{cos\alpha cos\beta\ -sin\alpha sin\beta}$$

$$= \frac{2sin^2\alpha cos\alpha cos\beta\ +2sin\alpha\ cos^2\alpha sin\beta\ -2sin\alpha sin\beta\ cos^2\beta - 2cos\alpha cos\beta\ sin^2\beta}{cos\alpha cos\beta\ -sin\alpha sin\beta}$$

$$= \frac{2cos\alpha\ cos\beta\ .[sin^2\alpha - sin^2\beta] + 2sin\alpha sin\beta\ .[cos^2\alpha - cos^2\beta]}{cos\alpha cos\beta\ -sin\alpha sin\beta}$$

$$= \frac{2cos\alpha cos\beta\ .[sin^2\alpha - sin^2\beta] + 2sin\alpha sin\beta\ .[(1-sin^2\alpha) - (1-sin^2\beta)]}{cos\alpha cos\beta\ -sin\alpha sin\beta}$$

$$= 2(sin^2\alpha - sin^2\beta).\frac{[cos\alpha cos\beta\ -sin\alpha sin\beta\ ]}{cos\alpha cos\beta\ -sin\alpha sin\beta} = 2(sin^2\alpha - sin^2\beta) = \mathbf{R.H.S}$$

2. **Conditional proofs involving Double Angle Identities**

a) If $tan^2\alpha = 2tan^2\beta + 1,\ prove\ that\ cos2\beta = 1 + 2cos2\alpha$

b) If $tan^2\theta = 2tan^2\varphi + 1,\ \ then\ show\ that\ cos2\theta + sin^2\varphi = 0$

c) If $tan\theta = \frac{b}{a}, prove\ that\ acos2\theta + bsin2\theta = a$

d) If $tan\theta = \frac{p}{q}, prove\ that\ pcos2\theta + qsin2\theta = \frac{p(3q^2-p^2)}{p^2+q^2}$

e) If $tan\theta = \lambda tan(A-\theta), show\ that\ (\lambda-1)sinA = (\lambda+1)sin(2\theta - A)$

f) If $tan^2\alpha - 2tan^2\beta = 1, prove\ that\ cos2\alpha + sin^2\beta = 0$

Solution

a) $tan^2\alpha = 2tan^2\beta + 1, to\ prove\ cos2\beta = 1 + 2cos2\alpha$

$\mathbf{L.H.S}: cos2\beta = \frac{cos^2\beta - sin^2\beta}{1} = \frac{cos^2\beta - sin^2\beta}{cos^2\beta + sin^2\beta}$

Dividing he numerator and denominator by $cos^2\beta$ to create $tan\beta$

$\Rightarrow cos2\beta = \frac{cos^2\beta/cos^2\beta - sin^2\beta/cos^2\beta}{cos^2\beta/cos^2\beta + sin^2\beta/cos^2\beta} = \frac{1-tan^2\beta}{1+tan^2\beta}$

But from $tan^2\alpha = 2tan^2\beta + 1 \Rightarrow tan^2\beta = \frac{tan^2\alpha - 1}{2}$

$\therefore cos2\beta = \frac{1 - \frac{tan^2\alpha - 1}{2}}{1 + \frac{tan^2\alpha - 1}{2}} = \frac{2 - tan^2\alpha + 1}{2 + tan^2\alpha - 1} = \frac{3 - tan^2\alpha}{1 + tan^2\alpha} = \frac{3 - tan^2\alpha}{sec^2\alpha}$

$= \frac{3 - \frac{sin^2\alpha}{cos^2\alpha}}{\frac{1}{cos^2\alpha}} = 3cos^2\alpha - sin^2\alpha = 3cos^2\alpha - (1 - cos^2\alpha)$

$= 4cos^2\alpha - 1, from\ double\ angle\ identities, cos2\alpha = 2cos^2\alpha - 1$

$\Rightarrow cos2\alpha = 4\left[\frac{cos2\alpha + 1}{2}\right] - 1 = 2cos2\alpha + 2 - 1 = 2cos2\alpha + 1 = \mathbf{R.H.S}$

b) $\tan^2\theta = 2\tan^2\varphi + 1, to\ prove\ cos2\theta + \sin^2\varphi = 0$

$\mathbf{L,H,S}: cos2\theta + sin^2\varphi = \frac{cos^2\theta - sin^2\theta}{1} + sin^2\varphi = \frac{cos^2\theta - sin^2\theta}{cos^2\theta + sin^2\theta} + sin^2\varphi$

Creating $tan\theta$ as in previous example

$cos2\theta + sin^2\varphi = \frac{1-tan^2\theta}{1+tan^2\theta} + sin^2\varphi$

From $tan^2\theta = 2tan^2\varphi + 1 \Rightarrow\ cos2\theta + sin^2\varphi = \frac{1-(2tan^2\varphi+1)}{1+(2tan^2\varphi+1)} + sin^2\varphi$

$cos2\theta + sin^2\varphi = \frac{-2tan^2\varphi}{2(1+tan^2\varphi)} + sin^2\varphi = -\frac{\left(\frac{sin^2\varphi}{cos^2\varphi}\right)}{sec^2\varphi} + sin^2\varphi$

$= -\frac{sin^2\varphi}{cos^2\varphi}.cos^2\varphi + sin^2\varphi = 0 = \mathbf{R.H.S}$

c) $\tan\theta = \frac{b}{a},\ to\ prove\ \ a\cos 2\theta + b\sin 2\theta = a$

$\mathbf{L.H.S}: acos2\theta + bsin2\theta = a.\frac{(cos^2\theta - sin^2\theta)}{1} + b.\frac{2sin\theta cos\theta}{1}$

$= a.\frac{cos^2\theta - sin^2\theta}{cos^2\theta + sin^2\theta} + b.\frac{2sin\theta cos\theta}{cos^2\theta + sin^2\theta}$

Dividing the numerator and denominator by $cos^2\theta$

$acos2\theta + bsin2\theta = a.\frac{1 - tan^2\theta}{1 + tan^2\theta} + b.\frac{2tan\theta}{1 + tan^2\theta}$

$= \frac{a\left(1 - \frac{b^2}{a^2}\right) + 2b.\frac{b}{a}}{1 + \frac{b^2}{a^2}} = \frac{\left(a - \frac{b^2}{a}\right).a^2}{a^2 + b^2} = \frac{a^2}{a^2 + b^2}.\left(\frac{a^2 - b^2 + 2b^2}{a}\right) = \frac{a^2}{a^2 + b^2}.\frac{a^2 + b^2}{a} = a$

d) $(d), (e)\ and\ (f)\ are\ left\ as\ an\ exercise$

Task

Proceeding as above,

a) $If\ sin3\theta = p, sin^2\theta = \frac{3}{4} - q, prove\ that\ p^2 + 16q^2 = 12q^2$

b) $If\ 2cos\theta = x + \frac{1}{x}, show\ that\ 2cos3\theta = x^3 + \frac{1}{x^3}$

Half Angle Identities

a) $\mathbf{sin\frac{\varphi}{2}}$

Using $\sin(\varphi + \theta) = \sin\varphi\cos\theta + \cos\varphi\sin\theta$

$\sin\varphi = \sin\left(\frac{\varphi}{2} + \frac{\varphi}{2}\right) = \sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + \cos\frac{\varphi}{2}\sin\frac{\varphi}{2}$

$$\mathbf{\sin\varphi = 2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}$$

Recall: $cos2\theta = cos^2\theta - sin^2\theta = 1 - 2sin^2\theta\ \ substituting\ \theta = \frac{\varphi}{2}$

$\therefore cos\varphi = cos^2\frac{\varphi}{2} - sin^2\frac{\varphi}{2} = 1 - 2sin^2\frac{\varphi}{2}$

$$\therefore \mathbf{\sin\frac{\varphi}{2} = \pm\sqrt{\frac{1 - cos\varphi}{2}}},$$

The "+" is used when $\frac{\varphi}{2}$ is in $1^{st}\ or\ 2^{nd}\ quadrant$ and otherwise the "–" is used

b) $\mathbf{cos\frac{\varphi}{2}}$

$$\left.\begin{aligned}\mathbf{cos\varphi} &= \mathbf{cos^2\varphi - sin^2\varphi}\\ &= \mathbf{2cos^2\frac{\varphi}{2} - 1}\\ &= \mathbf{1 - 2sin^2\frac{\varphi}{2}}\end{aligned}\right\} \quad \left[\because \mathbf{cos^2\frac{\varphi}{2} + sin^2\frac{\varphi}{2} = 1}\right]$$

From $cos\varphi = 2cos^2\varphi - 1$

$$\therefore \mathbf{\cos\frac{\varphi}{2} = \pm\sqrt{\frac{1 + cos\varphi}{2}}},$$

The "+" is used when $\frac{\varphi}{2}$ is in $1^{st}\ or\ 4^{th}\ quadrant$ and otherwise the "–" is used

c) $\boldsymbol{\tan\frac{\varphi}{2}}$

$\tan\varphi = \tan\left(\frac{\varphi}{2}+\frac{\varphi}{2}\right) = \frac{tan\frac{\varphi}{2}+tan\frac{\varphi}{2}}{1-tan^2\frac{\varphi}{2}}$

Also $\tan\frac{\varphi}{2} = \frac{sin\frac{\varphi}{2}}{cos\frac{\varphi}{2}} = \frac{\pm\sqrt{\frac{1-cos\varphi}{2}}}{\pm\sqrt{\frac{1+cos\varphi}{2}}}$

$\therefore \boldsymbol{\tan\frac{\varphi}{2} = \pm\sqrt{\frac{1-cos\varphi}{1+cos\varphi}}}$

Where $cos\varphi \neq -1$ *and* the "+" is used when $\frac{\varphi}{2}$ is in 1st *or* 3rd *quadrant* and otherwise the "–" is used

d) $\boldsymbol{\cot\frac{\varphi}{2}}$

$cot\frac{\varphi}{2} = \frac{1}{tan\frac{\varphi}{2}}$ $\quad\therefore \boldsymbol{\cot\frac{\varphi}{2} = \pm\sqrt{\frac{1+cos\varphi}{1-cos\varphi}}}$

Application

- Solution to trigonometric equations
- Proof of identities
- Elimination of parameters
- Simplifying trigonometric equations
- Integration
- Derivation of the $t-$ formuale

Derivation of the *t-formulae*

i) From $\cos\varphi = cos\left(\frac{\varphi}{2}+\frac{\varphi}{2}\right) = cos\frac{\varphi}{2}cos\frac{\varphi}{2} - sin\frac{\varphi}{2}sin\frac{\varphi}{2}$

$= \frac{cos^2\frac{\varphi}{2}-sin^2\frac{\varphi}{2}}{1} = \frac{cos^2\frac{\varphi}{2}-sin^2\frac{\varphi}{2}}{cos^2\frac{\varphi}{2}+sin^2\frac{\varphi}{2}}$

Dividing the Numerator and Denominator by $cos^2\frac{\varphi}{2}$

$\cos\varphi = \frac{cos^2\frac{\varphi}{2}/cos^2\frac{\varphi}{2} - sin^2\frac{\varphi}{2}/cos^2\frac{\varphi}{2}}{cos^2\frac{\varphi}{2}/cos^2\frac{\varphi}{2} + sin^2\frac{\varphi}{2}/cos^2\frac{\varphi}{2}} = \frac{1-tan^2\frac{\varphi}{2}}{1+tan^2\frac{\varphi}{2}}$ $\quad$ Let $t = \tan\frac{\varphi}{2}$

$\therefore \boldsymbol{\cos\varphi = \frac{1-t^2}{1+t^2}, t = \tan\frac{\varphi}{2}}$

In general $\boldsymbol{\cos(Angle) = \frac{1-t^2}{1+t^2}, t = \tan\left(\frac{Angle}{2}\right)}$

ii) From $\sin\varphi = \sin\left(\frac{\varphi}{2}+\frac{\varphi}{2}\right) = \sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + \cos\frac{\varphi}{2}\sin\frac{\varphi}{2}$

$= 2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}$ $\sin\varphi = \frac{2sin\frac{\varphi}{2}cos\frac{\varphi}{2}}{1} = \frac{2sin\frac{\varphi}{2}cos\frac{\varphi}{2}}{cos^2\frac{\varphi}{2}+sin^2\frac{\varphi}{2}}$

Dividing the Numerator and Denominator by $cos^2\frac{\varphi}{2}$

$\Rightarrow \sin\varphi = \frac{2sin\frac{\varphi}{2}cos\frac{\varphi}{2}/cos^2\frac{\varphi}{2}}{cos^2\frac{\varphi}{2}/cos^2\frac{\varphi}{2}+sin^2\frac{\varphi}{2}/cos^2\frac{\varphi}{2}} = \frac{2tan\frac{\varphi}{2}}{1-tan^2\frac{\varphi}{2}}$, let $t = \tan\frac{\varphi}{2}$

$\therefore \boldsymbol{\sin\varphi = \frac{2t}{1+t^2}, t = \tan\frac{\varphi}{2}}$

In general $\boldsymbol{\sin(Angle) = \frac{2t}{1+t^2}, t = \tan\left(\frac{Angle}{2}\right)}$

iii) From $\tan\varphi = \tan\left(\frac{\varphi}{2}+\frac{\varphi}{2}\right) = \frac{\tan\frac{\varphi}{2}+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}\tan\frac{\varphi}{2}} = \frac{2\tan\frac{\varphi}{2}}{1-\tan^2\frac{\varphi}{2}}$, let $t = \tan\frac{\varphi}{2}$

$\therefore \boldsymbol{\tan\varphi = \frac{2t}{1-t^2}}$, $\boldsymbol{t = \tan\frac{\varphi}{2}}$

In general $\tan(\boldsymbol{Angle}) = \boldsymbol{\frac{2t}{1-t^2}}$, $\boldsymbol{t = \tan\left(\frac{Angle}{2}\right)}$

$\boldsymbol{\cot\varphi = \frac{1}{\tan\varphi} = \frac{1-t^2}{2t}}$, $\boldsymbol{t =}$ $\boldsymbol{t = \tan\frac{\varphi}{2}}$

NOTE:

It is very important to note carefully the generalized $t-formulae$ of the trig ratios above and their definitions.

Application

- Solution to trigonometric equations
- Proof of identities
- Simplifying trigonometric equations
- Integration

These will be tackled later.

Proof of identities

Prove the following identities

a) $cosec\frac{1}{2}\theta = \pm\frac{\sqrt{2-2cos\theta}}{1-cos\theta} = \pm\frac{\sqrt{2+2cos\theta}}{sin\theta}$

b) $sec\frac{1}{2}\theta == \pm\frac{\sqrt{2+2cos\theta}}{1+cos\theta}$

c) $2\cot\frac{1}{2}A + \tan A = \tan A\cot^2\frac{1}{2}A$

d) $tan^2\frac{1}{2}x = -2\pm\sqrt{5}\ if\ tan\frac{1}{2}x = cosecx - sinx$

e) $2cot2A\cot\frac{1}{2}A = \frac{cosA}{1-cosA} - \frac{1+cosA}{cosA}$

f) $\tan^2\frac{\theta}{2} = \frac{2sin\theta - sin2\theta}{2sin\theta + sin2\theta}$

g) $cot\frac{1}{2}\varphi = \frac{sin\varphi}{1-cos\varphi}$

h) $\frac{tanA+secA-1}{tanA-secA+1} = \tan\left(\frac{\pi}{4}+\frac{A}{2}\right)$

i) $\frac{1+sin\theta}{1-\sin\theta} = tan^2\left(45^0+\frac{\theta}{2}\right)$

j) $\cos\varphi = \frac{cos\theta-e}{1-ecos\theta}\ if\ \tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}.\tan\frac{\varphi}{2}$

k) $\boldsymbol{Uneb2010}.Show\ that\ \tan\left(\frac{\pi}{4}+\frac{A}{2}\right) = \frac{1+sinA}{cosA}$

l) $\frac{1+sinx+cosx}{1+sinx-cosx} = \cot\frac{x}{2}$, $\frac{hence}{otherwise}$ $solve\ the\ equation\ \frac{1+sinx+cosx}{1+sinx-cosx} = \sqrt{2}$

m) $2cot\ \frac{A}{2} - sinA = cot^2\frac{A}{2}sinA$

Solution

a) $cosec\frac{1}{2}\theta = \pm\frac{\sqrt{2-2cos\theta}}{1-cos\theta} = \pm\frac{\sqrt{2+2cos\theta}}{sin\theta}$

R.H.S: $\pm\frac{\sqrt{2-2cos\theta}}{1-cos\theta} = \pm\frac{\sqrt{2-2\left(\frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}\right)}}{1-\left(\frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}\right)} = \pm\frac{\sqrt{\frac{2+2tan^2\frac{\theta}{2}-2+2tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}}}{\frac{1+tan^2\frac{\theta}{2}-1+tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}}$

$= \pm\frac{\sqrt{\frac{4tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}}}{\frac{2tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}} = \pm\frac{2tan\frac{\theta}{2}}{\sqrt{1+tan^2\frac{\theta}{2}}}.\frac{1+tan^2\frac{\theta}{2}}{2tan^2\frac{\theta}{2}}$

$= \pm\frac{1}{\sqrt{sec^2\frac{\theta}{2}}}.\frac{sec^2\frac{\theta}{2}}{tan\frac{\theta}{2}} = \frac{1}{\sec\frac{\theta}{2}}.sec^2\frac{\theta}{2}.\frac{1}{\tan\frac{\theta}{2}} = \sec\frac{\theta}{2}.\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$

$= \cos\frac{\theta}{2}\times\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \frac{1}{\sin\frac{\theta}{2}} = cosec\frac{\theta}{2}$

Also $\text{cosec}\ \frac{\theta}{2} = \pm\frac{\sqrt{2+2cos\theta}}{sin\theta}$

R.H.S: $\pm\frac{\sqrt{2+2cos\theta}}{sin\theta} = \frac{\sqrt{2}.\sqrt{1+cos\theta}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$, but $cos^2\frac{\theta}{2} = \frac{1+cos\theta}{2} \Rightarrow 1+cos\theta = 2cos^2\frac{\theta}{2}$

R.H.S $= \frac{\sqrt{\mathbf{2}}.\sqrt{2cos^2\frac{\theta}{2}}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\mathbf{2}\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{1}{\sin\frac{\theta}{2}} = \text{cosec}\ \frac{\theta}{2}$

b) $\sec\frac{1}{2}\theta = \pm\frac{\sqrt{2+2cos\,\theta}}{1+cos\theta}$

R.H.S $= \frac{\sqrt{2}.\sqrt{1+cos\theta}}{1+cos\theta} = \frac{\sqrt{\mathbf{2}}}{\sqrt{1+cos\theta}}$ **,** but $cos^2\frac{\theta}{2} = \frac{1+cos\theta}{2} \Rightarrow 1+cos\theta = 2cos^2\frac{\theta}{2}$

$\Rightarrow$ **R.H.S** $= \frac{\sqrt{\mathbf{2}}}{\sqrt{1+cos\theta}} = \frac{\sqrt{\mathbf{2}}}{\sqrt{2cos^2\frac{\theta}{2}}} = \frac{\mathbf{1}}{\cos\frac{\theta}{2}} = sec\frac{1}{2}\theta$

c) $2\cot\frac{1}{2}A + \tan A = \tan A\cot^2\frac{1}{2}A$

L.H.S: $2cot\frac{1}{2}A + tanA = \frac{2}{\tan\frac{A}{2}} + \frac{2\tan\frac{A}{2}}{1-tan^2\frac{A}{2}} = \frac{2-2tan^2\frac{A}{2}+2tan^2\frac{A}{2}}{\tan\frac{A}{2}.\left(1-tan^2\frac{A}{2}\right)}$

$= \frac{2}{\tan\frac{A}{2}.\left(1-tan^2\frac{A}{2}\right)}$ ..... $\left(\times\frac{\tan\frac{A}{2}.}{\tan\frac{A}{2}.}\right)$

$= \frac{2\tan\frac{A}{2}.}{1-tan^2\frac{A}{2}}.\frac{1}{\tan^2\frac{A}{2}.} = tanA.cot^2\frac{A}{2} =$ **R.H.S**

d) To prove $\tan^2\frac{1}{2}x = -2\pm\sqrt{5}$ given $\tan\frac{1}{2}x = \text{cosec}\,x - \sin x$

From $\tan\frac{1}{2}x = \text{cosec}x - \sin x = \frac{1}{sinx} - sinx = \frac{1-\sin^2 x}{sinx} = \frac{cos^2 x}{\sin x}$

But $cosx = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, $sinx = \frac{2\tan\frac{x}{2}}{1+tan^2\frac{x}{2}}$

$\therefore \tan\frac{1}{2}x = \frac{\left(\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}\right)^2}{\frac{2\tan\frac{x}{2}}{1+tan^2\frac{x}{2}}} = \frac{\left(1-tan^2\frac{x}{2}\right)^2}{2\tan\frac{x}{2}.\left(1+tan^2\frac{x}{2}\right)}$

$\Rightarrow 2\tan^2\frac{x}{2} = \frac{\left(1-tan^2\frac{x}{2}\right)^2}{\left(1+tan^2\frac{x}{2}\right)} \quad \therefore 2tan^2\frac{x}{2} + 2tan^4\frac{x}{2} = 1 - 2\tan^2\frac{x}{2} + \tan^4\frac{x}{2}$

$\Rightarrow \tan^4\frac{x}{2} + 4\tan^2\frac{x}{2} - 1 = 0$ Solving the equation $let\ u = tan^2\frac{x}{2}$

$\Rightarrow u^2 + 4u - 1 = 0 \ \therefore u = \frac{-4\pm\sqrt{16+4}}{2} = \frac{-4\pm\sqrt{20}}{2} = \frac{-4\pm2\sqrt{5}}{2} = -2 \pm \sqrt{5}$

$\therefore \tan^2\frac{x}{2} = = -2 \pm \sqrt{5}$

e) $2\cot 2A\cot\frac{1}{2}A = \frac{cosA}{1-cosA} - \frac{1+cosA}{cosA}$

R.H.S: $\frac{cosA}{1-cosA} - \frac{1+cosA}{cosA} = \frac{cos^2A-(1-cos^2A)}{cosA\,(1-cosA)} = \frac{2cos^2A-1}{cosA\,(1-cosA)} = \frac{cos\,2A}{cosA\,(1-cosA)}$

But $\sin^2\frac{A}{2} = \frac{1-cosA}{2}$ or $2sin^2\frac{A}{2} = 1 - cosA$

$\therefore \frac{cosA}{1-cosA} = \frac{cos\,2A}{cosA\,.2sin^2\frac{A}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\left(\times\frac{sin\,2A}{sin\,2A}\right)$

$= \frac{cos\,2A}{sin\,2A}.\frac{sin\,2A}{cosA\,.2sin^2\frac{A}{2}} = \cot 2A\,.\frac{2\sin\,AcosA}{cosA\,.2sin^2\frac{A}{2}} = cot2A.\frac{sinA}{\sin^2\frac{A}{2}}$

$= cot2A.\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin^2\frac{A}{2}} = cot2A.\frac{2\cos\frac{A}{2}}{\sin\frac{A}{2}} = cot2A.\cot\frac{A}{2} =$ **L.H.S**

f) (f), (g) $Left\ as\ an\ exercise$

h) $\frac{tanA+secA-1}{tanA-secA+1} = tan\left(\frac{\pi}{4}+\frac{A}{2}\right)$

L.H.S: $\frac{tanA+secA-1}{tanA-secA+1} = \frac{\frac{sinA}{cosA}+\frac{1}{cosA}-1}{\frac{sinA}{cocsA}-\frac{1}{cosA}+1} = \frac{sinA+1-cosA}{sinA-1+cosA}$

$= \frac{sinA+(1-cosA)}{sinA-(1-cosA)}$, $but\ 1 - cosA = 2sin^2\frac{A}{2}$

$\therefore \frac{sinA+(1-cosA)}{sinA-(1-cosA)} = \frac{sinA+2sin^2\frac{A}{2}}{sinA-2sin^2\frac{A}{2}} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}+2sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}-2sin^2\frac{A}{2}} = \frac{2\sin\frac{A}{2}\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)}{2\sin\frac{A}{2}\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)}$

$= \frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}$ Dividing the Numerator and Denominator by $\cos\frac{A}{2}$

$= \frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}} = \frac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}} = \frac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}} = \frac{\tan\frac{\pi}{4}+\tan\frac{A}{2}}{1-\tan\frac{\pi}{4}.\tan\frac{A}{2}} = \tan\left(\frac{\pi}{4}+\frac{A}{2}\right) =$ **R.H.S**

i) $\frac{1+sin\theta}{1-\sin\,\theta} = \tan^2\left(45^0+\frac{\theta}{2}\right)$

L.H.S: $\frac{1+sin\theta}{1-\sin\,\theta} = \frac{\left(1+\frac{2\tan\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}\right)}{\left(1-\frac{2\tan\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}\right)} = \frac{1+tan^2\frac{\theta}{2}+2\tan\frac{\vartheta}{2}}{1+tan^2\frac{\theta}{2}-2\tan\frac{\vartheta}{2}} = \frac{\left(1+\tan\frac{\theta}{2}\right)^2}{\left(1-\tan\frac{\theta}{2}\right)^2}$

$= \left(\frac{tan\,45^0+\tan\frac{\theta}{2}}{tan\,45^0-\tan\frac{\theta}{2}}\right)^2 = tan^2\left(45^0+\frac{\theta}{2}\right)$

j) $\cos\varphi = \frac{cos\theta-e}{1-ecos\theta}$ $if\ \tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}.\tan\frac{\varphi}{2}$

L. H. S: $\cos\varphi = \frac{1-tan^2\frac{\varphi}{2}}{1+tan^2\frac{\varphi}{2}}$ Now from $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}.\tan\frac{\varphi}{2}$, *squaring both sides*

$\tan^2\frac{\theta}{2} = \left(\frac{1-e}{1+e}\right)tan^2\frac{\varphi}{2}$ *or* $tan^2\frac{\varphi}{2} = tan^2\frac{\theta}{2}.\left(\frac{1+e}{1-e}\right)$

$\therefore \cos\varphi = \frac{1-tan^2\frac{\theta}{2}.\left(\frac{1+e}{1-e}\right)}{1+tan^2\frac{\theta}{2}.\left(\frac{1+e}{1-e}\right)} = \frac{1-e-(1+e).tan^2\frac{\theta}{2}}{1-e+(1+e).tan^2\frac{\theta}{2}}$

$= \frac{(1-e).cos^2\frac{\theta}{2} - (1+e)sin^2\frac{\theta}{2}}{(1-e).cos^2\frac{\theta}{2} + (1+e)sin^2\frac{\theta}{2}} = \frac{cos^2\frac{\theta}{2} - e.cos^2\frac{\theta}{2} - sin^2\frac{\theta}{2} - e.sin^2\frac{\theta}{2}}{cos^2\frac{\theta}{2} - e.cos^2\frac{\theta}{2} + sin^2\frac{\theta}{2} + e.sin^2\frac{\theta}{2}}$

$= \frac{\left(cos^2\frac{\theta}{2} - sin^2\frac{\theta}{2}\right) + e.\left(-sin^2\frac{\theta}{2} - cos^2\frac{\theta}{2}\right)}{\left(cos^2\frac{\theta}{2} + sin^2\frac{\theta}{2}\right) + e.\left(cos^2\frac{\theta}{2} - sin^2\frac{\theta}{2}\right)} = \frac{cos\theta - e.\left(cos^2\frac{\theta}{2} + sin^2\frac{\theta}{2}\right)}{1 - e.cos\theta}$

$= \frac{cos\theta - e}{1-e.cos\theta} = \mathbf{L.H.S}$

Task

Prove that $\frac{sin\alpha sin\beta}{cos\alpha + cos\beta} = \frac{2\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1-tan^2\frac{\alpha}{2}.tan^2\frac{\beta}{2}}$

Finding Exact value of Trigonometric Ratios of Special Angles

Show that

a) $sin^2\frac{\pi}{4} = \frac{2-\sqrt{2}}{2}$

b) $tan^2\frac{\pi}{8} = \frac{2-\sqrt{2}}{2+\sqrt{2}}$

c) $cos^2\frac{\pi}{8} = \frac{2+\sqrt{2}}{2}$

d) $sin^2\frac{5\pi}{8} = \frac{2-\sqrt{2-\sqrt{3}}}{4}$

e) $tan\frac{3\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$

f) $cos^2\frac{11\pi}{12} = \frac{2+\sqrt{3}}{4}$

Solution

a) $\sin^2\frac{\pi}{4} = \frac{2-\sqrt{2}}{2}$, $\frac{\pi}{8} radians = \frac{180^0}{8} = \frac{45^0}{2}$

From $sin^2\frac{\theta}{2} = \frac{1-cos\theta}{2}$, $let\ \theta = 45^0 = \frac{\pi}{4}$

$\Rightarrow sin^2\left(\frac{45^0}{2}\right) = sin^2\frac{\pi}{8} = \frac{1-cos\,45^0}{2} = \frac{1-\frac{1}{\sqrt{2}}}{2} = \frac{\sqrt{2}-1}{2\sqrt{2}}$ ..... $\left(\times\frac{\sqrt{2}}{\sqrt{2}}\right)$

$= \frac{2-\sqrt{2}}{4}$

b) $\tan^2\frac{\pi}{8} = \frac{2-\sqrt{2}}{2+\sqrt{2}}$

$\tan^2\frac{\pi}{8} = tan^2\left(\frac{45^0}{2}\right)$ $From\frac{sin^2\frac{\pi}{8}}{cos^2\frac{\pi}{8}} = tan^2\frac{\pi}{8}$

Using $sin^2\frac{\theta}{2} = \frac{1-cos\theta}{2}$, $cos^2\frac{\theta}{2} = \frac{1+cos\theta}{2}$, let $\theta = 45^0 = \frac{\pi}{4}$

$\Rightarrow \tan^2\frac{\pi}{8} = \frac{(1-cos\,45^0)/2}{(1+cos\,45^0)/2} = \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}-1}{\sqrt{2}+1}$ .....$\left(\times\frac{\sqrt{2}}{\sqrt{2}}\right)$

$\therefore \tan^2\frac{\pi}{8} = \frac{2-\sqrt{2}}{2+\sqrt{2}}$

c) Left as an exercise

d) $\sin^2\frac{5\pi}{8} = \frac{2-\sqrt{2-\sqrt{3}}}{4}$

$\frac{5\pi}{24} radians = \frac{5\times 180^0}{4} = \frac{75^0}{2}$

Using $sin^2\frac{\theta}{2} = \frac{1-cos\theta}{2}$, $let\ \theta = 75^0$

$\sin^2\frac{5\pi}{8} = sin^2\left(\frac{75^0}{2}\right) = \frac{1-cos\,75^0}{2}$

But $cos75^0 = cos(30^0 + 45^0) = cos30cos45 - sin30sin45$

$= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} - \frac{1}{2}\times\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\Rightarrow sin^2\frac{5\pi}{8} = sin^2\left(\frac{75^0}{2}\right) = \frac{1-\frac{\sqrt{3}-1}{2\sqrt{2}}}{2} = \frac{2\sqrt{2}-\sqrt{3}+1}{4\sqrt{2}}$ ..... $\left(\times\frac{\sqrt{2}}{\sqrt{2}}\right)$

$= \frac{4-\sqrt{6}+\sqrt{2}}{8} = \frac{4-(\sqrt{6}-\sqrt{2})}{8}$

From algebra $\sqrt{6} - \sqrt{2} = \sqrt{(\sqrt{6} - \sqrt{2})^2}$ to create the root in the required proof

$\therefore \sqrt{6} - \sqrt{2} = \sqrt{(6 - 2\sqrt{12} + 2)} = \sqrt{8 - 2.2\sqrt{3}}$

$= 2.\sqrt{2-\sqrt{3}}$

$\therefore \sin^2\frac{5\pi}{8} = \frac{4-(\sqrt{6}-\sqrt{2})}{8} = \frac{4-2.\sqrt{2-\sqrt{3}}}{8} = \frac{2-\sqrt{2-\sqrt{3}}}{4}$

e) $left\ as\ an\ exercise$

f) $cos^2\frac{11\pi}{12} = \frac{2+\sqrt{3}}{4}$

$\frac{11\pi}{12} radians = \frac{11\times 180^0}{12} = 165^0$

$\Rightarrow cos^2\frac{11\pi}{12} = cos^2 165^0 = \frac{2+\sqrt{3}}{4}$

But $165^0 = \frac{330^0}{2}$ $i.e. making\ use\ of\ half\ angles$

$\therefore cos^2\frac{11\pi}{12} = cos^2 165^0 = cos^2\left(\frac{330^0}{2}\right) = \frac{2+\sqrt{3}}{4}$

Using $cos^2\frac{\theta}{2} = \frac{1+cos\theta}{2}$ $let\ \theta = 330^0$

$\Rightarrow cos^2\frac{11\pi}{12} = \frac{1+\frac{\sqrt{3}}{2}}{2} = \frac{2+\sqrt{3}}{4}$

Task

Show that

i) $\tan^2\frac{\pi}{16} = \frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}$

ii) $\cos\frac{7\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}$

Solution to Trigonometric Equations

The use of half angle identities in solution to trigonometric equations is done using the $t-formulae$ for convenience.

The equations to be solved by $t-formulae$ usually take the form

$$\boldsymbol{a\cos(m\theta)+b\sin(m\theta)+c=0}$$

Note: The angle is the same for cosine and sine trig ratios, and *a, b*, and *c* are constants.

Example

1. *Solve the equation for* $0^0 \le \theta \le 360^0$

a) $3cos\theta + 4sin\theta = 2$

b) $2cos2\theta + 3sin2\theta - 2 = 0$

c) $2sin\theta + 7cos\theta = 4$

d) $3tan\theta - 2sec\theta = 4$

e) $5sin3\theta - 12cos3\theta = 6$

f) $\cos\frac{1}{2}\theta - 7\sin\frac{1}{2}\theta = 2$

Solution

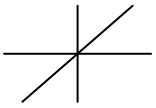
a) $3\cos\theta + 4\sin\theta = 2$

From $cos\theta = \frac{1-t^2}{1+t^2}$, $sin\theta = \frac{2t}{1+t^2}$, $t = \tan\frac{\theta}{2}$

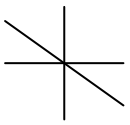
$\Rightarrow 3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) = 2$ .....$(\times(1+t^2))$

$\Rightarrow 3 - 3t^2 + 8t = 2 + 2t^2 \; or \; 5t^2 - 8t - 1 = 0$

$\therefore t = \frac{8\pm\sqrt{64+20}}{10} = \frac{8\pm9.165}{10} \Rightarrow t = 1.7165, -0.1165$

For $\tan\frac{\theta}{2} = 1.7165$  $\tan^{-1}(1.7165) = 59.8^0$

$\frac{\theta}{2} = 59.8^0, 239.8^0 \Rightarrow \theta = 119.6^0, 479.6^0$

For $\tan\frac{\theta}{2} = -0.1165$  $\tan^{-1}(0.1165) = 6.6^0$

$\frac{\theta}{2} = 173.4^0, 353.4^0 \Rightarrow \theta = 346.8^0, 706.8^0$

$\therefore \theta = 119.6^0, 346.8^0$

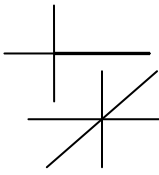
b) $2cos2\theta + 3sin2\theta - 2 = 0$

$\cos 2\theta = \frac{1-t^2}{1+t^2}$, $\sin 2\theta = \frac{2t}{1+t^2}$, $t = \tan\theta$ $\left[\because t = \tan\frac{Angle}{2}\right]$

$\Rightarrow 2\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right) - 2 = 0$..... $(\times(1+t^2))$

$\Rightarrow 2 - 2t^2 + 6t - 2 - 2t^2 = 0 \; or \; -4t^2 + 6t = 0 \; or \; -2t^2 - 3t = 0$

By factorization $t = 0, \frac{3}{2}$

For $\tan\theta = 0$  $\tan^{-1}(0) = 0^0, 180^0, 360^0$

For $\tan\theta = \frac{3}{2}$ $\tan^{-1}\left(\frac{3}{2}\right) = 56.3^0$

$\Rightarrow \theta = 56.3^0, 236.3^0$

$\therefore \theta = 0^0, 56.3^0, 180^0, 236.3^0, 360^0$

c) *Left as an exercise* $\quad$ ***Answer***: $\boldsymbol{\theta = 72.6^0, 319.3^0}$

d) *Left as an exercise* $\quad$ ***Answer***: $\boldsymbol{76.7^0, 209.6^0}$ $\left[\begin{matrix} HINT: 3tan\theta - 2sec\theta = 4 \\ \frac{3sin\theta}{cos\theta} - \frac{2}{cos\theta} = 4\ or \\ 3sin\theta - 4cos\theta - 2 = 0 \end{matrix}\right]$

e) $5\sin 3\theta - 12\cos 3\theta = 6$

$\cos 3\theta = \frac{1-t^2}{1+t^2}$, $\sin 3\theta = \frac{2t}{1+t^2}$, $t = \tan\frac{3}{2}\theta$ $\left[\because t = \tan\frac{Angle}{2}\right]$

$\Rightarrow 5\left(\frac{2t}{1+t^2}\right) - 12\left(\frac{1-t^2}{1+t^2}\right) = 6 \ldots\ldots\ldots\ldots\ldots\ldots \left(\times (1+t^2)\right)$

$\Rightarrow 10 - 12 + 12t^2 = 6 + 6t^2\ or\ 6t^2 + 10t - 18 = 0$

$or\ 3t^2 + 5t - 9 = 0\ \therefore t = \frac{-5 \pm \sqrt{25+108}}{6} \Rightarrow t = 1.0888, -2.7554$

For $\tan\frac{3\theta}{2} = 1.0888$ $\quad$ $\tan^{-1}(1.0888) = 47.4^0$

$\Rightarrow \frac{3\theta}{2} = 47.4^0.227.4^0, 407.4^0, 587.5^0 \quad \therefore \theta = \frac{2}{3}(47.4^0.227.4^0, 407.4^0, 587.5^0)$

$\therefore \theta = 31.6^0, 151.6^0, 271.6^0$

For $\tan\frac{3\theta}{2} = -2.7554$ $\quad$ $\tan^{-1}(2.7554) = 70.1^0$

$\Rightarrow \frac{3\theta}{2} = 109.9^0, 289.9^0, 469.9^0 \quad \therefore \theta = \frac{2}{3}(109.9^0, 289.9^0, 469.9^0)$

$\therefore \theta = 73.3^0, 193.3^0, 313.3^0$

$\therefore \theta = 31.6^0, 73.3^0, 151.6^0, 193.3^0, 271.6^0, 313.3^0$

f) *left as an exercise*. [HINT: use $t = \tan\frac{\theta}{4}$ $\quad$ **Ans**: No angle in this interval]

Solving the equation $a\cos(m\theta) + b\sin(m\theta) + c = 0$ using $R - formulae$

To solve this equation, first express

$$a\,cos(m\theta) + b\,sin(m\theta) \equiv Rsin(m\theta \pm \alpha)$$
$$\equiv Rcos(m\theta \pm \alpha), \alpha\ is\ Acute$$

Then solve the equation $Rcos(m\theta \pm \alpha) + c = 0$ *or* $Rsin(m\theta \pm \alpha)$.

Example

2. *Solve the above equations in the previous example using R – formulae.*

Solution

a) $3cos\theta + 4sin\theta = 2$

Let $3cos\theta + 4sin\theta \equiv Rsin(\theta + \alpha)$

$\equiv R[sin\theta cos\alpha + cos\theta sin\alpha]$

$\equiv (Rcos\alpha)\sin\theta + (Rsin\alpha)cos\theta$

Equating corresponding coefficients of trig ratios

For $cos\theta$: $3 = Rsin\alpha$ ……(1)

For $\sin\theta$: $4 = Rcos\alpha$ ……(2)

Solving (1) *and* (2) simultaneously for R *and* α

$(1)^2 + (2)^2$

$3^2 + 4^2 = R^2sin^2\alpha + R^2cos^2\alpha = R^2(sin^2\alpha + cos^2\alpha)$

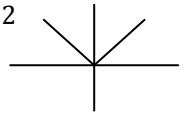
$25 = R^2$ $\quad \therefore R = 5$

(1) ÷ (2) to create $tan\alpha$ $\quad \therefore \alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.9^0$

$\frac{3}{4} = \frac{Rsin\alpha}{Rcos\alpha} = tan\alpha$

$\Rightarrow 3cos\theta + 4sin\theta = 5sin(\theta + 36.9^0) = 2$

$\Rightarrow sin(\theta + 36.9^0) = \frac{2}{5}$



$\sin^{-1}\left(\frac{2}{5}\right) = 23.6^0$

$\theta + 36.9^0 = 23.6^0, 156.4^0, 383.6^0 \Rightarrow \theta = (23.6^0, 156.4^0, 383.6^0) - 36.9^0$

$\therefore \theta = 119.5^0, 346.7^0$

b) $2cos2\theta + 3sin2\theta - 2 = 0$

Let $2cos2\theta + 3sin2\theta \equiv Rcos(2\theta - \alpha)$

$\equiv R[cos2\theta cos\alpha + sin2\theta sin\alpha]$

$\equiv (Rcos\alpha)cos2\theta + (Rsin\alpha)sin2\theta$

Equating corresponding coefficients of trig ratios

For $cos2\theta$: $2 = Rcos\alpha$ ……(1)

For $\sin 2\theta$: $3 = Rsin\alpha$ ……(2)

$(1)^2 + (2)^2$

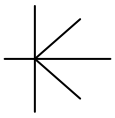
$2^2 + 3^2 = R^2\cos^2\alpha + R^2\sin^2\alpha = R^2(\cos^2\alpha + \sin^2\alpha)$

$13 = R^2$ $\quad \therefore R = \sqrt{13}$

(2) ÷ (1) to create $\tan\alpha$

$\frac{3}{2} = \frac{Rsin\alpha}{Rcos\alpha} = \tan\alpha$

Now $2cos2\theta + 3sin2\theta - 2 = 0 \Rightarrow \sqrt{13}\,cos(2\theta - 56.3^0) - 2 = 0$

$cos(2\theta - 56.3^0) = \frac{2}{\sqrt{13}}$  $\cos^{-1}\left(\frac{2}{\sqrt{13}}\right) = 56.3^0$

$\therefore 2\theta - 56.3^0 = \pm 56.3^0, 303.7^0, 416.3^0, 776.3^0, 663.7^0$

$\therefore 2\theta = (\pm 56.3^0, 303.7^0, 416.3^0, 776.3^0, 663.7^0) + 56.3^0$

$\therefore \theta = 0^0, 56.3^0, 180^0, 236.3^0, 360^0$

e) $5sin3\theta - 12cos3\theta = 6$

Let $5sin3\theta - 12cos3\theta \equiv Rsin(3\theta + \alpha)$

$\equiv R[sin3\theta cos\alpha + cos3\theta sin\alpha]$

$\equiv (Rcos\alpha)\sin 3\theta + (Rsin\alpha)cos3\theta$

Equating corresponding coefficients of trig ratios

For $sin3\theta$: $5 = Rcos\alpha$ ……(1)

For $\cos 3\theta$: $-12 = Rsin\alpha$ ……(2)

Solving (1) *and* (2) simultaneously for R *and* α

$(1)^2 + (2)^2$

$5^2 + 12^2 = R^2cos^2\alpha + R^2sin^2\alpha = R^2(cos^2\alpha + sin^2\alpha)$

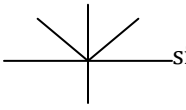
$169 = R^2$ $\boxed{\therefore R = 13}$

(2) ÷ (1) to create $tan\alpha$

$\frac{-12}{5} = \frac{Rsin\alpha}{Rcos\alpha} = \tan\alpha$ $\boxed{\therefore \alpha = -\tan^{-1}\left(\frac{12}{5}\right) = -67.4^0}$

$\therefore 5\sin 3\theta - 12\cos 3\theta = 13sin(3\theta + -67.4^0) = 6$

$= 13sin(3\theta - 67.4^0) = 6$

$sin(3\theta - 67.4^0) = \frac{6}{13}$  $\sin^{-1}\left(\frac{6}{13}\right) = 27.5^0$

$3\theta - 67.4^0 = 27.5^0, 152.5^0, 387.5^0, 512.5^0, 747.5^0, 872.5^0, 1107.5^0, 1232.5^0$

$\therefore 3\theta = (27.5^0, 152.5^0, 387.5^0, 512.5^0, 747.5^0, 872.5^0, 1107.5^0, 1232.5^0) + 67.4^0$

$\therefore \theta = 31.6^0, 73.3^0, 151.6^0, 193.3^0, 271.6^0, 313.3^0$

Task

1. *Solve* $3sinx + 4cosx = 2\ for -180^0 \leq x \leq +180^0$ (***Uneb* 2011**) ***Ans***: $-$ $\mathbf{29.6^0, 103.2^0}$
2. *Solve the equation* $\sqrt{3}\,cos2\theta - sin2\theta = \sqrt{2}$ *for any values of* θ *between* 0^0 *and* 360^0 (***Jjeb*2011**) ***Ans***: $\mathbf{7.5^0, 142.5^0, 187.5^0, 330^0}$
3. *Given that* $t = \tan\frac{\theta}{2}$, *solve the equation* $5sin\theta - 2cos2\theta = 1\ for\ 0^0 \leq \theta \leq 360^0$ ***Ans***: $\mathbf{32.5^0, 191.1^0}$
4. *Solve for* $0^0 < \theta < 360^0$ *in the equation* $10sin^2\theta - 5sin2\theta = 4$

$\boldsymbol{Ans}: \mathbf{63.4^0, 243.4^0, 161.6^0, 341.56^0}$ $\left[HINT: use\ sin^2\theta = \frac{1-cos\,2\theta}{2}\right]$

Proof of identities using t − formulae

Prove the following identities

a) $2\cot\frac{A}{2} - sinA = cot^2\frac{A}{2}sinA$

b) $2\cot\frac{A}{2} + tanA = cot^2\frac{A}{2}tanA$

c) $\cot\frac{1}{2}\varphi = \sin\frac{\varphi}{1-cos\varphi}$

Solution

a) $2\cot\frac{A}{2} - sinA = cot^2\frac{A}{2}\sin A$

$\mathbf{L.H.S}: 2\cot\frac{A}{2} - sinA = \frac{2}{\tan\frac{A}{2}} - \frac{2\tan\frac{A}{2}}{1+tan^2\frac{A}{2}} = \frac{2}{t} - \frac{2t}{1+t^2} = \frac{2+2t^2-2t^2}{t(1+t^2)} = \frac{2}{t(1+t^2)}$

$= \frac{2}{\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}.sec^2\frac{A}{2}}$ $\left[\because 1 + tan^2\frac{A}{2} = sec^2\frac{A}{2}\right]$

$= \frac{2\cos\frac{A}{2}}{\sin\frac{A}{2}.\,1/cos^2\frac{A}{2}} = 2\cot\frac{A}{2}.cos^2\frac{A}{2}$ ..... $\left(\times\frac{\sin\frac{A}{2}}{\sin\frac{A}{2}}\right)$

$= 2\cot\frac{A}{2}.\frac{\sin\frac{A}{2}.\cos\frac{A}{2}.\cos\frac{A}{2}}{\sin\frac{A}{2}} = \cot\frac{A}{2}.\left(2\sin\frac{A}{2}.\cos\frac{A}{2}\right).\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}$

$= \cot\frac{A}{2}.sinA.\cot\frac{A}{2} = cot^2\frac{A}{2}sinA = \mathbf{R.H.S}$

b) $2\cot\frac{A}{2} + tanA = cot^2\frac{A}{2}tanA$

$\mathbf{L.H.S}: 2\cot\frac{A}{2} + \tan A = \frac{2}{\tan\frac{A}{2}} + \frac{2\tan\frac{A}{2}}{1-tan^2\frac{A}{2}} = \frac{2}{t} + \frac{2t}{1-t^2} = \frac{2-2t^2+2t^2}{t(1-t^2)} = \frac{2}{t(1-t^2)}$

$= \frac{2}{\tan\frac{A}{2}.\left(1-tan^2\frac{A}{2}\right)} = \frac{2\cot\frac{A}{2}}{\left(cos^2\frac{A}{2}-sin^2\frac{A}{2}\right)/cos^2\frac{A}{2}}$

$= \frac{2\cot\frac{A}{2}.cos^2\frac{A}{2}}{cosA} = \cot\frac{A}{2}.\frac{2\cos\frac{A}{2}.\sin\frac{A}{2}.\cos\frac{A}{2}}{cosA.\sin\frac{A}{2}}$

$\cot\frac{A}{2}.\frac{sinA}{cosA}.\cot\frac{A}{2} = cot^2\frac{A}{2}tanA = \mathbf{R.H.S}$

c) *left as an exercise*

Stating the Maximum and Minimum values of
$a\cos(m\theta) + b\sin(m\theta) + c$ and values of θ for which they occur

First express $\boldsymbol{a\,cos(m\theta) + b\,sin(m\theta) + c}$ in the $\boldsymbol{R - formulae}$ and state the maximum and minimum values of the expression as it was done in $y = a \pm bcos\,(m\theta + \alpha)$ and $y = a \pm bcos\,(m\theta + \alpha)$

Examples

State the maximum and minimum values of the following trigonometric expression, and state the values of θ for which they occur between 0^0 and 360^0 inclusive

a. $2sin\theta - 5cos\theta$

b. $3cos\theta + 4sin\theta - 3$

c. $2cos2\theta + 3sin2\theta - 2$

d. $5sin3\theta - 12cos3\theta + 6$

e. $3\sqrt{2}cos(\theta + 45^0) + 7sin\theta - 1$

f. $cos(\theta + 60^0) - cos\theta - 0.5$

Solution

a. Let $2sin\theta - 5cos\theta \equiv Rcos(\theta + \alpha) = (Rcos\alpha)cos\theta - (Rsin\alpha)\sin\theta$

Comparing corresponding coefficients

For $sin\theta$: $2 = -Rsin\alpha$ ..... (1)

For $cos\theta$: $-5 = Rcos\alpha$ ..... (2)

Solving (1) *and* (2) simultaneously for R *and* α

$(1)^2 + (2)^2$

$2^2 + 5^2 = R^2sin^2\alpha R^2cos^2\alpha = R^2(sin^2\alpha + cos^2\alpha)$

$29 = R^2$ $\quad \therefore R = \sqrt{29}$

(1) ÷ (2) to create $\tan\alpha$ $\quad \therefore \alpha = \tan^{-1}\left(\frac{2}{5}\right) = 21.8^0$

$\frac{2}{-5} = \frac{-Rsin\alpha}{Rcos\alpha} = -\tan\alpha$

$\therefore 2sin\theta - 5cos\theta = \sqrt{29}\, cos(\theta + 21.8^0)$

$max\ (2sin\theta - 5cos\theta) = max\left[\sqrt{29}\, cos(\theta + 21.8^0)\right] = \sqrt{29}.max[cos(\theta + 21.8^0)]$

$= \sqrt{29}$

Finding the value of θ_{max}

$\sqrt{29}\, cos(\theta + 21.8^0) = \sqrt{29} \Rightarrow cos(\theta + 21.8^0) = 1$

$\Rightarrow \theta + 21.8^0 = 0^0, 360^0 \quad \therefore \theta_{max} = 338.2^0$

$min\ (2sin\theta - 5cos\theta) = min\left[\sqrt{29}\, cos(\theta + 21.8^0)\right] = \sqrt{29}.min[cos(\theta + 21.8^0)]$

$= -\sqrt{29}$

Finding the value of θ_{min}

$\sqrt{29}\, cos(\theta + 21.8^0) = -\sqrt{29} \Rightarrow cos(\theta + 21.8^0) = -1$

$\Rightarrow \theta + 21.8^0 = 180^0 \quad \therefore \theta_{min} = 158.2^0$

b. $3\cos\theta + 4\sin\theta - 3$

$3\cos\theta + 4\sin\theta \equiv R\sin(\theta + \alpha)$ *proceeding as above,*

$3\cos\theta + 4\sin\theta = 5\sin(\theta + 36.9^0)$ $\quad (Left\ for\ students\ to\ derive)$

$max\ (3\cos\theta + 4\sin\theta - 3) = max(5\sin\ (\theta + 36.9^0) - 3)$

$= max[5\sin(\theta + 36.9^0)] - min(3)$

$= 5(1) - 3 = 2$

Finding the value of θ_{max}

$5\sin(\theta + 36.9^0) - 3 = 2 \Rightarrow \sin(\theta + 36.9^0) = 1$

$\Rightarrow \theta + 36.9^0 = 90^0 \quad \therefore \theta_{max} = 53.1^0$

$min\ (3cos\theta + 4sin\theta - 3) = min(5sin(\theta + 36.9^0) - 3)$

$= min[5sin(\theta + 36.9^0)] - max(3)$

$= 5(-1) - 3 = -8$

Finding the value of θ_{min}

$5\ sin(\theta + 36.9^0) - 3 = -8 \Rightarrow sin(\theta + 36.9^0) = -1$

$\Rightarrow \theta + 36.9^0 = 270^0 \quad \therefore \theta_{min} = 233.1^0$

c. $2cos2\theta + 3sin2\theta - 2$

Let $2\cos 2\theta + 3\sin 2\theta \equiv R\cos(2\theta - \alpha)$ *proceeding as usual,*

$2\cos 2\theta + 3\sin 2\theta = \sqrt{13}\cos(2\theta - 56.3^0)$

$max\ (2\cos 2\theta + 3\sin 2\theta - 2) = max\left[\sqrt{13}\cos(2\theta - 56.3^0) - 2\right]$

$= \sqrt{13}\, max[\cos(2\theta - 56.3^0) - 2] \ = \sqrt{13} - 2$

Finding the value of θ_{max}

$\sqrt{13}\ cos(2\theta - 56.3^0) - 2 = \sqrt{13} - 2 \Rightarrow \cos(2\theta - 56.3^0) = 1$

$\Rightarrow 2\theta - 56.3^0 = 0^0, 360^0 \quad \therefore 2\theta = 56.3^0, 416.3^0$

$\therefore \theta_{max} = 28.2^0, 208.2^0$

$min\ (2cos2\theta + 3sin2\theta - 2) = min\left[\sqrt{13}\ cos(2\theta - 56.3^0) - 2\right]$

$= \sqrt{13}\, min[\ cos(2\theta - 56.3^0) - 2] = -\sqrt{13} - 2$

Finding the value of θ_{min}

$\sqrt{13}\ cos(2\theta - 56.3^0) - 2 = -\sqrt{13} - 2 \Rightarrow cos(2\theta - 56.3^0) = -1$

$\Rightarrow 2\theta - 56.3^0 = 180^0, 540^0$ *solving for* θ

$\therefore \theta_{min} = 118.2^0, 298.2^0$

d. *Left as an exercise*

e. *(e) and (f) use special angles and expand using identities to express in the form being dealt with, then proceed as in the above examples.*

Answer: $(\boldsymbol{e})\ \boldsymbol{max = 4, min = -6, \theta_{max} = 53.1^0, \theta_{min} = 233.1^0}$

$(\boldsymbol{f})\ \boldsymbol{max = 0.5, min = -1.5, \theta_{max} = 240^0, \theta_{min} = 60^0}$

NOTE:

Other applications mentioned and not tackled in this edition shall be handled in the 2nd *edition.*

Task

Express $3cos^2\theta + 4sin\theta cos\theta - 5sin^2\theta$ *in the form* $a + b\ cos(2\theta - \beta)$ *where a, b are constants and* β *is an acute angle. Hence prove tat the maximum and minimum values of the expression*

$9 + 3cos^2\theta + 4sin\theta cos\theta - 5sin^2\theta$ *are* $2\left(4 + \sqrt{5}\right)$ *and* $2\left(4 - \sqrt{5}\right)$ *respectively.*

$\left[\textbf{HINT: } cos^2\theta = \frac{1 + cos2\theta}{2}, \quad sin^2\theta = \frac{1 + cos2\theta}{2} \text{ and } 4sin\theta cos\theta = 2sin2\theta\right]$

THE FACTOR FORMULAE
(Sum and Product identities)

The factor formulae provides a means to convert products of certain trigonometric functions in to sums and conversely.
Factor formulae are derived using the sum and difference of identities for the sines and cosine function.
Recall
$cos(A+B) = cosAcosB - sinAsinB$
$cos(A-B) = cosAcosB + sinAsinB$
Adding gives
$cos(A+B) + cos(A-B) = 2cosAcosB$
Subtracting gives
$cos(A+B) - cos(A-B) = -2sinAsinB$
Also $sin(A+B) = sinAcosB + cosAsinB$
$sin(A-B) = sinAcosB - cosAsinB$
Adding gives
$sin(A+B) + sin(A-B) = 2sinAcosB$
Subtracting gives
$sin(A+B) - sin(A-B) = 2cosAsinB$

Thus if A and B are real(*measures of angles*), then

i) $cos(A+B) + cos(A-B) = 2cosAcosB$
ii) $cos(A+B) - cos(A-B) = -2sinAsinB$
iii) $sin(A+B) + sin(A-B) = 2sinAcosB$
iv) $sin(A+B) - sin(A-B) = 2cosAsinB$

The factor formulae can be converted in to an equivalent form by making the substitution of variables.
let $P = A + B$ ..... (1)
$Q = A - B$ ..... (2)
Solving for A and B
Adding: $\frac{P+Q}{2} = A$ ..... (3)
Subtracting: $\frac{P-Q}{2} = B$..... (4)
Substituting eqn. (1) to (4) in to the above factor formulae, then

$\mathbf{cos}P + \mathbf{cos}Q = \mathbf{2cos}\frac{P+Q}{2}\mathbf{cos}\frac{P-Q}{2}$	$\mathbf{sin}P + \mathbf{sin}Q = \mathbf{2sin}\frac{P+Q}{2}\mathbf{cos}\frac{P-Q}{2}$
$\mathbf{cos}P - \mathbf{cos}Q = -\mathbf{2sin}\frac{P+Q}{2}\mathbf{sin}\frac{P-Q}{2}$	$\mathbf{sin}P - \mathbf{sin}Q = \mathbf{2cos}\frac{P+Q}{2}\mathbf{sin}\frac{P-Q}{2}$

Application

- Solution to trigonometric equations
- Proof of identities
- Factorization and simplifying trigonometric equations

Solution to trigonometric equations

1. *Solve for* $0^0 \leq x \leq 360^0$ *the equations*

i) $sin3x + sinx = 0$

ii) $cosx + cos5x = 0$

iii) $cos4x - cosx = 0$

iv) $sin(x + 10^0) + sinx = 0$

v) $cos(x + 20^0) - cos(x - 70^0) = 0$

vi) $cos(2x + 10^0) + cos(2x - 10^0) = 0$

Solution

i) $sin3x + sinx = 0$

$2sin\left(\frac{3x+x}{2}\right)cos\left(\frac{3x-x}{2}\right) = 0 \Rightarrow 2sin2xcosx = 0$

$\Rightarrow sin2x = 0, cosx = 0$ $[\because a.b = 0\ iff\ a = 0, b = 0]$

Dealing with $sin2x = 0$

$2x = 0^0, 180^0, 360^0, 540^0, 900^0$

$\therefore x = 0^0, 90^0, 180^0 270^0, 450^0$

Dealing with $cosx = 0$

$\therefore x = 90^0, 270^0$

$\therefore x = 0^0, 90^0, 180^0, 270^0$

ii) $cosx + cos5x = 0$

$2cos\left(\frac{x+5x}{2}\right)cos\left(\frac{x-5x}{2}\right) \Rightarrow 2cos3xcos(-2x) = 0$

$\therefore cos3xcos2x = 0$ $[\because \cos(-2x) = cos2x] \Rightarrow cos3x = 0, cos2x = 0$

Dealing with $cos3x = 0$

$\Rightarrow 3x = 90^0, 270^0, 450^0, 540^0, 630^0, 810^0, 900^0, 990^0$

$\therefore x = 30^0, 90^0, 150^0, 180^0, 210^0, 270^0, 300^0, 330^0$

Dealing with $cos2x = 0$

$\Rightarrow 2x = 90^0, 270^0, 450^0, 540^0$

$\therefore x = 45^0, 135^0, 225^0, 270^0$

$\therefore x = 30^0, 45^0, 90^0, 135^0, 150^0, 180^0, 210^0, 225^0, 270^0, 300^0, 315^0, 330^0$

iii) Left as an exercise ***Answer***: $0^0, 72^0, 120^0, 144^0, 216^0, 240^0, 288^0, 360^0$

iv) $sin(x + 10^0) + sinx = 0$

$2\sin\left(\frac{x+10+x}{2}\right)\cos\left(\frac{x+10-x}{2}\right) = 0$ …… $(\div 2)$

$\Rightarrow \sin(x + 5^0)\, cos5^0 = 0$ ……$(\div cos5^0)$

$\Rightarrow \sin(x + 5^0) = 0 \ \ \therefore (x + 5^0) = 0^0, 180^0, 360^0$

$\therefore x = 175^0, 355^0$ $[reject -5^0 \ \ \because it\ is\ out\ or\ range]$

v) *Left as an exercise* ***Answer***: $\mathbf{25^0, 205^0}$

vi) *Left as an exercise* ***Answer***: $\mathbf{45^0, 135^0, 225^0, 315^0}$

2. *Solve the equations for values of* θ *from* 0^0 *to* 180^0 *inclusive.*

a. $cos\theta + cos3\theta + cos5\theta = 0$

b. $sin\theta - 2sin2\theta + sin3\theta = 0$

c. $sin\theta + cos2\theta - sin3\theta = 0$

d. $\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

Solution

This set of numbers is solved by factorization. Group any two trig ratios, apply factor formulae. If the third/ remaining trig ratio is a factor of the two ten factorize.

a. $cos\theta + cos3\theta + cos5\theta = 0$

Here we can group either group $(cos\theta + cos5\theta)$or $(cos3\theta + cos5\theta)$

$\Rightarrow cos\theta + (cos3\theta + cos5\theta) = 0$

$\Rightarrow cos\theta + 2cos4\theta\cos(-\theta) = 0\ or\ cos\theta[1 + 2cos4\theta] = 0\ [\because cos(-\theta) = cos\theta]$

$cos\theta = 0\ or\ 1 + 2cos4\theta = 0$

Dealing with $cos\theta = 0$

$\Rightarrow \theta = 90^0, 270^0$

Dealing with $cos4\theta = -\frac{1}{2}$

$\Rightarrow 4\theta = 120^0, 240^0, 480^0, 600^0, 840^0, 960^0, 1200^0, 1320^0$

$\Rightarrow \theta = 30^0, 60^0, 120^0, 150^0, 210^0, 240^0, 300^0, 330^0$

$\therefore \theta = 30^0, 60^0, 90^0, 120^0, 150^0$

b. $sin\theta - 2sin2\theta + sin3\theta = 0$

$sin\theta + sin3\theta - 2sin2\theta = 0 \Rightarrow 2sin2\theta\cos(-\theta) - 2sin2\theta = 0$

$2sin2\theta[cos\theta - 1] = 0$. From which $sin2\theta = 0\ or\ cos\theta - 1 = 0 \Rightarrow cos\theta = 1$

Dealing with $cos\theta = 1$

$\Rightarrow \theta = 0^0, 360^0$

Dealing with $sin2\theta = 0$

$\Rightarrow 2\theta = 0^0, 180^0, 360^0\ \ or\ \theta = 0^0, 90^0, 180^0$

$\therefore \theta = 0^0, 90^0, 180^0$

c. $left\ as\ an\ exercise\ \boldsymbol{Answer}: \mathbf{30^0, 45^0, 135^0, 150^0}$

d. $\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

$\left(\cos\frac{1}{2}\theta + \cos\frac{5}{2}\theta\right) + 2\cos\frac{3}{2}\theta = 0 \Rightarrow 2\cos\frac{3}{2}\theta\cos\left(-\frac{1}{2}\theta\right) + 2\cos\frac{3}{2}\theta = 0$

$2\cos\frac{3}{2}\theta\left[\cos\frac{1}{2}\theta + 1\right] = 0$ From which $\cos\frac{3}{2}\theta = 0\ or\ \cos\frac{1}{2}\theta + 1 = 0\ \Rightarrow$

$\cos\frac{1}{2}\theta = -1$

Dealing with $\cos\frac{3}{2}\theta = 0$

$\Rightarrow \frac{3}{2}\theta = 90^0, 270^0\ or\ \theta = \frac{2}{3}\times(90^0, 270^0)$

$\Rightarrow \theta = 60^0, 180^0$

Dealing with $\cos\frac{1}{2}\theta = 1$

$\Rightarrow \frac{1}{2}\theta = 180^0\ or\ \theta = 360^0$

$\therefore \theta = 60^0, 180^0$ $[reject\ 360^0\ \because it\ is\ out\ of\ range]$

NOTE: Factor formula is applied to *two trig ratios with the same coefficient.*

3. *Solve for* $0^0 < \theta < 360^0$ *the equation*

a. $sin3\theta + 2sin5\theta sin^2\theta + sin7\theta = 0$

b. $sin\theta + sin7\theta = sin4\theta$

c. $cos3\theta = cos(2\theta + 45^0)$

d. $2sin^2(\theta - 30^0) = cos2\theta + \frac{1}{2}$

Solution

a. $(a)\ and\ (b)\ left\ as\ an\ exercise$ $\boldsymbol{Answer}: (\boldsymbol{a})\mathbf{0^0, 36^0, 72^0, 90^0, 108^0, 144^0, 180^0, 216^0, 252^0\ 270^0, 288^0, 324^0}$

c. $cos3\theta = cos(2\theta + 45^0)$

$cos3\theta - cos(2\theta + 45^0) = 0 \Rightarrow -2sin\left(\frac{5\theta+45^0}{2}\right) sin\left(\frac{\theta-45^0}{2}\right) = 0$ From which

$sin\left(\frac{5\theta+45^0}{2}\right) = 0\ or\ sin\left(\frac{\theta-45^0}{2}\right) = 0$

Dealing with $sin\left(\frac{\theta-45^0}{2}\right) = 0 \Rightarrow \left(\frac{\theta-45^0}{2}\right) = 0^0, 180^0, 360^0 \ \therefore \theta = 45^0, 315^0$

Dealing with $sin\left(\frac{5\theta+45^0}{2}\right) = 0 \Rightarrow \left(\frac{5\theta+45^0}{2}\right) = 0^0, 180^0, 360^0, 540^0, 720^0, 900^0$

$\Rightarrow \boldsymbol{\theta} = \frac{[2\times(0^0, 180^0, 360^0, 540^0, 720^0, 900^0) - 45^0]}{\mathbf{5}} \ \therefore \theta = 63^0, 135^0, 207^0, 279^0, 351^0$

$\therefore \boldsymbol{\theta} = 45^0, 63^0, 135^0, 207^0, 279^0, 315^0, 351^0$

d. $2\sin^2(\theta - 30^0) = cos2\theta + \frac{1}{2}$

$\sin^2(\theta - 30^0) = \frac{1-cos\,2(\theta-30^0)}{2} = \frac{1-cos\,(2\theta-60^0)}{2}$

$\Rightarrow \frac{2.[1-cos\,(2\theta-60^0)]}{2} - cos2\theta = \frac{1}{2}\ or\ 1 - cos(2\theta - 60^0) - cos2\theta = \frac{1}{2}$

$cos(2\theta - 60^0) + cos2\theta = \frac{1}{2} \Rightarrow 2cos(2\theta - 30^0)cos(-30^0) = \frac{1}{2}$

$2.\frac{\sqrt{3}}{2}.cos(2\theta - 30^0) = \frac{1}{2} \Rightarrow cos(2\theta - 30^0) = \frac{1}{2\sqrt{3}}\ taking + ve\ root$

$2\theta - 60^0 = 73.2^0, 286.8^0, 433.2^0, 646.8^0 \Rightarrow 2\theta = (73.2^0, 286.8^0, 433.2^0, 646.8^0) + 60^0$

$\therefore \theta = \frac{(73.2^0, 286.8^0, 433.2^0, 646.8^0)+60^0}{2} = 51.6^0, 158.4^0, 231.6^0, 338.4^0$

Proof of identities

1. *Prove the following identities*

a. $cos^2A - cos^2B = sin(A + B)sin(B - A)$

b. $\frac{cosB + cosC}{sinB - sinC} = \cot\frac{B-C}{2}$

c. $\frac{SinB - sinC}{sinB + sinC} = \cot\frac{B+C}{2}\tan\frac{B-C}{2}$

d. $\frac{cos\,7t + cos\,5t}{sin\,7t - sin\,5t} = \frac{cosect}{sect}$

e. $\frac{sin\,4x + sin\,2x}{cos\,4x + cos\,2x} = \frac{1}{cot\,3x}$

f. $\frac{\sin 3x-\sin x}{\cos 3x+\cos x}=\tan x$

g. $\frac{\sin 3t+\sin t}{\sin 3t-\sin t}=\frac{2}{1-\tan^2 t}$

h. $\frac{\cos 9x+\cos 5x}{\sin 9x-\sin 5x}=\frac{1-\tan^2 x}{2\tan x}$

Solution

a. $\cos^2 A-\cos^2 B=\sin(A+B)\sin(B-A)$

$\mathbf{L.H.S}: \cos^2 A-\cos^2 B=(\cos A+\cos B)(\cos A-\cos B)$

$=\left[2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right].\left[-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$

$=-\left[2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A+B}{2}\right)\right].\left[2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$

$=-\sin\left[2\left(\frac{A+B}{2}\right)\right].\sin\left[2\left(\frac{A-B}{2}\right)\right]$

$=-\sin(A+B)\sin(A-B)=\sin(A+B)\sin(B-A)\quad[\because \sin(-\theta)=-\sin\theta]$

b. $\frac{\cos B+\cos C}{\sin B-\sin C}=\cot\frac{B-C}{2}$

$\mathbf{L.H.S}: \frac{\cos B+\cos C}{\sin B-\sin C}=\frac{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}=\frac{\cos\frac{B-C}{2}}{\sin\frac{B-C}{2}}=\cot\frac{B-C}{2}=\mathbf{R.H.S}$

c. $\frac{\sin B-\sin C}{\sin B+\sin C}=\cot\frac{B+C}{2}\tan\frac{B-C}{2}$

$\mathbf{L.H.S}: \frac{\sin B-\sin C}{\sin B+\sin C}=\frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}=\frac{\cos\frac{B+C}{2}}{\sin\frac{B+C}{2}}.\frac{\sin\frac{B-C}{2}}{\cos\frac{B-C}{2}}=\cot\frac{B+C}{2}\tan\frac{B-C}{2}=\mathbf{R.H.S}$

d. $\frac{\cos 7t+\cos 5t}{\sin 7t-\sin 5t}=\frac{\cos ect}{\sec t}$

$\mathbf{L.H.S}: \frac{\cos 7t+\cos 5t}{\sin 7t-\sin 5t}=\frac{2\cos\left(\frac{7t+5t}{2}\right)\cos\left(\frac{7t-5t}{2}\right)}{\cos\left(\frac{7t+5t}{2}\right)\sin\left(\frac{7t-5t}{2}\right)}=\frac{\cos 6t\cos t}{\cos 6t\sin t}=\frac{\cos ect}{\sec t}=\mathbf{R.H.S}$

e. $(e),(f)\ left\ as\ an\ exercise$

g. $\frac{\sin 3t+\sin t}{\sin 3t-\sin t}=\frac{2}{1-\tan^2 t}$

$\mathbf{L.H.S}: \frac{\sin 3t+\sin t}{\sin 3t-\sin t}=\frac{2\sin\left(\frac{3t+t}{2}\right)\cos\left(\frac{3t-t}{2}\right)}{\cos\left(\frac{3t+t}{2}\right)\sin\left(\frac{3t-t}{2}\right)}=\frac{\sin 2t\cos t}{\cos 2t\sin t}=\frac{2\sin t\cos t\ .\cos t}{\frac{1-\tan^2 t}{1+\tan^2 t}.\sin t}\quad\left[\begin{matrix}\because \sin 2t=2\cos t\sin t\\ \cos 2t=\frac{1-\tan^2 t}{1+\tan^2 t}\end{matrix}\right]$

$=\frac{2(1+\tan^2 t).\cos^2 t}{1-\tan^2 t}=\frac{2\sec^2 t.\cos^2 t}{1-\tan^2 t}=\frac{2\left(\frac{1}{\cos^2 t}.\cos^2 t\right)}{1-\tan^2 t}=\frac{2}{1-\tan^2 t}=\mathbf{R.H.S}$

h. $\frac{\cos 9x+\cos 5x}{\sin 9x-\sin 5x}=\frac{1-\tan^2 x}{2\tan x}$

$\mathbf{L.H.S}: \frac{\cos 9x+\cos 5x}{\sin 9x-\sin 5x}=\frac{2\cos\frac{9x+5x}{2}\cos\frac{9x-5x}{2}}{2\cos\frac{9x+5x}{2}\sin\frac{9x-5x}{2}}=\frac{\cos 7x\cos 2x}{\cos 7x\sin 2x}=\frac{\cos 2x}{\sin 2x}$

$=\frac{\left(\frac{1-\tan^2 t}{1+\tan^2 t}\right)}{2\tan x/(1-\tan^2 t)}=\frac{1-\tan^2 x}{2\tan x}=\mathbf{R.H.S}$

2. *Prove the following identities*

a. $\frac{\sin x+\sin 2x+\sin 3x}{\sin 2x}=2\cos x+1$

b. $\cos x+\sin 2x-\cos 3x=\sin 2x(2\sin x+1)$

c. $\frac{\cos 3\theta+\cos 5\theta+\cos 7\theta}{\cos 5\theta}=2\cos\theta+1$

d. $1+2\cos 2\theta+\cos 4\theta=4\cos^2\theta\cos 2\theta$

e. $\sin x-\sin(x+60^0)+\sin(x+120^0)=0$

f. $\cos x+\cos(x+120^0)+\cos(x+240^0)=0$

g. $sin\alpha + sin\left(\alpha + \frac{2\pi}{3}\right) + sin\left(\alpha + \frac{4\pi}{3}\right) = 0$

h. $cos\theta + cos\left(\theta + \frac{2\pi}{3}\right) + cos\left(\theta + \frac{4\pi}{3}\right) = 0$

Solution

a. $\frac{sinx + sin2x + sin3x}{sin2x} = 2cosx + 1$

Since $sin2x$ is a factor of$(sinx + sin3x)$, then

L.H.S: $\frac{sinx + (sin2x + sin3)x}{sin2x} = \frac{sin2x + (2sin2xcosx)}{sin2x} = \frac{sin2x[1 + 2cos2x]}{sin2x} = 2cosx + 1 = \mathbf{R.H.S}$

b. $cosx + sin2x - cos3x = sin2x(2sinx + 1)$

L.H.S: $(cosx - cos3x) + sin2x = -2sin2xsin(-x) + sin2x$

$= 2sin2xsinx + sin2x = sin2x(2sinx + 1) = \mathbf{R.H.S}$

c. Left as an exercise

d. $1 + 2cos2\theta + cos4\theta = 4cos^2\theta cos2\theta$

L.H.S: $1 + 2cos2\theta + cos4\theta = (1 + cos4\theta) + 2cos2\theta$

But $1 = cos0^0 = cos360^0$, Replacing 1 with $cos0^0$ to allow usage of factor formulae

$\Rightarrow 1 + 2cos2\theta + cos4\theta = (cos0 + cos4\theta) + 2cos2\theta = 2cos2\theta\cos(-2\theta) + 2cos2\theta$

$= 2cos2\theta(cos2\theta + 1) = 2cos2\theta(2cos^2\theta - 1 + 1) = 4cos^2\theta cos2\theta = \mathbf{R.H.S}$

e. $\sin x - \sin(x + 60^0) + \sin(x + 120^0) = 0$

Combining $sinx + sin(x + 120^0)$ to give $\sin(x + 60^0)$ as a factor

L.H.S: $sinx - sin(x + 60^0) + sin(x + 120^0)$

$= [sinx + sin(x + 120^0)] - sin(x + 60^0)$

$= 2\,sin(x + 60^0)\,cos(-60^0) - sin(x + 60^0)$

$= sin(x + 60^0) = [2cos60^0 - 1] = 0$

f. $Left\ as\ an\ exercise$

g. $sin\alpha + sin\left(\alpha + \frac{2\pi}{3}\right) + sin\left(\alpha + \frac{4\pi}{3}\right) = 0$

Re-writing in terms of degrees

$sin\alpha + sin(\alpha + 120^0) + sin(\alpha + 240^0) = 0$

$\Rightarrow sin\alpha + sin(\alpha + 120^0) + sin(\alpha + 240^0) = [sin\alpha + sin(\alpha + 240^0)] + sin(\alpha + 120^0)$

$= 2sin(\alpha + 120^0)\cos(-120^0) + sin(\alpha + 120^0) = sin(\alpha + 120^0)[2cos(-120^0) + 1] = 0$

3. *Prove the following identities*

a. $\frac{sin3Asin6A + sinAsin2A}{sin3Acos6A + sinAcos2A} = tan5A$

b. $\frac{sin3\theta cos4\theta - sin\theta cos2\theta}{sin\theta sin4\theta + cos\theta cos6\theta} = tan2\theta$

c. $\frac{sin11AsinA + sin7Asin3A}{cos11AsinA + cos7Asin3A} = tan8A$

d. $\frac{sin\theta sin2\theta + sin3\theta sin6\theta + sin4\theta sin13\theta}{sin\theta cos2\theta + sin3\theta cos6\theta + sin4\theta cos13\theta} = tan9\theta$

Solution

a. $\frac{sin3Asin6A + sinAsin2A}{sin3Acos6A + sinAcos2A} = tan5A$

Here express the product of trig ratios as a sum or difference of cosines/sines using the factor formulae.
The factor formulae to be used are;

If A and B are real(*measures of angles*), then

i) $\cos(A+B)+\cos(A-B)=2\cos A\cos B$
ii) $\cos(A+B)-\cos(A-B)=-2\sin A\sin B$
iii) $\sin(A+B)+\sin(A-B)=2\sin A\cos B$
iv) $\sin(A+B)-\sin(A-B)=2\cos A\sin B$

Using $cos(A+B)-cos(A-B)=-2sinAsinB\ or\ sinAsinB$

$$=-\frac{1}{2}[cos(A+B)-cos(A-B)]$$

$$\Rightarrow sin3Asin6A=-\frac{1}{2}[cos(3A+6A)-cos(3A-6A)]=-\frac{1}{2}[cos9A-cos(-3A)]$$

$$=-\frac{1}{2}[cos9A-cos3A]$$

$$\Rightarrow\ sinAsin2A=-\frac{1}{2}[cos(A+2A)-cos(A-2A)]=-\frac{1}{2}[cos3A-cos(-A)]$$

$$=-\frac{1}{2}[cos3A-cosA]$$

Using $sin(A+B)+sin(A-B)=2sinAcosB\ or\ sinAcosB$

$$=\frac{1}{2}[sin(A+B)+sin(A-B)]$$

$$\Rightarrow sin3Acos6A=\frac{1}{2}[sin(3A+6A)+sin(3A-6A)]=\frac{1}{2}[sin9A+sin(-3A)]$$

$$=\frac{1}{2}[sin9A-sin3A]\qquad[\because \sin(-3A)=-sin3A]$$

$$\Rightarrow sinAcos2A=\frac{1}{2}[sin(A+2A)+sin(A-2A)]=\frac{1}{2}[sin3A+sin(-A)]$$

$$=\frac{1}{2}[sin3A-sinA]$$

$$\therefore\ \frac{sin\,3Asin\,6A+sinAsin\,2A}{sin\,3Acos\,6A+sinAcos\,2A}=\frac{-\frac{1}{2}[cos\,9A-cos\,3A]+-\frac{1}{2}[cos\,3A-cosA\,]}{\frac{1}{2}[sin\,9A-sin\,3A]+\frac{1}{2}[sin\,3A-sinA\,]}=\frac{-cos\,9A+cos\,3A-cos\,3A+cosA}{sin\,9A-sin\,3A+sin\,3A-sinA}$$

$$=\frac{cosA-cos\,9A}{sin\,9A-sinA}=\frac{-2sin\,5Asin\,(-4A)}{2cos\,5Asin\,4A}=\frac{2sin\,5Asin\,4A}{2cos\,5Asin\,4A}=tan5A$$

b. $\frac{sin\,3\theta cos\,4\theta-sin\theta cos\,2\theta}{sin\theta sin\,4\theta+cos\theta cos\,6\theta}=tan2\theta$

L.H.S: $\frac{sin\,3\theta cos\,4\theta-sin\theta cos\,2\theta}{sin\theta sin\,4\theta+cos\theta cos\,6\theta}=\frac{\frac{1}{2}[sin\,7\theta+sin\,(-\theta)]-\frac{1}{2}[sin\,3\theta+sin\,(-\theta)]}{-\frac{1}{2}[cos\,5\theta-cos\,(-3\theta)]+\frac{1}{2}[cos\,7\theta+cos\,(-5\theta)]}$

$$=\frac{sin\,7\theta-sin\theta\ -sin\,3\theta+sin\theta}{cos\,3\theta-cos\,5\theta+cos\,7\theta+cos\,5\theta}=\frac{sin\,7\theta-sin\,3\theta}{cos\,7\theta+cos\,3\theta}=\frac{2cos\,5\theta sin\,2\theta}{2cos\,5\theta cos\,2\theta}=\frac{sin\,2\theta}{cos\,2\theta}=tan2\theta$$

c. $left\ as\ an\ exercise$

d. $\frac{sin\theta sin\,2\theta+sin\,3\theta sin\,6\theta+sin\,4\theta sin\,13\theta}{sin\theta cos\,2\theta+sin\,3\theta cos\,6\theta+sin\,4\theta cos\,13\theta}=tan9\theta$

L.H.S: $=\frac{-\frac{1}{2}[cos\,3\theta-cos\,(-\theta)]+-\frac{1}{2}[cos\,9\theta-cos\,(-3\theta)]+-\frac{1}{2}[cos\,17\theta-cos\,(-9\theta)]}{\frac{1}{2}[sin\,3\theta+sin\,(-\theta)]+\frac{1}{2}[sin\,9\theta+sin\,(-3\theta)]+\frac{1}{2}[sin\,17\theta+sin\,(-9\theta)]}$

$$=\frac{-cos\,3\theta+cos\theta\ -cos\,9\theta+cos\,3\theta-cos\,17\theta+cos\,9\theta}{sin\,3\theta-sin\theta\ +sin\,9\theta-sin\,3\theta+sin\,17\theta-sin\,9\theta}=\frac{cos\theta\ -cos\,17\theta}{sin\,17\theta-sin\theta}$$

$$=\frac{-2sin\,9\theta sin\,(-8\theta)}{2cos\,9\theta sin\,8\theta}=\frac{sin\,9\theta}{cos\,9\theta}.\frac{sin\,8\theta}{sin\,8\theta}=tan9\theta$$

PROOF OF IDENTITIES INVOLVING ANGLES OF A TRIANGLE
Example

1. *If A, B, and C are angles of a triangle, prove that*

a. $cosA + cosB + cosC - 1 = 4\,sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$

b. $sinA + sinB + sinC = 4\,cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}$

c. $sin2A + sin2B + sin2C = 4sinAsinBsinC$

d. $sin2A + sin2B - sin2C = 4cosAcosBsinC$

e. $\frac{cosA}{sinBsinC} + \frac{cosB}{sinCsinA} + \frac{cosC}{sinAsinB} = 2$

f. $\sin(B + C - A) + \sin(C + A - B) - \sin(A + B - C) = 4cosAcosBsinC$

g. $cos2A + cos2B + cos2C = -1 - 4cosAcosBcosC$

h. $cos4A - cos4B - cos4C = 4sin2Bsin2Ccos2A - 1$ (***Uneb* 2012**)

i. $cosA + cosB - cosC = -1 + 4\,cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$

j. $\frac{sin\,2A + sin\,2B + sin\,2C}{cosA + cosB + cosC - 1} = 8\,cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}$

k. $tanA + tanB + tanC = tanAtanBtanC$

l. $cot\frac{A}{2} + cot\frac{B}{2} + cot\frac{c}{2} = cot\frac{A}{2}cot\frac{B}{2}cot\frac{C}{2}$

m. $tan\frac{A}{2}tan\frac{B}{2} + tan\frac{B}{2}tan\frac{C}{2} + tan\frac{C}{2}tan\frac{A}{2} = 1$

n. $\frac{\cot A + \cot B}{tanA + tanB} + \frac{\cot B + \cot C}{tanB + tanC} + \frac{\cot C + \cot A}{tan\,C + tan\,A} = 1$

Solution

a. $cosA + cosB + cosC - 1 = 4\,sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$

L.H.S: $cosA + cosB + cosC - 1 = (cosA + cosB) + (cosC - 1)$

$= 2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right) + (cosC - cos0^0) = 2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right) + -2\,sin\frac{C}{2}sin\frac{C}{2}$

$= 2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right) - 2\,sin\frac{C}{2}sin\frac{C}{2}$, but $A + B = 180^0 - C \Rightarrow \frac{A+B}{2} = 90^0 - \frac{c}{2}$

$\mathbf{L.H.S} = 2cos\left(90^0 - \frac{c}{2}\right)cos\left(\frac{A-B}{2}\right) - 2\,sin\frac{C}{2}sin\frac{C}{2}$,

But $cos\left(90^0 - \frac{c}{2}\right) = cos90^0\,cos\frac{C}{2} + sin90^0\,sin\frac{C}{2} = sin\frac{C}{2}$

$\Rightarrow \mathbf{L.H.S} = 2\,sin\frac{C}{2}\left[cos\left(\frac{A-B}{2}\right) - sin\frac{C}{2}\right]$, but $A + B + C = 180^0$, $\frac{C}{2} = 90^0 - \left(\frac{A+B}{2}\right)$

$\Rightarrow \mathbf{L.H.S} = 2\,sin\frac{C}{2}\left[cos\left(\frac{A-B}{2}\right) - sin\left[90^0 - \left(\frac{A+B}{2}\right)\right]\right]$,

But $sin\left[90^0 - \left(\frac{A+B}{2}\right)\right] = sin90^0cos\left(\frac{A+B}{2}\right) - cos90^0sin\left(\frac{A+B}{2}\right) = cos\left(\frac{A+B}{2}\right)$

$\Rightarrow \mathbf{L.H.S} = 2\,sin\frac{C}{2}\left[cos\left(\frac{A-B}{2}\right) - cos\left(\frac{A+B}{2}\right)\right] 2\,sin\frac{C}{2}\left[-2sin\left(\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\right)sin\left(\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\right)\right]$

$= 4\,sin\frac{C}{2}\left[-sin\frac{A}{2}sin\left(-\frac{B}{2}\right)\right] = 4\,sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$

b. $sinA + sinB + sinC = 4\,cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}$

L.H.S: $(sinA + sinB) + sinC = 2sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right) + 2\,sin\frac{C}{2}sin\frac{C}{2}$

$\left[\because sinC = sinC + sin0 = 2\,sin\frac{C}{2}sin\frac{C}{2}\right]$, but $A+B + C = 180^0$, $\frac{C}{2} = 90^0 - \left(\frac{A+B}{2}\right)$

$\therefore sin\left(\frac{A+B}{2}\right) = sin\left(90^0 - \frac{c}{2}\right) = sin90^0\, cos\frac{C}{2} - cos90^0\, sin\frac{C}{2} = cos\frac{C}{2}$

$\Rightarrow$ **L.H.S** $= 2\,cos\frac{C}{2}\left[cos\left(\frac{A-B}{2}\right) + sin\frac{C}{2}\right]$, also $sin\frac{C}{2} = sin\left[90^0 - \left(\frac{A+B}{2}\right)\right] = cos\left(\frac{A+B}{2}\right)$

$\Rightarrow$ **L.H.S** $= 2\,cos\frac{C}{2}\left[cos\left(\frac{A-B}{2}\right) + cos\left(\frac{A+B}{2}\right)\right]$

$= 2\,cos\frac{C}{2}\left[2cos\left(\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\right)cos\left(\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\right)\right]$

$= 4\,cos\frac{C}{2}\left[cos\frac{A}{2}cos\left(-\frac{B}{2}\right)\right] = 4\,cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}$

c. $sin2A + sin2B + sin2C = 4sinAsinBsinC$

L.H.S: $(sin2A + sin2B) + sin2C = 2sin(A+B)cos(A-B) + 2cosCsinC$

But $sin(A+B) = si\,n(180^0 - C) = sin180^0cosC - cos180^0sinC = sinC$

$\therefore sin2A + sin2B + sin2C = 2sinCcos(A-B) + 2cosCsinC$

$= 2sinC[cos(A-B) + cosC]$

$But\ C = 180^0 - (A+B) \Rightarrow cosC = cos\big(180^0 - (A+B)\big)cos180^0cos(A+B) + sin180^0cos(A+B)$

L.H.S $= 2sinC[cos(A-B) - cos(A+B)]$

$= 2sinC\left[-2sin\left(\frac{A+B+A-B}{2}\right)sin\left(\frac{A+B-A+B}{2}\right)\right]$

$= 2sinC[-2sinAsin(-B)] = 4sinAsinBsinC$

d. $left\ as\ an\ exercise$

e. $\frac{cosA}{sinBsinC} + \frac{cosB}{sinCsinA} + \frac{cosC}{sinAsinB} = 2$

L.H.S: $\frac{cosA}{sinBsinC} + \frac{cosB}{sinCsinA} + \frac{cosC}{sinAsinB} = \frac{cosAsinA\ + cosBsinB\ + cosCsinC}{sinA\,sinBsinC}$

Multiplying Numerator and Denominator by 2 to create double angles of sine ratios

L.H.S $= \frac{2cosAsinA\ + 2cosBsinB\ + 2cosCsinC}{2sinAsinBsinC} = \frac{sin\,2A + sin\,2B + sin\,2C}{2sinAsinBsinC}$

Simplifying Numerator just like in (c) above, then $sin2A + sin2B + sin2C = 4sinAsinBsinC$

$\therefore$ **L.H.S** $= \frac{4sinAsinBsinC}{2sinAsinBsinC} =$ **2**

f. $sin(B + C - A) + sin(C + A - B) - sin(A + B - C) = 4cosAcosBsinC$

From $A + B + C = 180^0 \Rightarrow A + B = 180 - C, C + A = 180 - B,\ B + C = 180 - A$

$sin(B + C - A) = sin(180 - 2A) = sin180cos2A - cos180sin2A = sin2A$

$sin(C + A - B) = sin(180 - 2B) = sin180cos2B - cos180sin2B = sin2B$

$sin(A + B - C) = sin(180 - 2C) = sin180cos2C - cos180sin2C = sin2C$

$\therefore$ **L.H.S** $= sin2A + sin2B - sin2C = 2sin(A+B)cos(A-B) - 2cosCsinC$

But $sin(A+B) = sin(180^0 - C) = sin180^0cosC - cos180^0sinC$

$= sinC$

$Also\ C = 180^0 - (A+B)$

$\Rightarrow cosC = cos\big(180^0 - (A+B)\big) = cos180^0cos(A+B) + sin180^0cos(A+B)$

$= -cos(A+B)$

$\therefore\ sin2A + sin2B - sin2C = 2sinC[cos(A-B) + cos(A+B)]$

$= 2sinC[2cocsAcos(-B)]$

$= 4cosAcosBsinC$

g. $left\ as\ an\ exercise$

h. $cos4A - cos4B - cos4C = 4sin2Bsin2Ccos2A - 1$

$\mathbf{L.H.S} = -2sin(2A + 2B)sin(2A - 2B) + (cos0 - cos4A) - 1[\because cos0 - 1 = 1 - 1]$

$But\ A + B + C = 180^0 \Rightarrow 2A + 2B = 360^0 - 2C$

$sin(2A + 2B) = sin(360^0 - 2C) = sin360cos2C - cos360sin2C = -sin2C$

$\mathbf{L.H.S} = -[-2sin2Csin(2A - 2B)] + -2sin2Csin(-2C) - 1$

$= 2sin2C[sin(2A - 2B) + sin2C] - 1 = 2sin2C[(sin(2A - 2B) + sin2C)] - 1$

$= 2sin2C[2sin(A - B + C)cos(A - B - C)] - 1 = 2sin2C[2sin(180^0 - 2B)cos(2A - 180^0)]$

$But\ sin(180^0 - 2B) = sin180cos2B + cos180sin2B = -sin2B,$

$cos(2A - 180^0) = cos2Acos180 - sin180sin2A = -cos2A$

$\therefore \mathbf{L.H.S} = 2sin2C[-2sin2B.-cos2A] - 1 = 4sin2Bsin2Ccos2A - 1$

i. $cosA + cosB - cosC = -1 + 4\,cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$

$\mathbf{L.H.S} = 2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right) - cosC + (cos0 - 1)$

$= 2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right) + (cos0 - cosC) - 1$

$= 2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right) + \left[-2sin\frac{C}{2}sin\left(-\frac{C}{2}\right)\right] - 1$

$= 2cos\left(90^0 - \frac{c}{2}\right)cos\left(\frac{A-B}{2}\right) + 2sin\frac{C}{2}sin\frac{C}{2} - 1$

$= 2sin\frac{C}{2}cos\left(\frac{A-B}{2}\right) + 2sin\frac{C}{2}sin\frac{C}{2} - 1$

$= 2sin\frac{C}{2}\left[cos\left(\frac{A-B}{2}\right) + sin\frac{C}{2}\right] - 1,\ but\ sin\frac{C}{2} = sin\left(90^0 - \frac{A+B}{2}\right)$

$= cos\left(\frac{A+B}{2}\right)$

$\mathbf{L.H.S} = 2sin\frac{C}{2}\left[cos\left(\frac{A-B}{2}\right) + cos\left(\frac{A+B}{2}\right)\right] - 1$

$= 2sin\frac{C}{2}\left[2cos\left(\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\right)cos\left(\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\right)\right]$

$= 2sin\frac{C}{2}\left[2cos\frac{A}{2}cos\left(-\frac{B}{2}\right)\right] = 4cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$

j. $\frac{sin\,2A+sin\,2B+sin\,2C}{cosA+cosB+cosC-1} = 8\,cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}$

$\mathbf{L.H.S} = \frac{sin\,2A+sin\,2B+sin\,2C}{cosA+cosB+cos\,C-1} = \frac{4sinAsinBsinC}{2\,sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}}$ $\begin{bmatrix} proof\ of sin2A + sin2B + sin2C \\ and\ \ cosA + cosB + cosC - 1\ are \\ left\ for\ the\ students\ \ or\ refer\ to \\ (c)\ and\ (a)\ above \end{bmatrix}$

$\mathbf{L.H.S} = \frac{4\left[2\,sin\frac{A}{2}cos\frac{A}{2}\right]\left[2\,sin\frac{B}{2}cos\frac{B}{2}\right]\left[2\,sin\frac{C}{2}cos\frac{C}{2}\right]}{2\,sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}} = 8\,cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}$

k. $tanA + tanB + tanC = tanAtanBtanC$

$A + B + C = 180^0\ or\ A + B = 180^0 - C, Now\ introducing\ tangent\ ratio$

$tan(A + B) = tan(180^0 - C) \Rightarrow \frac{tanA+tanB}{1-tanAtanB} = \frac{tan\,180^0-tanC}{1+tan\,180^0tanC} = -tanC$

$[\because tan180^0 = 0]$

$tanA + tanB = -tanC + tanAtanBtanC \ \therefore tanA + tanB + tanC = tanAtanBtanC$

1. $cot\frac{A}{2} + cot\frac{B}{2} + cot\frac{c}{2} = cot\frac{A}{2}cot\frac{B}{2}cot\frac{C}{2}$

Again

$A + B + C = 180^0 \ or \frac{A}{2} + \frac{B}{2} = 90^0 - \frac{C}{2}$, *Now introducing cotangent ratio*

$cot\left(\frac{A}{2} + \frac{B}{2}\right) = cot\left(90^0 - \frac{C}{2}\right) \Rightarrow \frac{1}{tan\left(\frac{A}{2}+\frac{B}{2}\right)} = \frac{cos\left(90^0 - \frac{C}{2}\right)}{sin\left(90^0 - \frac{C}{2}\right)}$

$\Rightarrow \frac{1 - tan\frac{A}{2}tan\frac{B}{2}}{tan\frac{A}{2} + tan\frac{B}{2}} = \frac{cos\,90\,cos\frac{C}{2} + sin\,90\,sin\frac{C}{2}}{sin\,90\,cos\frac{C}{2} - cos\,90\,sin\frac{C}{2}} = \frac{sin\frac{C}{2}}{cos\frac{C}{2}} = tan\frac{C}{2}$

$\Rightarrow 1 - tan\frac{A}{2}tan\frac{B}{2} = tan\frac{A}{2}tan\frac{C}{2} + tan\frac{C}{2}tan\frac{B}{2} \ or \ tan\frac{A}{2}tan\frac{B}{2} + tan\frac{C}{2}tan\frac{B}{2} + tan\frac{A}{2}tan\frac{C}{2} = 1$

Dividing through by $tan\frac{A}{2}tan\frac{B}{2}\ tan\frac{C}{2}$

$\Rightarrow \frac{tan\frac{A}{2}tan\frac{B}{2}}{tan\frac{A}{2}tan\frac{B}{2}\ tan\frac{C}{2}} + \frac{tan\frac{C}{2}tan\frac{B}{2}}{tan\frac{A}{2}tan\frac{B}{2}\ tan\frac{C}{2}} + \frac{tan\frac{A}{2}tan\frac{C}{2}}{tan\frac{A}{2}tan\frac{B}{2}\ tan\frac{C}{2}} = \frac{1}{tan\frac{A}{2}tan\frac{B}{2}\ tan\frac{C}{2}}$

$\Rightarrow \frac{1}{tan\frac{C}{2}} + \frac{1}{tan\frac{A}{2}} + \frac{1}{tan\frac{B}{2}} = cot\frac{A}{2}cot\frac{B}{2}cot\frac{C}{2} \ \therefore$

$cot\frac{A}{2} + cot\frac{B}{2} + cot\frac{c}{2} = cot\frac{A}{2}cot\frac{B}{2}cot\frac{C}{2}$

m. *proceed as in* (l) *above. Left as an exercise. HINT: Introduce* $tan/\cot on\ \frac{A}{2} + \frac{B}{2} = 90^0 - \frac{C}{2}$

n. $\frac{\cot A + \cot B}{tanA + tanB} + \frac{\cot B + \cot C}{tanB + tanC} + \frac{\cot C + \cot A}{tan\ C + tan\ A} = 1$

$\mathbf{L.H.S} = \frac{\frac{1}{tanA} + \frac{1}{tanB}}{tanA + tanB} + \frac{\frac{1}{tanB} + \frac{1}{tanC}}{tanB + tanC} + \frac{\frac{1}{tanC} + \frac{1}{tanA}}{tanC + tanA}$

$= \frac{tanA + tanB}{tanAtanBtanC} + \frac{tanB + tanC}{tanAtanBtanC} + \frac{tanA + tanC}{tanAtanBtanC}$

$= \frac{tanA + tanB}{tanAtanBtanC} + \frac{tanB + tanC}{tanAtanBtanC} + \frac{tanA + tanC}{tanAtanBtanC}$

$= \frac{1}{tanAtanB} + \frac{1}{tanBtanC} + \frac{1}{tanAtanC}$ ..... (1)

From $A + B + C = 180^0 \ or \ A + B = 180^0 - C$, *Now introducing tangent ratio*

$\tan(A + B) = \tan(180^0 - C)$

$\Rightarrow \frac{tanA + tanB}{1 - tanAtanB} = \frac{tan\,180^0 - tanC}{1 + tan\,180^0 tanC} = -tanC \ [\because tan180^0 = 0]$

$tanA + tanB = -tanC + tanAtanBtanC \ \therefore tanA + tanB + tanC = tanAtanBtanC$

Dividing through by $tanAtanBtanC$

$\Rightarrow \frac{1}{tanBtanC} + \frac{1}{tanAtanC} + \frac{1}{tanAtanB} = 1$ ..... (2)

$\therefore$ (2) *in to* (1) , $\mathbf{L.H.S} = \mathbf{R.H.S}$

2. *If* $A, B, and\ C$ *are angles of a triangle, prove that*

a. $sin^2A + sin^2B - sin^2C = 2sinAsinBsinC$

b. $sin^2A + sin^2B + sin^2C = 2 + 2cosAcosBcosC$

c. $cos^2A + cos^2B + cos^2C = 1 - 2cosAcosBcosC$

d. $cos^2A + cos^2B - cos^2C = 1 - 2sinAsinBsinC$

e. $sin^2\frac{A}{2}+sin^2\frac{B}{2}-sin^2\frac{C}{2}=1-2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$

f. $sin^2\frac{A}{2}+sin^2\frac{B}{2}+sin^2\frac{C}{2}=1-2sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$

g. $cos^2\frac{A}{2}+cos^2\frac{B}{2}-cos^2\frac{C}{2}=2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$

h. $cos^2\frac{A}{2}+cos^2\frac{B}{2}+cos^2\frac{C}{2}=2\left(1+sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\right)$

Solution

When proving these identities, it is convenient to use half or double angle identities

$i.e\ sin^2A=\frac{1-cos\,2A}{2}\ and\ cos^2A=\frac{1+cos\,2A}{2}$

a. $sin^2A+sin^2B-sin^2C=2sinAsinBsinC$

$\mathbf{L.H.S}=\frac{1-cos\,2A}{2}+\frac{1-cos\,2B}{2}-\frac{1-cos\,2C}{2}=\frac{1}{2}[1-(cos2A+cos2B-cos2C)]$

$=\frac{1}{2}[1-(2cos(A+B)cos(A-B)-cos2C)]$

$=\frac{1}{2}[1-(2cos(180-C)cos(A-B)-cos2C)]$

$=\frac{1}{2}\left[1-\left(-2cosC.cos(A-B)-(2cos^2C-1)\right)\right]$

$=\frac{1}{2}[1+2cosC.cos(A-B)+2cos^2C-1]$

$=cosC.[cos(A-B)+cosC]\,,but\ cosC=cos[180^0-(A+B)]$

$\therefore sin^2A+sin^2B-sin^2C=cosC.[cos(A-B)-cos(A+B)]$

$=cosC.[-2sinAsin(-B)]=2sinAsinBsinC=\ \mathbf{R.H.S}$

b. $sin^2A+sin^2B+sin^2C=2+2cosAcosBcosC$

$\mathbf{L.H.S}=\frac{1-cos\,2A}{2}+\frac{1-cos\,2B}{2}+\frac{1-cos\,2C}{2}=\frac{1}{2}[3-(cos2A+cos2B+cos2C)]$

$=\frac{1}{2}[3-(2cos(A+B)cos(A-B)+cos2C)]$

$=\frac{1}{2}[3-(2cos(180-C)cos(A-B)+cos2C)]$

$=\frac{1}{2}\left[3-\left(-2cosC.cos(A-B)+(2cos^2C-1)\right)\right]$

$=\frac{1}{2}[3+2cosC.cos(A-B)-2cos^2C+1]$

$=2+cosC.[cos(A-B)-cosC]\,,but\ cosC=cos[180^0-(A+B)]$

$\therefore sin^2A+sin^2B+sin^2C=cosC.[cos(A-B)+cos(A+B)]$

$=2+cosC[2cosAcosB]=2+2cosAcosBcosC=\ \mathbf{R.H.S}$

c. $cos^2A+cos^2B+cos^2C=1-2cosAcosBcosC$

$\mathbf{L.H.S}=\frac{1+cos\,2A}{2}+\frac{1+cos\,2B}{2}+\frac{1+cos\,2C}{2}=\frac{1}{2}[3+cos2A+cos2B+cos2C]$

$=\frac{1}{2}[3+(2cos(A+B)cos(A-B)+2cos^2C-1)]$

$=\frac{1}{2}[3+(2cos(180-C)cos(A-B)+2cos^2C-1)]$

$=\frac{1}{2}[2+(2cos(180-C)cos(A-B)+2cos^2C)]$

$=1+\ -cosCcos(A-B)+cos^2C=1+cosC[cosC-cos(A-B)]$

$=1+cosC\left[cos\left(180^0-(A+B)\right)-cos(A-B)\right]$

$=1-cosC[cos(A+B)+cos(A-B)]$

$=1-cosC[2cosAcos(-B)]=1-2cosAcosBcosC=\mathbf{R.H.S}$

d. $Left\ as\ an\ ecercise$

e. $sin^2\frac{A}{2}+sin^2\frac{B}{2}-sin^2\frac{C}{2}=1-2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}$

$$\mathbf{L.H.S}=\frac{1-cosA}{2}+\frac{1-cosB}{2}-\frac{1-cosC}{2}=\frac{1}{2}[1-(cosA+cosB-cosC)]$$
$$=\frac{1}{2}\left[1-\left(2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)+cosC\right)\right]$$
$$=\frac{1}{2}\left[1-\left(2cos\left(90^0-\frac{C}{2}\right)cos\left(\frac{A-B}{2}\right)+cosC\right)\right]$$
$$=\frac{1}{2}\left[1-\left(2sin\frac{C}{2}cos\left(\frac{A-B}{2}\right)\right)+\left(1-2sin^2\frac{C}{2}\right)\right]$$
$$=1-sin\frac{C}{2}\left[cos\left(\frac{A-B}{2}\right)+sin\frac{C}{2}\right]$$
$$=1-sin\frac{C}{2}\left[cos\left(\frac{A-B}{2}\right)+sin\left[90^0-\frac{(A+B)}{2}\right]\right]$$
$$=1-sin\frac{C}{2}\left[cos\left(\frac{A-B}{2}\right)+cos\left(\frac{A+B}{2}\right)\right]=1-sin\frac{C}{2}\left[2\,cos\frac{A}{2}cos\left(-\frac{B}{2}\right)\right]$$
$$=1-2cos\frac{A}{2}cos\frac{B}{2}sin\frac{C}{2}=\mathbf{R.H.S}$$

f. $(f),(g)\ and\ (h)\ left\ as\ an\ exercise$

Task

1) *If* $A+B+C=180^0$, *prove that* $\frac{sin(B+C)+sin(A+C)sin(A+B)}{cos\left(\frac{\pi}{2}-B\right)cos\left(\frac{3\pi}{2}-C\right)\left(\frac{7\pi}{2}-A\right)}=1$

2) *If in a* ΔABC, $sin2A=sin2B+sin2C$, *prove that either B or C is a right angle*

Conditional proofs in factor formulae

1. *If* $sinA+sinB=a\ and\ cosA+cosB=b, prove\ that\ tan\left(\frac{A-B}{2}\right)=\sqrt{\left(\frac{4-a^2-b^2}{a^2+b^2}\right)}$

2. *Given that* $sinx+siny=\lambda_1, and\ cosx+cosy=\lambda_2$, *show that*

I. $tan\left(\frac{x+y}{2}\right)=\frac{\lambda_1}{\lambda_2}$

II. $cos(x+y)=\frac{{\lambda_2}^2-{\lambda_1}^2}{{\lambda_2}^2+{\lambda_1}^2}$ $\quad(\boldsymbol{uneb1994})$

3. *If* $sin\theta+sin2\theta=a\ and\ cos\theta+cos2\theta=b, prove\ that$
$(a^2+b^2)(a^2+b^2-3)=2b$

Solution

1. $sinA+sinB=a\ ,\ cosA+cosB=b, To\ prove\ that\ tan\left(\frac{A-B}{2}\right)=\sqrt{\left(\frac{4-a^2-b^2}{a^2+b^2}\right)}$

$from\ \mathbf{R.H.S}\ \sqrt{\left(\frac{4-a^2-b^2}{a^2+b^2}\right)}=\sqrt{\left(\frac{4-(a^2+b^2)}{a^2+b^2}\right)}$

$$a^2+b^2=(sinA+sinB)^2+(cosA+cosB)^2$$
$$=(sin^2A+2sinAsinB+sin^2B)+(cos^2A+2cosAcosB+cos^2B)$$
$$=(sin^2A+cos^2A)+(sin^2B+cos^2B)+2(sinAsinB+cosAcosB)$$
$$=1+1+2cos(A-B)=2+2cos(A-B)$$

$$\therefore\sqrt{\left(\frac{4-a^2-b^2}{a^2+b^2}\right)}=\sqrt{\frac{4-[2+2cos(A-B)]}{2+2cos(A-B)}}=\sqrt{\frac{2-2cos(A-B)}{2+2cos(A-B)}}=\sqrt{\frac{1-cos(A-B)}{1+cos(A-B)}}\ ,now\ let\ A-B=\theta$$

$$\Rightarrow\sqrt{\frac{1-cos(A-B)}{1+cos(A-B)}}=\sqrt{\frac{1-cos\theta}{1+cos\theta}}=\sqrt{\frac{2sin^2\frac{\theta}{2}}{2cos^2\frac{\theta}{2}}}=tan\frac{\theta}{2}$$

$\therefore \sqrt{\left(\frac{4-a^2-b^2}{a^2+b^2}\right)} = \tan\left(\frac{A-B}{2}\right)$

2. $sinx + siny = \lambda_1, and\ cosx + cosy = \lambda_2$, $To\ show\ that\ tan\left(\frac{x+y}{2}\right) = \frac{\lambda_1}{\lambda_2}$

R.H.S : $\frac{\lambda_1}{\lambda_2} = \frac{sinx+siny}{cosx+cosy} = \frac{2sin\left(\frac{x+y}{2}\right)cos\left(\frac{x-y}{2}\right)}{2cos\left(\frac{x+y}{2}\right)cos\left(\frac{x-y}{2}\right)} = \frac{sin\left(\frac{x+y}{2}\right)}{cos\left(\frac{x+y}{2}\right)} = \tan\left(\frac{x+y}{2}\right)$

$To\ show\ that\ cos(x+y) = \frac{{\lambda_2}^2-{\lambda_1}^2}{{\lambda_2}^2+{\lambda_1}^2}$

R.H.S: $\frac{{\lambda_2}^2-{\lambda_1}^2}{{\lambda_2}^2+{\lambda_1}^2} = \frac{(cosx+cosy)^2-(sinx+siny)^2}{(cosx+cosy)^2+(sinx+siny)^2}$

$= \frac{(cos^2x+2cosxcosy+cos^2y)-(sin^2x+2sinxsiny+sin^2y)}{(cos^2x+2cosxcosy+cos^2y)+(sin^2x+2sinxsiny+sin^2y)}$

$= \frac{(cos^2x-sin^2x)+(cos^2y-sin^2y)+2(cosxcosy-sinxsiny)}{1+1+2cos(x-y)} = \frac{cos2x+cos2y+2cos(x+y)}{2+2cos(x-y)}$

$= \frac{2cos(x+y)cos(x-y)+2cos(x+y)}{2+2cos(x-y)} = \frac{cos(x+y)[cos(x-y)+1]}{[cos(x-y)+1]} = cos(x+y)$

3. *Left as an exercise.*

FINDING ANGLES

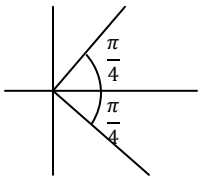
(The inverse notation

A given trig equation may have *infinite number* of solutions if no restriction for the angles is done. $e.g. -\pi\ to + \pi\ or\ 0\ to\ 2\pi\ e.t.c$

When the solution is infinitely stated, the solution is called *general solution.*

Example

1. *Solve* $\boxed{\cos x = \frac{\sqrt{2}}{2}}$



Taking $\frac{\sqrt{2}}{2} = +ve\ value$, then

$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^0 = \frac{\pi}{4}$

$\Rightarrow x = \frac{\pi}{4} \pm 2\pi n\ , -\frac{\pi}{4} \pm 2\pi n$

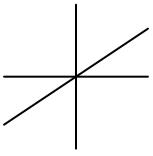
$\therefore \left\{x: x = \frac{\pi}{4} \pm 2\pi n\ or\ -\frac{\pi}{4} \pm 2\pi n\right\}$, where $n \in \mathbb{Z}. i.e, n\ is\ an\ integer.$

Thus this infinite set is denoted as

$$\boxed{\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\ or\ arc\ cos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \pm 2\pi n\ or\ -\frac{\pi}{4} \pm 2\pi n}$$

NOTE: $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) \neq \left[cos\left(\frac{\sqrt{2}}{2}\right)\right]^{-1}$

2. *Solve* $\boxed{\tan x = \frac{\sqrt{3}}{2}}$



Taking $\frac{\sqrt{3}}{2} = +ve\ value$, then

$tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^0 = \frac{\pi}{6}$

$\Rightarrow x = \frac{\pi}{6} \pm \pi n$

$\therefore \left\{x: x = \frac{\pi}{6} \pm \pi n\right\}$

Thus this infinite set is denoted as

$$\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \; or \; arc\tan\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \pm \pi n$$

3. *Solve* $\sin x = \frac{1}{\sqrt{2}}$

Taking $\frac{1}{\sqrt{2}} = +ve\ value$, then

$sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^0 = \frac{\pi}{4}$

$\Rightarrow x = \frac{\pi}{4} + 2\pi n \; or \; \left(\pi - \frac{\pi}{4}\right) + 2\pi n = (2n+1)\pi - \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4} + \pi n, for\ n\ is\ even\ or - \frac{\pi}{4} + \pi n\ for\ n\ is\ odd$

$\therefore \left\{x: \pi n + (-1)^n \frac{\pi}{4}\ for\ n \in \mathbb{Z}\right\}$

The solution set is denoted as

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \; or \; arc\sin\left(\frac{1}{\sqrt{2}}\right) = \pi n + (-1)^n \frac{\pi}{4}$$

In general, the key to identifying all members of a set like $\sin^{-1}(x) = y \; or \; \cot^{-1}(x) = y$ is to find the real members of y such that $0 \le y \le 2\pi \; or \; 0 \le y \le 360^0$ and then using $2\pi n \; or \; \pi n$, period for the function to determine the remaining members

Example

1. *Evaluate*

i) $sin\left(arc\ cos\frac{\sqrt{2}}{2}\right)$

ii) $\sec\left(arc\ tan(1)\right)$

iii) $\sin\left(arc\ cos\left(\frac{5}{13}\right)\right)$

iv) $\sin\left(arc\ sin\left(\frac{4}{5}\right)\right)$

v) $\sec\left(arc\ sin\left(-\frac{7}{25}\right)\right)$

vi) $\cot\left(arc\ cosec\left(\frac{25}{24}\right)\right)$

vii) $\cos\left(arc\ cot(-4)\right)$

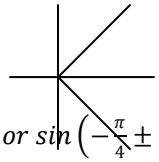
Solution

i) $\sin\left(arc\cos\frac{\sqrt{2}}{2}\right) = \sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Dealing with the bracket

Let $x = cos^{-1}\left(\frac{\sqrt{2}}{2}\right) \; or \; cosx = \frac{\sqrt{2}}{2}$

Now solving $cosx = \frac{\sqrt{2}}{2}$ in general solution



$cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^0 = \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4} \pm 2\pi n\,, -\frac{\pi}{4} \pm 2\pi n$

$\therefore sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = sin\left(\frac{\pi}{4} \pm 2\pi n\right)\ or\ sin\left(-\frac{\pi}{4} \pm 2\pi n\right)$

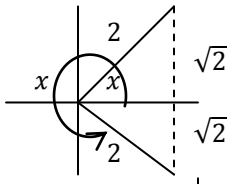
$= sin\left(\frac{\pi}{4}\right)\ or\ sin\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\ or\ -\frac{\sqrt{2}}{2}$

Alternatively

$\sin\left(arc\ cos\frac{\sqrt{2}}{2}\right) = \sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Dealing with the bracket, and taking $\frac{\sqrt{2}}{2} = +ve$

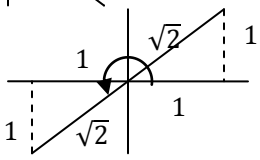
Let $x = cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\ or\ cosx = \frac{\sqrt{2}}{2}$



$\therefore In\ 1^{st} quadrant, sinx = +\frac{\sqrt{2}}{2}$

$\therefore In\ 4^{th} quadrant, sinx = -\frac{\sqrt{2}}{2}$

ii) $sec\left(arc\ tan(1)\right)$



$sec\left(arc\ tan(1)\right) = sec(\tan^{-1}(1))$

Let $\tan^{-1}(1) = \theta\ or\ tan\theta = 1$

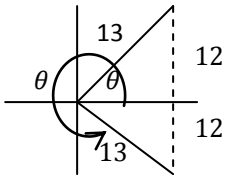
$\Rightarrow sec(\tan^{-1}(1)) = sec\theta = \frac{1}{cos\theta}$

$In\ 1^{st} quadrant\ cos\theta = \frac{1}{\sqrt{2}}\ \therefore sec\theta = \sqrt{2}$

$In\ 3^{rd} quadrant\ cos\theta = -\frac{1}{\sqrt{2}}\ \therefore sec\theta = -\sqrt{2}$

$\therefore sec\left(arc\ tan(1)\right) = \pm\sqrt{2}$

iii) $\sin\left(arc\ cos\left(\frac{5}{13}\right)\right) = \sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$



Let $\cos^{-1}\left(\frac{5}{13}\right) = \theta\ or\ cos\theta = \frac{5}{13}$

$\therefore sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = sin\theta$

In 1st quadrant, $sin\theta = \frac{12}{13}$

In 4th quadrant, $sin\theta = -\frac{12}{13}$

$\therefore\ sin\left(arc\ cos\left(\frac{5}{13}\right)\right) = \pm\frac{12}{13}$

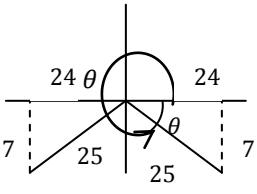
iv) $Left\ as\ an\ exercise\ \boldsymbol{Answer}: \frac{4}{5}$

v) $sec\left(arc\ sin\left(-\frac{7}{25}\right)\right) = sec\left(\sin^{-1}\left(-\frac{7}{25}\right)\right)$

Let $\sin^{-1}\left(-\frac{7}{25}\right) = \theta\ or\ \sin\theta = -\frac{7}{25} \Rightarrow sec\theta = \frac{1}{cos\theta}$

In 3rd quadrant, $cos\theta = -\frac{24}{25}\ \therefore sec\theta = -\frac{25}{24}$

In 4th quadrant, $cos\theta = +\frac{24}{25}\ \ \therefore sec\theta = +\frac{25}{24}$

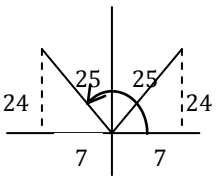


vi) $\cot\left(arc\ \text{cosec}\left(\frac{25}{24}\right)\right) = cot\left(\ \text{cosec}^{-1}\left(\frac{25}{24}\right)\right)$

Let $\text{cosec}^{-1}\left(\frac{25}{24}\right) = \theta\ or\ \text{cosec}\theta = \frac{25}{24}$

But $cosec\theta = \frac{1}{sin\theta} \Rightarrow sin\theta = \frac{24}{25}$

$\therefore\ cot\left(arc\ cosec\left(\frac{25}{24}\right)\right) = cot\theta = \frac{1}{tan\theta}$



In 1st quadrant, $tan\theta = \frac{24}{7}\ \therefore cot\theta = \frac{7}{24}$

In 2nd quadrant, $tan\theta = -\frac{24}{7}\ \therefore cot\theta = -\frac{7}{24}$

$\therefore cot\left(arc\ cosec\left(\frac{25}{24}\right)\right) = \pm\frac{7}{24}$

vii) *Left as an exercise*

2. *Evaluate*

i) $cos\left(arc\ sin\left(\frac{3}{5}\right) + arc\ tan\left(\frac{12}{13}\right)\right)$

ii) $sin\left(arc\ cos\left(\frac{3}{5}\right) + arc\ sec(3)\right)$

iii) $sin\left(2\ arc\ sin\left(\frac{4}{5}\right)\right)$

iv) $cos\left(2\ arc\ cot\left(\frac{5}{13}\right)\right)$

v) $tan\left(\frac{1}{2}\ arc\ cos\left(\frac{1}{2}\right)\right)$

vi) $-2sin(arc\ tan(1)).sin\left(arc\ tan(\sqrt{3})\right)$

vii) $2cos(arc\ sec(2))$.

Solution

i) $\cos\left(arc\ sin\left(\frac{3}{5}\right) + arc\ tan\left(\frac{12}{13}\right)\right) = \cos\left(\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{12}{13}\right)\right)$

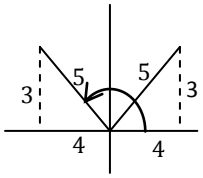
Let $\sin^{-1}\left(\frac{3}{5}\right) = \alpha\ \Rightarrow sin\alpha = \frac{3}{5}\ and\ \tan^{-1}\left(\frac{12}{13}\right) = \beta \Rightarrow tan\beta = \frac{12}{13}$

$\therefore\ cos\left(\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{12}{13}\right)\right) = cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta$

For $sin\alpha = \frac{3}{5}$

In 1st *quadrant* $\cos\alpha = +\frac{4}{5}$

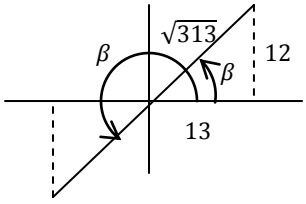
In 2nd *quadrant* $\cos\alpha = -\frac{4}{5}$



For $\tan\beta = \frac{12}{13}$

In 1st *quadrant* $\cos\beta = +\frac{13}{\sqrt{313}}$

In 3rd *quadrant* $\cos\beta = -\frac{13}{\sqrt{313}}$



$\Rightarrow \cos\left(\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{12}{13}\right)\right)$

$= \pm\frac{4}{5}.\pm\frac{13}{\sqrt{313}} - \frac{3}{5}.\pm\frac{12}{\sqrt{313}} = \pm\frac{52}{5\sqrt{313}} \pm \frac{36}{5\sqrt{313}}$

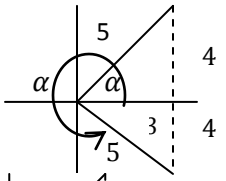
ii) $\sin\left(arc\cos\left(\frac{3}{5}\right) + arc\ \text{sec}(3)\right) = \sin(cos^{-1}\left(\frac{3}{5}\right) + \sec^{-1}(3)\ )$

Let $cos^{-1}\left(\frac{3}{5}\right) = \alpha \Rightarrow \cos\alpha = \frac{3}{5}$ *and* $\sec^{-1}(3) = \beta \Rightarrow sec\beta = 3, sec\beta = \frac{1}{cos\beta} \therefore$ $\cos\beta = \frac{1}{3}$

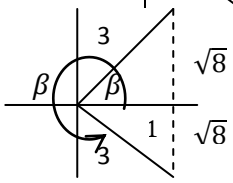
$\Rightarrow \sin\left(arc\cos\left(\frac{3}{5}\right) + arc\sec(3)\right) = sin(\alpha+\beta) = sin\alpha cos\beta + cos\alpha sin\beta$

For $cos\alpha = \frac{3}{5} \Rightarrow sin\alpha = \pm\frac{4}{5}$

$(In\ 1^{st}\ and\ 4^{th}\ quadrant)$



For $\cos\beta = \frac{1}{3} \Rightarrow \sin\beta = \pm\frac{2}{3}\sqrt{2}$

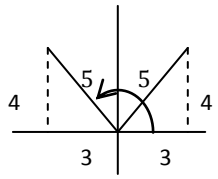


$\therefore \sin\left(arc\cos\left(\frac{3}{5}\right) + arc\sec(3)\right) = \pm\frac{4}{5}.\frac{1}{3} + \frac{1}{3}.\pm\frac{2}{3}\sqrt{2} = \pm\frac{4}{15} \pm \frac{2}{3}\sqrt{2}$

iii) $\sin\left(2\ arc\ sin\left(\frac{4}{5}\right)\right) = \sin\left(2\ \sin^{-1}\left(\frac{4}{5}\right)\right)$

Let $\theta = \sin^{-1}\left(\frac{4}{5}\right) \Rightarrow sin\theta = \frac{4}{5}$

$\cos\theta = \pm\frac{3}{5}$



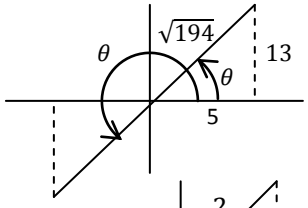
$\Rightarrow \sin\left(2\ arc\sin\left(\frac{4}{5}\right)\right) = \sin 2\theta = 2\sin\theta\cos\theta$

$= 2.\pm\frac{3}{5}.\frac{4}{5} = \pm\frac{24}{25}$

iv) $\cos\left(2\ arc\cot\left(\frac{5}{13}\right)\right) = \cos\left(2\cot^{-1}\left(\frac{5}{13}\right)\right)$

Let $\theta = \cot^{-1}\left(\frac{5}{13}\right) \Rightarrow \cot\theta = \frac{5}{13}\ or \tan\theta = \frac{13}{5}$

$\cos\theta = \pm\frac{5}{\sqrt{194}}$

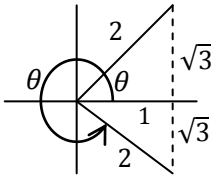


$\therefore \cos\left(2\ arc\cot\left(\frac{5}{13}\right)\right) = \cos 2\theta = 2\cos^2\theta - 1$

$= 2\left(\pm\frac{5}{\sqrt{194}}\right)^2 - 1 = -\frac{72}{97}$

v) $\tan\left(\frac{1}{2}\ arc\cos\left(\frac{1}{2}\right)\right) = \tan\left(\frac{1}{2}\ \cos^{-1}\left(\frac{1}{2}\right)\right)$

Let $\theta = \cos^{-1}\left(\frac{1}{2}\right) \Rightarrow cos\theta = \frac{1}{2}$



$\Rightarrow \tan\left(\frac{1}{2}\ arc\cos\left(\frac{1}{2}\right)\right) = \tan\frac{1}{2}\theta$

$= \pm\sqrt{\left(\frac{1-cos\theta}{1+cos\theta}\right)}$ $[see\ half\ angle\ identities]$

$= \pm\sqrt{\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)} = \pm\frac{1}{\sqrt{3}}$

vi) $-2\sin\ (arc\tan(1)).\sin\ \left(arc\tan\left(\sqrt{3}\right)\right) = -2\sin\ (\tan^{-1}(1)).\sin\ \left(\tan^{-1}\left(\sqrt{3}\right)\right)$

Let $tan^{-1}(1) = \alpha \Rightarrow tan\alpha = 1 \; and \; \tan^{-1}(\sqrt{3}) = \beta \Rightarrow tan\beta = \sqrt{3}$

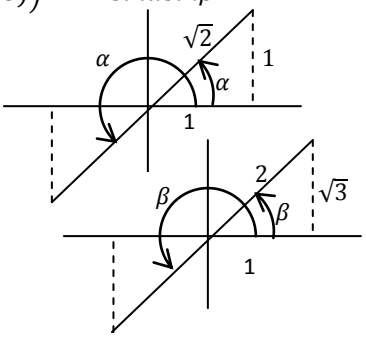
$\Rightarrow -2sin(arc\, tan(1)).sin\left(arc\, tan(\sqrt{3})\right) = -2sin\alpha sin\beta$

For $\tan\alpha = 1$

$\Rightarrow sin\alpha = \pm\frac{1}{\sqrt{2}}$

For $\tan\beta = \sqrt{3}$

$\Rightarrow \sin\beta = \pm\frac{\sqrt{3}}{2}$



$\therefore -2sin(arc\, tan(1)).sin\left(arc\, tan(\sqrt{3})\right) = -2.\pm\frac{1}{2}.\pm\frac{\sqrt{3}}{2} = \pm\frac{\sqrt{6}}{2}$

vii) $Left\ as\ an\ exercise$

NOTE:

In the above examples, the angles in **TWO** quadrants have been used in the evaluation of trigonometric functions. Now let us look at inverse of trigonometric functions requiring only the *principal value, the numerically smallest value.*

Conventional definitions

1. The unique number y such that $y\epsilon\, sin^{-1}\, x, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, is called *the principal value of arc sinx and is denoted as* $y = Arc\, sinx$.
The capital "A" denotes the Principal value.
The $Arc\ sin$ function is
$\{(x, y): y = Arc\, sinx, |x| \leq 1, x\epsilon\mathbb{R}\}$

2. *The principal value of* $Arc\, cosx, -1 \leq x \leq +1$ is the unique number $y\epsilon\, cos^{-1}\, x, 0 \leq y \leq \frac{\pi}{2}$. the principal value is denoted $Arc\, cosx$.
The $Arc\ cos$ function is
$\{(x, y): y = Arc\, cosx, |x| \leq 1, x\epsilon\mathbb{R}\}$

3. The unique member $y, -\frac{\pi}{2} < y < \frac{\pi}{2}, y\epsilon\, tan^{-1}\, x, x\epsilon\mathbb{R}$, is the *principal value of arc tanx; it is denoted Arc tanx.*
The *Arc tan* function is
$\{(x,y): y = Arc\ tanx, x\epsilon\mathbb{R}\}$
The unique member $y, 0 < y < \pi, y\epsilon \cot^{-1} x, x\epsilon\mathbb{R}$, is the principal value of $arc\ cotx$; denoted $Arc\ cotx$.
The *Arc cot* function is
$\{(x,y): y = Arc\ cotx, x\epsilon\mathbb{R}\}$

4. The *principal values of arc secx and arc cosecx are denoted by*
$Arc\ secx = Arc\ cos\frac{1}{x}\ , |x| \geq 1$
$Arc\ cosecx = Arc\ sin\frac{1}{x}\ , |x| \geq 1$

Note:
Much emphasis must be put on the above definitions $i.e$ angle ranges for each trigonometric function.

Examples

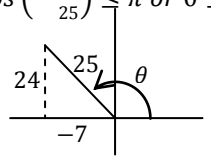
1. *Evaluate*

a. $sin\left[Arc\ cos\left(-\frac{7}{25}\right)\right]$
b. $cos\left[Arc\ sin\left(-\frac{1}{5}\right)\right]$
c. $tan[Arc\ tan4]$
d. $cot\left[Arc\ cos\left(-\frac{11}{61}\right)\right]$
e. $sec[Arc\ cosec4]$

Solution

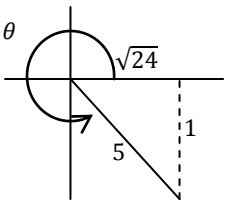
a. $\sin\left[Arc\ cos\left(-\frac{7}{25}\right)\right]$

Let $Arc\ cos\left(-\frac{7}{25}\right) = \theta, cos\theta = -\frac{7}{25}$. *Since* $0 \leq Arc\ cos\left(-\frac{7}{25}\right) \leq \pi$ *or* $0 \leq \theta \leq \pi$,
Then θ lies in2^{nd} quadrant
$\therefore \sin\left[Arc\ cos\left(-\frac{7}{25}\right)\right] = \sin\theta = +\frac{24}{25}$



b. $\cos\left[Arc\ sin\left(-\frac{1}{5}\right)\right]$

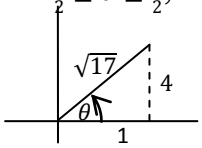
Let $Arc\ sin\left(-\frac{1}{5}\right) = \theta, sin\theta = -\frac{1}{5}$. But $-\frac{\pi}{2} \leq Arc\ sin\left(-\frac{1}{5}\right) \leq \frac{\pi}{2}$ *or* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$,
and since $sin\,\theta = -\frac{1}{5}$, then θ *lies in* 4th quadrant
$\therefore\ cos\left[Arc\ sin\left(-\frac{1}{5}\right)\right] = cos\theta = +\frac{\sqrt{24}}{5}$



c. $\tan[Arc\ tan4]$

Let $Arc\ tan4 = \theta, tan\theta = 4$. But $-\frac{\pi}{2} \le Arc\ tan\ 4 \le \frac{\pi}{2}$ *or* $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, hence θ is a measure in 1st quadrant

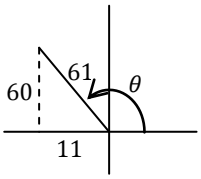
$\therefore \tan[Arc\ tan4] = \tan\theta = +4$



d. $\cot\left[Arc\ cos\left(-\frac{11}{61}\right)\right]$

Let $Arc\ cos\left(-\frac{11}{61}\right) = \theta, cos\theta = -\frac{11}{61}$. Since $0 \le Arc\ cos\left(-\frac{11}{61}\right) \le \pi$ *or* $0 \le \theta \le \pi$, *T*hen θ lies in2^{nd} quadrant

$\therefore\ cot\left[Arc\ cos\left(-\frac{11}{61}\right)\right] = cot\theta = \frac{1}{tan\theta} = \frac{1}{-\frac{60}{11}} = -\frac{11}{60}$



e. *left as an exercise* ***Answer***: $\frac{4}{\sqrt{11}}$

2. *Evaluate*

a. $sin\left[Arc\ cos\left(\frac{4}{5}\right) + \frac{\pi}{2}\right]$

b. $cos\left[Arc\ cos\left(-\frac{1}{2}\right) - Arc\ sin\left(\frac{3}{5}\right)\right]$

c. $sin\left[Arc\ cos\left(\frac{1}{2}\right) - Arc\ sin\left(\frac{5}{13}\right)\right]$

d. $tan\left[Arc\ sin\left(\frac{1}{\sqrt{10}}\right) + Arc\ cos\left(-\frac{4}{5}\right)\right]$

e. $cos\left[Arc\ tan\left(-\frac{4}{3}\right) - Arc\ sin\left(\frac{11}{61}\right)\right]$

f. $cot\left[Arc\ cot\left(\frac{2}{3}\right) + Arc\ cot\left(\frac{1}{3}\right)\right]$

g. $cos\left[\frac{1}{2}Arc\ sin\left(-\frac{15}{17}\right)\right]$

h. $tan\left[2Arc\ cot\left(\frac{24}{7}\right)\right]$

i. $tan\left[\frac{1}{2}Arc\ sec\left(-\frac{13}{5}\right)\right]$

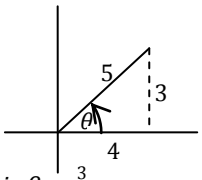
Solution

a. $\sin\left[Arc\ cos\left(\frac{4}{5}\right) + \frac{\pi}{2}\right]$, let $Arc\ cos\left(\frac{4}{5}\right) = \theta \Rightarrow cos\theta = \frac{4}{5}$

Since $0 \le Arc\ cos\left(\frac{4}{5}\right) \le \pi\ or\ 0 \le \theta \le \pi$, then θ is in the 1st *quadrant*

$\sin\left[Arc\ cos\left(\frac{4}{5}\right) + \frac{\pi}{2}\right] = \sin\left[\theta + \frac{\pi}{2}\right]$

$= sin\theta cos\frac{\pi}{2} + cos\theta sin\frac{\pi}{2} = cos\theta = +\frac{4}{5}$



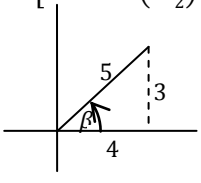
b. $cos\left[Arc\ cos\left(-\frac{1}{2}\right) - Arc\ sin\left(\frac{3}{5}\right)\right]$

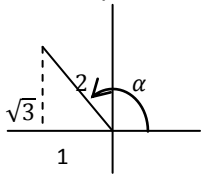
Let $Arc\ cos\left(-\frac{1}{2}\right) = \alpha, cos\alpha = -\frac{1}{2}$ and $Arc\ sin\left(\frac{3}{5}\right) = \beta \Rightarrow sin\beta = \frac{3}{5}$

Since $0 \le\ Arc\ cos\left(-\frac{1}{2}\right) \le \pi,\ -\frac{\pi}{2} \le Arc\ sin\left(\frac{3}{5}\right) \le \frac{\pi}{2}$, then

α lies in2^{nd} quadrant, β *in* 1st quadrant

$\Rightarrow\ cos\left[Arc\ cos\left(-\frac{1}{2}\right) - Arc\ sin\left(\frac{3}{5}\right)\right] = cos[\alpha - \beta] = cos\alpha cos\beta + sin\alpha sin\beta$





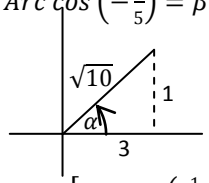
$$= -\frac{1}{2}\times\frac{4}{5}+\frac{\sqrt{3}}{2}\times\frac{3}{5} = \frac{3\sqrt{3}-4}{10}$$

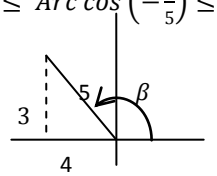
c. Left as an exercise **Ans**: $\frac{12\sqrt{3}-5}{26}$

d. $tan\left[Arc\ sin\left(\frac{1}{\sqrt{10}}\right)+Arc\ cos\left(-\frac{4}{5}\right)\right]$

Let $Arc\ sin\left(\frac{1}{\sqrt{10}}\right)=\alpha\ , sin\alpha=\frac{1}{\sqrt{10}}\ for -\frac{\pi}{2}\le Arc\ sin\left(\frac{1}{\sqrt{10}}\right)\le\frac{\pi}{2}$,

$Arc\ cos\left(-\frac{4}{5}\right)=\beta\Rightarrow cos\beta=-\frac{4}{5}\ for\ 0\le\ Arc\ cos\left(-\frac{4}{5}\right)\le\pi$





$$\Rightarrow \tan\left[Arc\sin\left(\frac{1}{\sqrt{10}}\right)+Arc\cos\left(-\frac{4}{5}\right)\right]=\tan[\alpha+\beta]=\frac{tan\alpha+tan\beta}{1-tanatan\beta}$$

$$=\frac{\frac{1}{3}-\frac{3}{4}}{1+\frac{1}{3}\times\frac{3}{4}}=-\frac{5}{12}\div\frac{15}{12}=-\frac{1}{3}$$

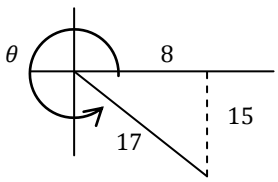
e. Left as an exercise. **Ans:** $\frac{136}{305}$

f. *Try it out*

g. $cos\left[\frac{1}{2}Arc\ sin\left(-\frac{15}{17}\right)\right], let\ Arc\ sin\left(-\frac{15}{17}\right)=\theta\Rightarrow sin\,\theta=-\frac{15}{17}\ , -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$cos\left[\frac{1}{2}Arc\ sin\left(-\frac{15}{17}\right)\right]=cos\frac{1}{2}\theta=\pm\sqrt{\frac{1+cos\theta}{2}}$

Since the angle is in the 4th quadrant,



$cos\frac{1}{2}\theta=+ve$

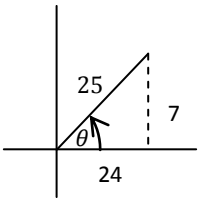
$$\Rightarrow cos\frac{1}{2}\theta=+\sqrt{\frac{1+\frac{8}{17}}{2}}=+\frac{25}{\sqrt{34}}$$

h. $\tan\left[2Arc\ cot\left(\frac{24}{7}\right)\right]$

Let $Arc\ cot\left(\frac{24}{7}\right)=\theta\Rightarrow cot\theta=\frac{24}{7}\ or\ tan\theta=\frac{7}{24}, 0<\theta<\pi$

$$\therefore \tan\left[2Arc\ cot\left(\frac{24}{7}\right)\right]=\tan 2\theta=\frac{2tan\theta}{1-tan^{\,2}\theta}$$

$$=\frac{2\times\frac{7}{24}}{1-\left(\frac{7}{24}\right)^2}=\frac{336}{527}$$



i. Left as an exercise.

3. *Show that*

i) $sin[Arc\ cosx+Arc\ siny]=\sqrt{(1-x^2)(1-y^2)}+xy$

ii) $Arc\ cotx=\frac{\pi}{2}-Arc\ tanx$

iii) $Arc\ tan\left(\frac{1}{2}\right)+Arc\ tan\left(\frac{1}{3}\right)=\frac{\pi}{4}$

iv) $cos(Arc\ sinu)=\sqrt{1-u^2}\ , |u|\le 1$

v) $tan[Arc\ tan(a)+Arc\ tan(1)]=\frac{1+a}{1-a}$

vi) $tan[Arc\ tan(b)-Arc\ tan(a)]=\frac{b-a}{1+b.a}$

vii) $sin[Arc\ sinx+Arc\ siny]=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

viii) $Arc\ tan\left(-\frac{3}{4}\right) + Arc\ sin\left(-\frac{7}{25}\right) = Arc\ cos\left(\frac{3}{5}\right)$

ix) $2Arc\ tan\left(\frac{1}{3}\right) + Arc\ tan\left(\frac{1}{7}\right) = \frac{\pi}{4}$

Solution

i) $sin[Arc\ cosx + Arc\ siny] = \sqrt{(1-x^2)(1-y^2)} + xy$

Let $Arc\ cosx = \alpha\ , cos\alpha = x\ But\ sin\alpha = \sqrt{1-cos^2\alpha} = \sqrt{1-x^2}$

$Arc\ siny = \beta\ , sin\beta = y\ , cos\beta = \sqrt{1-sin^2\beta} = \sqrt{1-y^2}$

$Now\ sin[\alpha+\beta] = sin\alpha cos\beta + cos\alpha sin\beta = \sqrt{1-x^2}.\sqrt{1-y^2} + xy$

$= \sqrt{(1-x^2)(1-y^2)} + xy = \mathbf{R.H.S}$

ii) $Arc\ cotx = \frac{\pi}{2} - Arc\ tanx$

Let $\mathbf{R.H.S} = A = \frac{\pi}{2} - Arc\ tanx, let\ Arc\ tanx = \theta \Rightarrow tan\theta = x$

$\therefore A = \frac{\pi}{2} - \theta$, now introducing cotangent on both sides

$cotA = cot\left(\frac{\pi}{2} - \theta\right) = \frac{cos\left(\frac{\pi}{2}-\theta\right)}{sin\left(\frac{\pi}{2}-\theta\right)} = \frac{sin\theta}{cos\theta} = tan\theta$

$\Rightarrow cotA = tan\theta = x\ or A = \cot^{-1}x = \mathbf{L.H.S}$

iii) $Arc\ tan\left(\frac{1}{2}\right) + Arc\ tan\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Let $Arc\ tan\left(\frac{1}{2}\right) + Arc\ tan\left(\frac{1}{3}\right) = D\ , Arc\ tan\left(\frac{1}{2}\right) = \alpha\ ,$

$tan\alpha = \frac{1}{2}\ , Arc\ tan\left(\frac{1}{3}\right) = \beta\ or\ tan\beta = \frac{1}{3}$

$\Rightarrow \alpha + \beta = D$, now introducing tangent/sine/cosine

$\Rightarrow tanD = tan(\alpha+\beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha tan\beta} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1\ \ \therefore D = \tan^{-1}(1) = \frac{\pi}{4} = \mathbf{R.H.S}$

i) $left\ as\ an\ exercise$

ii) $tan[Arc\ tan(a) + Arc\ tan(1)] = \frac{1+a}{1-a}$

$Let\ Arc\ tan(a) = \alpha\ or\ tan\alpha = a\ , Arc\ tan(1) = \beta\ or\ tan\beta = 1$

Now $tan[Arc\ tan(a) + Arc\ tan(1)] = tan(\alpha+\beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha tan\beta} = \frac{1+a}{1-a} = \mathbf{R.H.S}$

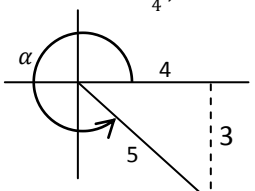
iii) $Left\ as\ an\ exercise$

iv) $Left\ as\ an\ exercise$

v) $Arc\ tan\left(-\frac{3}{4}\right) + Arc\ sin\left(-\frac{7}{25}\right) = Arc\ cos\left(\frac{3}{5}\right)$

let $Arc\ tan\left(-\frac{3}{4}\right) + Arc\ \sin\left(-\frac{7}{25}\right) = D, Arc\ \tan\left(-\frac{3}{4}\right) = \alpha\ or \tan\alpha = -\frac{3}{4},$

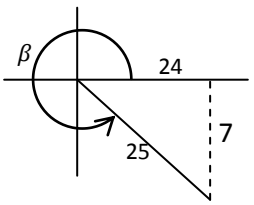
$Arc\ sin\left(-\frac{7}{25}\right) = \beta\ or\ sin\beta = -\frac{7}{25},$



Now$\alpha + \beta = D$, introducing cosine

$cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta = cosD$

But for $\tan\alpha = -\frac{3}{4}\ , -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$



For $sin\beta = -\frac{7}{25}\ , -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

$\therefore cosD = \frac{4}{5}\times\frac{24}{25} - \left(-\frac{3}{5}\right)\left(-\frac{7}{25}\right) = \frac{96-21}{125} = \frac{3}{5}$

$\therefore cosD = \frac{3}{5}, D = \cos^{-1}\left(\frac{3}{5}\right) = \mathbf{R.H.S}$

Note

The convetional defination of inverse of trigonometric functions have been made of use in this example.

ix. $2Arc\,tan\left(\frac{1}{3}\right) + Arc\,tan\left(\frac{1}{7}\right) = \frac{\pi}{4}$

$let\ \mathbf{L.H.S} = 2Arc\,tan\left(\frac{1}{3}\right) + Arc\,tan\left(\frac{1}{7}\right) = y$

Let $Arc\,tan\left(\frac{1}{3}\right) = \alpha, tan\alpha = \frac{1}{3}, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\ and$

$Arc\,tan\left(\frac{1}{7}\right) = \beta\ or\ tan\beta = \frac{1}{7}, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

$\Rightarrow 2\alpha + \beta = y, now\ introducing\ tangent\ on\ both\ sides$

$\therefore tan(2\alpha + \beta) = tany \Rightarrow tany = \dfrac{tan2\alpha + tan\beta}{1 - tan2\alpha tan\beta}$

$Simplifying\ tan2\alpha = \frac{2tan\alpha}{1-tan^2\alpha} = \frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} = \frac{2}{3}\times\frac{9}{8} = \frac{3}{4}$

$\therefore \tan y = \frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}} = \frac{25}{28} \div \frac{25}{28} = 1 \Rightarrow y = \mathbf{L.H.S} = \tan^{-1}(1) = 45^0 = \frac{\pi}{4} = \mathbf{R.H.S}$

REVISION TASK

Show that

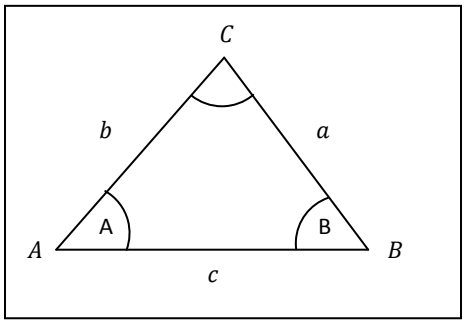
a) $tan^{-1}(-x) = -tan^{-1}(x)$
b) $cos^{-1}(-x) = \pi - cos^{-1}(x) = -cos^{-1}(x)$
c) $sin^{-1}(-x) = -sin^{-1}(x)$
d) $cos^{-1}(x) + sin^{-1}(x) = \frac{\pi}{2}$
e) $cos^{-1}\left(\frac{63}{65}\right) + 2\,tan^{-1}\left(\frac{1}{5}\right) = sin^{-1}\left(\frac{3}{5}\right)$
f) $2\,tan^{-1}\left(\frac{1}{3}\right) + tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$
g) $tan^{-1}\,1 + tan^{-1}\,2 + tan^{-1}\,3 = \pi$
h) $If\ cot^{-1}\left(\frac{1}{a}\right) + cot^{-1}\left(\frac{1}{b}\right) + cot^{-1}\left(\frac{1}{c}\right) = \frac{\pi}{2}\ then\ bc + ca + ab = 1$
i) $If\ cos^{-1}(a) + cos^{-1}(b) + cos^{-1}(c) = \pi, then\ a^2 + b^2 + c^2 = 1$
j) $sin\left[2\,tan^{-1}\left(\frac{1}{4}\right) + cot^{-1}\left(\frac{23}{7}\right) = \frac{1}{\sqrt{2}}\right]$
k) $tan^{-1}\,x + tan^{-1}\,y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$
l) $2\,tan^{-1}\,x = sin^{-1}\left(\frac{2x}{1+x^2}\right)$
m) $sin^{-1}\,x = cosec^{-1}\frac{1}{x}$
n) $cos^{-1}\,x = sec^{-1}\frac{1}{x}$
o) $tan^{-1}\,x = cot^{-1}\frac{1}{x}$

RELATIONS BETWEEN SIDES AND ANGLES OF A TRIANGLE

Solution to triangles

Euler's notation of Triangle

Consider the figure below



If A , B and C denote angles of a triangle, then by Euler's notation, a, b *and* c denote the sides of the sides opposite to these angles.

When three elements, one at least of which is a side, are known, the other three can be found. The process of calculating for the other unknown elements of a triangle when the three of the elements are given, is termed as *solution to a triangle*.

For any triangle $A + B + C = 180^0$, this also is used in solving a triangle.

Laws used in soling a triangle

These include:

- *Sine rule*
- *Cosine rule*
- *Tangent rule*

These are useful in dealing with non-right triangles.

THE SINE RULE

States that

$$\frac{\boldsymbol{a}}{\boldsymbol{sinA}} = \frac{\boldsymbol{b}}{\boldsymbol{sinB}} = \frac{\boldsymbol{c}}{\boldsymbol{sinC}} = \boldsymbol{2R},$$

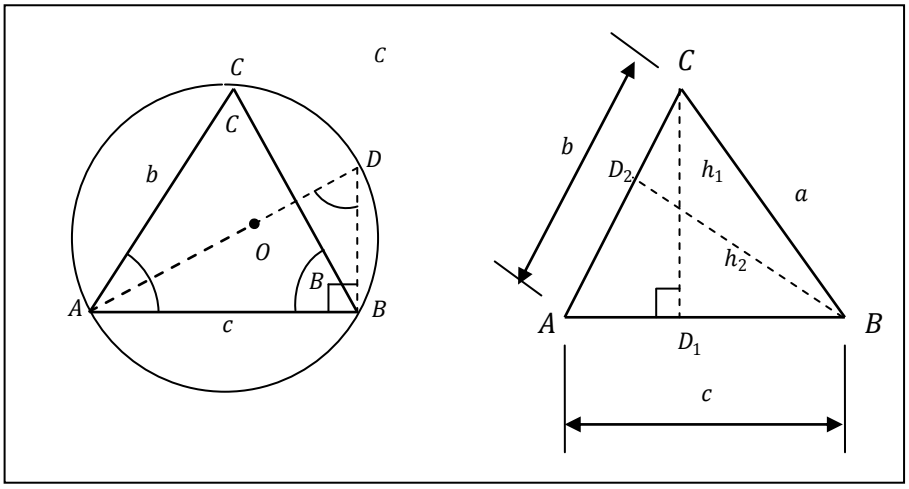
$R = radius\ of\ circumscribing\ circle\ of\ triangle$

Or

The sides of a triangle are proportional to the sines of angles opposite to them

Proof: consider *two* cases

Case I: Acute angle



Consider ΔABC with acute angles. h_1 and h_2 are altitudes from C and B respectively.

From $fig.a$; in ΔABD

$sinC = \frac{AB}{AD} = \frac{c}{2R} \quad \therefore 2R = \frac{c}{sinC}$ .....(1)

From $fig.b$; in ΔACD_1; $sinA = \frac{h_1}{b} \quad \therefore h_1 = b\sin A$.....(2)

In ΔAD_1B, $sinB = \frac{h_1}{a} \quad \therefore h_1 = asinB$ .....(3)

$Eqns$: (1) = (3)

$bsinA = asinB \Rightarrow \frac{b}{sinB} = \frac{a}{sinA}$ .....(4)

In ΔAD_2B; $sinA = \frac{h_2}{c} \Rightarrow h_2 = csinA$ .....(5)

In $\Delta CD_2B; sinC = \frac{h_2}{a} \Rightarrow h_2 = asinC$ .....(6)

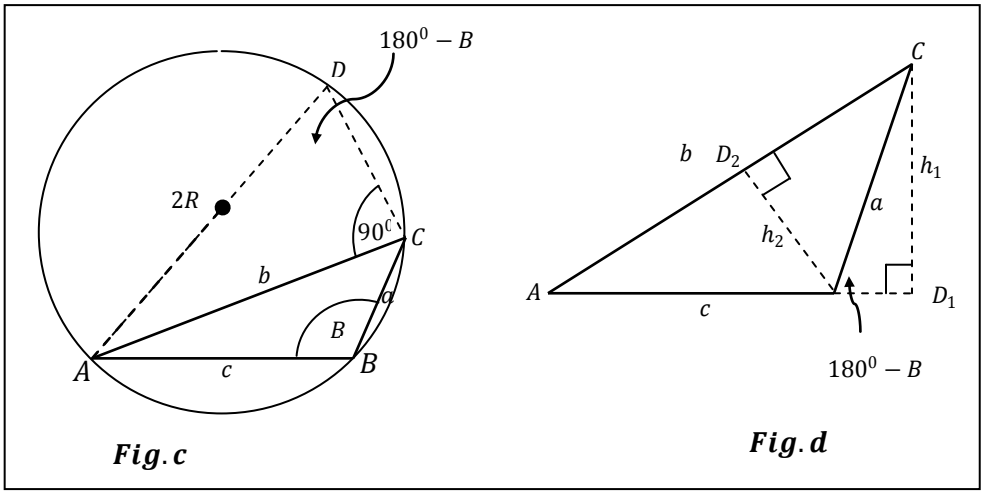
$Eqns$: (5) = (6)

$csinA = asinC \Rightarrow \frac{a}{sinA} = \frac{c}{sinC}$ .....(7)

Now $Eqns$: (7), (4) and (1) are the same. $i.e$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \text{, the } \textit{sine rule}$$

Case II: Obtuse angle



Consider ΔABC with $< B$ obtuse, h_1, h_2 are altitudes from C and B respectively. In *fig C*, in ΔADC

$sinD = \frac{b}{2R}$, $2R = \frac{b}{sinD}$. But $sinD = sin(180 - B) = sinB$

$\therefore 2R = \frac{b}{sinB}$ ..... (1)

In *fig. c*, in ΔAD_1C

$sinA = \frac{h_1}{b}$, $h_1 = bsinA$ ..... (2)

In $\Delta BCD, sin(180 - B) = sinB = \frac{h_1}{a}$, $h_1 = asinB$ ..... (3)

$Eqns.(2) = (3) \Rightarrow bsinA = asinB$, $\frac{a}{sinA} = \frac{b}{sinB}$ ..... (4)

In $fig.d, in\ \Delta ABD_2$

$sinA = \frac{h_2}{c}, h_2 = csinA$ ..... (5)

In ΔBD_2C

$sinC = \frac{h_2}{a}$, $h_2 = asinC$ ..... (6)

$Eqns.(5) = (6) \Rightarrow csinA = asinC$, $\frac{a}{sinA} = \frac{c}{sinC}$ ..... (7)

Now $Eqns.(7), (4)\ and\ (1)$ are the same

$$\boxed{\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2R\ , the\ sine\ rule}$$

When to apply sine rule

- *One side and two angles are given.* Here get the 3rd angle by use of $A + B + C = 180^0$
- *Two sides and non-included angle are given i.e. the non-included angle is opposite to either given side.* Here sometimes an *ambiguous case arises, so take care when dealing with this. See examples on sine rule.*

When not to apply sine rule

- Three sides are given
- Two sides and their included angle

THE COSINE RULE

States that

In any triangle, the square of a side is equal to the sum of their squares of the other sides minus twice the product of these sides times the cosine of the measure of their included angle.$i.e.$ In a ΔABC

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc.cosA}$$
$$\boldsymbol{b^2 = a^2 + c^2 - 2ac.cosA}$$
$$\boldsymbol{c^2 = a^2 + b^2 - 2ab.cosA}$$

Proof

Two cases are to be considered

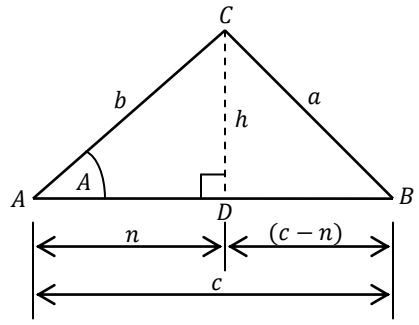


Fig. I Acute Angled Triangle

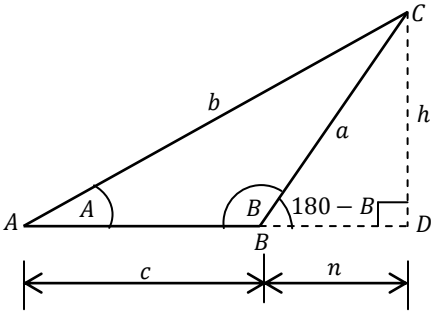


Fig. II Acute Angled Triangle

Considering $\boldsymbol{Fig.I}$

In ΔADC, $h^2 = b^2 - n^2$ ..... (1)

In ΔCDB, $h^2 = a^2 - (c-n)^2 = a^2 - c^2 + 2cn - n^2$..... 2)

Subtracting (1) and (2) gives

$0 = b^2 - a^2 + c^2 - 2cn$, but from ΔACD, $cosA = \frac{n}{b}$, $n = b.cosA$

$$\therefore \boldsymbol{a^2 = b^2 + c^2 - 2bc.cosA,}\ \textit{the cosine rule}$$

Considering $\boldsymbol{Fig.II}$

In ΔBCD, $h^2 = a^2 - n^2$..... (1)

In ΔADC, $h^2 = b^2 - (c+n)^2 = b^2 - c^2 - 2cn - n^2$..... (2)

Subtracting (1) and (2) gives

$0 = a^2 - b^2 + c^2 + 2cn$, but from ΔBDC, $cos(180 - B) = -cosB = \frac{n}{a}$, $n = a.cosB$

$\therefore \boldsymbol{b^2 = a^2 + c^2 - 2ac.cosA}$, *the cosine rule*

When to apply cosine rule

- *Two sides and the included angle given*
- *Three sides are given*

When not to apply cosine rule

- Two sides and non-included angle given
- One side and two of the angles given

Application of cosine and sine rule

- Solution to triangles
- Proof of identities related to triangles
- Finding area of a triangle

General examples on *sine rule*

1. *Solve the triangle if*

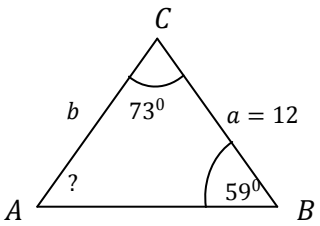
i) $a = 12, B = 59^0, C = 73^0$

ii) $A = 75.6^0, b = 5.6, C = 48.3^0$

Solution

NOTE: *The angles in this question are all acute.*

i)



Finding $< A$

$< A + 73 + 59 = 180$

$\therefore < A = 48^0$

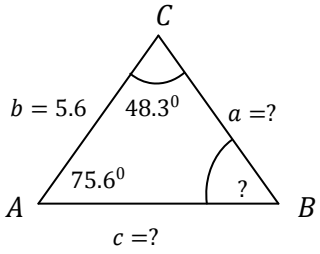
From $\frac{a}{sinA} = \frac{c}{sinB} = \frac{c}{sinC} \Rightarrow \frac{12}{sin\,48} = \frac{b}{sin\,59} = \frac{c}{sin\,73}$

$\Rightarrow c = \frac{12 sin\,73}{sin\,48} = 15.4$

$\Rightarrow b = \frac{12 sin\,59}{sin\,48} = 13.8$

$\therefore A = 48^0, b = 13.8, c = 15.4$

ii)



Finding $< B$

$< B + 48.3 + 75.6 = 180$

$< B = 56.1^0$

From $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 75.6} = \frac{5.6}{\sin B} = \frac{c}{\sin 48.3}$

$\Rightarrow a = \frac{5.6 \sin 75.6}{\sin 56.1} = 6.5$

$\Rightarrow c = \frac{5.6 sin\, 48.3}{sin\, 56.1} = 5.0$

$\therefore < B = 56.1^0 , a = 6.5 , c = 5.0$

2. *Solve the triangle if*

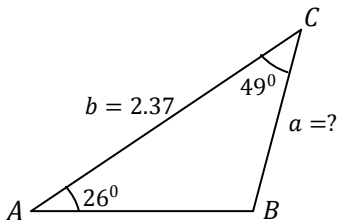
a) $A = 26^0 , b = 2.37 , C = 49^0$

b) $a = 136 , B = 104.2^0 , C = 43.1^0$

Solution

NOTE: *one of the angles given is obtuse*

a)



Finding $< B$

$< B + 26 + 49 = 180$

$< B = 105^0$

?

By sine rule $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} \Rightarrow \frac{136}{sin\, 32.7} = \frac{b}{sin\, 104.2} = \frac{c}{sin\, 43.1}$

$\therefore b = \frac{136 sin\, 104.2}{sin\, 32.7} = 244.04$

$c = \frac{136 sin\, 43.1}{32.7} = 172.01$

$\therefore < A = 32.7^0 , b = 244.05 , c = 172.01$

Task

1. *If in* $\Delta ABC , A = 34^0 , C = 120^0$ *and* $b = 200m$*, how long are the sides BC and AB?* **Ans:** $\boldsymbol{AB = 395m , BC = 255m}$
2. *Solve the triangle if* $A = 71^0 , B = 42^0 , c = 15.$
   Ans $\boldsymbol{a = 15.41, b = 10.90, C = 67^0}$

The ambiguous case

Two sides and non- included angle given

Let A, b and a be the given elements in ΔABC.

The different cases that might arise are given below for $A < 90^0$ and for $A \geq 90^0$

Case I: for $A < 90^0$

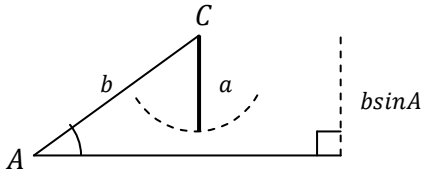


Fig. No solution: $\boldsymbol{a < b \sin A}$

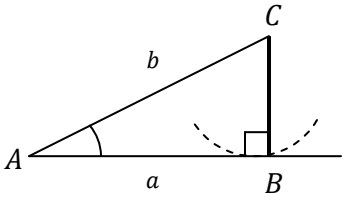


Fig. One solution: $\boldsymbol{a = b \sin A}$

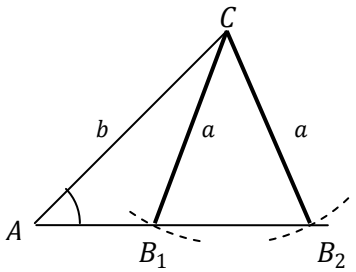


Fig. Two solutions: $b \sin A < a < b$

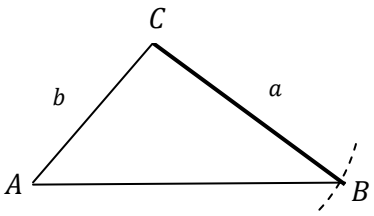


Fig. One solution: $a \geq b$

Case II: For $A \geq 90^0$

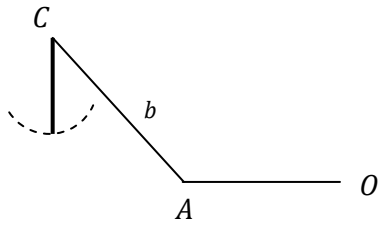


Fig. No solution: $a \geq b$

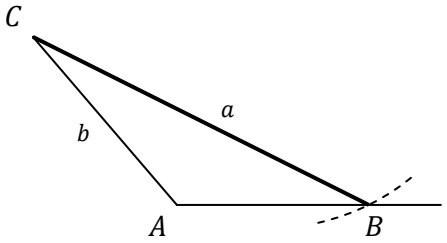


Fig. One solution: $a > b$

NOTE: sine rule can be used to solve a triangle for the data set is ambiguous, but first determine whether the solution exists.

Example

1. *Given ΔABC with $A = 30^0$, $a = 15$, $b = 20$, find B, C*

Solution

$\sin 30^0 = \frac{1}{2}$ and $b\sin 30^0 = 10$, hence $b\sin 30^0 < a < b$ *. i. e* $10 < 15 < 20$

There are two solutions.

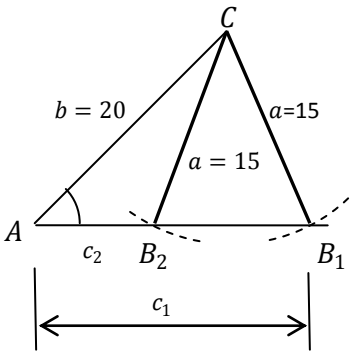
Solution I

$\frac{a}{\sin 30} = \frac{b}{\sin B_1}$, for ΔAB_1C

$\Rightarrow \frac{15}{1/2} = \frac{20}{\sin B_1}$, $sinB_1 = \frac{10}{15}, B_1 = 41.8^0$

$\therefore < ACB_1 = 180 - (30 + 41.8) = 108.2^0$

and $c = \frac{sin\,108.2^0 \times 15}{sin\,30^0} = 28.5$



Solution II

Since $< AB_2C$ *and* $< CB_2B_1$ are

Supplementary and $< CB_2B_1 = < CB_1B_2 = 41.8^0$

$\therefore < AB_2C = 180 - 41.8 = 138.2^0$

Now $< ACB_2 = 180 - (30 + 138.2) = 11.8^0$

In $\Delta\, ACB_2$, $\frac{a}{sinA} = \frac{c_2}{sinC} \Rightarrow \frac{15}{\sin 30} = \frac{c_2}{\sin 11.8}$ *or* $c_2 = \frac{15 \times sin\,11.8}{sin\,30} = 6.2$

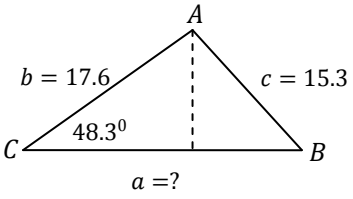
2. *Solve the triangle completely*

a. $b = 17.6$, $C = 48.3^0$, $c = 15.3$

b. $B = 129^0$, $b = 7.89, c = 4.56$

Solution

a. By sine rule $\frac{a}{sinA} = \frac{17.6}{sinB} = \frac{15.3}{sin\,48.3}$



From the given data, $c < b$ and

$17.6sin48.3 = 13.14 < c$

$\therefore bsinC < c < b$, hence two solutions

From ΔAB_1C ,

$\frac{b}{sin\,B_1} = \frac{c}{sinC} \Rightarrow \frac{17.6}{sin\,B_1} = \frac{15.3}{sin\,48.3}$

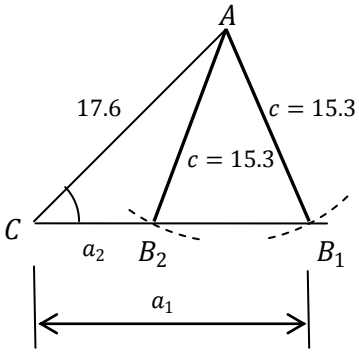
$\therefore B_1 = 59.2^0 =< AB_1C$

Also $< A = 180 - (48.3 + 59.2) = 72.5^0$

But $< AB_2B_1 =< AB^1C = 59.2^0$

Then $< CB_2A = 180 - 59.2 = 120.8^0$

From ΔACB_2 , $\frac{a_2}{sinA} = \frac{b}{sin\,B_2} \Rightarrow \frac{a_2}{sinA} = \frac{17.6}{sin\,120.8}$



$< CB_2A = 180 - (48.3 + 120.8) = 10.9^0$

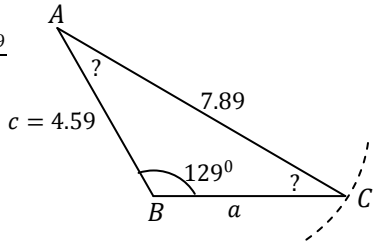
$\therefore a_2 = \frac{17.6sin10.9}{sin120.8} = 3.87$

From ΔACB_1 , $\frac{c}{sinC} = \frac{a_1}{sinA} \Rightarrow \frac{15.3}{sin\,48.3} = \frac{a_1}{sin\,72.5}$ $\therefore a_1 = 19.54$

$\therefore B = 59.2^0, A = 72.5^0, a = 19.54$; $B = 120.8^0, A = 10.9^0, a = 3.87$

b. Since $< B = 129^0 > 90^0$, then looking at $b > c$ then there is only one solution.

By sine rule $\frac{a}{sinA} = \frac{7.89}{sin\,129} = \frac{4.59}{sinC} \Rightarrow sinC = \frac{4.59sin\,129}{7.89}$



$\therefore C = 26.9^0$

$\therefore < A = 180 - (26.9 + 129) = 24.1^0$

Also $\frac{a}{sin\,24.1} = \frac{7.89}{sin\,129} \Rightarrow a = 4.15$

$\therefore C = 26.9^0$, $A = 24.1^0, a = 4.15$

3. *Solve the triangle completely in which*

$b = 15.6m$, $c = 12.3m$, $C = 34.3^0$

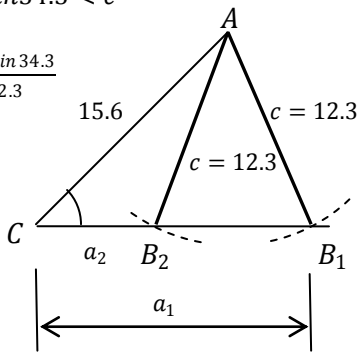
Solution

By sine rule $\frac{a}{sinA} = \frac{15.6}{sinB} = \frac{12.3}{sin\,34.3}$

From the given data, $c < b$ and $bsinC = 15.6sin34.3 < c$

$\therefore bsinC < c < b$, hence two solutions.

From ΔAB_1C , $\frac{a_1}{sinA} = \frac{15.6}{sin\,B_1} = \frac{12.3}{sin\,34.3} \Rightarrow sinB_1 = \frac{15.6sin\,34.3}{12.3}$



$\therefore B_1 = 45.6^0$

$< CAB_1 = A = 180 - (34.3 + 45.6) = 100.1^0$

$\therefore a_1 = \frac{12.3 \times sin100.1}{sin34.3} = 21.49$

$From\ \Delta ACB_2$, $\frac{12.3}{sin34.3} = \frac{a_2}{sin < CAB_2}$

$< CB_2A = 180 - 45.6^0 = 134.4^0$

$< CAB_2 = 180 - (34.3 + 134.4) = 11.3^0$

$\therefore \frac{12.3}{sin34.3} = \frac{a_2}{sin11.3} \Rightarrow a_2 = 4.28$

$\therefore B = 45.6^0, A = 100.1^0, a = 21.49m$; $B = 134.4^0, A = 11.3^0, a = 4.28$

Task

Solve the triangle if $b = 2.718, c = 3.142, B = 54.3^0$

Answer: $\boldsymbol{A = 55.9^0, C = 69.9^0, a = 2.770}$; $\boldsymbol{A = 15.6^0, C = 110.2^0, a = 0.897}$

General examples on cosine rule

Solve the triangle in which

i) $b = 10.67m, c = 21.7m$ *and* $A = 44.5^0$

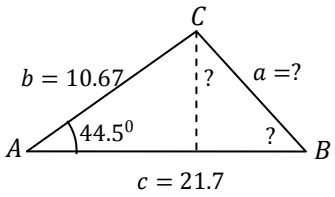
ii) $a = 5m, a = 8m$ *and* $C = 40^0$

iii) $a = 9, c = 5$ *and* $B = 120^0$

iv) $a = 1, b = 2$ *and* $c = 2$

v) $A = 3, b = 7$ *and* $c = 5$

Solution



i) By cosine rule $a^2 = b^2 + c^2 - 2bc.cosA$

$a^2 = 10.67^2 + 21.7^2 - 2 \times 10.67 \times 21.7cos44.5$

$\therefore a^2 = 254.4483085 \Rightarrow a = 15.97m$

The other angles may be got using sine rule.

$\frac{15.97}{sin\,44.5} = \frac{10.67}{sinB}$ *or* $sinB = \frac{10.67 sin\,44.5}{15.97}$ $\therefore B = 27.9^0$

$\therefore < C = 180 - (27.9 + 44.5) = 107.6^0$

$\therefore < C = 107.6^0, 27.9^0, a = 15.97$

ii) *Left as an exercise* **Ans**: $\boldsymbol{c = 5.26, A = 37.7^0, B = 102.3^0}$

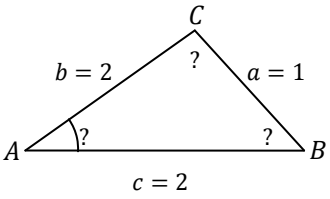
iii) *Left as an exercise* **Ans**: $\boldsymbol{b = 12.29, A = 39.4^0, C = 20.6^0}$

iv) from $a^2 = b^2 + c^2 - 2bc.cosA$

$\therefore cosA = \frac{2^2+2^2-1^2}{2\times2\times2} \Rightarrow A = 28.96^0$

$cosB = \frac{a^2+c^2-b^2}{2.a.c} = \frac{1^2+2^2-2^2}{2\times1\times2} \Rightarrow B = 75.52^0$

$\therefore < C = 180 - (28.96 + 75.52) = 75.52^0$



v) *Left as an exercise*

APPLICATION OF SINE RULE AND COSINE RULE

Proof of identities related to triangles

Examples on SINE RULE

In a ΔABC , prove that

i) $\frac{b-c}{a} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}$

ii) $\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$

iii) $\frac{a+b-c}{a+b+c} = tan\frac{A}{2}tan\frac{B}{2}$

iv) $\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = tanBcotC$

v) $sin\frac{1}{2}(B-C) = \frac{b-c}{a}.\cos\frac{A}{2}$

vi) $\frac{a^2-b^2}{c^2} = \frac{sin\,(A-B)}{sin\,(A+B)}$

Solution

i) $\frac{\boldsymbol{b-c}}{\boldsymbol{a}} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}$, $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2R$

$\frac{b-c}{a} = \frac{2RsinB-2RsinC}{2RsinA} = \frac{sinB-sinC}{sinA} = \frac{2cos\left(\frac{B+C}{2}\right)sin\left(\frac{B-C}{2}\right)}{2\,sin\frac{A}{2}cos\frac{A}{2}}$;

$\frac{B+C}{2} = 90 - \frac{A}{2}$ $\therefore cos\left(\frac{B+C}{2}\right) = cos\left(90-\frac{A}{2}\right) = sin\frac{A}{2}$

$\therefore \frac{b-c}{a} = \frac{2\,sin\frac{A}{2}sin\left(\frac{B-C}{2}\right)}{2\,sin\frac{A}{2}cos\frac{A}{2}} = \frac{sin\left(\frac{B-C}{2}\right)}{cos\frac{A}{2}}$

ii) $\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$

$\frac{a+b}{a-b} = \frac{2RsinA+2RsinB}{2RsinA-2RsinB} = \frac{sinA+sinB}{sinA-sinB} = \frac{2sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)}{2\,cos\left(\frac{A+B}{2}\right)sin\left(\frac{A-B}{2}\right)} = \frac{sin\left(\frac{A+B}{2}\right)}{cos\left(\frac{A+B}{2}\right)}.\frac{cos\left(\frac{A-B}{2}\right)}{sin\left(\frac{A-B}{2}\right)}$

$\therefore \frac{a+b}{a-b} = tan\left(\frac{A+B}{2}\right).cot\left(\frac{A-B}{2}\right) = \frac{tan\left(\frac{A+B}{2}\right)}{tan\left(\frac{A-B}{2}\right)}$

iii) $\frac{a+b-c}{a+b+c} = tan\frac{A}{2}tan\frac{B}{2}$

$\frac{a+b-c}{a+b+c} = \frac{2RsinA+2RsinB-2RsinC}{2RsinA+2RsinB+2RsinC} = \frac{sinA+sinB-sinC}{inA+sinB+sinC} = \frac{2sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)-2sin\frac{C}{2}cos\frac{C}{2}}{2\,sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)+2sin\frac{C}{2}cos\frac{C}{2}}$

But $sin\left(\frac{A+B}{2}\right) = sin\left(90-\frac{C}{2}\right) = cos\frac{C}{2}$

$\therefore \frac{a+b-c}{a+b+c} = \frac{2cos\frac{C}{2}cos\left(\frac{A-B}{2}\right)-2sin\frac{C}{2}cos\frac{C}{2}}{2\,cos\frac{C}{2}cos\left(\frac{A-B}{2}\right)+2sin\frac{C}{2}cos\frac{C}{2}} = \frac{cos\left(\frac{A-B}{2}\right)-sin\frac{C}{2}}{cos\left(\frac{A-B}{2}\right)+sin\frac{C}{2}}$

But $sin\frac{C}{2} = sin\left(90-\left(\frac{A+B}{2}\right)\right) = cos\left(\frac{A+B}{2}\right)$

$\Rightarrow \frac{a+b-c}{a+b+c} = \frac{cos\left(\frac{A-B}{2}\right)-cos\left(\frac{A+B}{2}\right)}{cos\left(\frac{A-B}{2}\right)+cos\left(\frac{A+B}{2}\right)} = \frac{-2sin\left[\frac{\left(\frac{A-B}{2}\right)+\left(\frac{A+B}{2}\right)}{2}\right]sin\left[\frac{\left(\frac{A-B}{2}\right)-\left(\frac{A+B}{2}\right)}{2}\right]}{2cos\left[\frac{\left(\frac{A-B}{2}\right)+\left(\frac{A+B}{2}\right)}{2}\right]cos\left[\frac{\left(\frac{A-B}{2}\right)-\left(\frac{A+B}{2}\right)}{2}\right]} = \frac{-2sin\left(\frac{A}{2}\right)sin\left(-\frac{B}{2}\right)}{2cos\left(\frac{A}{2}\right)cos\left(-\frac{B}{2}\right)}$

$\therefore \frac{a+b-c}{a+b+c} = \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \tan\frac{A}{2}\tan\frac{B}{2}$

iv) $\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = tanBcotC$

$\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{(2RsinA)^2+(2RsinB)^2-(2RsinC)^2}{(2RsinA)^2-(2RsinB)^2+(2RsinC)^2} = \frac{\sin^2 A+\sin^2 B-\sin^2 C}{\sin^2 A-\sin^2 B+\sin^2 C}$

From double angle identities

$sin^2A = \frac{1-cos\,2A}{2}, sin^2B = \frac{1-cos\,2B}{2}, sin^2C = \frac{1-cos\,2C}{2}$

$\Rightarrow \frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{\frac{1-cos\,2A}{2}+\frac{1-cos\,2B}{2}-\frac{1-cos\,2C}{2}}{\frac{1-cos\,2A}{2}-\frac{1-cos\,2B}{2}+\frac{1-cos\,2C}{2}} = \frac{1-cos\,2A+1-cos\,2B-1+cos\,2C}{1-cos\,2A-1+cos\,2B+1-cos\,2C}$

$= \frac{1-[cos\,2A+cos\,2B-cos\,2C]}{1-[cos\,2A-cos\,2B+cos\,2C]} = \frac{1-[2cos(A+B)cos(A-B)-cos\,2C]}{1-[-2sin(A+B)sin(A-B)+cos\,2C]}$

But $cos(A+B) = cos(180-C) = -cosC$;

Also $sin(A+B) = sin(180-C) = sinC$;

$\therefore \frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{1-[-2cosC.cos(A-B)-(2cos^2C-1)]}{1-[-2sinC.sin(A-B)+(1-2sin^2C)]} = \frac{1-[-2cosC(cos(A-B)+cosC)+1]}{1-[-2sinC(sin(A-B)+sinC)+1]}$

$= \frac{1+2cosC.[cos(A-B)+cosC]-1}{1+2sinC.[sin(A-B)+sinC]-1} = \frac{cosC}{sinC}.\left[\frac{cos(A-B)+cosC}{sin(A-B)+sinC}\right]$

$= cotC.\left[\frac{2cos\left(\frac{A-B+C}{2}\right)cos\left(\frac{A-B-C}{2}\right)}{2sin\left(\frac{A-B+C}{2}\right)cos\left(\frac{A-B-C}{2}\right)}\right]$

But $\frac{A-B+C}{2} = \frac{A+C}{2}-\frac{B}{2} = \frac{180-B}{2}-\frac{B}{2} = 90-B, \therefore cos\left(\frac{A-B+C}{2}\right) = cos(90-B) = sinB$

Also $sin\left(\frac{A-B+C}{2}\right) = sin(90-B) = cosB$

$\therefore \frac{a^2+b^2-c^2}{a^2-b^2+c^2} = cotC.\frac{sinB}{cosB} = tanB.cotC$

v) Left as an exercise. Refer to (i) above.

vi) $\frac{a^2-b^2}{c^2} = \frac{sin(A-B)}{sin(A+B)}$

$\frac{a^2-b^2}{c^2} = \frac{(2RsinA)^2-(2RsinB)^2}{(2RsinC)^2} = \frac{sin^2A-sin^2B}{sin^2C} = \frac{\frac{1-cos\,2A}{2}+\frac{1-cos\,2B}{2}}{sin^2C} = \frac{cos\,2B-cos\,2A}{2sin^2C}$

$\frac{a^2-b^2}{c^2} = \frac{-2sin(B+A)sin(B-A)}{2sinC.sinC}$

But $sin(B+A) = sin(180-C) = sinC$

$\therefore \frac{a^2-b^2}{c^2} = \frac{-2sinC.sin(B-C)}{2sinC.sinC} = \frac{sin(B-C)}{sinC} = \frac{sin(B-C)}{sin(A+B)}$

Task

For any ΔABC, prove that $tan\frac{1}{2}(A-B) = \left(\frac{a-b}{a+b}\right).cot\frac{C}{2}$

Examples on COSINE RULE

In ΔABC, prove that

i) $a^2 = (b-c)^2 + 4bc.sin^2\frac{1}{2}A$

ii) $a^2+b^2+c^2 = 2(bc.cosA + ac.cosB + ab.cosC)$

iii) $\frac{b^2-c^2}{a^2} = \frac{sin(B-C)}{sin(B+C)}$

iv) $\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = tanBcotC$

Solution

i) $a^2 = (b-c)^2 + 4bc.sin^2\frac{1}{2}A$

$\mathbf{R.H.S}: (b-c)^2 + 4bc.sin^2\frac{1}{2}A = b^2 - 2bc + c^2 + 4bc\left(\frac{1-cosA}{2}\right)$

$= b^2 - 2bc + c^2 + 2bc(1-cosA) = b^2 + c^2 - 2bc.cosA$

But from cosine rule $a^2 = b^2 + c^2 - 2bc.cosA$

$\therefore$ $\mathbf{R.H.S} = a^2 = \mathbf{L.H.S}$

ii) $a^2 + b^2 + c^2 = 2(bc.cosA + ac.cosB + ab.cosC)$

$a^2 = b^2 + c^2 - 2bc.cosA$

$b^2 = a^2 + c^2 - 2ac.cosB$ } *Adding the equations*

$c^2 = a^2 + b^2 - 2ab.cosC$

$a^2 + b^2 + c^2 = (2a^2 + 2b^2 + 2c^2) - 2(bc.cosA + ac.cosB + ab.cosC)$

$\therefore a^2 + b^2 + c^2 = 2(bc.cosA + ac.cosB + ab.cosC)$

iii) $\frac{b^2-c^2}{a^2} = \frac{sin(B-C)}{sin(B+C)}$

From $b^2 = a^2 + c^2 - 2ac.cosB \Rightarrow b^2 - c^2 = a^2 - 2ac.cosB$ ……($\div a^2$)

$\frac{b^2-c^2}{a^2} = 1 - 2.\frac{c}{a}.cosB$

From sine rule, $c = 2RsinC$, $a = 2RsinA$

$\therefore \frac{b^2-c^2}{a^2} = 1 - 2.\frac{2RsinC}{2RsinA}.cosB = 1 - \frac{2sinCcosB}{sinA} = \frac{sinA - 2sinCcosB}{sinA}$

But $sinA = sin\big(180 - (B+C)\big) = sin(B+C)$

$\therefore \frac{b^2-c^2}{a^2} = \frac{sin(B+C) - 2sinCcosB}{sin(B+C)} = \frac{sinBcosC + cosBsinC - 2sinCcosB}{sin(B+C)}$

$= \frac{sinBcosC - cosBsinC}{sin(B+C)} = \frac{sin(B-C)}{sin(B+C)}$

iv) $\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \tan B \cot C$

From the cosine rule

$c^2 = a^2 + b^2 - 2ab.cosC$ *or* $a^2 + b^2 = c^2 + 2ab.cosC$

$b^2 = a^2 + c^2 - 2ac.cosB$ *or* $a^2 + c^2 = b^2 + 2ac.cosB$

$\Rightarrow \frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{c^2+2ab.cosC-c^2}{b^2+2ac.cosB-b^2} = \frac{2ab.cosC}{2ac.cosB} = \frac{b}{c}.\frac{cosC}{cosB}$

From sine rule, $c = 2RsinC$, $b = 2RsinB$

$\therefore \frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{2RsinB}{2RsinC}.\frac{cosC}{cosB} = \frac{sinB}{cosB}.\frac{sinC}{cosC} = \tan B \cot C$

NOTE:

When the proof is involving sides and angles of a triangle, the application of the suitable rule can be determined by the sides of the triangle.$i.e$. For sides being squared, use cosine rule otherwise use sine rule.

THE TANGENT RULE

For any ΔABC with Euler's notation

$$\tan\frac{1}{2}(B-C)=\left(\frac{b-c}{b+c}\right).\cot\frac{A}{2}$$
$$\tan\frac{1}{2}(A-B)=\left(\frac{a-b}{a+b}\right).\cot\frac{C}{2}$$

Proof: this is based on sine rule.

$\tan\frac{1}{2}(B-C)=\left(\frac{b-c}{b+c}\right).\cot\frac{A}{2}$

$\frac{b-c}{b+c}=\frac{2RsinB-2RsinC}{2RsinB+2RsinC}=\frac{sinB-sinC}{sinB+sinC}=\frac{2cos\left(\frac{B+C}{2}\right)sin\left(\frac{B-C}{2}\right)}{2sin\left(\frac{B+C}{2}\right)cos\left(\frac{B-C}{2}\right)}$

But $\frac{B+C}{2}=90-\frac{A}{2}$

$\therefore \cos\left(\frac{B+C}{2}\right)=\cos\left(90-\frac{A}{2}\right)=\sin\frac{A}{2}$, and $\therefore \sin\left(\frac{B+C}{2}\right)=\sin\left(90-\frac{A}{2}\right)=\cos\frac{A}{2}$

$\therefore \frac{b-c}{b+c}=\frac{2sin\frac{A}{2}sin\left(\frac{B-C}{2}\right)}{2cos\frac{A}{2}cos\left(\frac{B-C}{2}\right)}=tan\frac{A}{2}.tan\left(\frac{B-C}{2}\right)$

Now $\left(\frac{b-c}{b+c}\right).cot\frac{A}{2}=tan\frac{A}{2}.tan\left(\frac{B-C}{2}\right).cot\frac{A}{2}=tan\frac{1}{2}(B-C)=$ **L.H.S**

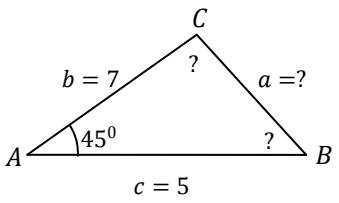
Application of Tangent rule

Used in solving a triangle when two sides and included angle are given

Example

Given that in any ΔABC , $tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}.cot\frac{A}{2}$, *solve the triangle with sides* 5 *and* 7 *and the included angle* 45^0 (***uneb* 1999**)

Solution



Let $b=7$, $c=5$ to avoid $(b-c)<0$, $A=45^0$

By tangent rule,

$\tan\left(\frac{B-C}{2}\right)=\frac{7-5}{7+5}.\cot\frac{45}{2}=\frac{2}{12}\times\frac{1}{tan\,22.5}=\frac{1}{6tan\,22.5}$

$\therefore \frac{B-C}{2}=\tan^{-1}\left(\frac{1}{6tan\,22.5}\right)=21.9^0$

$\Rightarrow \frac{B-C}{2}=21.9^0$ ..... (1)

But $A+B+C=180^0$ $\Rightarrow \frac{B+C}{2}=90-\frac{A}{2}=90-\frac{45}{2}=67.5^0$

$\therefore \frac{B+C}{2}=67.5^0$ ..... (2)

Adding (1) and (2),

$B=89.4^0$

Subtracting (1) and (2),

$C=45.6^0$

From sine rule $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\Rightarrow\frac{a}{\sin 45}=\frac{7}{\sin 89.4}$

$\therefore a=\frac{7\sin 45}{\sin 89.4}=4.95$

$\therefore\ B = 89.4^0, C = 45.6^0, a = 4.95$

Note

When using the tangent rule, identify included angle between the two given sides. Equations

$\frac{B+C}{2} = 90 - \frac{A}{2}$, and another containing $\frac{B-C}{2}$ from the tangent rule are usually solved simultaneously for B and C if A is given.

Task

Solve the triangle in which $b = 10.67m$, $c = 21.7m$, $A = 44.8^0$

***Answer*:** $\boldsymbol{a = 16.0m, C = 107.2^0, B = 28^0}$

AREA OF A TRIANGLE

Recall: Area of a triangle, $\Delta = \frac{1}{2}.b.h$, where b = measure of one side and h = altitude to that side.

Consider the fig. below

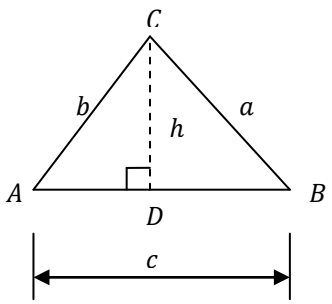


Fig. Acute angled triangle

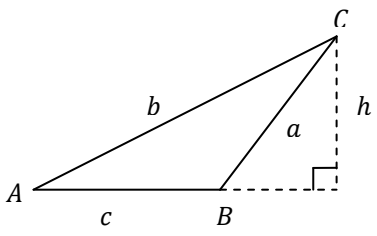


Fig. Obtuse angled triangle

Area, $\Delta = \frac{1}{2}b.h$, for both

Three cases arise on area of a triangle

a) **To determine the area of a triangle with *two sides* and *an included angle* given**

Let A, b, and c be given measures.

$Area, \Delta = \frac{1}{2}.c.h$. See figure above.

But from trigonometry, $h = bsinA$

$$\therefore Area, \Delta = \frac{1}{2}.b.c\sin A$$

Similarly,

$$Area, \Delta = \frac{1}{2}.a.c\sin B$$
$$Area, \Delta = \frac{1}{2}.a.b\sin C$$

Example

Find the area of a BC , given that $B = 40^0, a = 5$ and $c = 10$

Solution

$Area\ , \Delta = \frac{1}{2}.a.c\sin B = \frac{1}{2} \times 5 \times 10 \times \sin 40 = 16\ sq.units$

b) **To find area of a triangle given *one side* and *two or three angles***

Let A, B and C be given measures of angles

$\therefore Area\ , \Delta = \frac{1}{2}.b.c\sin A$

Now finding b

From sine rule $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} \Rightarrow b = \frac{c\sin B}{\sin C}$ $where\ C = 180 - (A + B)$, now known,

$\therefore b = \frac{c\sin B}{\sin C}$

$$\therefore\ Area\ , \Delta = \frac{1}{2}c^2.\frac{\sin A\sin B}{\sin C}$$

Similarly

$$Area\ , \Delta = \frac{1}{2}b^2.\frac{\sin A\sin C}{\sin B}$$

$$Area\ , \Delta = \frac{1}{2}a^2.\frac{\sin B\sin C}{\sin A}$$

Example

Find the area of ΔABC given that $a = 10, A = 45^0, B = 60^0, C = 35^0$

Solution

Using

$Area\ , \Delta = \frac{1}{2}a^2.\frac{sinBsinC}{sinA} \Rightarrow Area\ , \Delta = \frac{1}{2} \times 10^2 \times \frac{sin\,60\,sin\,35}{sin\,85} = 25\ sq.units$

c) **To find area of a triangle given only the *three sides***

The above two cases can be made of use by transforming this question in to any of the cases *(a)* and *(b)* above.

The angles can be obtained using the cosine rule, and then using the sine rule if necessary.

Alternative approach

This requires *half angle identities. i.e.*

$$\boldsymbol{sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}} \qquad \boldsymbol{cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}}$$

$$\boldsymbol{sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}} \qquad \boldsymbol{cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}}$$

$$\boldsymbol{sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}} \qquad \boldsymbol{cos\frac{B}{2} = \sqrt{\frac{s(s-c)}{ab}}}$$

Where $\boldsymbol{s = semi\ perimeter = \frac{1}{2}(a + b + c)}$

Proof of half angle identities

Proof is based on half angle identities and cosine rule.

$\sin^2\frac{A}{2}=\frac{1-cosA}{2}, a^2=b^2+c^2-2bc.\cos A \Rightarrow \cos A=\frac{b^2+c^2-a^2}{2bc}$

$\sin^2\frac{A}{2}=\frac{1-\left(\frac{b^2+c^2-a^2}{2bc}\right)}{2}=\frac{2bc-(b^2+c^2)+a^2}{4bc}=\frac{a^2-[b^2-2bc+c^2]}{4bc}$

$=\frac{a^2-(b-c)^2}{4bc}=\frac{(a+b-c)(a-(b-c))}{4bc}=\frac{(a+b-c)(a+c-b)}{4bc}$ $\left[\begin{array}{l}\therefore By\ difference\ of\ two\ squares\\ i.e. A^2-B^2=(A+B)(A-B)\end{array}\right]$

But $s=\frac{1}{2}(a+b+c)$..... $(*)$

Subtracting c on both sides

$\Rightarrow s-c=\frac{1}{2}(a+b+c)-c=\frac{1}{2}(a+b-c)\ or\ 2(s-c)=a+b-c$

Subtracting b on both sides of $eqn.(*)$

$\Rightarrow s-b=\frac{1}{2}(a+b+c)-b=\frac{1}{2}(a+c-b)\ or\ 2(s-b)=a+c-b$

$\therefore \sin^2\frac{A}{2}=\frac{2(s-c).2(s-b)}{4bc}=\frac{(s-b)(s-c)}{bc}\Rightarrow \boldsymbol{\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}}$

Prove the other two half angle identities of sines following the above derivation of $\sin\frac{A}{2}$.

$\cos^2\frac{A}{2}=\frac{1+cosA}{2}, a^2=b^2+c^2-2bc.\cos A \Rightarrow cosA=\frac{b^2+c^2-a^2}{2bc}$

$\cos^2\frac{A}{2}=\frac{1+\left(\frac{b^2+c^2-a^2}{2bc}\right)}{2}=\frac{2bc+b^2+c^2-a^2}{4bc}=\frac{[b^2-2bc+c^2]-a^2}{4bc}$

$=\frac{(b+c)^2-a^2}{4bc}=\frac{(b+c+a)(b+c-a)}{4bc}$ $\left[\begin{array}{l}\therefore By\ difference\ of\ two\ squares\\ i.e. A^2-B^2=(A+B)(A-B)\end{array}\right]$

From $s=\frac{1}{2}(a+b+c)\Rightarrow 2s=a+b+c$

Subtracting a on both sides

$\Rightarrow s-a=\frac{1}{2}(a+b+c)-a=\frac{1}{2}(b+c-a)\ or\ 2(s-a)=b+c-a$

$\therefore cos^2\frac{A}{2}=\frac{2s.2(s-a)}{4bc}=\frac{s(s-a)}{bc}\Rightarrow \boldsymbol{cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}}$

Prove the other two half angle identities of cosines as above.

Now area of a triangle is obtained as follows:

From case (a) above,

$\therefore Area, \Delta=\frac{1}{2}.b.c\sin A=\frac{1}{2}.b.c.2\sin\frac{A}{2}\cos\frac{A}{2}=b.c.\sin\frac{A}{2}\cos\frac{A}{2}$

But $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ and $cos\ \frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$

$\therefore Area, \Delta=b.c\times\sqrt{\frac{(s-b)(s-c)}{bc}}\times\sqrt{\frac{s(s-a)}{bc}}=\sqrt{s(s-a)(s-b)(s-c)}$

$$\boxed{\therefore \boldsymbol{Area, \Delta=\sqrt{s(s-a)(s-b)(s-c)}}}$$

This formula is called ***Heron's formula*** *or* ***Hero's formula of area of triangle.***

Example

1. *Find the area of a ΔABC if $a = 7, b = 8$ and $c = 9$*

Solution

Using $\therefore Area, \Delta = \sqrt{s(s-a)(s-b)(s-c)}$ and $s = \frac{1}{2}(a+b+c)$

$s = \frac{1}{2}(7+8+9) = 12$

$\therefore Area, \Delta = \sqrt{12.(12-7)(12-8)(12-9)} = 12\sqrt{5}\ sq.units$

2. (a) *prove that in any ΔABC, $\frac{b^2-c^2}{a^2} = \frac{\sin(B-C)}{\sin(B+C)}$*

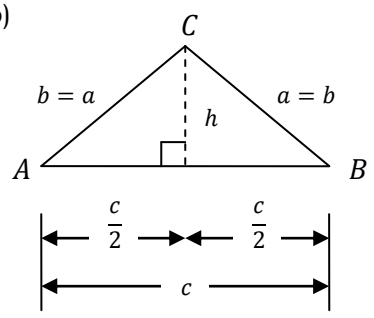
(b) *Show that for any isosceles triangle ABC with $AB = c$, the base, area of the triangle is given by $\Delta = \frac{1}{2}c\sqrt{s(s-c)}$, where* s *is half perimeter of the triangle.*

Given that $\Delta = \sqrt{3}$ and $s = 4$, determine the sides of the triangle. (***Uneb* 1992**)

Solution

a) See notes on cosine/sine rule application

b)



Given: $AC = BC = a = b$

$Area, \Delta = \frac{1}{2}\overline{AB}.h = \frac{1}{2}c.b\sin A$

$= \frac{1}{2}bc.2\sin\frac{A}{2}\cos\frac{A}{2}$

$\therefore \Delta = bc.\sin\frac{A}{2}\cos\frac{A}{2}$ .....(1)

From $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, and squaring *eqn.*(1)

$\Delta^2 = (bc)^2.\sin^2\frac{A}{2}\cos^2\frac{A}{2}$

$s - b = \frac{1}{2}(a+b+c) - b = \frac{1}{2}(a+c-b)$

But $a = b$

$\therefore s - b = \frac{1}{2}(a+c-a) = \frac{1}{2}c$

$s - c = \frac{1}{2}(a+b+c) - c = \frac{1}{2}(a+b-c)$

Also $a = b$

$\therefore s - c = \frac{1}{2}(a+a-c) = \frac{1}{2}(2a-c)$

$s - a = \frac{1}{2}(a+b+c) - a = \frac{1}{2}(b+c-a)$

But $a = b$

$\therefore s - a = \frac{1}{2}(a+c-a) = \frac{1}{2}c$

Also from $s = \frac{1}{2}(a+b+c)$ and $a = b \Rightarrow s = \frac{1}{2}(a+a+c) = \frac{1}{2}(2a+c)$

$\therefore \Delta^2 = (bc)^2.\sin^2\frac{A}{2}\cos^2\frac{A}{2}$, becomes

$\Delta^2 = b^2c^2.\left[\frac{\frac{1}{2}c\times\frac{1}{2}(2a-c)}{bc}\right].\left[\frac{\frac{1}{2}c\times\frac{1}{2}(2a+c)}{bc}\right] = \frac{\frac{1}{4}c^2(2a-c).\frac{1}{4}(2a+c)}{b^2c^2}.b^2c^2$

$= \frac{1}{16}c^2.(2a-c)(2a+c)$

But $s = \frac{1}{2}(2a+c) \Rightarrow 2s = 2a+c$ and also

$s - c = \frac{1}{2}(2a-c) \Rightarrow 2(s-c) = 2a-c$

$\therefore \Delta^2 = \frac{1}{16}c^2 \times 2s \times 2(s-c) = \frac{1}{4}c^2.s(s-c)$

$\therefore \Delta = \frac{1}{2}c\sqrt{s(s-c)}$

Given that $\Delta = \sqrt{3}, s = 4, a = b = ?, c = ?$

From $\Delta = \frac{1}{2}c\sqrt{s(s-c)}$, squaring both sides

$\Delta^2 = \frac{1}{4}c^2.s(s-c) \Rightarrow 4\Delta^2 = c^2.4(4-c)\ or\ \Delta^2 = c^2(4-c)$

But $\Delta^2 = \left(\sqrt{3}\right)^2 = 3$, $\therefore 3 = 4c^2 - c^3\ or\ c^3 - 4c^2 + 3 = 0$

Solving this polynomial equation,

Let $f(c) = c^3 - 4c^2 + 3 = 0$

For $f(1) = 1 - 4 + 3 = 0 \Rightarrow c - 1$ is a factor of $f(c)$

By long division

$$
\begin{array}{r|l}
c-1 & c^2 - 3c - 3 \\
\hline
 & c^3 - 4c^2 + 0c + 3 \\
 & \underline{c^3 - c^2} \\
 & \quad -3c^2 + 3 \\
 & \quad \underline{-3c^2 + 3c} \\
 & \qquad -3c + 3 \\
 & \qquad \underline{-3c + 3} \\
 & \qquad\quad 0
\end{array}
$$

$\therefore f(c) = (c^2 - 3c - 3)(c-1) = 0$

Solving $c^2 - 3c - 3 = 0$

$c = \frac{3\pm\sqrt{9+12}}{2} = \frac{3\pm4.6}{2}$

$\therefore c = 1, 3.8 \qquad [c = -0.8\ is\ rejected]$

Now from $s = \frac{1}{2}(a+b+c) = \frac{1}{2}(2a+c)$,

For $s = 4, c = 1 \Rightarrow 4 = \frac{1}{2}(2a+1)\ or\ a = 3.5 = b$

For $s = 4, c = 3.8 \Rightarrow 4 = \frac{1}{2}(2a+3.8)\ or\ a = 2.1 = b$

Half angle identities proofs of a triangle

In any ΔABC , show that

a) $sinA = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

b) $(a+b+c)\left(tan\frac{A}{2}+tan\frac{B}{2}\right) = 2c.cotC$

c) $1 - tan\frac{A}{2}.tan\frac{B}{2} = \frac{c}{s}\ ; s = \frac{1}{2}(a+b+c)$

d) $tan\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}$

e) $Area\ , \Delta = s^2 tan\frac{A}{2}tan\frac{B}{2}tan\frac{C}{2};\ s = \frac{1}{2}(a+b+c)$

f) $\frac{1}{a}cos^2\frac{A}{2} + \frac{1}{b}cos^2\frac{B}{2} + \frac{1}{c}cos^2\frac{C}{2} = \frac{(a+b+c)^2}{4abc}$

g) $Area\ , \Delta = \frac{1}{2}a^2\frac{sinBsinC}{sin(B+C)}$

h) $abc = 4R\Delta\ , where\ R = radius\ of\ circumscribed\ circle\ and\ \Delta = area\ of\ \Delta ABC$

Solution

a) $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$, but $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

$\therefore sinA = 2.\sqrt{\frac{(s-b)(s-c)}{bc}}.\sqrt{\frac{s(s-a)}{bc}} = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

b) $(a+b+c)\left(tan\frac{A}{2}+tan\frac{B}{2}\right) = 2c.cotC$

From $s = \frac{1}{2}(a+b+c) \Rightarrow 2s = a+b+c$

Also $\tan\frac{A}{2} = \frac{sin\frac{A}{2}}{cos\frac{A}{2}} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Similarly $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$ and $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

$\therefore (a+b+c)\left(tan\frac{A}{2}+tan\frac{B}{2}\right) = 2s\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\right]$

$= 2s\left[\frac{\sqrt{(s-b)^2(s-c).s}+\sqrt{s.(s-a)^2(s-c)}}{\sqrt{s^2(s-a)(s-b)}}\right] = 2s\left[\frac{(s-b)\sqrt{s(s-c)}+(s-a)\sqrt{s(s-c)}}{s\sqrt{(s-a)(s-b)}}\right]$

$= \left[\frac{2s\sqrt{s(s-c)}[s-b+s-a]}{s\sqrt{(s-a)(s-b)}}\right] = \frac{2\sqrt{s(s-c)}.[2s-(a+b)]}{\sqrt{(s-a)(s-b)}} = \frac{2\sqrt{s(s-c)}.[2s-(2s-c)]}{\sqrt{(s-a)(s-b)}}$

$= \frac{2\sqrt{s(s-c)}.c}{\sqrt{(s-a)(s-b)}} = 2c.\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = 2c.\frac{1}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}} = 2c.\frac{1}{tan\frac{C}{2}} = 2c.cot\frac{C}{2}$

c) $1 - tan\frac{A}{2}.tan\frac{B}{2} = \frac{c}{s}\ ; s = \frac{1}{2}(a+b+c)$

$1 - tan\frac{A}{2}.tan\frac{B}{2} = 1 - \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.\sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = 1 - \sqrt{\frac{(s-a)(s-b)(s-c)^2}{s^2(s-a)(s-b)}}$

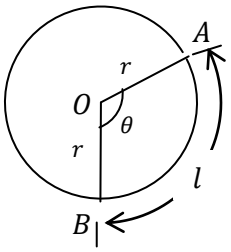
$= 1 - \sqrt{\frac{(s-c)^2}{s^2}} = 1 - \frac{s-c}{s} = \frac{s-s+c}{s} = \frac{c}{s}$

d) $(d), (e), (f)\ and\ (h)$, are left as an exercise.

RADIANS

The unit of angular measure is a *radian,*

Length of an arc of a circle



$\theta \propto l \Rightarrow \theta = kl\,, k = a\ constant$

When $\theta = 360^0\,, then\ l = circumfrence\ of\ a\ circle = 2\pi r$

$\therefore l = k\theta\ becomes\ 2\pi r = k \times 360^0\ or\ k = \frac{\pi r}{180^0}$

Thus $l = \frac{\pi r\theta}{180^0}$

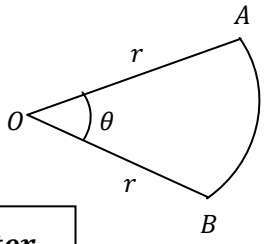
Length of an arc $= \left(\frac{\pi}{180}\right).r\theta$

Area of a sector of circle

$Area\,,\ A \propto \theta \Rightarrow A = k\theta\,, k = constant$

When $\theta = 360^0\,, area\,, A = \pi r^2$

$\therefore k = \frac{A}{\theta} = \frac{\pi r^2}{360^0}$



$\therefore$ $\boldsymbol{Area, A = \left(\frac{\pi}{360^0}\right) r^2\theta = Area\ of\ a\ sector}$

NOTE:

From length of an arc, $= \left(\frac{\pi}{180}\right).r\theta$, if $l = r = radius\ of\ a\ circle, then\ \theta =$ $1\ radians$

$r = \frac{\pi}{180}.r\ \theta\ or\ 1\ \pi\ rad = 180^0$

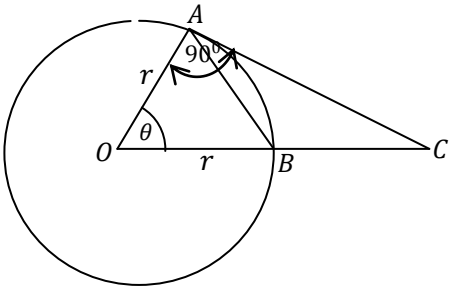
$\therefore$ $\boldsymbol{1\pi\ radian = 180^0}$

$\boldsymbol{Length\ of\ an\ arc = r.\theta}$

$\boldsymbol{Area\ of\ a\ sector = \frac{1}{2}r^2\theta}$

Trigonometric ratios of small angles in Radians

Consider the figure



Let $A_1 = area\ of\ \Delta OAB$

$A_2 = area\ of\ sector\ OAB$

$A_3 = area\ of\ \Delta AOC$

, and $\Delta OAC\ is\ aright\ triangle$

$A_1 = \frac{1}{2}.\overline{OA}.\overline{OB}.\sin\theta = \frac{1}{2}r^2\sin\theta$

$A_2 = \frac{1}{2}r^2\theta$

$A_3 = \frac{1}{2}.\overline{OA}.\overline{AC} = \frac{1}{2}r.r\tan\theta$

$= \frac{1}{2}r^2\tan\theta$

From the figure,

$A_1 < A_2 < A_3 \Rightarrow \frac{1}{2}r^2 sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2 tan\theta$

Dividing through by $\frac{1}{2}r^2\theta$

$\Rightarrow sin\theta < \theta < tan\theta$ .....$(\div sin\theta)$

$\Rightarrow 1 < \frac{\theta}{sin\theta} < \frac{sin\theta}{cos\theta}.\frac{1}{sin\theta}$

$\Rightarrow 1 < \frac{\theta}{sin\theta} < \frac{1}{cos\theta}$

Now as $\theta \to 0$, $cos\theta \to 1$, and $\therefore \frac{\theta}{sin\theta}$ will lie between 1 $and \frac{1}{cos\theta}$

$\Rightarrow 1 < \frac{\theta}{sin\theta} < 1\ or \frac{\theta}{sin\theta} \to 1\ as\ \theta \to 0$

$\therefore \theta \to \sin\theta\ \ as\ \theta \to 0$

$$\boxed{\therefore \sin\theta \approx \theta}$$

$\cos\theta = 1 - 2sin^2\frac{\theta}{2}$ $[From\ Half\ angle\ identities]$

$As\ \theta \to 0$, $cos\theta \to 1$, $\sin\frac{\theta}{2} \to \frac{\theta}{2}$

$\therefore \cos\theta \to 1 - 2.\left(\frac{\theta}{2}\right)^2 = 1 - \frac{1}{2}\theta^2$

$$\boxed{\therefore \cos\theta \approx 1 - \frac{1}{2}\theta^2}$$

$\tan\theta = \frac{sin\theta}{cos\theta} \approx \frac{\theta}{1-\frac{1}{2}\theta^2} = \frac{2\theta}{2-\theta^2}$

Now if θ is very small, then $\theta^2 \approx 0\ .i.e\ more\ smaller\ than\ \theta$

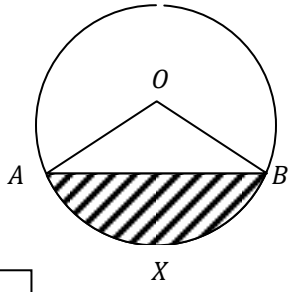
$\therefore \tan\theta \approx \frac{2\theta}{\theta} = \theta$

$$\boxed{\therefore \tan\theta \approx \theta}$$

The above enclosed results are very vital in differentiation from first principles and in approximation of trigonometric functions.

Area of a segment

Consider the figure below



$Area\ of\ segment = Area\ of\ shaded\ region\ AXB\ (minor)$

$= Area\ of\ sector\ OAXB - Area\ of \Delta OAB$

$\Rightarrow Area\ of\ segment\ (minor) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 sin\theta$

$$\therefore \boldsymbol{Area\ of\ segment\ (minor) = \frac{1}{2}r^2(\theta - sin\theta)}$$

$Area\ of\ segment(AYB) = Area\ of\ circle - Area\ of\ minor\ segment\ (major)$

$$= \pi r^2 - \frac{1}{2}r^2(\theta - sin\theta)$$

$$\therefore Area\ of\ major\ segment = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{r^2(2\pi - \theta + sin\theta)}$$

Conversion of Radians to Degrees

1) *Convert*

i) $2\ radians$

ii) $4\pi\ radians\ to\ degrees.$

Solution

i) Using $1\pi\ rad = 180^0\ or\ 1\ rad = \frac{180^0}{\pi}$

$\Rightarrow 2\ radians = \frac{180^0}{\pi} \times 2 = \frac{360^0}{\pi}$

ii) $4\pi\ radians = \frac{180^0}{\pi} \times 4\pi = 720^0$

2) *Convert*

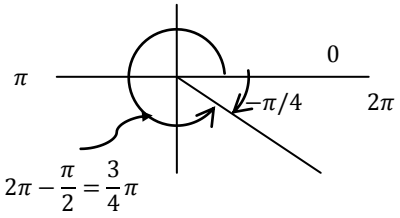
i) -45^0

ii) $120^0\ to\ radians$

Solution

i) $1\pi\ rad = 180^0\ or\ 1^0 = \frac{\pi}{180^0}$

$\Rightarrow -45^0 = \frac{\pi}{180^0} \times -45^0 = -\frac{\pi}{4}\ radians$



$\therefore -45^0 = \frac{3}{4}\pi\ radians$

ii) $1\pi\ rad = 180^0\ , 120^0 = \frac{\pi}{180^0} \times 120^0 = \frac{2}{3}\pi\ radians$

$\therefore 120^0 = \frac{2}{3}\pi\ radians$

Note

Angle measurement in *clockwise direction* give rise to *negative angles,* now concluding in terms of positive angles, proceed as in (i) above.

3) *Convert*

i) $-30^0\ to\ radians$

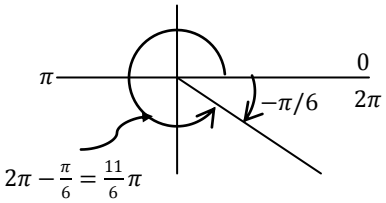
ii) $-\frac{\pi}{3}\ radians\ to\ degrees$

Solution

i) $1\pi\ rad = 180^0\ or\ 1^0 = \frac{\pi}{180^0}$

Now $-30^0 = \frac{\pi}{180^0} \times -30 = -\frac{\pi}{6}$

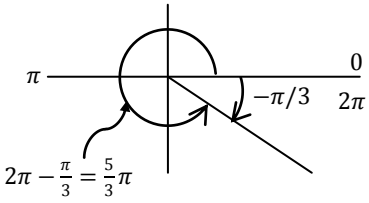
$\therefore -30^0 = 2\pi - \frac{\pi}{6} = \frac{11}{6}\pi$



ii) $1\pi\ rad = 180^0$

$-\frac{\pi}{3} rad = 180 \times -\frac{1}{3} = -60^0$

$\therefore -\frac{\pi}{3} rad = \frac{5}{3} \times 180 = 300^0$



DIFFERENTIATION

Definition:

The process of finding the gradient function of a curve is *Differentiation.*

Note:

Gradient function is also called the derived function or derivative

Denotion:

If $y = f(x)$ is a curve, then the gradient function of the curve is given by $\frac{dy}{dx} = f'(x)$

Gradient of a curve at any point

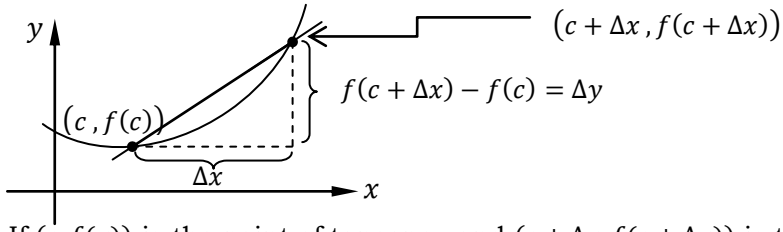
Definition:

The gradient of a curve at any point is the gradient of the tangent to the curve at that point.

When the tangent of a line is determined by means of limits leads to the definition of *derivative of a function.*

The slope of a tangent line at $P(c, f(c))$ can be approximated using a *secant line*

Consider the figure below



If $(c, f(x))$ is the point of tangency and $(c + \Delta x, f(c + \Delta x))$ is the second point on the graph of $f(x)$, the slope of the secant line through the points is given by;

$$m_{sec} = \frac{f(c+\Delta x)-f(c)}{(c+\Delta x)-c} = \frac{f(c+\Delta x)-f(c)}{\Delta x}$$

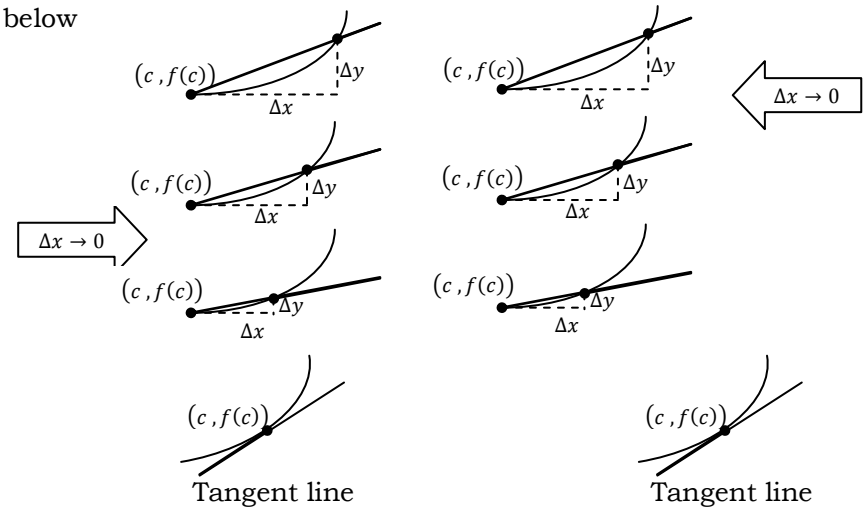
$\frac{\text{change in y}}{\text{change in x}}$

Slope of the secant line

The right hand side of this equation is a *Differential Quotient.*

The denominator Δx is the *change in* x and the numerator is the *change in* $\Delta y. i. e.\ \Delta y = f(c + \Delta x) - f(c).$

To approximate accurately the slope of the tangent line, $\Delta x \to 0$ as in fig. below



Definition of Tangent line with slope m
If $f(x)$ is defined on an open interval containing c and if the limit
$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} = m$, exists, then the line passing through $(c\,, f(c))$ with slope m is the tangent line to the graph of $f(x)$ at the point $(c\,, f(c))$.

The slope of the tangent line to the graph of $f(x)$ at the point $(c\,, f(c))$ is also called *slope of the graph of* $f(x)$ *at c.*

NOTE
Since the tangent of a line is to be determined by limits the four basic limit theorems involving a combination of functions are listed below. In each theorem we assume that
$\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists.

<u>Theorem 1:</u> SUM THEOREM OF LIMITS
The limit $\lim_{x \to a}[f(x) + g(x)]$ exists, and
$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

<u>Theorem 2:</u> CONSTANT MULTIPLE THEOREM FOR LIMITS
For any real number, the limit $\lim_{x \to a} c.f(x)$ exists, and
$\lim_{x \to a} c.f(x) = c \lim_{x \to a} f(x)$

<u>Theorem 3:</u> PRODUCT THEOREM FOR LIMITS
The limit $\lim_{x \to a}[f(x).g(x)]$ exists, and

$\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$

Theorem 4: QUOTIENT THEOREM FOR LIMITS

If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} f(x)/g(x)$ exists, and $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Other limits to be encountered are:

i) $\lim_{x\to a} c = c$
ii) $\lim_{x\to a} x = a$
iii) $\lim_{x\to a} cx = ca$
iv) $\lim_{x\to a} x^n = a^n$
v) $\lim_{x\to a} \sqrt{x} = \sqrt{a}$ for $a > 0$
vi) $\lim_{x\to 0} \sin x = 0$
vii) $\lim_{x\to 0} \cos x = 1$
viii) $\lim_{x\to a} \frac{1}{x^n} = \frac{1}{a^n}$ for $a \neq 0$
ix) $\lim_{x\to a} x^{1/n} = a^{1/n}$

Example 1

The Slope of the Graph of a Linear Function

Find the slope of the graph $f(x) = 2x - 3$ *at the point* $(2\,,1)$

Solution

To find the slope of the graph of $f(x)$ when $c = 2$, apply the definition of the slope of the tangent as follows

$$\lim_{\Delta x\to 0} \frac{f(2+\Delta x)-f(2)}{\Delta x} = \lim_{\Delta x\to 0} \frac{f[2(2+\Delta x)-3]-f[2(2)-3]}{\Delta x}$$
$$= \lim_{\Delta x\to 0} \frac{4+2\Delta x-3-4+3}{\Delta x} = \lim_{\Delta x\to 0} \frac{2\Delta x}{\Delta x}$$
$$= \lim_{\Delta x\to 0} 2 = 2$$

Thus the slope of $f(x) = 2x - 3$ is $m = 2$

Example 2

Tangent lines to the Graph of Non linear Function

Find the slope of the tangent line to the graph of $f(x) = x^2 + 1$ *at the points* $(0\,,1)$ *and* $(-1\,,2)$

Solution

Let $\big(x\,, f(x)\big)$ represent an arbitrary point on the graph of$f(x)$. The slope of the tangent at $\big(x\,, f(x)\big)$ is given by

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{(x+\Delta x)^2+1-(x^2+1)}{\Delta x}$$
$$\lim_{\Delta x\to 0} \frac{x^2+2x\Delta x+(\Delta x)^2+1-x^2-1}{\Delta x} = \lim_{\Delta x\to 0} \frac{2x\Delta x+(\Delta x)^2}{\Delta x}$$
$$\lim_{\Delta x\to 0}(2x + \Delta x) = 2x$$

Thus slope at any point $\big(x\,, f(x)\big)$ on the graph of $f(x)$ is $m = 2x$

At the point $(0\,,1)\,, m = 2(0) = 0$ *and at* $(-1\,,2)\,, m = 2(-1) = -2$

THE DERIVATIVE OF A FUNCTION

Definition of the Derivative of a Function

The *derivative* of $f(x)$ at x is given by

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

, provided the limit exists

Other notations of derivative of a function $y = f(x)$

$$f'(x), \frac{dy}{dx}, y', \frac{d}{dx}[f(x)], D_x[y]$$

The notation $\frac{dy}{dx}$ using limit notation is;

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

DIFFERENTIATING FROM FIRST PRINCIPLES
(The Differential Coefficient)

The process of calculating the ratio of the incremental change in a function of y of x to the incremental change in x, $.i.e.$ determining an expression for $\frac{\delta y}{\delta x}$ and then finding the limiting value of this ratio as δx approaches zero ($i.e.\ \delta x \to 0$) is known as *Differentiating From First Principles.*

NOTE

For a general function $y = f(x)$, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

Examples

1. *Differentiate from first principles* $f(x) = x^3 + 2x$

Solution

Using the limit process

$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3+2(x+\Delta x)-(x^3+2x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3+2x+2\Delta x-x^3-2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3+2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(3x^2+3x\Delta x+(\Delta x)^2+2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + (\Delta x)^2 + 2 = 3x^2 + 2$$

Alternatively

Let $f(x) = y = x^3 + 2x$

Let δy be small increment in y which gives rise to a small increment δx in x

$y + \delta y = (x + \delta x)^3 + 2(x + \delta x)$

$\delta y = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 + 2x + 2\delta x - x^3 - 2x$

$\delta y = 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 + 2\delta x$

Dividing through by δx

$\frac{\delta y}{\delta x} = 3x^2 + 3x\delta x + (\delta x)^2 + 2$

Introducing limits on both sides

$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} 3x^2 + 3x\delta x + (\delta x)^2 + 2$

$\therefore \frac{dy}{dx} = f'(x) = 3x^2 + 2$

2. *Differentiate from first principles $f(x) = \sqrt{x}$. Hence find the slope of the graph at the points $(1, 1)$ and $(4, 2)$*

Solution

Using $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$, Rationalizing the Numerator

$= \lim_{\Delta x \to 0} \left[\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\right]\left[\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}\right]$

$= \lim_{\Delta x \to 0} \frac{x+\Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$

$= \lim_{\Delta x \to 0} \frac{1}{(\sqrt{x+\Delta x} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$

At the point $(1, 1)$, the slope is $f'(1) = \frac{1}{2}$

At the point $(4, 2)$, the slope is $f'(4) = \frac{1}{4}$

3. *Find from first principles the derivative with respect to t for the function $\frac{2}{t}$*

Solution

Let $y = \frac{2}{t}$

$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$

$= \lim_{\Delta t \to 0} \frac{\frac{2}{t+\Delta t} - \frac{2}{t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{2t - 2(t+\Delta t)}{(t+\Delta t)t.\Delta t}$

$= \lim_{\Delta t \to 0} \frac{-2\Delta t}{(t+\Delta t)t.\Delta t} = \lim_{\Delta t \to 0} \frac{-2}{(t+\Delta t)t} = -\frac{2}{t^2}$

Basic Rules in Differentiation

> **Theorem 1: Constant Rule**
> The derivative of a constant function is 0 *. i. e.* If c is a real number then $\frac{d}{dx}(c) = 0$

Proof:

Let $f(x) = c$, then by limit definition of the derivative

Theorem 2: The Power Rule
If n is a rational number, then the function $f(x) = x^n$ is differentiable and $\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(c) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$
$= \lim_{\Delta x \to 0} \frac{c-c}{\Delta x} = 0$

Proof:
If n is a positive integer greater than1, the binomial expansion produces the following

$\frac{d}{dx}(x^n) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$
$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}(\Delta x)^2 + \cdots\ldots + (\Delta x)^n - x^n}{\Delta x}$
$= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \frac{n(n-1)}{2!}(\Delta x)^2 + \cdots\ldots + (\Delta x)^n}{\Delta x}$
$= \lim_{\Delta x \to 0} \frac{\Delta x\left[nx^{n-1} + \frac{n(n-1)}{2!}(\Delta x)^{2-1} + \cdots\ldots + (\Delta x)^{n-1}\right]}{\Delta x} = nx^{n-1}$

NOTE:

1. For $n = 1, \frac{d}{dx}(x) = 1$
2. The binomial expansion is used in proving a special case of the above Power Rule. $i.e.$

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}(\Delta x)^2 + \cdots\ldots + (\Delta x)^n$$

Theorem 3: The Constant Multiple Rule
If $f(x)$ is a differentiable function and c is a real number, then $c.f(x)$ is also differentiable and $\frac{d}{dx}\big(c.f(x)\big) = c.f'(x)$

Proof:
$\frac{d}{dx}\big(c.f(x)\big) = \lim_{\Delta x \to 0} \frac{c.f(x+\Delta x) - c.f(x)}{\Delta x}$
$= \lim_{\Delta x \to 0} \frac{c[f(x+\Delta x) - f(x)]}{\Delta x} = c \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$
$= c.f'(x)$

NOTE:
The constant Multiple Rule states that a constant can be factored out of the differentiation process, even if it appears in the denominator.
$\frac{d}{dx}\big(c.f(x)\big) = c.\frac{d}{dx}\big(f(x)\big) = c.f'(x)$
$\frac{d}{dx}\left(\frac{f(x)}{c}\right) = \frac{1}{c}.\frac{d}{dx}\big(f(x)\big) = \frac{1}{c}.f'(x)$

Theorem 4: The Sum and Difference Rules
If the sum (or difference) of two differentiable functions is differentiable and is sum(or difference) of their derivatives.
$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$; Sum Rule
$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$; Difference Rule

Proof:

$$\frac{d}{dx}[f(x)+g(x)] = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-[f(x)+g(x)]}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x)}{\Delta x}\right]$$
$$= \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]$$
$$= \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]+\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right] = f'(x)+g'(x)$$

The sum and the difference can be extended to any number of finite functions. e.g.
If $F(x) = f(x)+g(x)-h(x)-k(x)$, then
$F'(x) = f'(x)+g'(x)-h'(x)-k'(x)$

Examples
Find the derivative of:

a) $f(x) = x^3$
b) $g(x) = \sqrt[3]{x}$
c) $y = \frac{1}{x^2}$
d) $y = \frac{2}{x}$
e) $f(x) = x^3 - 4x + 5$
f) $y = -\frac{x^4}{2} + 3x^3 - 2x$
e) $f(t) = \frac{4t^2}{5}$
f) $y = \frac{1}{2\sqrt[3]{x^2}}$
g) $y = -\frac{3}{2}x$

Solution
The following order can be used .i.e.

Given function	Rewrite	Differentiate	Simplify
$f(x) = x^3$		$f'(x) = 3x^2$	
$g(x) = \sqrt[3]{x}$	$g(x) = x^{\frac{1}{3}}$	$g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$	$g'(x) = \frac{1}{3\sqrt[3]{x^2}}$
$y = \frac{1}{x^2}$	$y = x^{-2}$	$\frac{dy}{dx} = (-2)x^{-3}$	$\frac{dy}{dx} = -\frac{2}{x^3}$

Given function	Rewrite	Differentiate	Simplify
$y=\frac{2}{x}$	$y=2x^{-1}$	$\frac{dy}{dx}=(-2)x^{-2}$	$\frac{dy}{dx}=-\frac{2}{x^2}$
$f(t)=\frac{4t^2}{5}$	$f(t)=\frac{4t^2}{5}$	$f'(t)=\frac{4}{5}(2)t$	$f'(t)=\frac{8}{5}t$
$y=\frac{1}{2\sqrt[3]{x^2}}$	$y=\frac{1}{2}x^{-\frac{2}{3}}$	$\frac{dy}{dx}=\left(\frac{1}{2}\right)\left(-\frac{2}{3}\right)x^{-\frac{5}{3}}$	$\frac{dy}{dx}=-\frac{1}{3\sqrt[3]{x^5}}$
$y=\frac{1}{x^2}$	$y=x^{-2}$	$\frac{dy}{dx}=(-2)x^{-3}$	$\frac{dy}{dx}=-\frac{2}{x^3}$
$y=-\frac{3}{2}x$	$y=-\frac{3}{2}x$	$\frac{dy}{dx}=\left(-\frac{3}{2}\right)(1)$	$\frac{dy}{dx}=-\frac{3}{2}$
$f(x)=x^3-4x+5$		$f'(x)=3x^2-4$	$f'(x)=3x^2-4$
$y=-\frac{x^4}{2}+3x^3-2x$		$\frac{dy}{dx}=-\frac{4}{2}x^3+3(3x^2)-2$	$\frac{dy}{dx}=-2x^3+9x^2-2$

The Product Rule and the Quotient Rule

Theorem 5: The Product Rule
The product of two differentiable functions, $f(x)$ and $g(x)$, is itself differentiable. Moreover, the derivative of $f(x).g(x)$ is the first function times the derivative of the second, plus the second function times the derivative of the first.
$\frac{d}{dx}[f(x).g(x)]=f(x).g'(x)+g(x).f'(x)$

Proof:
$\frac{d}{dx}[f(x).g(x)]=\lim_{\Delta x\to 0}\frac{f(x+\Delta x).g(x+\Delta x)-[f(x).g(x)]}{\Delta x}$
Now adding and subtacting $f(x+\Delta x).gx$ (*Take note of this*)
$\frac{d}{dx}[f(x).g(x)]=\lim_{\Delta x\to 0}\frac{f(x+\Delta x).g(x+\Delta x)-f(x+\Delta x).gx+f(x+\Delta x).gx-[f(x).g(x)]}{\Delta x}$
$=\lim_{\Delta x\to 0}\left[f(x+\Delta x).\frac{g(x+\Delta x)-g(x)}{\Delta x}+g(x).\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$
$=\lim_{\Delta x\to 0}\left[f(x+\Delta x).\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]+\lim_{\Delta x\to 0}\left[g(x).\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$
$=\lim_{\Delta x\to 0}f(x+\Delta x).\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}+\lim_{\Delta x\to 0}g(x).\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$
$\therefore\ \frac{d}{dx}[f(x).g(x)]=f(x).g'(x)+g(x).f'(x)$
Extending the Product Rule to cover more than $2-terms$ .e.g if $f(x)$, $g(x)$ and $h(x)$ are differentiable functions of x, then

$\frac{d}{dx}[f(x).g(x).h(x)]=f'(x).g(x)h(x)+g'(x).f(x).h(x)+h'(x)f(x).g(x)$

NOTE:

The alternative statement of product rule is when $f(x) = u$ and $g(x) = v$ is:

$\frac{d}{dx}(u.v) = v\frac{du}{dx} + u\frac{dv}{dx}$

Proving from first principles

Let $y = u.v$

If $\delta y\,, \delta u\,, \delta v$ are increments in $y\,, u$ and v respectively, corresponding to an increment δx in x, then

$y + \delta y = (u + \delta u).(v + \delta v)$

$= uv + u\delta v + v\delta u + \delta u.\delta v$

By subtraction

$\delta y = uv + u\delta v + v\delta u + \delta u.\delta v - uv$

Dividing through by δx

$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u.\frac{\delta v}{\delta x}$

As $\delta x \to 0\,, \delta u \to 0$ and $\frac{\delta u}{\delta x}\,, \frac{\delta v}{\delta x}\,, \frac{\delta y}{\delta x}$ tend respectively to $\frac{du}{dx}\,, \frac{dv}{dx}$ and $\frac{dy}{dx}$

$$\boxed{\frac{d}{dx}(u.v) = v\frac{du}{dx} + u\frac{dv}{dx}}$$

Example

Find the derivative of;

a) $h(x) = (3x - 2x^2)(5 + 4x)$

b) $y = (1 + x^2)(1 - 2x^2)$

Solution

a) $h'(x) = \overbrace{(3x - 2x^2)}^{\text{First}}.\overbrace{\frac{d}{dx}(5 + 4x)}^{\text{Derivative of } 2^{nd}} + \overbrace{(5 + 4x)}^{\text{Second}}.\overbrace{\frac{d}{dx}(3x - 2x^2)}^{\text{Derivative of } 1^{st}}$

$= (3x - 2x^2).(4) + (5 + 4x).(3 - 4x)$

$= 12x - 8x^2 + 15 - 8x - 16x^2$

$= -24x^2 + 4x + 15$

Alternatively; Not using Product Rule

$h(x) = (3x - 2x^2)(5 + 4x) = -8x^3 + 2x^2 + 15x$

$h'(x) = \frac{d}{dx}[-8x^3 + 2x^2 + 15x] = -24x^2 + 4x + 15$

b) $y = (1 + x^2)(1 - 2x^2)$

$\frac{dy}{dx} = (1 + x^2).\frac{d}{dx}(1 - 2x^2) + (1 - 2x^2).\frac{d}{dx}(1 + x^2)$

$= (1 + x^2)(-4x) + (1 - 2x^2).(2x)$

$= -8x^3 - 2x$

The Quotient Rule

Theorem 6: the Quotient Rule

The quotient $\frac{f(x)}{g(x)}$ of two differentiable functions $f(x)$ and $g(x)$, itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of $\frac{f(x)}{g(x)}$ is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator divided by the square of the denominator.

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2}$, $g(x) \neq 0$

Proof:

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{\Delta x\to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)}-\frac{f(x)}{g(x)}}{\Delta x} = \lim_{\Delta x\to 0} \frac{g(x).f(x+\Delta x)-f(x).g(x+\Delta x)}{\Delta x.g(x).g(x+\Delta x)}$

Now adding and subtracting $f(x).g(x)$ (T*ake note of this)*

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{\Delta x\to 0} \frac{g(x).f(x+\Delta x)-f(x).g(x)+f(x).g(x)-f(x).g(x+\Delta x)}{\Delta x.g(x).g(x+\Delta x)}$

$$= \frac{\lim_{\Delta x\to 0} g(x)\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]-\lim_{\Delta x\to 0} f(x)\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]}{\lim_{\Delta x\to 0} g(x).g(x+\Delta x)}$$

$$= \frac{g(x)\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]-f(x)\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]}{g(x)\lim_{\Delta x\to 0}.g(x+\Delta x)}$$

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2}$

Example

Find the derivative of;

a) $y = \frac{5x-2}{x^2+1}$ b) $f(x) = \frac{1-3x^2}{2+4x^2}$

Solution

a) $\frac{dy}{dx} = \frac{(x^2+1).\frac{d}{dx}(5x-2)-(5x-2).\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{(x^2+1)(5)-(5x-2)(2x)}{(x^2+1)^2}$

$= \frac{5x^2+5-10x^2-4x}{(x^2+1)^2} = \frac{-5x^2+4x+5}{(x^2+1)^2}$

b) $f'(x) = \frac{(2+4x^2).\frac{d}{dx}(1-3x^2)-(1-3x^2).\frac{d}{dx}(2+4x^2)}{(2+4x^2)^2} = \frac{(2+4x^2).(-6x)-(1-3x^2).(8x)}{(2+4x^2)^2}$

$= \frac{-12x-24x^3-8x+24x^3}{(2+4x^2)^2} = -\frac{20x}{[2(1+2x^2)]^2} = -\frac{5x}{(1+2x^2)^2}$

Remark

Prove that $\frac{d}{dx}[x^n] = nx^{n-1}$ *for a negative integer exponent*

Solution

Let $n = -k$

$\Rightarrow \frac{d}{dx}[x^n] = \frac{d}{dx}\left[\frac{1}{x^k}\right]$

By quotient rule,

$\frac{d}{dx}[x^{-k}] = \frac{x^k.(0)-(1).(kx^{k-1})}{x^{2k}} = -kx^{-k-1}$

$\therefore \frac{d}{dx}[x^{-k}] = -kx^{-k-1} = nx^{n-1}$

The Chain Rule

Theorem 7: The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and;

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Or equivalently $\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$

NOTE

The chain rule requires the General Power Rule. This rule is applicable when differentiating functions of the form

$y = [u(x)]^n$

Theorem 8: The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$\frac{dy}{dx} = n[u(x)]^{n-1}.\frac{du}{dx}$

Or equivalently

$\frac{d}{dx}[u^n] = nu^{n-1}.u'$

Proof:

Because $y = u^n$, apply chain rule.

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{d}{du}(u^n).\frac{du}{dx}$

Example

Find the derivative of:

a) $f(x) = (3x - 2x^2)^3$ b) $y = (1 - 3x^2)^5$

Solution

a) $f(x) = (3x - 2x^2)^3$, let $u = 3x - 2x^2$

$\Rightarrow f(x) = u^3$

Applying the General Power Rule

$$\Rightarrow f'(x) = \overbrace{3}^{n}.\overbrace{(3x - 2x^2)^2}^{u^{n-1}}.\overbrace{\frac{d}{dx}(3x - 2x^2)}^{u'}$$

$$= 3(3x - 2x^2)(3 - 4x)$$

b) $y = (1 - 3x^2)^5$, let $u = 1 - 3x^2$

Applying the General Power Rule

$\frac{dy}{dx} = 5(1 - 3x^2)^4 . \frac{d}{dx}(1 - 3x^2)$

$= 5(1 - 3x^2)^4 . (-6x) = -30x(1 - 3x^2)^4$

Differentiating Functions involving Radicals

Find $\frac{dy}{dx}$ *if* $y = \sqrt[3]{(x^2 - 1)^2}$. *Hence find all points on the curve for which* $\frac{dy}{dx} = 0$

Solution

Rewriting the function

$y = (x^2 - 1)^{2/3}$, let $u = x^2 - 1$

Applying the General Power Rule

$$\Rightarrow \frac{dy}{dx} = \overbrace{\frac{2}{3}}^{n} . \overbrace{(x^2 - 1)^{-1/3}}^{u^{n-1}} . \overbrace{\frac{d}{dx}(x^2 - 1)}^{u'}$$

$= \frac{2}{3}(x^2 - 1)^{-1/3}(2x) = \frac{4x}{3\sqrt[3]{x^2 - 1}}$ is the point when $\frac{dy}{dx} = 0$

Differentiating Quotients with Constant Numerator

Differentiate $g(t) = -\frac{7}{(2t-3)^2}$

Solution

Re-writing the function

$g(t) = -\frac{7}{(2t-3)^2} = -7.(2t - 3)^{-2}$

Applying the General Power Rule

$g'(t) = -7.(-2).(2t - 3)^{-3}.(2)$

$= 28(2t - 3)^{-3} = \frac{28}{(2t-3)^3}$

Simplifying Derivatives

The knowledge of simplifying indices is necessary

a) **Simplifying by Factoring out the Least Powers**

Find $f'(x)$ *if* $f(x) = x^2\sqrt{1 - x^2}$

Solution

$f(x) = x^2\sqrt{1 - x^2} = f(x) = x^2(1 - x^2)^{1/2}$	Rewrite
$f'(x) = x^2 . \frac{d}{dx}(1 - x^2)^{1/2} + (1 - x^2)^{\frac{1}{2}} . \frac{d}{dx}(x^2)$	Product Rule
$= x^2 . \frac{1}{2} . (1 - x^2)^{-1/2} . (-2x) + (1 - x^2)^{\frac{1}{2}} . (2x)$	General Power Rule
$= -x^3 . (1 - x^2)^{-1/2} + 2x . (1 - x^2)^{\frac{1}{2}}$	Simplify
$= x(1 - x^2)^{-1/2}[-x^2 . (1) + 2 . (1 - x^2)]$	Factoring

$= \frac{x(2-3x^2)}{\sqrt{1-x^2}}$ Simplify

b) **Simplifying Derivative of a Quotient**

Given that $f(x) = \frac{x}{\sqrt[3]{x^2+4}}$, *find* $f'(x)$

Solution

$f(x) = \frac{x}{\sqrt[3]{x^2+4}} = \frac{x}{(x^2+4)^{1/3}}$ Rewrite

$f'(x) = \frac{(x^2+4)^{1/3}.(1)-x.\frac{1}{3}.(x^2+4)^{-2/3}.(2x)}{(x^2+4)^{2/3}}$ Quotient Rule

$= \frac{1}{3}.(x^2+4)^{-2/3}\left[\frac{3(x^2+4)-2x^2}{(x^2+4)^{2/3}}\right]$ Factoring

$= \frac{x^2+12}{3(x^2+4)^{4/3}}$ Simplifying

c) **Simplifying Derivative of a Power**

Find $\frac{dy}{dx}$ *if* $y = \left(\frac{3x-1}{x^2+3}\right)^2$

Solution

$\frac{dy}{dx} = 2.\left(\frac{3x-1}{x^2+3}\right).\frac{d}{dx}\left(\frac{3x-1}{x^2+3}\right)$ General Power Rule

$= \left[\frac{2(3x-1)}{x^2+3}\right].\left[\frac{(x^2+3).3-(3x-1).2x}{(x^2+3)^2}\right]$ Quotient Rule

$= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3).(x^2+3)^2}$ Simplify

$= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}$ Simplify

Task

1. Find the first derivative of the algebraic function

a) $y = (2x-7)^3$, **Ans**: $\frac{dy}{dx} = 6(2x-7)^2$

b) $g(x) = 3(4-9x)^4$, **Ans**: $g'(x) = -108(4-9x)^3$

c) $f(x) = (9-x^2)^{\frac{2}{3}}$, **Ans**: $f'(x) = \frac{2}{9}(9-x^2)^{\frac{-1}{3}} \cdot (-2x) = \frac{-4x}{3(9-x^2)^{\frac{2}{3}}}$

d) $y = \sqrt[3]{(9x^2+4)}$, **Ans**: $\frac{dy}{dx} = \frac{1}{3}(9x^2+4)^{-2/3}.(18x) = \frac{6x}{(9x^2+4)^{-2/3}}$

e) $f(t) = \frac{1}{(t-3)^2}$, **Ans**: $f'(t) = -2(\mathrm{t}-3)^{-3}(1) = -\frac{2}{(\mathrm{t}-3)^3}$

f) $y = \frac{1}{\sqrt{x+2}}$, **Ans**: $\frac{dy}{dx} = -\frac{1}{2(x+2)^{3/2}}$

g) $y = \frac{x}{\sqrt{x^2+2}}$, **Ans**: $\frac{dy}{dx} = \frac{1}{(x^2+2)^{3/2}}$

2. *Evaluate the derivative of the function at the given point;*

Function	**Point**	**Answer**
$s(t) = \sqrt{t^2+2t+8}$	$(2\,,4)$	$\frac{3}{4}$
$f(x) = \frac{3}{x^3-4}$	$\left(-1\,,\frac{5}{3}\right)$	$-\frac{9}{25}$
$f(t) = \frac{3t+2}{t-1}$	$(0\,,-2)$	-5

3. a) *Given that* $y = \sqrt[3]{\frac{x-1}{(x^2-1)^2}}$, *show that* $\frac{dy}{dx} = \frac{1-3x}{3(x+1)^{5/3}(x-1)^{4/3}}$

b) *Given that* $y = \sqrt{\frac{(x+1)^3}{x+2}}$, *show that* $\frac{dy}{dx} = \frac{(2x+5)\sqrt{x+1}}{2(x+2)^{3/2}}$

4. a) *P is the point* (x, y) *and Q is the point* $(x + \delta x, y + \delta y)$*on the graph* $y = \sqrt{x}$. *Show that* $\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$, *and hence find the gradient of the curve at the point P.*

b) *A curve is defined by* $y = f(x)$. *Show from first principles that* $\frac{dy}{dx} = \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x)-f(x)}{\delta x}\right\}$. *Hence find the gradient function of the curve;*

i) $y = \frac{1}{x^2}$ **Ans:** $\frac{dy}{dx} = -\frac{2}{x^3}$

ii) $y = \frac{1}{\sqrt[3]{8-x}}$ **Ans:** $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{(8-x)^2}}$

5. *Differentiate from first principles*

a) $y = \frac{1}{x^2+1}$

b) $y = \frac{1}{x^2+x-2}$

c) $y = \frac{2}{\sqrt[3]{8-x^2}}$

d) $y = \sqrt[3]{x^2+1}$

Higher–order Derivatives

When a curve $y = f(x)$ is differentiated once, $\frac{dy}{dx}$ is of first order. When it is differentiated twice, $\frac{d^2y}{dx^2}$ is of second order.

If it is differentiated thrice, $\frac{d^3y}{dx^3}$ is of third order.

In general if it is differentiated $n-$ times, i.e. $\frac{d^ny}{dx^n}$ is of $n-$ order.

First derivative: $f'(x) = \frac{dy}{dx} = \frac{d}{dx}[f(x)]$.

Second derivative: $f''(x) = \frac{d^2y}{dx^2} = \frac{d^2}{dx^2}[f(x)]$.

Third derivative: $f'''(x) = \frac{d^3y}{dx^3} = \frac{d^3}{dx^3}[f(x)]$

n^{th} derivative: $f^n(x) = \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}[f(x)]$.

Example

1. *Find:* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *if* $y = x^3 + 4x^2 - 3$.

Solution

$\frac{dy}{dx} = 3x^2 + 8x$

$\frac{d^2y}{dx^2} = 6x + 8$

Differentiating Implicit functions

When function is expressed explicitly as a function of the other, this function is an explicit function: e.g. $y = 3x^2 - 5$. The variable y is written explicitly as a function of .

Now if a function cannot be expressed explicitly as a function of the other, it is called an implicit function. e.g. $x^2 - 2y^3 + 4y = 2$, where it is very difficult to express y as a function of x explicitly.

To differentiate implicit functions i.e. finding $\frac{dy}{dx}$, differentiate the terms involving x as usual, but when you differentiate terms involving y, you must apply chain rule because you are assuming that y is defined implicitly as a function of x.

The following theorem can be used.

Theorem

If $= y^n$, then $\frac{dz}{dx} = ny^{n-1}.\frac{dy}{dx}$

If $z = x^m y^n$, then $\frac{dz}{dx} = x^m.\frac{d}{dx}(y^n) + y^n.\frac{d}{dx}(x^m)$

$= x^m.n.y^{n-1}.\frac{dy}{dx} + my^n.x^{m-1}$

Example

Differentiate with respect to x

a) $y = x^3$ c) $z = x + 3y$

b) $z = y^3$ d) $x.y^2$

Solution

a) $\frac{dy}{dx} = \frac{d}{dx}[x^3] = 3x^2$ Variables agree, use simple Power Rule

Variables agree

b) $\frac{dz}{dx} = \frac{d}{dx}[y^3] = 3\overbrace{y^2}^{nu^{n-1}}.\overbrace{\frac{dy}{dx}}^{u'} = 3y^2.\frac{dy}{dx}$ Variables disagree, use Chain Rule

Variables disagree

c) $\frac{dz}{dx} = \frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}$ Apply Chain Rule on $\frac{d}{dx}(3y) = 3\frac{dy}{dx}$

d) $\frac{dz}{dx} = \frac{d}{dx}[x.y^2] = x.\frac{d}{dx}[y^2] + y^2.\frac{d}{dx}[x]$ Apply Product Rule

$= x.\left(2y\frac{dy}{dx}\right) + y^2.(1)$ Apply Chain Rule

$= 2xy\frac{dy}{dx} + y^2$ Simplify

Guidelines for Implicit Differentiation of an equation

1. Differentiate both sides of the equation with respect to x
2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation and move all the other terms to the right side of the equation
3. Factor $\frac{dy}{dx}$ out of the left side of the equation
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by the left hand factor that does not contain $\frac{dy}{dx}$

Example

Find $\frac{dy}{dx}$ *given that* $y^3 + y^2 - 5y - x^2 = -4$

Solution

1. Differentiate both sides of the equation w.r.t x

$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}(-4)$

$\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[5y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$

$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$

2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation

$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$

3. Factor $\frac{dy}{dx}$ out of the left side of the equation

$(3y^2 + 2y - 5)\frac{dy}{dx} = 2x$

4. Solve for $\frac{dy}{dx}$ by dividing both sides by $(3y^2 + 2y - 5)$

$\therefore \frac{dy}{dx} = \frac{2x}{3y^2+2y-5}$

Finding the Slope of a Graph Implicitly

1. *Determine the slope of the line too the graph of* $x^2 + 4y^2 = 4$ *at the point* $\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$

Solution

Differentiating the graph w.r.t.x

$2x + 8y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{4y}$

Slope at $\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$ is $-\frac{\sqrt{2}}{4.1/\sqrt{2}} = \frac{1}{2}$

2. *Determine the slope of the graph* $3(x^2 + y^2)^2 = 100xy$ *at the point* $(3, 1)$

Solution

$\frac{d}{dx}[3(x^2+y^2)^2] = \frac{d}{dx}(100xy)$

$3.(2).(x^2+y^2).\left(2x+2y\frac{dy}{dx}\right) = 100\left[x.\frac{dy}{dx}+y\right]$

$12x(x^2+y^2)+12y(x^2+y^2).\frac{dy}{dx} = 100x\frac{dy}{dx}+100y$

$12y(x^2+y^2).\frac{dy}{dx}-100x\frac{dy}{dx} = 100y-12x(x^2+y^2)$

$[12y(x^2+y^2)-100x]\ \frac{dy}{dx} = 100y-12x(x^2+y^2)$

$\therefore \frac{dy}{dx} = \frac{100y-12x(x^2+y^2)}{12y(x^2+y^2)-100x} = \frac{25y-3x(x^2+y^2)}{3y(x^2+y^2)-25x}$

At $(3,1)$, slope of the graph is

$\frac{dy}{dx} = \frac{25(1)-3(3)((3)^2+(1)^2)}{3(1)((3)^2+(1)^2)-25(3)} = \frac{25-90}{30-75} = \frac{13}{9}$

Task

Find $\frac{dy}{dx}$ by implicit differentiation and evaluate the derivative at the indicated point

Equation	**Point**	**Answer**
$xy = 4$	$(-4,-1)$	$-y/x, -1/4$
$y^2 = \frac{x^2-9}{x^2+9}$	$(3,0)$	$\frac{18x}{(x^2+y^2)y}$, undefined
$x^{2/3}+y^{2/3} = 5$	$(8,1)$	$-\sqrt[3]{y/x}$, $-\frac{1}{2}$

Finding the Second Derivative Implicitly

Given that $y^2+x^2 = 25$, find $\frac{d^2y}{dx^2}$

Solution

$2y\frac{dy}{dx}+2x = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

Differentiating w.r.t.x

$\frac{d^2y}{dx^2} = \frac{y(-1)-(-x).\frac{dy}{dx}}{y^2} = \frac{-y+x.(-x/y)}{y^2}$

$= \frac{y^2-x^2}{y^3} = -\left(\frac{x^2+y^2}{y^3}\right) = -\frac{25}{y^3}$

Task

1. *If $x^2+y^2 = y$, show that $\frac{dy}{dx} = \frac{2x}{1-2y}$. Hence show also that $\frac{d^2y}{dx^2} = \frac{2(1-8y)}{(1-2y)^3}$*
2. *Find $\frac{d^2y}{dx^2}$ in terms of x and y if*
   a) $x^2+xy = 5$ **Ans:** $10/x^3$
   b) $x^2-y^2 = 16$ **Ans:** $-16/y^3$
3. *Find the equations of the tangent line and normal to the circle at the indicated points. (The normal line at a point is perpendicular to the tangent line at the point.)*

Circle: $x^2 + y^2 = 25$	**Point:** $(-3\,,4)\ ,\ (4\,,3)$
Answer:	
At $(-3\,,4)$	**Tangent:** $3x - 4y + 25 = 0$ **Normal:** $4x + 3y = 0$
At $(4\,,3)$	**Tangent:** $4x + 3y - 25 = 0$ **Normal:** $3x - 4y = 0$

Proofs in Implicit Differentiation

a) *If* $y = \sqrt{5x^2 + 3}$, *show that* $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$

b) *If* $y = \sqrt[3]{3 - x^2}$, *show that* $3y\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 + 2 = 0$

c) *If* $y = \frac{1}{\sqrt{x^2-3}}$, *show that* $\frac{d^2y}{dx^2} + \frac{6}{y}\frac{dy}{dx} = \frac{1}{y^5}$

d) *If* $y^2 - 2y\sqrt{1 + x^2} + x^2 = 0$, *show that* $\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$

Solution

a) $y = \sqrt{5x^2 + 3} \Rightarrow y^2 = 5x^2 + 3$

$\Rightarrow 2y\frac{dy}{dx} = 10x \Rightarrow y\frac{dy}{dx} = 5x$

Differentiating w.r.t. x

$y.\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}.\frac{d}{dx}(y) = 5$

$y\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{dy}{dx} = 5$

$\therefore y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$

b) Left as an exercise

c) Left as an exercise

d) $y^2 - 2y\sqrt{1 + x^2} + x^2 = 0$

Rewriting and differentiating w.r.t.x

$y^2 - 2y(1 + x^2)^{1/2} + x^2 = 0$

$\Rightarrow 2y\frac{dy}{dx} - 2\left[(1 + x^2)^{\frac{1}{2}}.\frac{dy}{dx} + y.\frac{1}{2}(1 + x^2)^{-1/2}.2x\right] + 2x = 0$

$\Rightarrow 2y\frac{dy}{dx} - 2\sqrt{1 + x^2}\frac{dy}{dx} - \frac{2xy}{\sqrt{1+x^2}} + 2x = 0$

$\Rightarrow 2\left[y - \sqrt{1 + x^2}\right]\frac{dy}{dx} - \frac{2xy}{\sqrt{1+x^2}} + 2x = 0$

$\Rightarrow \left[y - \sqrt{1 + x^2}\right]\frac{dy}{dx} = \frac{xy}{\sqrt{1+x^2}} - x = \frac{x\left[y-\sqrt{1+x^2}\right]}{\sqrt{1+x^2}} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$

DIFFERENTIATING PARAMETRIC EQUATIONS

If two equations $x = f(t)$ and $y = \varphi(t)$ are functions of the same variable t, t is called a *parameter*.

To differentiate *.i.e.* to find $\frac{dy}{dx}$, chain rule is used.

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$$

For the second derivative

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}}$$

Example

1. *Find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *if:*

a) $x = t^3 + t^2 \;\; y = t^2 + t$

b) $x = \frac{2t}{t+2}$, $y = \frac{3t}{t+3}$

c) $x = \frac{t}{1-t}$, $y = \frac{t^2}{1-t}$

Solution

a) $x = t^3 + t^2 \Rightarrow \frac{dx}{dt} = 3t^2 + 2t$

$\Rightarrow \frac{dy}{dt} = 2t + 1$

$\therefore \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = (2t+1) \times \frac{1}{3t^2+2t} = \frac{2t+1}{3t^2+2t}$

$\therefore \frac{dy}{dx} = \frac{2t+1}{3t^2+2t}$

To find $\frac{d^2y}{dx^2}$, proceed using the formula $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{2t+1}{3t^2+2t}\right).\frac{dt}{dx}$

$= \frac{(3t^2+2t).(2)-(2t+1).(6t+2)}{(3t^2+2t)^2} . \frac{1}{(3t^2+2t)}$

$= \frac{6t^2+4t-(12t^2+4t+6t+2)}{(3t^2+2t)^3} = \frac{-6t^2-6t+2}{(3t^2+2t)^3}$

b) Left as an exercise

c) Left as an exercise

2. *Find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at the point* $\left(\frac{1}{2},\frac{1}{2}\right)$ *on the curve defined by parametric equations*

$x = \frac{t}{1+t}$, $y = \frac{t^3}{1+t}$

Solution

$x = \frac{t}{1+t} \Rightarrow \frac{dx}{dt} = \frac{1+t-t}{(1+t)^2} = \frac{1}{(1+t)^2}$

$y = \frac{t^3}{1+t} \Rightarrow \frac{dy}{dt} = \frac{(1+t).3t^2-t^3}{(1+t)^2} = \frac{3t^2+2t^3}{(1+t)^2}$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3t^2+2t^3}{(1+t)^2} \times \frac{(1+t)^2}{1}$

$\therefore \frac{dy}{dx} = 3t^2 + 2t^3$

Now to obtain the slope at $\left(\frac{1}{2}, \frac{1}{2}\right)$ a suitable value of t is needed.

Using $x = \frac{t}{1+t}$, $y = \frac{t^3}{1+t}$

The general point on the curve is $\left(\frac{t}{1+t}, \frac{t^3}{1+t}\right)$

Since $\left(\frac{1}{2}, \frac{1}{2}\right)$ is on the curve, then $\frac{t}{1+t} = \frac{1}{2} \Rightarrow 2t = 1 + t \ , \ t = 1$

$\therefore \frac{dy}{dx}_{at\ \left(\frac{1}{2},\ \frac{1}{2}\right)} = \frac{dy}{dx}_{at\ t=1} = 3(1)^2 + 2(1)^3 = 5$

Note

If $\frac{1}{2} = \frac{t^3}{1+t}$, the y-value being used in evaluation of t, t value would be $t = 1$. Thus you can use any equation in evaluating the $t-$value.

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}$

$= \frac{d}{dt}(3t^2 + 2t^3).\frac{(1+t^2)}{1}$

$= (6t + 6t^2).(1 + t^2) = 6t(1 + t)(1 + t^2)$

Since $t = 1, \frac{d^2y}{dx^2} = 6(1)(1 + 1)(1 + 1^2) = 24$

$\therefore \frac{d^2y}{dx^2}_{at\ t=1} = 24$

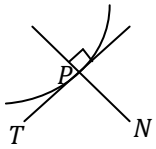
Task

Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *and evaluate each at the specified value of the parameter.*

Parametric equations	**Point**	**Answers**
$x = 2t\ , y = 3t - 1$	$t = 3$	$\frac{dy}{dx} = \frac{3}{2}\ , \frac{d^2y}{dx^2} = 0$
$x = t + 1\ , y = t^2 + 3t$	$t = -1$	$\frac{dy}{dx} = 2t + 3\ , \frac{d^2y}{dx^2} = 2$ At $t = 1, \frac{dy}{dx} = 1\ , \frac{d^2y}{dx^2} = 2$
$x = (t^2 - 1)^2\ , y = t^3$	$t = \sqrt{2}$	$\frac{dy}{dx} = \frac{3t}{4(t^2-1)}\ , \frac{d^2y}{dx^2} = -\frac{3(1+t^2)}{16t(t^2-1)^3}$ At $t = \sqrt{2}, \frac{dy}{dx} = \frac{3\sqrt{2}}{4}\ , \frac{d^2y}{dx^2} = \frac{9\sqrt{2}}{32}$
$x = \frac{t^2}{1+t^3}\ , y = \frac{t^3}{1+t^3}$		$\frac{dy}{dx} = \frac{3t}{2-t^3}$, for $t = 1, \frac{dy}{dx} = 2\ , \frac{d^2y}{dx^2} = 48$

TANGENTS AND NORMALS

Definition: A normal to a curve at a point is the straight line through the point at right angles to the tangent at the point.



$\overline{PT}$ = Tangent at P

$\overline{PN}$ = Normal at P

From the previous discussion, if $P(x_1, y_1)$ is a point on the curve, then

Slope of the tangent at $P(x_1, y_1) = \frac{dy}{dx}_{at\ P(x_1, y_1)}$

Slope of the normal at $P(x_1, y_1) = -\frac{1}{\frac{dy}{dx}_{at\ P(x_1, y_1)}}$

Example

1. *Find the equation of the tangent and normal to the curve* $y = 3x^2 - x + 1$ *at* $x = 0$

Solution

$y = 3x^2 - x + 1$, for $x = 0, y = 1 \ \therefore$ point is $(0, 1)$

Tangent

$\frac{y-1}{x-0} = m_1 \ , \ m_1 = \frac{dy}{dx}_{at\ (0,1)}$

$\frac{dy}{dx} = 6x - 1 \ \therefore m_1 = 6(0) - 1 = -1$

$\frac{y-1}{x-0} = -1 \ \therefore y = -x + 1$

Normal

$\frac{y-1}{x-0} = m_2 \ , \ m_2 = -\frac{1}{m_1} = -\frac{1}{-1} = 1$

$\therefore \frac{y-1}{x-0} = 1 \Rightarrow y = x + 1$

2. *Find the equation of the tangents to the curve* $y = x^3 - 6x^2 + 12x + 12$ *which are parallel to the line* $y = 3x$

Solution

Let the required tangent be $y = mx + c$

Now the tangent is parallel to the line $y = 3x \Rightarrow m = 3$.i.e. for parallel lines $m_1 = m_2$

Now slope of the tangent at a point $P(x, y) = \frac{dy}{dx}$

$\Rightarrow 3 = 3x^2 - 12x + 12 \ or \ -4x + 3 = 0$

$\Rightarrow (x - 1)(x - 3) = 0 \ \therefore x = 1, 3$

Now required tangents are at points $x = 1, x = 3$

For $x = 1, y = ?, x = 3, y = ?$

For $x = 1, y = 1 - 6 + 12 + 12 = 19 \ \therefore (1, 19)$

For $x = 3, y = 27 - 54 + 36 + 12 = 3 \ \therefore (3, 3)$

Required tangents

$\frac{y-19}{x-1} = 3 \Rightarrow 3x - y + 16 = 0$

$\frac{y-3}{x-3} = 3 \Rightarrow 3x - y - 3 = 0$

3. *Find the equation of the normal to the curve* $x = \frac{t}{1+t}$, $y = \frac{t^3}{1+t}$ *at the point* $\left(\frac{1}{2}, \frac{1}{2}\right)$

Solution

Point: $\left(\frac{1}{2}, \frac{1}{2}\right)$

Normal's equation

$\frac{y-1/2}{x-1/2} = m'$

But $m' = -\frac{1}{slope\ \ of\ \ the\ tangent\ \ at\ \left(\frac{1}{2},\frac{1}{2}\right)}$

Slope of the tangent at $P(x, y) = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$x = \frac{t}{1+t} \Rightarrow \frac{dx}{dt} = \frac{(1+t)-t}{(1+t)^2} = \frac{1}{(1+t)^2}$

$y = \frac{t^3}{1+t} \Rightarrow \frac{dy}{dt} = \frac{(1+t).3t^2 - t^3}{(1+t)^2} = \frac{3t^2+2t^3}{(1+t)^2}$

$\therefore \frac{dy}{dx} = \frac{3t^2+2t^3}{(1+t)^2} \times \frac{(1+t)^2}{1} = 3t^2 + 2t^3$

To obtain the slope of the tangent at $\left(\frac{1}{2}, \frac{1}{2}\right)$, a suitable value of t is needed

Using $\left(\frac{1}{2}, \frac{1}{2}\right) \equiv \left(\frac{t}{1+t}, \frac{t^3}{1+t}\right)$

$\Rightarrow \frac{1}{2} = \frac{t}{1+t} \Rightarrow 2t = 1 + t \quad \therefore t = 1$

$\therefore$ Slope of the tangent$= 3(1)^2 + 2(1)^3 = 5$

$\therefore$ Slope of the normal, $m' = -\frac{1}{5}$

Equation of the normal becomes

$\frac{y-1/2}{x-1/2} = -\frac{1}{5} \Rightarrow \frac{2y-1}{2x-1} = -\frac{1}{5}$

$\Rightarrow 10y - 5 = -2x + 1 \quad \therefore x + 5y - 3 = 0$

Task

1. Find the equation of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at the point $(1, 4)$.

At what points of the curve is the tangent parallel to the line $4x + y - 3 = 0$? **Ans:** $5x - y - 1 = 0$, $2, 4, 4, -8$.

2. Find the equation of the tangent of the curve $y = x^3 + \frac{x^2}{2} + 1$ at the point $\left(-1, \frac{1}{2}\right)$. Find the coordinates of another point on the curve where the tangent is parallel to that at point $\left(-1, \frac{1}{2}\right)$.

Ans: $4x - 2y + 5 = 0$: $\left(\frac{2}{3}, \frac{41}{27}\right)$.

3. Find the equation of the tangent at the point $(1\,,-1)$ to the curve $y = x^3 - 4x^2 + 2$. What are the coordinates of the point where the tangent meets the curve again? Find the equation of the tangent at this point. **Ans:** $y = -5x + 4\,,(2\,,-6)\,,y = -4x + 2$

4. Find the coordinates of the point of intersection of the line $x - 3y = 0$ with the curve $y = x(1 - x^2)$. If these points are in order $P,O\,,Q$, prove that the tangents to the curve at P and Q are parallel, and the tangent at O is perpendicular to them. **Ans:** $\left(-\sqrt{\frac{2}{3}}\,,\ -\frac{1}{3}\sqrt{\frac{2}{3}}\right),(0\,,0)\,,\left(\sqrt{\frac{2}{3}}\,,\ \frac{1}{3}\sqrt{\frac{2}{3}}\right)$

5. Find the equation of a tangent at the point $P(3\,,9)$ to the curve $y = x^3 - 6x^2 + 15x - 9$. If O is the origin, and N is the foot of the perpendicular from P to the x-axis, prove that the tangent at P passes through the midpoint of ON. Find the coordinates of another point on the curve, and the tangent at which is parallel to the tangent at $(\,3\,,9)$. **Ans:** $6x - y - 9 = 0\,,(1\,,1)$

6. A tangent to the parabola $x^2 = 16y$ is perpendicular to the line $x - 2y - 3 = 0$. Find the equation of this tangent and the coordinates of the point of contact.
Ans: $2x + y + 16 = 0\,,(-16\,,16)$

7. Show that the tangent to the curve $y = 4 - 2x - x^2$ at the points $(-1\,,4)$ and $\left(\frac{1}{2}\,,2\frac{1}{2}\right)$ respectively pass through the point $\left(-\frac{1}{4}\,,5\frac{1}{2}\right)$

8. Find the value of constant c so that the tangent at the origin of the coordinates of the curve $y = x(c - x^2)$ makes an angle of 45^0 with the $x - axis$

APPLICATION OF DIFFERENTIATION

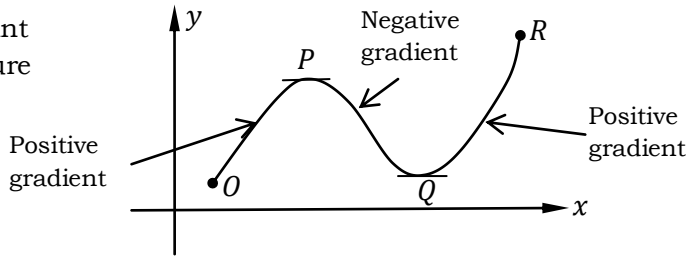
- Applied in Minimum and Maximum Problems.
- Determination of Turning Points of a curve or function.
- Small increments/changes.
- Small increment percentage.
- Finding Related Rates.
- Velocity, Acceleration and Displacement.

Turning points /stationary points of a curve

The turning points of a curve/function include:

- Maximum point
- Minimum point
- Inflexion point

Consider the figure

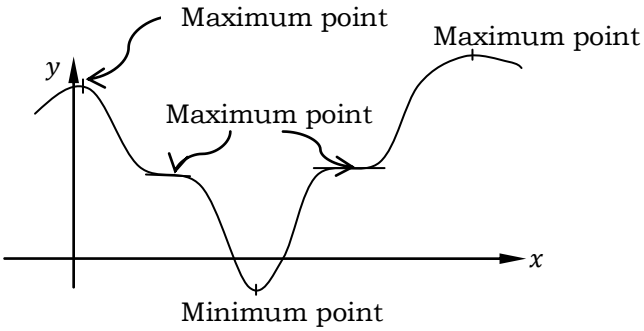


The gradient (or rate of change) of the curve changes from positive between O and P to negative between P and Q, and then positive again between Q and R. At point P, the gradient is zero, and as x increases, the gradient of the curve changes from positive just before P to negative just after. Such a point is called a maximum point. At point Q, the gradient is also zero and as x increases, the gradient of the curve changes from negative just before Q to positive just after such a point is called a minimum point. Points P and Q are given the general name of turning points.

NOTE

If the gradient on either side of that point is the same, such a point is called a point of inflexion.

Consider the figure.



Methods of determining Nature of Turning Points

- The first Derivative Test
- The second Derivative Test

Procedure for finding and distinguishing between stationary points

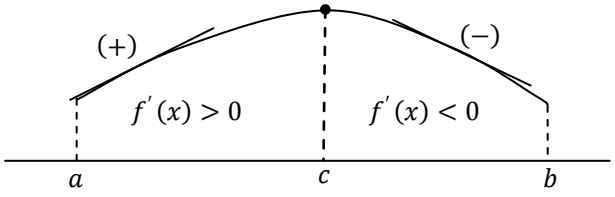
1) Given $y = f(x)$, determine $\frac{dy}{dx}$ $(i.e\ f'(x))$.
2) Let $\frac{dy}{dx} = 0$, and solve for the values of x
3) Substitute the values of x into the original equation, $y = f(x)$, to find the corresponding y-ordinate values. This establishes the coordinates of the stationary points.
4) Determine the nature of the stationary points using the above derivative methods.

The First Derivative Test

Here determine the sign of the gradient of the curve just before and just after the stationary points.

If the sign change for the gradient of the curve is;

a) Positive to negative – the point is a maximum one

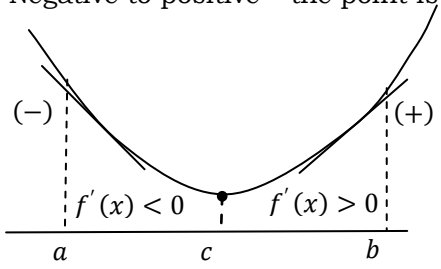


$(c, f(c))_{Maximum}$

NOTE:

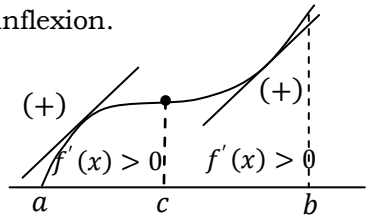
$c - a$ or $b - c$ should be a positive decimal number less than 1

b) Negative to positive – the point is a minimum one.

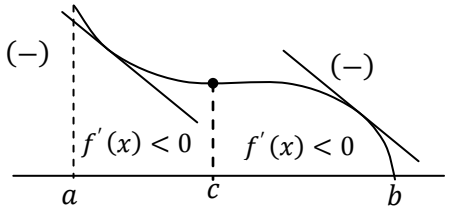


$(c, f(c))_{Minimum}$

c) Positive to positive or negative to negative – the point is a point of inflexion.



$(c, f(c))_{Inflexion}$



$(c, f(c))_{Inflexion}$

The second Derivative Test.

Let $y = f(x)$ be the curve such that $\frac{dy}{dx} = f'(x) = 0 \; at \; x = c. i. e, f'(c) = 0$

1) If $f''(c) > 0$, then $f(c)$ is a minimum one
2) if$f''(c) < 0$, then $f(c)$ is a maximum one
3) if $f''(c) = 0$, then the test fails. In such cases, you can use the First Derivative Test.

Example

Determine the nature of turning points of the curve, and hence sketch the curve.

a) $y = 3x^2 - 6x$.

b) $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$.

c) $f(x) = -3x^5 + 5x^3$.

Solution

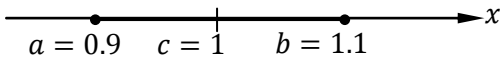
i) Since $y = 3x^2 - 6x$, $\frac{dy}{dx} = 6x - 6$.

ii) At turning points, $\frac{dy}{dx} = 0$, hence $6x - 6 = 0 \; \therefore x = 1$

iii) When $x = 1$, $y = 3(1)^2 - 6(1) = -3$

∴Ccoordinates of turning points $(1\,, -3)$

iv) Using the First Derivative Test.

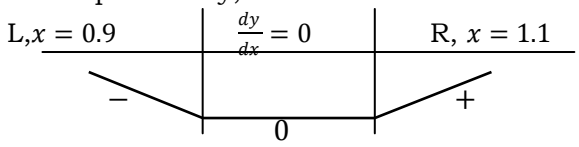


$\frac{dy}{dx}_{at\ x=0.9} = 6(0.9) - 6 = -0.6\,.i.e.$ Negative

$\frac{dy}{dx}_{at\ x=1.1} = 6(1.1) - 6 = 0.6\,.i.e.$ Positive

Since the gradient of the curve is negative just before the turning point and positive after $.i.e.(- \cup +), (1\,, -3)$ is *a minimum*

In a simplified way, a table can be used



$\therefore (1\,, -3)_{Minimum}$

$\left.\frac{dy}{dx}\right|_{at\ x=0.9} = 6(0.9) - 6 = -0.6\; ie. negative$

$\left.\frac{dy}{dx}\right|_{at\ x=1.1} = 6(1.1) - 6 = 0.6\; ie. positve$

Alternatively: Second Derivative Test

$y = 3x^2 - 6x$

$\frac{dy}{dx} = 6x - 6 \; \therefore \frac{d^2y}{dx^2} = 6$

Since $\frac{d^2y}{dx^2} = 6$ is a constant of any value of $x \; .i.e. \; x = 1$, if $\frac{d^2y}{dx^2} > 0$,then $(1\,, 3)$ must be *a minimum.*

To sketch the curve, the following steps can be followed.

1) Obtain the intercepts
2) Obtain the turning points
3) Obtain the nature of turning points
4) Sketch the curve

Intercepts

For x −intercept, $y = 0$

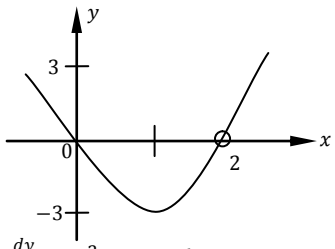
$0 = 3x^2 - 6x \;\; or \; 3x(x-2) = 0 \;\; \therefore x = 0,2$

$\therefore (0,0)$ and $(2,0)$ are intercepts

For y −intercept, $x = 0 \;\; \therefore (0,0)$ is still an intercept.

Turning points and their nature are $\therefore (1\,,-3)_{Minimum}$

Sketch: $y = 3x^2 - 6x$



b) Since $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}\,, \frac{dy}{dx} = x^2 - x - 6$

Turning points

For turning points, $\frac{dy}{dx} = 0$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0 \;\; \therefore x = 3\,, -2$

For $x = 3\,, y = \frac{(3)^3}{3} - \frac{(3)^2}{2} - 6(3) + \frac{5}{3} = -11\frac{5}{6}$

For $x = -2\,, y = \frac{(-2)^3}{3} - \frac{(-2)^2}{2} - 6(-2) + \frac{5}{3} = 9$

$\therefore$ Turning points are: $(-2\,,9)\,, \left(3\,, -11\frac{5}{6}\right)$

Nature of turning points

Using the second derivative test

Since $\frac{dy}{dx} = x^2 - x - 6$, then $\frac{d^2y}{dx^2} = 2x - 1$

Now when $x = -2\,, \frac{d^2y}{dx^2} = 2(-2) - 1 = -5$ which is negative

$\therefore (-2\,,9)$ is a maximum point

When $x = 3\,, \frac{d^2y}{dx^2} = 2(3) - 1 = 5$ which is positive

$\therefore \left(3\,, -11\frac{5}{6}\right)$ is a minimum point

Intercepts

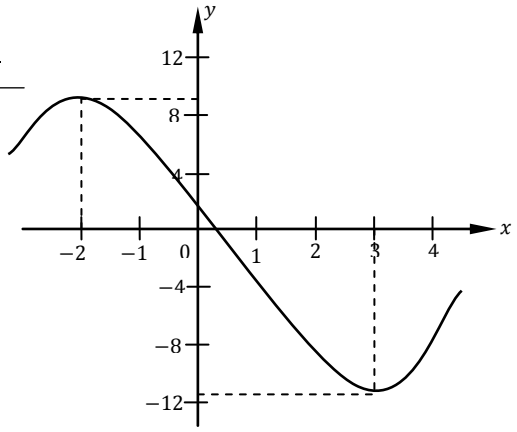
$x -$ intercept , $y = 0$

$\Rightarrow 0 = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3} \;\; or \;\; 2x^3 - 3x^2 - 36x + 10 = 0$

$\therefore x = 4.939\,, -3.712\,, 0.273$

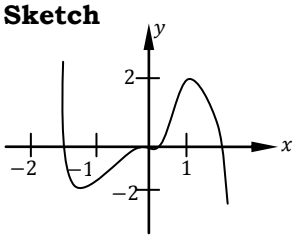
$y-$ Intercept, $x=0\,, y=\frac{5}{3}$

Sketch: $y=\frac{x^3}{3}-\frac{x^2}{2}-6x+\frac{5}{3}$



c)

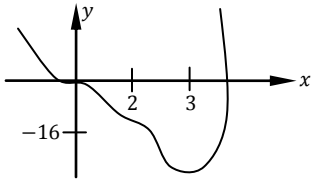
Point	**Nature**	**Sketch**
$(-1\,,-2)$	Minimum	
$(1\,,2)$	Maximum	
$(0\,,0)$	Inflexion	



Task

Determine the nature of turning points of the curve $y=x^4-3x^3$. *Hence sketch*

Point	**Nature**	**Sketch**
$(2\,,-16)$	Inflexion	
$(0\,,0)$	Inflexion	
$(3\,,-27)$	Minimum	



Applied Minimum and Maximum Problems

The common application of differentiation is determination of minimum and maximum values.

Common terms used here are:

- Greatest profit
- Least cost
- Least time
- Optimum size
- Least size
- Greatest distance

Problem Solving Strategy for Applied Minimum and Maximum Problems

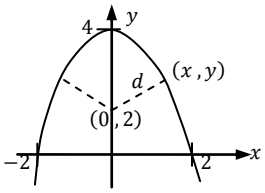
1. Assign symbols to all *given* quantities and quantities *to be determined*
2. Write *a primary equation* for the quantity that is to be maximized(or minimized).here a review of several useful formulas from geometry is needed
3. Reduce the primary equation to one having a single independent variable. This may involve the use of *secondary equations* relating the independent variables of the primary equation.
4. Determine the domain of the primary equation . *i. e.* determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques

Finding the Minimum Distance

Which points on the graph $y = 4 - x^2$ *are closest to the point* $(0\,,2)$*?*

Solution

Sketching $y = 4 - x^2$



From the fig. there are two points at minimum distance from the point $(0\,,2)$. Taking the general point on the curve $(x\,,y)$

d = Distance between two points

$d = \sqrt{(x-0)^2 + (y-2)^2}$ (Primary equation)

Using secondary equation $y = 4 - x^2$

$d = \sqrt{(x-0)^2 + (4-x^2-2)^2} = \sqrt{x^4 - 3x^2 + 4}$

For d to be small, $\frac{d}{dx}(d) = 0$

$\Rightarrow \frac{d}{dx}(d) = \frac{1}{2}(x^4 - 3x^2 + 4)^{-1/2}.(4x^3 - 6x)$

$\therefore 4x^3 - 6x = 0 \;\; or \;\; 2x(2x^2 - 3) = 0$

$\therefore x = 0\,, \pm\sqrt{\frac{3}{2}}$

Investigating the values of x

Using first derivative test,

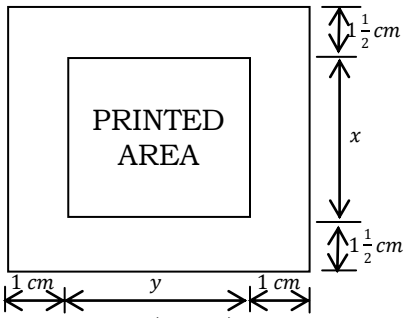
At $x = 0$ yields a maximum value, where as both $x = \pm\sqrt{\frac{3}{2}}$ yield minimum distance.

Hence, the closest points are: $\left(\sqrt{3/2}\,, 5/2\right)$ and $\left(-\sqrt{3/2}\,, 5/2\right)$

Finding the Minimum Area

1. *A rectangular page is to contain* $24cm^2$ *of print. The margins at the top and bottom of the page are each* $1\frac{1}{2}cm$*. The margins on each side are* $1\ cm$*. What should the dimensions of the page be so that the least amount of paper is used?*

Solution



Let A be the area to be minimized

$A = (x+3)(y+2)$ (Primary equation)

The printed area inside the margins is given by:

$24 = xy$ (Primary equation)

Solving the equation produces

$y = \frac{24}{x}$. Substitution in to the primary equation

$A = (x+3)\left(\frac{24}{x}+2\right) = 30 + 2x + \frac{72}{x}$

Because x must be positive, we are interested in only in values of A when $x > 0$

For minimum area, $\frac{dA}{dx} = 0$

$\Rightarrow 2 - \frac{72}{x^2} = 0 \ \ or \ \ x^2 = 36 \ \ \therefore x = +6$

Investigating $x = +6$

By second derivative, $\frac{d^2A}{dx^2} = \frac{144}{x^3}$

Now when $x = +6$, $\frac{d^2A}{dx^2} = \frac{144}{6^3} > 0$. Hence $x = 6$ gives minimum area.

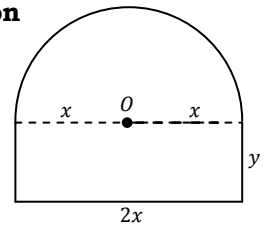
So the dimensions of the paper are:

$x + 3 = 9\ cm$ by $y + 2 = 6\ cm$

Finding the Minimum Area

2. *A window has the form of a rectangle surmounted by a semicircle. If the perimeter is* $40\ cm$*, find the dimensions so that the greatest amount of light may be admitted*

Solution



Let the radius of semicircle be x and width of window be y

Area of the frame for light to be admitted = Area of semicircle + Area of the rectangle

$A = \frac{\pi x^2}{2} + 2xy$ (Primary equation)

Perimeter of boundary = Perimeter of the rectangle + Perimeter of semicircle

$40 = (2x + y + y) + \frac{1}{2} \times 2\pi x$

$40 = 2x + 2y + \pi x$ (Secondary equation)

From this equation

$y = \frac{40-2x-\pi x}{2}$

∴ Primary equation becomes

$A = \frac{\pi x^2}{2} + 2x\left(\frac{40-2x-\pi x}{2}\right) = \frac{\pi x^2}{2} + 40x - 2x^2 - \pi x^2$

$= 40x - \frac{\pi x^2}{2} - 2x^2 = 40x - \frac{x^2}{2}(\pi + 4)$

For maximum area, $\frac{dA}{dx} = 0$

$\Rightarrow 40 - x(\pi + 4) = 0 \quad \therefore x = \frac{40}{\pi+4}$

Investigating $x = \frac{40}{\pi+4}$ to give the greatest area

$\Rightarrow \frac{d^2A}{dx^2} = -(\pi + 4) < 0$

∴ For maximum area, $x = \frac{40}{\pi+4}$

Dimensions are:

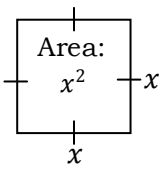
$y = \frac{40-2x-\pi x}{2} = \frac{40-\frac{80}{\pi+4}-\frac{40\pi}{\pi+4}}{2}$

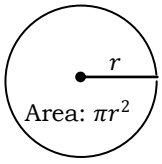
$= 20 - \frac{40}{\pi+4} - \frac{20\pi}{\pi+4} = \frac{20\pi+80-40-20\pi}{\pi+4} = \frac{40}{\pi+4}$

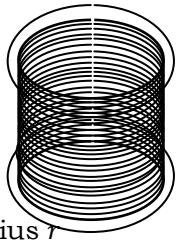
∴ Radius of semicircle = Height of the rectangle = $\frac{40}{\pi+4}$

3. *Four feet of a wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?*

Solution







Let the square be of side x, and circle of radius r

A = Area of square + Area of circle

$A = x^2 + \pi r^2$ (Primary equation)

Because the total amount of wire = 4 ft

4 = Perimeter of square + Circumference of the circle

$\Rightarrow 4 = 4x + 2\pi r$ (Secondary equation)

$\Rightarrow r = \frac{2(1-x)}{\pi}$ Substituting in to the primary equation

$A = x^2 + \pi\left(\frac{2(1-x)}{\pi}\right)^2 = x^2 + \frac{4(1-x)^2}{\pi}$

$\therefore A = \frac{1}{\pi}[(\pi+4)x^2 - 8x + 4]$

For maximum total area, $\frac{dA}{dx} = 0$

$\therefore \frac{dA}{dx} = \frac{1}{\pi}[2x(\pi+4) - 8] = 0$

$\therefore x = 4/(\pi+4) \approx 0.56$

Investigating whether $x = 4/(\pi+4)$ give maximum area

$\frac{d^2A}{dx^2} = \frac{2(\pi+4)}{\pi} > 0$

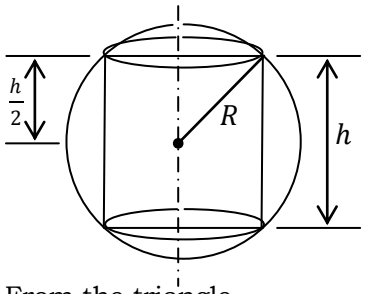
$\Rightarrow x = 4/(\pi+4)$ gives a minimum area.

But the minimum area of a square $.i.e.\ x^2$ is zero, so taking $x = 0$, the maximum area occurs when $x = 0.i.e.$ all the wire is used for a circle.

4. Inscribing a Cylinder in to a Sphere

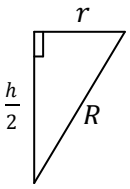
A cylinder of volume V is to be cut from a sphere of radius R. show that the maximum volume of the cylinder is $V = \frac{4\pi R^3}{3\sqrt{3}}$

Solution



Let r = radius of cylinder and
h = height of cylinder
Volume of cylinder
$V = \pi r^2 h$ Primary equation
Since r and h are varying, expressing the primary equation in terms of one variable

From the triangle



$r^2 = R^2 - \frac{h^2}{4}$ Secondary equation

$\therefore V = \pi h\left(R^2 - \frac{h^2}{4}\right) = \pi\left(hR^2 - \frac{h^3}{4}\right)$, R = constant

Now for maximum volume, $\frac{dV}{dh} = 0$

$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right) = 0 \quad \therefore 4R^2 - 3h^2 = 0$

$\Rightarrow h = \sqrt{4R^2/3} = \frac{2R}{\sqrt{3}}$

Investigating $h = \frac{2R}{\sqrt{3}}$ for maximum volume

$\frac{d^2V}{dh^2} = \pi\left(-\frac{6h}{4}\right)$, since $h = +\frac{2R}{\sqrt{3}} \Rightarrow \frac{d^2V}{dh^2} = -ve$

Thus height of the cylinder with maximum volume is $h = \frac{2R}{\sqrt{3}}$

Now maximum volume = $V = \pi\left(R^2.\frac{2R}{\sqrt{3}} - \frac{1}{4}.\left(\frac{2R}{\sqrt{3}}\right)^3\right)$

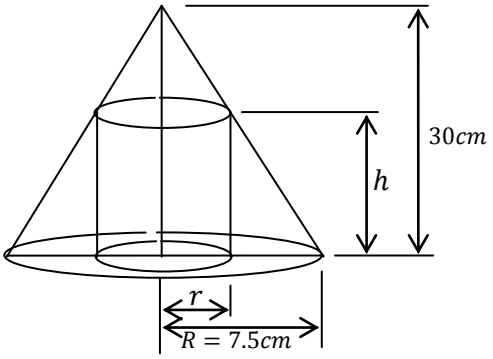
$= \pi\left(\frac{2R^3}{\sqrt{3}} - \frac{8R^3}{12\sqrt{3}}\right) = \pi\left(\frac{2R^3}{\sqrt{3}} - \frac{2R^3}{3\sqrt{3}}\right)$

$= \pi\left(\frac{6R^3 - 2R^3}{3\sqrt{3}}\right) = \frac{4\pi R^3}{3\sqrt{3}}$

5. Inscribing a Cylinder in to a Cone

Determine the height and radius of a cylinder of maximum volume which can be cut from a cone of height $30cm$ *and base radius* $7.5cm$

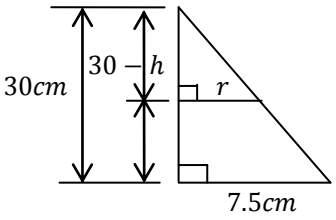
Solution



Volume of cylinder

$V = \pi r^2 h$ Primary equation

Obtaining the secondary equation



By similar triangles

$\frac{30}{7.5} = \frac{30-h}{r} \Rightarrow 4r = 30 - h$

$\therefore h = 30 - 4r$

$V = \pi r^2(30 - 4r) = \pi(30r^2 - 4r^3)$

For maximum volume, $\frac{dV}{dr} = 0$

$\frac{dV}{dr} = \pi(60r - 12r^2) = 0$

$\Rightarrow 12r(5 - r) = 0 \quad \therefore r = 0, 5$

Investigating the values of r

$\frac{d^2V}{dr^2} = \pi(60 - 24r)$

For $r = 0$, $\frac{d^2V}{dr^2} = \pi\big(60 - 24(0)\big) = 60\pi > 0$

For $r = 5$, $\frac{d^2V}{dr^2} = \pi\big(60 - 24(5)\big) = -60\pi < 0$

Hence $r = 5$ gives the maximum value

From $h = 30 - 4r$, taking $r = 5$

$h_{max} = 30 - 4(5) = 10$

$\therefore$ Radius$= 5cm$, $h = 10cm$

6. Finding the Maximum Volume

Write down the expression for volume, V and surface area, S of a cylinder of radius r *and height* h.

If the surface area is constant, show that the volume of the cylinder will be maximum when $h = 2r$.

Solution

$V = \pi r^2 h$, $S = 2\pi r^2 + 2\pi rh$ (Taking cylinder of closed ends)

Given: $S = 2\pi r^2 + 2\pi rh$ Secondary equation

Required: $V = \pi r^2 h$ Primary equation

Expressing the primary equation in terms of one variable

From $S = 2\pi r^2 + 2\pi rh \Rightarrow h = \left(\frac{S-2\pi r^2}{2\pi r}\right)$

$\Rightarrow V = \pi r^2 \left(\frac{S-2\pi r^2}{2\pi r}\right) = \frac{r}{2}(S - 2\pi r^2)$

$\Rightarrow V = \frac{rS}{2} - \pi r^3$

For maximum volume, $\frac{dV}{dr} = 0$

$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2 = 0$ (S = constant)

$r^2 = \frac{S}{6\pi}$ $\therefore S = 6\pi r^2$

Now from $h = \left(\frac{S-2\pi r^2}{2\pi r}\right)$, $S = 6\pi r^2$

$\Rightarrow h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = 2r \therefore h = 2r$

7. *A right circular cylinder has a given surface S (including the ends) and a maximum volume V. show that its height is equal to its diameter of the base.*

Solution

Given: $S = 2\pi r^2 + 2\pi rh$ Secondary equation

$V = \pi r^2 h$ Primary equation

From the secondary equation

$h = \left(\frac{S-2\pi r^2}{2\pi r}\right)$

$\therefore V = \pi r^2 \left(\frac{S-2\pi r^2}{2\pi r}\right) = \frac{rS}{2} - \pi r^3$

For maximum volume, $\frac{dV}{dr} = 0$

$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2 = 0$ (S = constant)

$r^2 = \frac{S}{6\pi}$ $\therefore S = 6\pi r^2$ *or* $r = \sqrt{\frac{S}{6\pi}}$

Now from $h = \left(\frac{S-2\pi r^2}{2\pi r}\right)$, $S = 6\pi r^2$

$\Rightarrow h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = 2r \therefore h = 2r =$ Diameter

Task

1. A chemical factory wishes to make a cylindrical container of thin metal to hold $10cm^3$, using the least possible area of metal. If the outside surface is $S\ cm^2$, and the radius $r\ cm$, show that $S = 2\pi r^2 + \frac{20}{r}$, and hence find the required radius and height. (leave π in your answer)

Ans: $r = \sqrt[3]{5/\pi}, h = 2\sqrt[3]{5/\pi}$

2. A circular cylinder open at the top is to be made so as to have a volume of $1\ m^3$. If $r\ (m)$ is the radius of the base, prove that the total outside

surface area is $(\pi r^2 + 2/r)\, m^3$. Hence prove that this surface is minimum when the height equals to the radius of the base. **Ans:** $h = \sqrt[3]{1/\pi} = r$

3. A piece of wire of length of $17l$ is cut in to two parts of lengths, x and $17l - x$. The former is bent into a shape of a square, and the latter in to a rectangle of which the base is double the height.
Find an expression for the sum of the two areas of these figures. Prove that the only value of x for which this sum is a maximum or minimum is $8l$

4. A right circular cylinder is inscribed in a given sphere. Show that when the area of the curved surface is greatest, the height of the cylinder is equal to its diameter.

5. A right circular cylinder is inscribed in a sphere of given radius a. Show that the volume of the cylinder is $\pi h\left(a^2 - \frac{h^2}{4}\right)$ where h is the height of the cylinder. Find the ratio of the height to the radius of the cylinder when its volume is greatest. **Ans:** $\sqrt{2}:1$

6. Find the dimensions of a rectangle of greatest area which can be inscribed in a circle of radius r. **Ans:** $r\sqrt{2}$

7. A piece of wire, which forms the circumference of a circle of $0.12m$ radius, is cut and bent so as to form two new circles. Find the radius of each circle in order that the sum of the areas of the two circles shall be small as possible. **Ans:** $each: 0.06m$

8. A prism of square section contains $64cm^3$ of clay, the side of the square being $x\ m$. Express the length of the prism in terms of x and find the total area of its faces. Show that the total area is a minimum when the prism is a cube.
Ans: $Length = \frac{64}{x^2}\ ;\ \ Area = \frac{256}{x} + 2x^2$

9. A closed cylindrical can has a height h and base radius r. The volume is $0.01m^3$. show that $h = 1/100\pi r^2$. Show further that S, surface area, is given by $S = 2\pi r^2 + 1/50r$. Hence find the value of r for which S is minimum. **Ans:** $(200\pi)^{-1/3}$

10. A cylinder is cut from a sphere of radius $5\ cm$. If the height of the cylinder is $2h$, show that the volume of the cylinder is $2\pi h(25 - h^2)$, assuming that the curved surfaces of te cylinder reach the surface of the sphere. Find the maximum volume of the cylinder. **Ans:** $500\pi\sqrt{3}/9$

11. A cylinder is inscribed in to a cone of height h. If the volume of the cylinder is maximum show that its height is $h/3$

12. Show that the volume of the biggest right circular cone that can be inscribed in a sphere of given radius is $8/27$ times that of the sphere.

13. A given quantity of metal is to be cast in to a half cylinder with rectangular base and semi circular ends. Show that in order that the

total surface area may be minimum, the ratio of the length of the cylinder to the diameter of its semi circular ends is $\pi(\pi - 2)$

14. Show that the semi vertical angle of a cone of maximum volume and given slant height is $\tan^{-1}(\sqrt{2})$

15. One corner of rectangular sheet of paper of width a is folded so as to reach the opposite edge of the sheet. Find the minimum length of the increase. **Ans:** $3\sqrt{3}a/4$

16. An open box is to be made from a rectangular piece of sheet metal $12cm \times 18cm$ by cutting out equal squares from each corner and folding up the sides. Find the dimensions of the box of largest volume that can be in this manner. **Ans:** $8 + 2\sqrt{7}, 2 + 2\sqrt{7}, 5 - \sqrt{7}$

17. Show that the surface area A, the volume V and radius r of a cylinder are connected by the equation $A = \frac{2V}{r} + 2\pi r^2$

i) Find the minimum possible surface area of a cylinder if its volume is to be $100\pi\ cm^3$. **Ans:** $255.8cm^3$

ii) A cylindrical block of ice is melting at a rate proportional to its surface area. Find the ratio of the length to radius if a cylindrical block of given volume is to melt as slowly as possible. **Ans:** 1: 2

18. A cylinder of height x fits exactly into a cone of height h and base radius r. Show that its volume is given by $v = \pi r^2\left(1 - \frac{x}{h}\right)^2$ show that the maximum volume of the cylinder occurs when $x = \frac{1}{3}h$. Show also that the maximum curved surface area of the cylinder occurs when $x = \frac{1}{2}h$.

Finding Related Rates

This involves the use of chain rule, when a quantity is varied with respect to time or another quantity.

Guidelines for solving Related-Rate problems

1. Identify all given quantities and quantities to be determined. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the chain rule, implicitly differentiate both sides of the equation with respect to time t.
4. After completing step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve the required rate of change.

Example

1. **Ripples in a pond.**
   A pebble is dropped into a calm pond, causing ripples inform of concentric cycles. The radius r of the outer ripple is increasing at a constant rate of 1cm/s. when the radius is 4cm, at what rate is the total area A of the disturbed water changing?

   Solution
   Given the equation: $A = \pi r^2$
   Given the equation: $\frac{dr}{dt} = 1$
   Find: $\frac{dA}{dt}$ when $r = 4$
   Differentiating $A = \pi r^2$ w.r.t. t
   $\frac{dA}{dt} = \pi . 2r . \frac{dr}{dt}$
   $= 2\pi(4)(1) = 8\pi$
   When $r = 4$, the area is changing at a rate $8\pi \text{cm}^2/\text{s}$

2. **An inflating balloon**
   Air is being pumped into a spherical balloon at a rate of $4.5\text{cm}^2/\text{min}$ find the change of radius when the radius is cm^2
   Solution
   Let v be the volume of the balloon
   v the radius
   Given rate: $\frac{dv}{dt} = +\frac{9}{2}$ (constant rate)
   Find: $\frac{dr}{dt}$ when $r = 2$
   To find the rate of change of radius, you must find an equation that relates the radius r to the volume v
   $v = \frac{4}{3}\pi r^3$
   Implicit differentiation with respect to t
   $\frac{dv}{dt} = 4\pi r^2 . \frac{dr}{dt}$
   Solving for $\frac{dr}{dt}$
   $\frac{dr}{dt} = \frac{1}{4\pi r^2} . \frac{dv}{dt}$
   Finally when = 2 , the rate of change of the radius is
   $\frac{dr}{dt} = \frac{1}{16\pi}\left(\frac{9}{2}\right) \backsimeq 0.09\ cm/min$

3. *The surface area of the sphere is decreasing at a rate of $0.9\ m^2/s$, when the radius is 0.6m. Calculate the rate of change of volume of the sphere at this instant.*

Solution

Given rate: $\frac{dA}{dt} = -0.9m^3/s$

Given equation: $A = 4\pi r^2$

Find: $\frac{dV}{dt}$ when $r = 0.6m$

Obtaining the relation between V, volume of the sphere and A, area of the sphere

$V = \frac{4}{3}\pi r^3 = 4\pi r^2.\frac{r}{3} = \frac{Ar}{3}$

Differentiating implicitly w.r.t time t

$\frac{dV}{dt} = \frac{1}{3}\left[r.\frac{dA}{dt} + A.\frac{dr}{dt}\right]$

$= \frac{r}{3}.\frac{dA}{dt} + \frac{A}{3}.\frac{dr}{dt}$

But $A = 4\pi r^2$

Differentiating implicitly w.r.t time t

$\frac{dA}{dt} = 4\pi.2r.\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{8\pi r}.\frac{dA}{dt}$

$\frac{dV}{dt} = \frac{r}{3}.\frac{dA}{dt} + \frac{A}{3}.\frac{1}{8\pi r}.\frac{dA}{dt}$

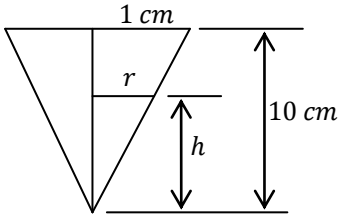
$= \frac{dA}{dt}\left(\frac{r}{3} + \frac{A}{24\pi r}\right) = \frac{dA}{dt}\left(\frac{r}{3} + \frac{4\pi r^2}{24\pi r}\right)$

$= \frac{dA}{dt}\left(\frac{r}{3} + \frac{r}{6}\right) = \frac{r}{2}.\frac{dA}{dt}$

$\therefore \frac{dV}{dt} = \frac{0.6}{2}(-0.9) = -0.27\ m^3/s$

4. *A vessel in the shape of a right circular cone of height* 10 *cm and base radius* 1 *cm catching the drips from a tap leaking at the rate of* 0.1 cm^3/s. *Find the rate at which the surface area of the water is increasing when the water is half way up the cone.*

Solution



Let water in the cone at any time t be at a height h and radius r

Given: $\frac{dV}{dt} = +0.1\ cm^3/s$

Given equation: $V = \frac{\pi r^2 h}{3}$

Find: $\frac{dA}{dt}$ *when* $h = \frac{1}{2} \times 10 = 5\ cm$

Differentiating volume w.r.t. time t

$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2.\frac{dh}{dt} + h.2r.\frac{dr}{dt}\right]$

$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2.\frac{dh}{dt} + 2hr.\frac{dr}{dt}\right]$ ..... (1)

Now $A = \pi r^2$

Differentiating area w.r.t. time t

$\frac{dA}{dt} = 2\pi r.\frac{dr}{dt}$ ..... (2)

Also by similar triangles

$\frac{1}{r} = \frac{10}{h} \Rightarrow h = 10r$

Differentiating area w.r.t. time t

$\frac{dh}{dt} = 10.\frac{dr}{dt}$ ..... (3)

Now eqn. (3) in to eqn. (1)

$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2.10.\frac{dr}{dt} + 2hr.\frac{dr}{dt}\right] = \frac{dr}{dt}\left[\frac{10\pi r^2}{3} + \frac{2\pi hr}{3}\right]$

$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{\frac{10\pi r^2 + 2\pi hr}{3}}$

Eqn. (4) in to eqn. (2)

$\therefore \frac{dA}{dt} = \frac{6\pi r.\frac{dV}{dt}}{10\pi r^2 + 2\pi hr} = \frac{3.\frac{dV}{dt}}{5r + h}$

$= \frac{3.\frac{dV}{dt}}{5.\frac{h}{10} + h} = \frac{30}{15h}.\frac{dV}{dt}$

$= \frac{2}{h}.\frac{dV}{dt} = \frac{2}{5}.(0.1) = 0.04\ cm^2/s$

Alternatively

Given: $\frac{dV}{dt} = +0.1\ cm^3/s$

Given equation: $V = \frac{\pi r^2 h}{3}$

Find: $\frac{dA}{dt}$ *when* $h = \frac{1}{2} \times 10 = 5\ cm$

Expressing volume in terms of one variable say h

By similar triangles

$\frac{1}{r} = \frac{10}{h} \Rightarrow r = \frac{h}{10}$

$\therefore V = \frac{\pi r^2 h}{3} \Rightarrow V = \frac{\pi h^2.h}{300} = \frac{\pi h^3}{300}$

Differentiating volume w.r.t. time t

$\frac{dV}{dt} = \frac{\pi h^2}{100}.\frac{dh}{dt} \Rightarrow 0.1 = \frac{\pi h^2}{100}.\frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{10}{\pi h^2}$ ..... (1)

Now $A = \pi r^2$

Expressing this area in terms of h

$A = \frac{\pi h^2}{100}$

Differentiating area w.r.t. time t

$\frac{dA}{dt} = \frac{\pi h}{50}.\frac{dh}{dt}$ ..... (2)

Eqn. (1) in to eqn. (2)

$\frac{dA}{dt} = \frac{\pi h}{50}.\frac{10}{\pi h^2} = \frac{1}{5h}$

$\therefore \frac{dA}{dt} = \frac{1}{5\times 5} = 0.04\ cm^2/s$

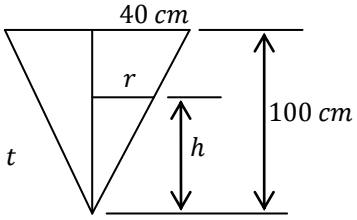
5. *A container is in the form of an inverted circular cone, its height is* $100\ cm$ *and base radius* $40\ cm$*. The container is full of water and has a small hole at its vertex. Water is flowing through the hhole at a rate of* $100\ cm^3/s$*. Find the rate at which the water level in the container is falling when the height of water in the container is halved.*

Solution

Given: $\frac{dV}{dt} - 10\ cm^3/s$

Given equation: $V = \frac{\pi r^2 h}{3}$

Find: $\frac{dh}{dt}$ *when* $h = \frac{1}{2} \times 100 = 50\ cm$



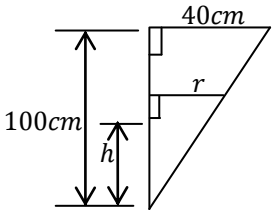
Differentiating volume implicitly w.r.t time t

$V = \frac{\pi r^2 h}{3} \Rightarrow \frac{dV}{dt} = \frac{\pi}{3}\left[r^2.\frac{dh}{dt} + h.2r.\frac{dr}{dt}\right]$

$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2.\frac{dh}{dt} + 2hr.\frac{dr}{dt}\right]$

$\Rightarrow r^2.\frac{dh}{dt} = \frac{3}{\pi}.\frac{dV}{dt} - 2hr.\frac{dr}{dt}$ ..... (1)

Obtaining the secondary equation



By similar triangles

$\frac{40}{r} = \frac{100}{h} \quad \therefore r = \frac{2}{5}h$

Now differentiating $r = \frac{2}{5}h$ implicitly w.r.t. time t

$\frac{dr}{dt} = \frac{2}{5}.\frac{dh}{dt}$ ..... (2)

Eqn. (2) in to eqn. (1)

$\Rightarrow r^2.\frac{dh}{dt} = \frac{3}{\pi}.\frac{dV}{dt} - 2hr.\frac{2}{5}.\frac{dh}{dt}$

But for $h = 50\ cm, r = \frac{2}{5} \times 50 = 20\ cm$

$\Rightarrow (20)^2.\frac{dh}{dt} = \frac{3}{\pi}.(-10) - 2(50)(20).\frac{2}{5}.\frac{dh}{dt}$

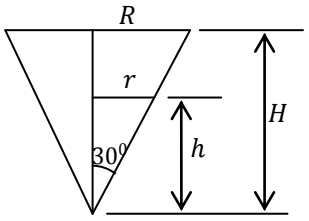
$\Rightarrow -\frac{30}{\pi} = 400.\frac{dh}{dt} + 800.\frac{dh}{dt}$

$\Rightarrow -\frac{30}{\pi} = 1200.\frac{dh}{dt}$

$\therefore \frac{dh}{dt} = -\frac{1}{40\pi}\ cm/s$

6. *A vessel containing water is in the form of an inverted hollow cone with a semi vertical angle of* 30^0. *There is a small hole at the vertex of the cone and water is running out at the rate of* $3\ cm^3/s$. *Find the rate at which the surface area in contact with water is changing when there are* $81\pi cm^3$ *of water remaining in the cone.*

Solution



Given rate: $\frac{dV}{dt} = -3$

Given equation: $V = \frac{\pi r^2 h}{3}$

Taking height of the water in the cone to be h and radius r at any time t

Find: $\frac{dS}{dt}$ when $V = 81\pi cm^3$

Differentiating volume implicitly w.r.t time t

$V = \frac{\pi r^2 h}{3} \Rightarrow \frac{dV}{dt} = \frac{\pi}{3}\left[r^2.\frac{dh}{dt} + h.2r.\frac{dr}{dt}\right]$

$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2.\frac{dh}{dt} + 2hr.\frac{dr}{dt}\right]$

$\Rightarrow \frac{3}{\pi}.\frac{dV}{dt} = r^2.\frac{dh}{dt} + 2hr.\frac{dr}{dt}$ ..... (1)

Obtaining the secondary equation

$\tan 30^0 = \frac{r}{h} \Rightarrow r = h\tan 30^0$

$\therefore r = \frac{h}{\sqrt{3}}$

Now differentiating w.r.t. time t

$\frac{dr}{dt} = \frac{1}{\sqrt{3}}.\frac{dh}{dt}$ ..... (2)

Eqn. (2) in to eqn. (1)

$\Rightarrow \frac{3}{\pi}.\frac{dV}{dt} = r^2.\frac{dh}{dt} + 2hr.\frac{1}{\sqrt{3}}.\frac{dh}{dt}$ ..... (3)

Now $S = \pi rl = \pi r\sqrt{r^2 + h^2}$

But $r = \frac{h}{\sqrt{3}}$

$\Rightarrow S = \pi\frac{h}{\sqrt{3}}.\sqrt{\frac{h^2}{3} + h^2} = \frac{2\pi h^2}{\sqrt{3}}$

Differentiating surface area implicitly w.r.t time t

$\frac{dS}{dt} = \frac{4\pi h}{\sqrt{3}}.\frac{dh}{dt}$ ..... (4)

From eqn. (3)

$\Rightarrow \frac{3}{\pi}.\frac{dV}{dt} = r^2.\frac{dh}{dt} + 2hr.\frac{1}{\sqrt{3}}.\frac{dh}{dt} = \left(r^2 + \frac{2hr}{\sqrt{3}}\right).\frac{dh}{dt}$

$\Rightarrow \frac{3\sqrt{3}}{\pi}.\frac{dV}{dt} = \left(r^2\sqrt{3} + 2hr\right).\frac{dh}{dt}$

Expressing this equation in terms of only one variable say h

$\Rightarrow \frac{3\sqrt{3}}{\pi}.\frac{dV}{dt} = \left(\left(\frac{h}{\sqrt{3}}\right)^2\sqrt{3} + 2h\left(\frac{h}{\sqrt{3}}\right)\right).\frac{dh}{dt}$

$\Rightarrow \frac{3\sqrt{3}}{\pi}.\frac{dV}{dt} = \left(\frac{h^2}{\sqrt{3}} + \frac{2h^2}{\sqrt{3}}\right).\frac{dh}{dt}$

$\Rightarrow \frac{3\sqrt{3}}{\pi}.\frac{dV}{dt} = \frac{3h^2}{\sqrt{3}}.\frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{3\sqrt{3}}{\pi}.\frac{dV}{dt}.\frac{\sqrt{3}}{3h^2} = \frac{3}{\pi h^2}.\frac{dV}{dt}$

Now eqn. (4) becomes

$\frac{dS}{dt} = \frac{4\pi h}{\sqrt{3}}.\frac{3}{\pi h^2}.\frac{dV}{dt} = \frac{4}{h}.\frac{dV}{dt}$

Now $V = \frac{\pi r^2 h}{3}$, $r = \frac{h}{\sqrt{3}}$

$\Rightarrow V = \frac{\pi h^3}{9}$

But $V = 81\pi \Rightarrow 81\pi = \frac{\pi h^3}{9}$

From which $h = 9$

$\therefore \frac{dS}{dt} = \frac{4}{h}.\frac{dV}{dt} = \frac{4}{9}.(-3)$

$\therefore \frac{dS}{dt} = -\frac{4}{3}\ cm^2/s$

Alternatively

At any time t, volume, $V = \frac{\pi r^2 h}{3}$

Now both r and h are variables, expressing $V = \frac{\pi r^2 h}{3}$ in terms of two variables

$\tan 30° = \frac{r}{h} \Rightarrow \frac{h}{\sqrt{3}} = r$

$\therefore\ V = \frac{\pi h}{3}.\frac{h^2}{3} = \frac{\pi r^3 h}{9}$

Now $\frac{dv}{dh} = \frac{\pi h^2}{3}$ ..... (1)

Net surface at any time t is

$S = \pi r l$

But $\cos 30° = \frac{h}{l} \Rightarrow l = hsec\ 30°$

$\therefore S = \pi.\frac{h}{\sqrt{3}}.h.\sec 30$

$= \frac{\pi h^2}{\sqrt{3}}.\frac{1}{\cos 30°}$

$= \frac{\pi h^2}{\sqrt{3}}.\frac{1}{\frac{\sqrt{3}}{2}} = \frac{2\pi h^2}{3}$

$\therefore \frac{ds}{dh} = \frac{4\pi h}{3}$

Now $\frac{dV}{dt} = -3$ (volume is decreasing)

And $\frac{ds}{dt}\ is$ required

Since there is no direct relationship between S and V, we use a 3-step link as follows

$\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{4\pi h}{3} \times \frac{3}{\pi h^2} \times (-3) = -\frac{12}{h}$

But $V = 81\pi = \frac{\pi h^3}{9}\ \ \therefore h = 9$

$\therefore \frac{dS}{dt} = -\frac{12}{9} = -\frac{4}{3}\ cm^2/s$

$\therefore$ Surface area is decreasing $\frac{4}{3}\ cm^2/s$

Task

1. Sand falls on to horizontal ground at the rate of 9m^3/min and forms a heap in the shape of a right circular cone vertical angle60°. Show that 10 seconds after the sand begins to fall, the rate at which the radius of the base of the pile is increasing is $3^{\frac{1}{2}}.\left(\frac{4}{\pi}\right)^{\frac{1}{3}}$ cm/min
2. A container in the shape of a right circular cone of height 20cm and radius 5cm is held vertex downward and filled with water which the drips out from the vertex at the rate of change of the height of the water in the cone when it is half empty (measured by volume) **Ans:** -0.1011 cm^3/s
3. A hemisphere bowl of radius a cm is initially full of water. The water runs out through a small hole at the bottom of the bowl at a constant rate such that it empties the bowl in 24s. Given that when the depth of

the bowl is xcm and the volume of the water is $\frac{1}{3}\pi x^2(3a-x)\text{cm}^3$, show that the depth of the water at that instant is decreasing at a rate $a^3[36x(2a-x)]^{-1}\ \text{cms}^{-1}$.

Find how long it will take for the depth of water to be $\frac{1}{3}a\ cm$ and the rate at which the depth is decreasing at that instant

Solution

Given: volume of hemispherical bowl

$$= \frac{2}{3}\pi r^3 = \frac{2}{3}\pi a^3$$

Volume rate $= \frac{dv}{dt} = \frac{\frac{2}{3}\pi r^3}{24} = = \frac{1}{36}\pi a^3 \text{cm}^3/\text{s}$

Primary equation : $V = \frac{\pi x^2(3a-x)}{3}, a$ =constant

Differentiating implicitly with respect to time, t

$\frac{dV}{dt} = \frac{\pi}{3}[6ax - 3x^2].\frac{dx}{dt}$

Required: $\frac{dx}{dt}$ *when heigght is* x

$\therefore \frac{dx}{dt} = \frac{3}{\pi(6ax-3x^2)}.\frac{dV}{dt}$

$= \frac{3}{\pi(6ax-3x^2)}.\frac{1}{36}\pi a^3 = \frac{a^3}{36(2ax-x^2)}$

$= \frac{a^3}{36(x[2a-x])} = a^3[36x(2a-x)]^{-1}$

Find t =? when $x = \frac{1}{3}a\ cm$

$\frac{dh}{dt}$ *when* $x = \frac{1}{3}a\ cm.$

Obtaining $\frac{dh}{dt}$ *when* $x = \frac{1}{3}a$

$\frac{dx}{dt} = \frac{a^3}{36x(2a-x)}$

$\frac{dx}{dt} = \frac{a^3}{\frac{36x}{3}\left(2a-\frac{x}{3}\right)} = \frac{a^2}{12\left(\frac{5a}{3}\right)} = \frac{a}{20}cm/s$

Using volume rate $= \frac{\text{Volume}}{\text{Time}}$

Volume of water at $x = \frac{a}{3}cm$

$V = \frac{\pi x^2(3a-x)}{3}$

$\therefore$ Volume $= \frac{\pi a^2}{27}\left(3a - \frac{a}{3}\right) = \frac{8\pi a^3}{81}\ cm^3$

Let t = Time taken to rerach a depth of $x = \frac{a}{3}$

Volume reduced in $t(s) = \frac{2\pi a^3}{3} - \frac{8\pi a^3}{81} = \frac{46\pi a^3}{81}$

$\therefore$ Time $= \frac{46\pi a^3}{81} \div \frac{\pi a^3}{36} = \frac{46}{81} \times 36 = 20.444\ s$

Task

1. A hemispherical bowl is being filled with water at a uniform rate. When the height of water is $h\ cm$, the volume is $\pi\left(rh^2 - \frac{h^3}{3}\right)cm$, $r\ cm$ being the radius of the hemisphere. Find the rate at which the water level is rising when it is half way to the top, given that $r = 6$ and that the bowl fills in 1 minute. **Ans:** $\frac{4}{45}\ cm/s$

2. An inverted right circular cone of vertical angle 120° is collecting water from a tap at a steady rate of $18\pi\ cm^3/min$. find :

i) The depth of water after 12 min.

ii) The rate of increase of the depth at this instant. **Ans:** i) 6 cm

ii) $\frac{1}{6}cm/s$

3. When the depth of water in a hemispherical bowl of radius r is x, the volume of water is $\frac{1}{3}\pi x^2(3r-x)$. A hemispherical bowl of radius 12 cm is initially full of water. The water runs out of the hole at rate of $96\ \pi\ cm^3 s^{-1}$. When the depth of water is $x\ cm$,show that the depth is decreasing at rate of $\frac{48}{x(24-x)}cms^{-1}$, find the rate at which the depth is decreasing when :

i) The bowl is full

ii) The depth is $6cm$ **Ans:** $\frac{39}{20}cms^{-1}$.

4. A hemispherical bowl of radius $a\ cm$ is filled with water at rate of $\frac{1}{12}\pi a^3 cm^3 s^{-1}$. When the depth of water in the hemispherical bowl of radius r is x, the volume of water is $\frac{1}{3}\pi x^2(3r-x)$. find the rate at which the level is rising when the depth of water is $\frac{1}{3}a\ cm$

Ans: $\frac{39}{20}\ cms^{-1}$

5. A flask in the shape of a cone is fixed with its vertex pointing downwards. The cone is filling with a liquid at a rate which at any instant is proportional to the surface area of the liquid at that instant show that the rate of increase in the depth of the liquid is constant.

Small changes or Increments

Let $p(x,y)$ be a general point on the curve$y=f(x)$, and $Q(x+\delta x, y+\delta y)$ is a point on the curve close to P

Then δx = small increase in x and δy = corresponding increase in y

From gradient of PQ $.i.e.$

$\lim_{x\to 0}\frac{\delta y}{\delta x}=\frac{dy}{dx}$

So when x is so small the

$\frac{\delta y}{\delta x}\backsimeq\frac{dy}{dx}$ $\quad\boxed{\therefore \delta y\backsimeq\frac{dy}{dx}.\delta x}$

This approximation is used to estimate a function close to a known value i.e. $\boxed{y+\delta y=y+\frac{dy}{dx}.\delta x}$

Example

Find an approximation of

a) $\sqrt{9.01}$

b) $\ln(1.1)$

c) $\sin 31°$ and $\sin 29°$ (Take $1°\backsimeq 0.0175$)

Solution

a) $\sqrt{9.01}=\sqrt{9+0.01}$ $\quad\therefore x=9\ ,dx=0.01$

Let $y=\sqrt{x}$

Required: $y+\delta y=y+\frac{dy}{dx}.\delta x$

$\Rightarrow y = x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$]

$\therefore \sqrt{9.01} \backsimeq \sqrt{x} + \frac{1}{2\sqrt{x}}.\delta x$

$\backsimeq \sqrt{9} + \frac{1}{2\sqrt{9}}.0.01$

$\backsimeq 3 + \frac{0.01}{6}$

b) $\ln(1.1) = \ln(1 + 0.1) \quad \therefore x = 1\,,\ dx = 0.1$

Let $y = \ln x$

Required: $y + \delta y = y + \frac{dy}{dx}.\delta x$

$\frac{dy}{dx} = \frac{1}{x}$

$\therefore \ln 1.1 = \ln 1 + \left(\frac{\delta x}{x}\right)$

$= \ln 1 + \frac{0.1}{1} = 0 + 0.1$

c) $\sin 31° = \sin(30 + 1)° \quad \therefore x = 30^0\,, \delta x = 1^0 = 0.0175$

Let $y = \sin x$

$\frac{dy}{dx} = \cos x$

$\therefore \sin 31° = \sin(x + \delta x) = \sin x + (\cos x).\delta x$

$= \sin 30^0 + \cos 30^0 \times 0.0175$

$= 0.5 + 0.866 \times 0.0175 = 0.5152$

Also $\sin 29° = \sin(30 - 1)° \quad \therefore x = 30^0\,, \delta x = -1^0 = -0.0175$

Let $y = \sin x$

$\frac{dy}{dx} = \cos x$

$\therefore \sin 29° = \sin(x + \delta x) = \sin x + (\cos x).\delta x$

$= \sin 30^0 + \cos 30^0 \times (-0.0175)$

$= 0.5 - 0.866 \times 0.0175 = -0.4848$

NOTE:

In general, the formula used for approximation is;

$$f(x + \Delta x) \simeq f(x) + f'(x).\Delta x$$

Example

Evaluate (i) $\sqrt[4]{624}$ (ii) $\sqrt[3]{28}$

Solution

i) $\sqrt[4]{624} \approx 4.99799879888 \approx 5$

Now $5^4 = 625$

$\Rightarrow \sqrt[4]{624} = \sqrt[4]{(625 - 1)} \Rightarrow x = 625\,, \Delta x = -1$

Let $f(x) = \sqrt[4]{x} = x^{1/4}\,,$

$\Rightarrow f'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4\sqrt[4]{x^3}}$

Using $f(x + \Delta x) \approx f(x) + f'(x).dx$

$f(x + \Delta x) = \sqrt[4]{x} + \frac{1}{4\sqrt[4]{x^3}}.\Delta x$

$\Rightarrow \sqrt[4]{624} = \sqrt[4]{625} + \frac{1}{4\sqrt[4]{(625)^3}} \times (-1)$

$= 5 - \frac{1}{4(5)^3} = 5 - 0.002$

$\therefore \sqrt[4]{624} \simeq 4.998$

ii) Left as an exercise

Small Percentage Increment

When x increases by $r\%$, then the increase in x, δx is given by:

$\%ge\ change/increase = \frac{Increase}{Original} \times 100$

$\Rightarrow r = \frac{\delta x}{x} \times 100$

Thus $\boxed{\delta x = \frac{r}{100}.x}$

NOTE:

Suggested steps in solving numbers on small percentage increment

1. Write the formula for the percentage error with respect to the given variable
2. Make the small increment the subject in the above formula
3. Write the formula involving the required percentage error in that quantity
4. Use the formula $\delta y \approx (dy/dx).\delta x$
5. Substitute the obtained increment in to the required percentage formula.

Example

1. *An error of* 3% *is made in measuring the radius r of a sphere. What is the resulting percentage error in calculating the volume?*

Solution

$\%ge$ error in $r = \frac{\delta r}{r} \times 100 \Rightarrow \delta r = \frac{3}{100} r = 0.03r$

Now required $\%ge$ error in volume $= \frac{\delta V}{V} \times 100$

But $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} \simeq \frac{\delta V}{\delta r} = 4\pi r^2 \Rightarrow \delta V = \left(\frac{dV}{dr}\right).\delta r$

$\therefore \delta V = 4\pi r^2.\delta r$

$\%ge$ error in volume $= 4\pi r^2.\delta r/V \times 100$

$= \frac{4\pi r^2.\delta r}{\frac{4}{3}\pi r^3} \times 100 = \frac{3}{r}\delta r \times 100$

$= \frac{3}{r} \times 0.03r \times 100 = 9\%$

2. *Find the increase in area of a circle when its radius changes from* $5m$ *to* $5.1m$

Solution

Change in radius $= \delta r = 5.1 - 5 = 0.1\ m$

Required: $\delta A = ?$

But $A = \pi r^2$

$\delta A = \left(\frac{dA}{sr}\right).\delta r = 2\pi r.\delta r$

$\delta A = 2\pi \times 5 \times (0.1) = \pi\ m^2$

3. *The period T of a simple pendulum is calculated from the formula* $T = 2\pi\sqrt{\frac{l}{g}}$*, where* l *is the length of the pendulum and* g *is the acceleration due to gravity. Find the percentage change in the period caused by lengthening the pendulum by* 2%*.*

Solution

Given: $\%ge$ change in $l = \frac{\delta l}{l} \times 100$

$\therefore \delta l = \frac{2}{100} l = 0.02l$

Required: $\%ge$ change in $T = \frac{\delta T}{T} \times 100$

$T = 2\pi\sqrt{\frac{l}{g}} = \frac{2\pi}{\sqrt{g}}.l^{1/2}$

Differentiating and using the formula

$\delta T = \left(\frac{dT}{dl}\right).\delta l$

$\Rightarrow \delta T = \frac{2\pi}{\sqrt{g}}.\frac{1}{2}.l^{-1/2}.\delta l = \frac{\pi}{\sqrt{g}}.\frac{1}{\sqrt{l}}.\delta l = \frac{\pi.\delta l}{\sqrt{l.g}}$

$\therefore \%ge$ Change in $T = \frac{\pi.\delta l}{\sqrt{l.g}} / 2\pi\sqrt{\frac{l}{g}} \times 100$

$= \frac{\delta l}{\sqrt{l.g}} \times \frac{\sqrt{g}}{2\sqrt{l}} \times 100$

$= \frac{\delta l}{2l} \times 100 = \frac{0.02l}{2l} \times 100 = 1\%$

4. *The side of a square can be measured accurately to* $o.1mm$*, what is the possible error in the area of the square whose side is measured to be* $200mm$*?*

Solution

Given: error in side $x = \delta x = 200mm$

Area, $A = x^2$

$dA \simeq 2x.\delta x = 2 \times 0.1 \times 200$

$= 40mm^2$

5. *.find the approximate percentage change in the square of a quantity when the quantity itself changes by* 0.1%*. hence calculate an approximate value of* $(10.01)^2$*.*

Solution

Let $y = x^2$, $x =$ quantity

$\delta y \simeq 2x.\delta x$

But $\delta x = \frac{0.1}{100} x = 0.001x$

$\therefore \delta y \simeq 2x.(0.001x) = 0.002x^2$

Required: $\%ge$ *change in* $y = \frac{\delta y}{y} \times 100$

$= \frac{0.002x^2}{x^2} \times 100 = 0.2\%$

Approximate value of $(10.01)^2$

Using $f(x+\Delta x) \approx f(x) + f'(x).dx$, taking

$x + \Delta x = 10 + 0.1 \quad \therefore x = 10\,,\ \Delta x = 0.1$

$f(x+\Delta x) \simeq x^2 + dy$

$\simeq 10^2 + 0.2 = 100.2$

6. *If* $y = x^3$*, find the approximate increase in* y *due to the increase of* 0.1% *in* x.

Solution

Given: $\frac{\delta x}{x} \times 100 = 0.1 \Rightarrow \delta x = 0.001x$

Required: $\%ge$ change in $y = \frac{\delta y}{y} \times 100$

But $\delta y \simeq \left(\frac{dy}{dx}\right).\delta x$

$\delta y \simeq 3x^2.\delta x = 3x^2 \times 0.001x = 0.003x^3$

$\therefore \%ge$ Change in $y = \frac{\delta y}{y} \times 100 = \frac{0.003x^3}{x^3} \times 100 = 0.3\%$

7. *The radius of a cylinder is reduced by* 3% *and its height increased by* 4%. *Determine the approximate percentage change in*

a) Its volume
b) In curved surface area

Solution

a) $\%ge$ change in volume$= \frac{\delta V}{V} \times 100$

Given: $\%ge$ change in radius $= \frac{dr}{r} \times 100$

$\therefore \delta r = \frac{3}{100} \times r = 0.03r$

Since there is a decrease in radius, then

$\delta r = -0.03r$

Given: $\%ge$ change in height $= \frac{dh}{h} \times 100$

$\therefore \delta h = \frac{4}{100} \times r = 0.04h$

Since there is an increase in height, then

$\delta h = +0.04h$

But $V = \pi r^2 h$

Differentiating implicitly w.r.t. any variable, say r (Note this step)

$\frac{dV}{dr} = \pi\left[r^2.\frac{dh}{dr} + h.\frac{d}{dr}(r^2)\right]$

$\frac{dV}{dr} = \pi\left[r^2.\frac{dh}{dr} + 2hr\right]$,now multiplying through by dr

$\Rightarrow \frac{dV}{dr} = \pi[r^2.dh + 2hr.dr]$

By small changes

$\therefore \delta V \simeq \pi[r^2.\delta h + 2hr.\delta r]$

$\simeq \pi[r^2.(0.04h) + 2hr.(-0.003r)]$

$\simeq \pi[0.04hr^2 - 0.06hr^2]$

$\simeq -0.02\pi r^2 h$

$\therefore$ $\%ge$ Change in volume $= \frac{-0.02\pi r^2 h}{\pi r^2 h} \times 100 = -2\%$

$\therefore$ Volume reduces by 2%

b) $\%ge$ change in curved surface area$= \frac{\delta A}{A} \times 100$

$A = 2\pi rh$

Differentiating implicitly w.r.t any variable, say h

$\frac{dA}{dh} = 2\pi\left[r.\frac{dh}{dh} + h.\frac{dr}{dh}\right]$

$= 2\pi\left[r + h.\frac{dr}{dh}\right]$, multiplying through by dh

$dA = 2\pi[r.dh + h.dr]$

By small increments,

$\delta A = 2\pi[r.\delta h + h.\delta r]$

$\therefore$ $\%ge$ Increase in curved area $= \frac{2\pi[r.\delta h + h.\delta r]}{2\pi rh} \times 100$

$= \frac{2\pi[r.(0.04h) + h.(-.003r)]}{2\pi rh} \times 100$

$= \frac{2\pi[0.04h.r - 0.03h.r]}{2\pi rh} \times 100 = 0.01 \times 100 = 1\%$

$\therefore$ Curved area increases by 1%

8. *Pressure P and volume V are related by the expression $PV^3 = C$ where C is a constant. Find the approximate percentage change in C when P is increased by* 2% *and volume decreased by* 0.8%

Solution

Given: $\delta P = \frac{2}{100P} = 0.02P$

For increase, $\delta P = +0.02P$

$\delta V = \frac{0.8}{100V} = 0.008V$

For a decrease, $\delta V = -0.008V$

Now $PV^3 = C$

Differentiating implicitly w.r.t. any variable, say P

$\frac{dC}{dP} = p.\frac{d}{dP}(V^3) + V^3.\frac{dP}{dP}$

$= \frac{dC}{dP} = 3PV^2.\frac{dV}{dP} + V^3$, multiplying through by dP

$dC = 3PV^2 dV + V^3.dP$

By small changes

$\delta C = 3PV^2\delta V + V^3.\delta P$

Required: $\%ge$ change in $C = \frac{\delta C}{C} \times 100$

$= \frac{3PV^2\delta V + V^3.\delta P}{PV^3} \times 100$

$= \frac{3PV^2(-0.008V) + V^3.(+0.02P)}{PV^3} \times 100$

$= \frac{-0.024 + 0.02}{1} \times 100 = -0.004 \times 100 = -0.4\%$

$\therefore$ $\%ge$ Change (Reduction) in $C = 0.4\%$

NOTE:

For more details on small percentage increments go to "Mathematics For an A-level Student" Volume I. Topic: Percentages and Errors in Binomial Expansion.

Task

1. The modulus of rigidity G is given by $G = \frac{R^4\theta}{L}$ where R is the radius, θ is the angle of twist and L is the length. Find the approximate percentage in G when R is measured 1.5% too large and θ is measured 5% too small. **Ans:** 1%

2. The volume of a cone is given by $V = \frac{\pi r^2 h}{3}$. If the volume increases by $\frac{2\pi}{3}\ cm^3/min$ and height decreases by $0.03\ cm/min$, find the rate of change of radius when $r = 10cm$ and $h = 5\ cm$ **Ans:** 5

3. If the radius of a spherical balloon increases from $1\ cm$ to$1.02\ cm$, find the approximate increase in volume. **Ans:** $0.08\pi\ cm^3$

4. If $R = ar^n$ and an error of x% is made in measuring r, prove that an error of nx% will result in R.

5. The volume of a sphere is increased by 3%. Find the percentage increase in the radius. **Ans:** 1%

6. If L is the length of a pendulum and $t(s)$ is the time of one complete swing. It is known that $l = kt^2$. If the length of the pendulum increases by x%, x being small, find the corresponding increase in the time of the swing. **Ans:** $0.5x$%

7. One side of a rectangle is three times the other. If the perimeter increases by 2% what is the percentage increase in the area? **Ans:** 4%

8. The base radius of the right circular cone increases and the volume changes by 2%. If the height remains constant, find the percentage increase in the circumference of the base. (**Uneb 2011**) **Ans:** 1%

9. An error of $2\frac{1}{2}$% is made in the measurement of the area of a circle. What percentage error results in the radius and circumference? **Ans:** 1.25% ; 1.25%

10. A group of sociologists has made a survey of unemployment in a certain London borough. The group claims that the number u of unemployment

in the borugh at any time is related to the rate of inflation by the formula $u^3 = k/r^2$ (k is a constant). Using this formula, find the percentage change in u corresponding to a 6% increase in r. **Ans:** 4% decrease.

11. Given that $y = 2 + k/x$ (k is a constant). Find the percentage error in y due to a 3% error in x when;

i) $y = 0.5$ ii) $y = 1$ iii) $y = 1.5$

i) **Ans:** 9% ii) **Ans:** 3% iii) **Ans:** 1%

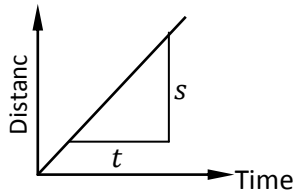
Acceleration and Velocity

Constant Velocity

When a car moves a distance $s\ (m)$ in a time $t(s)$ along a straight road, the *velocity, V* is constant.

$V = \frac{s}{t}\ m/s$

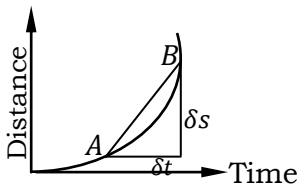
This is the gradient of the distance/time graph



Here V =constant= $\frac{s}{t}$

Variable Velocity

If however, the velocity of the car is not constant then the distance/time graph will not be a straight line.



For variable velocity we are concerned with *average velocity*

Definition

Average velocity= $\frac{Total\ \ distance\ \ \ travelled}{Total\ \ time\ \ taken} = \frac{Increase\ \ in\ s}{Increase\ \ in\ t}$

$\therefore$ Average velocity over time $\delta t = \frac{\delta s}{\delta t}$

Now as $\delta t \to 0$, the chord AB becomes a tangent, such that the velocity at point A is $V_A = \frac{ds}{dt}$

Velocity at an Instant

Thus, the velocity of a car at any instant is the gradient of the distance/time graph. Thus if the expression of distance is in terms of time t, then the velocity is obtained by differentiating the expression.

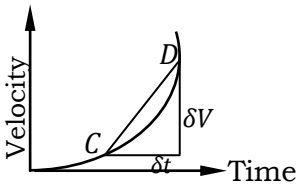
Definition:

Velocity at an instant is the limit of the average velocity for the interval following that instant, as the interval tends to zero. $i.e.$

$V = \lim_{as\ \delta t \to 0}(\delta s/\delta t) = ds/dt$

Variable Acceleration

The acceleration of a car is defined as the rate of change of velocity.



If δV is the change in v, and δt is the corresponding change in time, then $a = \lim_{as\ \delta t \to 0}(\delta V/\delta t)$, chord CD becomes a tangent at point C, the acceleration is given by $a = dV/dt$

Hence acceleration of the car at any instant is given by the gradient of velocity/time graph.

Now if the expression of velocity is known in terms of time t then the acceleration is obtained by differentiating the expression.

$\therefore$ Acceleration, $a = \frac{dV}{dt}$

However, $V = \frac{ds}{dt}$

Hence $a = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$

Example

1. *If a billiard ball is dropped from a height of* $100\ feet$*, its height, s at time t is given by the position function* $s = -16t^2 + 100$*, where s is measured in feet and t is measured in seconds. Find the average velocity over the time intervals*

a) $1 \le t \le 2$ b) $1 \le t \le 1.5$ c) $1 \le t \le 1.1$

Solution

a) $s_{t=1} = -16 + 100 = 84$
$s_{t=2} = -64 + 100 = 36$
} For $1 \le t \le 2$, the object falls from a height of $84ft$ to a height of $36ft$

$\therefore$ Average velocity $= \frac{36-84}{2-1} = -48ft/sec$

b) $s_{t=1} = -16 + 100 = 84$
$s_{t=1.5} = -36 + 100 = 64$
} For $1 \le t \le 1.5$, the object falls from a height

$\therefore$ Average velocity $= \frac{64-84}{1.5-1} = -40ft/sec$

c) For interval $1 \le t \le 1.1$, Average velocity $= \frac{80.64-84}{1.1-1} = -33.6ft/sec$

NOTE:
Average velocity is *negative*, indicating that the object is moving downward.

2. *At time $t = 0$, a diver jumps from a diving board that is $32 feet$ above the water the position of the diver is given by $s(t) = -16t^2 + 16t + 32$ where s is measured in feet and t is measured in seconds.*

a) *When does the diver hit the water?*
b) *What is the diver's velocity at impact?*

Solution

a) When the diver hits the water, $s = 0$
$0 = -16t^2 + 16t + 32$, solving $t = -1\,, 2$
Taking $t = 2s$, the diver takes $2\ seconds$

b) Velocity, $V = \frac{ds}{dt} = -32t + 16$
Now when $t = 2\,, V = -64 + 16 = -48\ ft/sec$

3. *The distance $x(m)$ moved by a car in a time $t(s)$ is given by $x = 3t^3 - 2t^2 + 4t - 1$. Determine the velocity and acceleration when*

a) $t = 0$ and b) $t = 1.5s$

Solution

Distance: $x = 3t^3 - 2t^2 + 4t - 1 \quad m$
Velocity: $V = \frac{dx}{dt} = 9t^2 - 4t + 4 \quad m/s$
Acceleration: $a = \frac{d^2x}{dt^2} = 18t - 4 \quad m/s^2$

a) When $t = 0$
Velocity, $V = \frac{dx}{dt} = 9(0)^2 - 4(0) + 4 = 4m/s$
Acceleration, $a = \frac{d^2x}{dt^2} = 18(0) - 4 = -4m/s^2$

b) When time $t = 1.5s$
Velocity, $V = \frac{dx}{dt} = 9(1.5)^2 - 4(1.5) + 4 = 18.25m/s$
Acceleration, $a = \frac{d^2x}{dt^2} = 18(1.5) - 4 = 23m/s^2$

4. *A particle moves along a straight line so that after $t(s)$, its distance from O affixed point on the line is $s(m)$ where $s = t^3 - 3t^2 + 2t$*

a) *When is the particle at O?*
b) *What is its velocity and acceleration at these times*
c) *What is the average velocity after the first second?*
d) *What is the average acceleration between $t = 0$ and $t = 2$?*

Solution

Given: Distance: $s = t^3 - 3t^2 + 2t \ m$

Velocity: $V = \frac{ds}{dt} = 3t^2 - 6t + 2 \ m/s$

Acceleration: $a = \frac{d^2s}{dt^2} = 6t - 6 \ m/s^2$

a) A particle is at O if $s = 0$

$\therefore 0 = t^3 - 3t^2 + 2t \ \Rightarrow t(t-1)(t-2) = 0$

$\therefore t = 0, 1, 2$

b) At $t = 0, V = 3(0)^2 - 6(0) + 2 = 2 \ m/s$

At $t = 0, a = 6(0) - 6 = -6 \ m/s^2$

At $t = 1, V = 3(1)^2 - 6(1) + 2 = -1 \ m/s$

At $t = 1, a = 6(1) - 6 = 0 \ m/s^2$

At $t = 2, V = 3(2)^2 - 6(2) + 2 = 2 \ m/s$

At $t = 2, a = 6(2) - 6 = 6 \ m/s^2$

c) Average Velocity $= \frac{s_{t=1} - s_{t=0}}{1-0} = \frac{(1)^3 - 3(1)^2 + 2(1)}{1} = 0 \ m/s$

d) Average Acceleration $= \frac{V_{t=1} - V_{t=0}}{2-0} = \frac{2-2}{2} = 0 \ m/s^2$

Task

1. $x = \frac{20}{3}t^3 - \frac{23}{2}t^2 + 6t + 5$ represents the distance $x \ metres$, moved by a body in $t \ seconds$. Determine:

a) The velocity and acceleration at the start

b) The velocity and acceleration when $t = 3s$

c) The value of t when the body is at rest

d) The values of t when the acceleration is $37 \ m/s$ and

e) The distance travelled in the third second

Ans: a) $6m/s$, $-23m/s^2$ b) $117m/s$, $97 \ m/s^2$ c) $t = \frac{3}{4}s \ or \frac{2}{3}s$

d) $1\frac{1}{2}s$ e) $75\frac{1}{6}m$

2. A missile fired from the ground level rises $x \ (m)$ vertically upwards in t seconds and $x = 100t - \frac{25}{2}t^2$. Find:

a) The initial velocity of the missile **Ans:** $100m/s$

b) The time taken when the height of the missile is maximum. **Ans:** $4s$

c) The maximum height reached. **Ans:** $200m$

d) The velocity with which the missile strikes the ground .**Ans:** $-100m/s$

3. At any time t seconds the distance $x \ metres$ of a particle moving in a straight line from a fixed point is given by: $x = 4t + \ln(1 - t)$. Determine

a) The initial velocity and acceleration **Ans:** $3 \ m/s$; $-1 \ m/s^2$

b) The velocity and acceleration after $t = 1.5s$. **Ans:** $6m/s$; $-4m/s^2$

c) The time taken when the velocity is zero. **Ans:** $\frac{3}{4}s$

DIFFERENTIATING TRIGONOMETRIC FUNCTIONS

Differentiating Trigonometric Functions from First Principles

The following should be noted when differentiating trigonometric functions.

1) All angles of the trigonometric ratios must be in radians. If the angle is not in radians, use the conversion $1\pi = 180^0$
2) The knowledge of limits is needed. *i.e.*

For small angle θ in radians, $\sin\theta \simeq \theta$ *or* $\frac{\sin\theta}{\theta} \simeq 1$, $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$, $\tan\theta \simeq \theta$

3) Factor formula is applied to simplify trigonometric expressions

1. *Differentiate from first principles*

a) $\sin x$

b) $\cos x$ c) $\tan x$

Solution

a) Let $y = \sin x$

Let x change by a small amount δx and the corresponding change in y be δy then;

$y + \delta y = \sin(x + \delta x)$

$\delta y = \sin(x + \delta x) - y$

$= \sin(x + \delta x) - \sin x$

$= 2\cos\left(x + \frac{\delta x}{2}\right).\sin\left(\frac{\delta x}{2}\right)$

Dividing through by δx

$\frac{\delta y}{\delta x} = 2\cos\left(x + \frac{\delta x}{2}\right).\frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}$

Introducing limits on both sides

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left\{2\cos\left(x + \frac{\delta x}{2}\right).\frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}\right\}$, putting $\theta = \frac{\delta x}{2} \Longrightarrow \delta x = 2\theta$

$= \lim_{\theta\to 0}\left\{2\cos(x + \theta).\frac{\sin(\theta)}{2\theta}\right\}$, but $\frac{\sin\theta}{\theta} \simeq 1$

$= \cos x$

$\therefore \frac{d}{dx}(\sin x) = \cos x$

b) Let $y = \cos x$

Let x change by a small amount δx and the corresponding change in y be δy then;

$y + \delta y = \cos(x + \delta x)$

$\delta y = \cos(x + \delta x) - y$

$= \cos(x + \delta x) - \cos x$

$= -2\sin\left(x + \frac{\delta x}{2}\right).\sin\left(\frac{\delta x}{2}\right)$

Dividing through by δx

$\frac{\delta y}{\delta x} = 2\sin\left(x+\frac{\delta x}{2}\right).\frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}$

Introducing limits on both sides

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left\{2\sin\left(x+\frac{\delta x}{2}\right).\frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}\right\}$, putting $\theta = \frac{\delta x}{2} \Longrightarrow \delta x = 2\theta$

$= \lim_{\theta\to 0}\left\{2\sin(x+\theta).\frac{\sin(\theta)}{2\theta}\right\}$, but $\frac{\sin\theta}{\theta} \simeq 1$

$= \sin x$

$\therefore \frac{d}{dx}(\cos x) = \sin x$

c) Let $y = \tan x$

Let x change by a small amount δx and the corresponding change in y be δy then;

$y + \delta y = \tan(x+\delta x)$

$\delta y = \tan(x+\delta x) - y$

$= \tan(x+\delta x) - \tan x$

$= \frac{\tan x+\tan\delta x}{1-\tan x.\tan\delta x} - \tan x$

$= \frac{\tan x+\tan\delta x-\tan x+\tan^2 x.\tan\delta x}{1-\tan x.\tan\delta x}$

$= \frac{\tan\delta x+\tan^2 x.\tan\delta x}{1-\tan x.\tan\delta x} = \frac{\tan\delta x(1+\tan^2 x)}{1-\tan x.\tan\delta x}$

$= \frac{\tan\delta x(\sec^2 x)}{1-\tan x.\tan\delta x}$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{\frac{\tan\delta x}{\delta x}(\sec^2 x)}{1-\tan x.\tan\delta x}$

Introducing limits on both sides

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left\{\frac{\frac{\tan\delta x}{\delta x}(\sec^2 x)}{1-\tan x.\tan\delta x}\right\}$, putting $\delta x = \theta$

$= \lim_{\theta\to 0}\left\{\frac{\frac{\tan\theta}{\theta}(\sec^2 x)}{1-\tan x.\tan\theta}\right\}$, but $\frac{\tan\theta}{\theta} \simeq \frac{\theta}{\theta} = 1$

$= \sec^2 x$

$\therefore \frac{d}{dx}(\tan x) = \sec^2 x$

Alternatively

$y + \delta y = \tan(x+\delta x)$

$\delta y = \tan(x+\delta x) - y$

$= \tan(x+\delta x) - \tan x$

$= \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}$

$= \frac{\sin(x+\delta x)\cos x-\cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}$

$= \frac{\sin(x+\delta x-x)}{\cos(x+\delta x)\cos x} = \frac{\sin(\delta x)}{\cos(x+\delta x)\cos x}$

Divide through by δx

$\frac{\delta y}{\delta x} = \frac{\frac{\sin(\delta x)}{\delta x}}{\cos(x+\delta x)\cos x}$

Introducing limits on both sides

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left\{\frac{\frac{\sin(\delta x)}{\delta x}}{\cos(x+\delta x)\cos x}\right\}$, putting $\delta x = \theta$

$\lim_{\theta\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\theta\to 0}\left\{\frac{\frac{\sin(\theta)}{\theta}}{\cos(x+\theta)\cos x}\right\}$, but $\frac{\sin\theta}{\theta} \simeq 1$

$\frac{dy}{dx} = \frac{1}{\cos x\cos x} = \sec^2 x$

$\therefore \frac{d}{dx}(\tan x) = \sec^2 x$

2. *Differentiate from fist principles*

a) $y = \sin 3x$

b) $y = \cos 2x$ c) $y = \tan\frac{1}{2}x$

Solution

a) $y = \sin 3x$

Let x change by a small amount δx and the corresponding change in y be δy then;

$y + \delta y = \sin 3(x + \delta x)$

$\delta y = \sin 3(x + \delta x) - y$

$= \sin 3(x + \delta x) - \sin 3x$

$= 2\cos\left(3x + \frac{3\delta x}{2}\right).\sin\left(\frac{3\delta x}{2}\right)$

Dividing through by δx

$\frac{\delta y}{\delta x} = 2\cos\left(3x + \frac{3\delta x}{2}\right).\frac{\sin\left(\frac{3\delta x}{2}\right)}{\delta x}$

Introducing limits on both sides

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left\{2\cos\left(3x + \frac{3\delta x}{2}\right).\frac{\sin\left(\frac{3\delta x}{2}\right)}{\delta x}\right\}$, putting $\theta = \frac{3\delta x}{2} \Longrightarrow \delta x = \frac{2}{3}\theta$

$= \lim_{\theta\to 0}\left\{2\cos(3x+\theta).\frac{\sin(\theta)}{\frac{2}{3}\theta}\right\} = \lim_{\theta\to 0}\left\{3\cos(3x+\theta).\frac{\sin(\theta)}{\theta}\right\}$, but $\frac{\sin\theta}{\theta} \simeq 1$

$= 3\cos 3x$

$\therefore \frac{d}{dx}(\sin 3x) = 3\cos 3x$

b) Left as an exercise. **Ans:** $-2\sin 2x$

c) Left as an exercise. **Ans:** $\frac{1}{2}\sec^2\frac{1}{2}x$

Differentiating Reciprocals of Cosine, Sine and Tangent

1. *Differentiate from first principles*

a) $\sec x$

b) $\operatorname{cosec} x$ c) $\cot x$

Solution

a) let $y = \sec x = \frac{1}{\cos x}$

Let x change by a small amount δx and the corresponding change in y be δy then;

$y + \delta y = \frac{1}{\cos(x+\delta x)}$

$\delta y = \frac{1}{\cos(x+\delta x)} - y$

$= \frac{1}{\cos(x+\delta x)} - \frac{1}{\cos x}$

$= \frac{\cos x - \cos(x+\delta x)}{\cos(x+\delta x)\cos x} = \frac{-2\sin\left(x+\frac{\delta x}{2}\right).\sin\left(-\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}$

$= \frac{2\sin\left(x+\frac{\delta x}{2}\right).\sin\left(\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}$

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{2\sin\left(x+\frac{\delta x}{2}\right).\frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}}{\cos(x+\delta x)\cos x}$

Introducing limits on both sides

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left\{\frac{2\sin\left(x+\frac{\delta x}{2}\right).\frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}}{\cos(x+\delta x)\cos x}\right\}$, putting $\theta = \frac{\delta x}{2} \Longrightarrow \delta x = 2\theta$

$= \lim_{\theta\to 0}\left\{\frac{2\sin(x+\theta).\frac{\sin(\theta)}{2\theta}}{\cos(x+\theta)\cos x}\right\} = \lim_{\theta\to 0}\left\{\frac{\sin(x+\theta).\frac{\sin(\theta)}{\theta}}{\cos(x+\theta)\cos x}\right\}$, but $\frac{\sin\theta}{\theta} \simeq 1$

$= \frac{\sin x}{\cos x\cos x} = \tan x\sec x$

$\therefore \frac{d}{dx}(\sec x) = \tan x\sec x$

b) left as an exercise: **Ans:** $-\operatorname{cosec} x\cot x$

c) left as an exercise: **Ans:** $-\operatorname{cosec}^2 x$

2. *Differentiate from first principles*

a) $\operatorname{cosec}\frac{1}{2}x$

b) $\sec 3x$ c) $\cot 4x$

Solution

a) Let $y = \operatorname{cosec}\frac{1}{2}x = \frac{1}{\sin\left(\frac{1}{2}x\right)}$

Let x change by a small amount δx and the corresponding change in y be δy then;

$y + \delta y = \frac{1}{\sin\left(\frac{x+\delta x}{2}\right)}$

$\delta y = \frac{1}{\sin\left(\frac{x+\delta x}{2}\right)} - \frac{1}{\sin\left(\frac{1}{2}x\right)}$

$= \frac{\sin\left(\frac{1}{2}x\right) - \sin\left(\frac{x+\delta x}{2}\right)}{\sin\left(\frac{x+\delta x}{2}\right).\sin\left(\frac{1}{2}x\right)} = \frac{2\cos\left(\frac{x}{2}+\frac{\delta x}{4}\right).\sin\left(-\frac{\delta x}{4}\right)}{\sin\left(\frac{x+\delta x}{2}\right).\sin\left(\frac{1}{2}x\right)} = \frac{-2\cos\left(\frac{x}{2}+\frac{\delta x}{4}\right).\frac{\sin\left(\frac{\delta x}{4}\right)}{\delta x}}{\sin\left(\frac{x+\delta x}{2}\right).\sin\left(\frac{1}{2}x\right)}$

Divide through by δx

$\frac{\delta y}{\delta x} = \frac{2\cos\left(\frac{x}{2}+\frac{\delta x}{4}\right).\frac{\sin\left(-\frac{\delta x}{4}\right)}{\delta x}}{\sin\left(\frac{x+\delta x}{2}\right).\sin\left(\frac{1}{2}x\right)}$

Introducing limits on both sides

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left\{\frac{-2\cos\left(\frac{x}{2}+\frac{\delta x}{4}\right).\frac{\sin\left(\frac{\delta x}{4}\right)}{\delta x}}{\sin\left(\frac{x+\delta x}{2}\right).\sin\left(\frac{1}{2}x\right)}\right\}$, putting $\frac{\delta x}{4} = \theta \Longrightarrow \delta x = 4\theta$

$= \lim_{\theta\to 0}\left\{\frac{-2\cos\left(\frac{x}{2}+\theta\right).\frac{\sin(\theta)}{4\theta}}{\sin\left(\frac{x+4\theta}{2}\right).\sin\left(\frac{1}{2}x\right)}\right\} = \lim_{\theta\to 0}\left\{\frac{-\cos\left(\frac{x}{2}+\theta\right).\frac{\sin(\theta)}{2\theta}}{\sin\left(\frac{x}{2}+2\theta\right).\sin\left(\frac{1}{2}x\right)}\right\}$

$\frac{dy}{dx} = \frac{-\frac{1}{2}.\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right).\sin\left(\frac{x}{2}\right)} = -\frac{1}{2}\cot\frac{x}{2}\operatorname{cosec}\frac{x}{2}$

$\therefore \frac{d}{dx}\left(\operatorname{cosec}\frac{1}{2}x\right) = -\frac{1}{2}\cot\frac{x}{2}\operatorname{cosec}\frac{x}{2}$

b) Left as an exercise. **Ans:** $3\sec 3x\tan 3x$

c) Left as an exercise. **Ans:** $-4\operatorname{cosec}^2 3x$

Extension of Differentiating From First Principles

Differentiate from first principles

a) $y = \sin^2 x$

b) $y = \cos^2(x^2)$ c) $y = \tan^2(x^2+1)$

Solution

a) $y = \sin^2 x$

Let x change by a small amount δx and the corresponding change in y be δy then;

$y + \delta y = \sin^2(x+\delta x)$

$\delta y = \sin^2(x+\delta x) - y$

$\delta y = \sin^2(x+\delta x) - \sin^2 x$

$= [\sin(x+\delta x) + \sin x].[\sin(x+\delta x) - \sin x]$

$= 2\sin\left(x+\frac{\delta x}{2}\right)\cos\left(\frac{\delta x}{2}\right).2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$

$= \left[2\sin\left(x+\frac{\delta x}{2}\right)\cos\left(x+\frac{\delta x}{2}\right)\right].2\cos\left(\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$

$= \sin(2x+\delta x).\sin(\delta x)$

Divide through by δx

$\frac{\delta y}{\delta x} = \sin(2x+\delta x).\frac{\sin(\delta x)}{\delta x}$

Introducing limits on both sides

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left\{\sin(2x+\delta x).\frac{\sin(\delta x)}{\delta x}\right\}$, putting $\delta x = \theta$

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\theta\to 0}\left\{\sin(2x+\theta).\frac{\sin(\theta)}{\theta}\right\} = \sin 2x$

$\therefore \frac{d}{dx}(\sin^2 x) = \sin 2x$

b) $y = \cos^2(x^2)$

Let x change by a small amount δx and the corresponding change in y be δy then;

$y + \delta y = \cos^2(x + \delta x)^2$

$\delta y = \cos^2(x + \delta x)^2 - y$

$\delta y = \cos^2(x + \delta x)^2 - \cos^2(x^2)$

$= [\cos(x + \delta x)^2 + \cos x^2].[\cos(x + \delta x)^2 - \cos x^2]$

$= 2\cos\left[\frac{(x+\delta x)^2+x^2}{2}\right]\cos\left[\frac{(x+\delta x)^2-x^2}{2}\right].\left[-2\left[\sin\left[\frac{(x+\delta x)^2+x^2}{2}\right]\right]\sin\left[\frac{(x+\delta x)^2-x^2}{2}\right]\right]$

$= 2\cos\left[\frac{(x+\delta x)^2+x^2}{2}\right]\sin\left[\frac{(x+\delta x)^2+x^2}{2}\right].-2\cos\left[\frac{(x+\delta x)^2-x^2}{2}\right]\sin\left[\frac{(x+\delta x)^2-x^2}{2}\right]$

$= \sin[(x + \delta x)^2 + x^2]\,[-\sin[(x + \delta x)^2 - x^2]]$

$= \sin[(x + \delta x)^2 + x^2]\,[-\sin[(2x + \delta x)(\delta x)]]$

$= -\sin[(x + \delta x)^2 + x^2].\sin[(\delta x)(2x + \delta x)]$

Divide through by δx

$\frac{\delta y}{\delta x} = -\sin[(x + \delta x)^2 + x^2].\frac{\sin[(\delta x)(2x+\delta x)]}{\delta x}$

Introducing limits on both sides

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left\{-\sin[(x + \delta x)^2 + x^2].\frac{\sin[(\delta x)(2x+\delta x)]}{\delta x}\right\}$

$= \lim_{\delta x\to 0}\{-\sin[(x + \delta x)^2 + x^2]\}.\lim_{\delta x\to 0}\left(\frac{\sin[(\delta x)(2x+\delta x)]}{\delta x}\right)$

$= -\sin[x^2 + x^2].\lim_{\delta x\to 0}\left(\frac{(\delta x)(2x+\delta x)}{\delta x}\right) = -2x.\sin(2x^2)$

$\therefore \frac{d}{dx}[\cos^2(x^2)] = -2x.\sin(2x^2)$

c) Left as an exercise. **Ans:** $4x\tan(x^2 + 1).\sec^2(x^2 + 1)$

Differentiating Inverse of Trigonometric Functions from First Principles

Differentiate from first principles

a) $\sin^{-1} x$

b) $\cos^{-1} x$ c) $\tan^{-1} x$

Solution

a) Let $y = \sin^{-1} x \Longrightarrow x = \sin y$

Let x change by a small amount δx and the corresponding change in y be δy then;

$x + \delta x = \sin(y + \delta y)$

Making δx the subject

$\delta x = \sin(y + \delta y) - x$

$\delta x = \sin(y + \delta y) - \sin y$

Dividing through by δy

$\frac{\delta x}{\delta y} = 2\cos\left(y + \frac{\delta y}{2}\right).\frac{\sin\left(\frac{\delta y}{2}\right)}{\delta y}$

$\therefore \frac{\delta y}{\delta x} = \frac{1}{2\cos\left(y+\frac{\delta y}{2}\right).\frac{\sin\left(\frac{\delta y}{2}\right)}{\delta y}}$

Introducing limits on both sides

$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}}\left(\frac{1}{2\cos\left(y+\frac{\delta y}{2}\right).\sin\left(\frac{\delta y}{2}\right)}\right)$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

b) Left as an exercise. **Ans:** $-\frac{1}{\sqrt{1-x^2}}$

c) Left as an exercise. **Ans:** $\frac{1}{1+x^2}$

Important General Deductions of Differentiating Trigonometric Functions

1. *Differentiate* $sin[f(x)]$ *w.r.t. x*

Solution

Let $y = \sin[f(x)]$

$u = f(x)$ ..... (1)

$\Longrightarrow y = \sin u$ ..... (2)

Writing Chain Rule from eqn. (1) and eqn. (1)

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$y = \sin u \Longrightarrow \frac{dy}{du} = \cos u$

$u = f(x) \Longrightarrow \frac{du}{dx} = f'(x)$

$\therefore \frac{dy}{dx} = \cos u \,.f'(x) = \cos f(x)\,.f'(x)$

$$\therefore \frac{d}{dx}\{\sin[f(x)]\} = f'(x).\cos[f(x)]$$

2. *Differentiate* $cos[f(x)]$ *w.r.t. x*

Proceeding as above,

$$\therefore \frac{d}{dx}\{\cos[f(x)]\} = -f'(x).\sin[f(x)]$$

3. *Differentiate* $\tan[f(x)]$ *w.r.t. x*

It can be deduced as in (1) above that

$$\therefore \frac{d}{dx}\{\tan[f(x)]\} = f'(x).\sec^2[f(x)]$$

4. *Differentiate* $\sec[f(x)]$ *w.r.t. x*

Solution

Let $y = \sec[f(x)]$

$u = f(x)$ ..... (1)

$\Rightarrow y = \sec u$ …… (2)

Writing Chain Rule from eqn. (1) and eqn. (1)

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$y = \sec u \Rightarrow \frac{dy}{du} = \sec u \,.\tan u$

$u = f(x) \Rightarrow \frac{du}{dx} = f'(x)$

$\therefore \frac{dy}{dx} = \sec u \,.\tan u \,.f'(x) = \sec[f(x)]\,.\tan[f(x)]\,.f'(x)$

$$\therefore \frac{\boldsymbol{d}}{\boldsymbol{dx}}\{\mathbf{sec}[\boldsymbol{f}(\boldsymbol{x})]\} = \boldsymbol{f}'(\boldsymbol{x}).\mathbf{sec}[\boldsymbol{f}(\boldsymbol{x})]\,.\mathbf{tan}[\boldsymbol{f}(\boldsymbol{x})]$$

5. *Differentiate* $\sec[f(x)]$ *w.r.t. x*

Proceeding as in (4) above

$$\therefore \frac{\boldsymbol{d}}{\boldsymbol{dx}}\{\mathbf{cosec}[\boldsymbol{f}(\boldsymbol{x})]\} = -\boldsymbol{f}'(\boldsymbol{x}).\mathbf{cosec}[\boldsymbol{f}(\boldsymbol{x})]\,.\mathbf{cot}[\boldsymbol{f}(\boldsymbol{x})]$$

6. *Differentiate* $\cot[f(x)]$ *w.r.t. x*

Proceeding as in the above seductions

$$\therefore \frac{\boldsymbol{d}}{\boldsymbol{dx}}\{\mathbf{cot}[\boldsymbol{f}(\boldsymbol{x})]\} = -\boldsymbol{f}'(\boldsymbol{x}).\mathbf{cosec^2}[\boldsymbol{f}(\boldsymbol{x})]$$

Examples

1. *Differentiate w.r.t.x*

a) $y = \sin\left(4x - \frac{\pi}{6}\right)$ c) $y = \sec(3x^2 + 1)$

b) $y = \cos(4x^2 + 3x)$ d) $y = \sin[\tan 2x]$ e) $y = \cot\sqrt{\frac{x}{1+x^2}}$

Solution

a) $y = \sin\left(4x - \frac{\pi}{6}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(4x - \frac{\pi}{6}\right).\cos\left(4x - \frac{\pi}{6}\right)$

$= 4\cos\left(4x - \frac{\pi}{6}\right)$

b) $y = \cos(4x^2 + 3x)$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(4x^2 + 3x).[-\sin(4x^2 + 3x)]$

$= -(8x + 3)\sin(4x^2 + 3x)$

c) $y = \sec(3x^2 + 1)$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(3x^2 + 1).\sec(3x^2 + 1).\tan(3x^2 + 1)$

$= 6x.\sec(3x^2 + 1).\tan(3x^2 + 1)$

d) $y = \sin[\tan 2x]$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\tan 2x).\cos[\tan 2x]$

$= \frac{d}{dx}(2x).\sec^2 2x\,.\cos[\tan 2x]$

$= 2\sec^2 2x\,.\cos[\tan 2x]$

e) $y = \cot\sqrt{\frac{x}{1+x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{x}{1+x^2}}\right).\left[-\operatorname{cosec}^2\sqrt{\frac{x}{1+x^2}}\right]$

Dealing with $\frac{d}{dx}\left(\sqrt{\frac{x}{1+x^2}}\right)$

$\frac{d}{dx}\left(\sqrt{\frac{x}{1+x^2}}\right) = \frac{d}{dx}\left(\frac{x^{1/2}}{(1+x^2)^{1/2}}\right)$

$= \frac{(1+x^2)^{\frac{1}{2}}.\frac{1}{2}.x^{-1/2} - x^{\frac{1}{2}}.\frac{1}{2}.(1+x^2)^{-1/2}.2x}{\left[(1+x^2)^{1/2}\right]^2} = \frac{\frac{1}{2}.x^{-1/2}.(1+x^2)^{-1/2}[1+x^2-2x^2]}{1+x^2}$

$= \frac{\frac{1}{2}.x^{-1/2}.(1+x^2)^{-1/2}(1-x^2)}{1+x^2} = \frac{1-x^2}{2x^{1/2}.(1+x^2)^{1/2}.(1+x^2)}$

$= \frac{1-x^2}{2\sqrt{x}.(1+x^2)^{3/2}} = \frac{1-x^2}{2\sqrt{x(1+x^2)^3}}$

$\therefore \frac{dy}{dx} = -\frac{1-x^2}{2\sqrt{x(1+x^2)^3}}.\operatorname{cosec}^2\sqrt{\frac{x}{1+x^2}} = \frac{x^2-1}{2\sqrt{x(1+x^2)^3}}.\operatorname{cosec}^2\sqrt{\frac{x}{1+x^2}}$

2. *Differentiate w.r.t.x*

a) $6\cos x - 8(x^2+x)$ c) $4x^2\sin x - 3x^2\cos x$ e) $\frac{\sin x}{x^2+\cos x}$

b) $x^2.\sin 2x$ d) $3x\sin x\cos x$ f) $\frac{x\cos x}{(x+1)\sin x}$

Solution

a) Let $y = 6\cos x - 8(x^2+x)$

$y = 6\cos x - 8x^2 - 8x$

$\Rightarrow \frac{dy}{dx} = 6\frac{d}{dx}(\cos x)8.\frac{d}{dx}(x^2) - 8\frac{d}{dx}(x)$

$\frac{dy}{dx} = -6\sin x - 16x - 8$

b) Let $y = x^2.\sin 2x$

By Product Rule

$\frac{dy}{dx} = x^2.\frac{d}{dx}(\sin 2x) + \sin 2x\,.\frac{d}{dx}(x^2)$

$= 2x^2\cos 2x + 2x\sin 2x$

c) Let $y = 4x^2\sin x - 3x^2\cos x$

$\frac{dy}{dx} = 4.\frac{d}{dx}(x^2\sin x) - 3.\frac{d}{dx}(x^2\cos x)$

By Product Rule

$\frac{dy}{dx} = 4x^2.\frac{d}{dx}(\sin x) + 4\sin x\,.\frac{d}{dx}(x^2) - \left[3x^2.\frac{d}{dx}(\cos x) + 3\cos x\,.\frac{d}{dx}(x^2)\right]$

$= x^2\cos x + 8x\sin x + 3x^2\sin x - 6x\cos x$

$= (x^2 - 6x)\cos x + (3x^2 + 8x)\sin x$

d) Let $y = 3x\sin x\cos x$

By Product Rule

$\frac{dy}{dx} = 3\left[x\sin x\,.\frac{d}{dx}(\cos x) + x\cos x\,.\frac{d}{dx}(\sin x) + \sin x\cos x\,.\frac{dx}{dx}\right]$

$\frac{dy}{dx} = 3[-x\sin x\sin x + x\cos x\,.\cos x + \sin x\cos x]$

$= 3[-x\sin^2 x + x\cos^2 x + \sin x\cos x] = 3\left[(\cos^2 x - \sin^2 x) + \frac{1}{2}\sin 2x\right]$

$= 3\cos 2x + \frac{3}{2}\sin 2x$

NOTE

$\frac{d}{dx}(u.v.w) = vw.\frac{du}{dx} + uw.\frac{dv}{dx} + uv.\frac{dw}{dx}$, is the extended product rule

e) Let $y = \frac{\sin x}{x^2+\cos x}$; By Quotient Rule

$\frac{dy}{dx} = \frac{(x^2+\cos x).\cos x - \sin x.(2x - \sin x)}{(x^2+\cos x)^2} = \frac{x^2\cos x + \cos^2 x - 2x\sin x + \sin^2 x}{(x^2+\cos x)^2}$

$\therefore \frac{dy}{dx} = \frac{x^2\cos x - 2x\sin x + 1}{(x^2+\cos x)^2}$

f) Let $y = \frac{x\cos x}{(x+1)\sin x}$, By Quotient Rule

$\frac{dy}{dx} = \frac{[(x+1)\sin x].\frac{d}{dx}(x\cos x) - x\cos x.\frac{d}{dx}[(x+1)\sin x]}{(x+1)^2\sin^2 x}$

$= \frac{[(x+1)\sin x].(-x\sin x + \cos x) - x\cos x.[\sin x + (x+1)\cos x]}{(x+1)^2\sin^2 x}$

$= \frac{-x(x+1)\sin^2 x + (x+1)\cos x\sin x - x\cos x\sin x - x(x+1)\cos^2 x}{(x+1)^2\sin^2 x}$

$= \frac{-x(x+1)[\sin^2 x + \cos^2 x] + \cos x\sin x(x+1-x)}{(x+1)^2\sin^2 x} = \frac{-x(x+1) + \cos x\sin x}{(x+1)^2\sin^2 x}$

$= \frac{\cos x\sin x - x(x+1)}{(x+1)^2\sin^2 x}$

Task

Differentiate w.r.t. x

a) $\frac{\sin x}{x^2+\cos x}$ b) $\frac{x\cos x}{x+3}$ c) $\frac{x+\sin x}{1+\cos x}$ d) $\frac{x}{x+\sin x}$

e) $\frac{x\sin x}{\cos x+\sin x}$

f) $\frac{x^2\sin x}{(x+1)(x^2-1)}$ g) $8x^2(1+\sin x)(1+\cos x)$ h) $x\cos x + 3(x+1)(x-1)$

Answers

a) $\frac{1+x(x\cos x - 2\sin x)}{x^2+\cos x}$ b) $\frac{3\cos x - (x+3)x\sin x}{(x+3)^2}$ c) $\frac{3+2\cos 2x + x\sin x}{(1+\cos x)^2}$ d) $\frac{\sin x - x\cos x}{(x+\sin x)^2}$

e) $\frac{x+\sin x(\sin x+\cos x)}{(\cos x+\sin x)^2}$

f) $\frac{\cos x(x^5+x^4-x^3-x^2) - \sin x(x^4+x^2+2x)}{(x+1)^2(x^2-1)^2}$ g) $16x(1+\cos x)(1+\sin x) + 8x^2(\cos x - \sin x)(\cos x + \sin x + 1)$

h) $\cos x - x\sin x + 6x$

PROOFS INVOLVING THE ABOVE THEOREMS

1. *If* $y = \frac{\sin x}{x}$, *show that* $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (x^2+2)y = 0$

2. *If* $y = \frac{\cos x}{x}$, *show that* $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = 0$

3. *If* $y = \sin x$, *show that* $\frac{d^2y}{dx^2} = -y$, $\frac{d^4y}{dx^4} = y$

4. *If* $y = \sec\theta$, *show that* $\frac{d^2y}{dx^2} - y(2y^2-1)$

Solution

1. $y = \frac{\sin x}{x} \Longrightarrow y.x^2 = \sin x$

Differentiating w.r.t x

$x^2\frac{dy}{dx} + 2xy = \cos x$

Now differentiating w.r.t x

$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 2\left(x\frac{dy}{dx} + y\right) = -\sin x$

$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = -\sin x = -y.x^2$

$\therefore x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (x^2+2)y = 0$

2. Left as an exercise

3. $y = \sin x \Longrightarrow \frac{dy}{dx} = \cos x$

$\Longrightarrow \frac{d^2y}{dx^2} = -\sin x = -y$

Differentiating w.r.t x

$\frac{d^3y}{dx^3} = -\frac{dy}{dx}$

Differentiating again w.r.t x

$\frac{d^4y}{dx^4} = -\frac{d^2y}{dx^2} = -(-y) = y$

$\therefore \frac{d^4y}{dx^4} = y$

4. Left as an exercise

Differentiating $\sin^n[f(x)]$, $\cos^n[f(x)]$, $\tan^n[f(x)]$ w.r.t x

a) Let $y = \sin^n[f(x)] = [\sin[f(x)]]^n$

$u = \sin[f(x)] \Longrightarrow \frac{du}{dx} = f'(x).\cos[f(x)]$

$\Longrightarrow y = u^n \Longrightarrow \frac{dy}{du} = nu^{n-1}$

By Chain Rule

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$= nu^{n-1}.f'(x).\cos[f(x)]$

$= n\sin^{n-1}[f(x)].f'(x).\cos[f(x)]$

$$\therefore \frac{d}{dx}\{\sin^n[f(x)]\} = n\sin^{n-1}[f(x)].f'(x).\cos[f(x)]$$

b) Let $y = \cos^n[f(x)]$

Proceeding as in (a) above

$$\therefore \frac{d}{dx}\{\cos^{n}[f(x)]\} = -n\cos^{n-1}[f(x)]\,.\,f'(x).\sin[f(x)]$$

c) Let $y = \tan^{n}[f(x)]$

Proceeding as in (a) above

$$\therefore \frac{d}{dx}\{\tan^{n}[f(x)]\} = -n\tan^{n-1}[f(x)]\,.\,f'(x).\sec^{2}[f(x)]$$

Examples

a) *Differentiate w.r.t x*

a) $y = \cos^4\left(2x + \frac{\pi}{3}\right)$ b) $y = \sqrt{\sin 4x}$ c)

$y = \tan^2(x^2 \tan x^2)$ d) $y = \sin^2\sqrt{x}$

e) $y = \frac{1+\sin^2 x}{1-\sin^2 x}$ f) $y = \sqrt{4 + 3\sin 2x}$

Solution

a) $y = \cos^4\left(2x + \frac{\pi}{3}\right) = \left[\cos\left(2x + \frac{\pi}{3}\right)\right]^4$

$\frac{dy}{dx} = 4\left[\cos\left(2x + \frac{\pi}{3}\right)\right]^3 . \frac{d}{dx}\left[\cos\left(2x + \frac{\pi}{3}\right)\right]$

$= 4\cos^3\left(2x + \frac{\pi}{3}\right).\frac{d}{dx}\left(2x + \frac{\pi}{3}\right).\left[-\sin\left(2x + \frac{\pi}{3}\right)\right]$

$= 4\cos^3\left(2x + \frac{\pi}{3}\right).\left[-2\sin\left(2x + \frac{\pi}{3}\right)\right]$

$= -8\cos^3\left(2x + \frac{\pi}{3}\right)\sin\left(2x + \frac{\pi}{3}\right)$

b) $y = \sqrt{\sin 4x} = [\sin 4x]^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}[\sin 4x]^{-\frac{1}{2}}.\frac{d}{dx}(\sin 4x)$

$= \frac{1}{2}[\sin 4x]^{-\frac{1}{2}}.4.\cos 4x$

$= \frac{2\cos 4x}{\sqrt{\sin 4x}}$

c) $y = \tan^2(x^2 \tan x^2) = [\tan(x^2 \tan x^2)]^2$

$\frac{dy}{dx} = 2[\tan(x^2 \tan x^2)].\frac{d}{dx}[\tan(x^2 \tan x^2)]$

$= 2\tan(x^2 \tan x^2).\frac{d}{dx}(x^2 \tan x^2).\sec(x^2 \tan x^2)$

$= 2\tan(x^2 \tan x^2).\sec(x^2 \tan x^2)\,[x^2.2x.\sec^2 x^2 + 2x.\tan x^2]$

$= 4x.\tan(x^2 \tan x^2).\sec(x^2 \tan x^2)\,[x^2.\sec^2 x^2 + \tan x^2]$

d) $y = \sin^2\sqrt{x} = \left[\sin x^{1/2}\right]^2$

$\frac{dy}{dx} = 2\left[\sin x^{\frac{1}{2}}\right].\frac{d}{dx}\left(\sin x^{1/2}\right)$

$= 2\sin\sqrt{x}.\frac{d}{dx}\left(x^{1/2}\right).\cos x^{1/2}$

$= 2\sin\sqrt{x}.\cos\sqrt{x}.\frac{1}{2}.x^{-1/2} = \frac{\sin(2\sqrt{x})}{2\sqrt{x}}$

e) $y = \frac{1+\sin^2 x}{1-\sin^2 x}$

$\frac{dy}{dx} = \frac{(1-\sin^2 x).\frac{d}{dx}(1+\sin^2 x)-(1+\sin^2 x).\frac{d}{dx}(1-\sin^2 x)}{(1-\sin^2 x)^2}$

$= \frac{(1-\sin^2 x).2\sin x\cos x-(1+\sin^2 x).(-2\sin x\cos x)}{(1-\sin^2 x)^2} = \frac{2\sin x\cos x(1-\sin^2 x+1+\sin^2 x)}{(\cos^2 x)^2}$

$= \frac{2\sin x\cos x(2)}{\cos^4 x} = \frac{4\sin x}{\cos^3 x} = 4\tan x\sec^2 x$

f) $y = \sqrt{4+3\sin 2x} = [4+3\sin 2x]^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}[4+3\sin 2x]^{-\frac{1}{2}}.\frac{d}{dx}[4+3\sin 2x]$

$= \frac{1}{2}(4+3\sin 2x)^{-\frac{1}{2}}.6.\cos 2x = \frac{3\cos 2x}{\sqrt{4+3\sin 2x}}$

Task

Differentiate w.r.t x

a) $\sin^3 x$ b) $\tan^3(3x-4)$ c) $\sin^3 x\tan 2x$

d) $(1-\cos^4 x)(4\cos^4 x)$

e) $\left(\frac{\cos x}{1+\sin x}\right)^3$ f) $\frac{\sin^2 x}{2+\sin^2 x}$ g) $\left(\frac{\sin 2x}{1+\cos 2x}\right)^2$ h) $\sqrt{\frac{1+\cos x}{1-\cos x}}$

Answers

a) $3\sin^2 x\cos x$ b) $9\tan^2(3x-4)\sec^2(3x-4)$

c) $3\sin^2 x\cos x\tan 2x + 2\sin^3 x\sec^2 2x$

d) $8\cos^7 x\sin x$ e) $-\frac{3\cos^2 x}{(1+\sin x)^3}$ f) $\frac{4\sin x\cos x}{(2+\sin^2 x)^2}$

g) $\frac{4\sin 2x}{(1+\cos x)^2}$ h) $-\frac{1}{1-\cos x}$

PROOFS INVOLVING THE ABOVE THEOREMS

a) *If* $y = (\tan x+\sec x)^m$ *where* $m>0$, *show that* $\frac{dy}{dx} = my\sec x$

b) *If* $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$, *show that* $\frac{dy}{dx} = \frac{1}{1-\sin x}$

c) *If* $y = \tan x + \frac{1}{3}\tan^3 x$, *show that* $\frac{dy}{dx} = \sec^4 x$

d) *If* $y = \tan^2 x$, *prove that* $\frac{d^2y}{dx^2} = 2(1+y)(1+3y)$

e) *If* $y = \sqrt{1+\sin x}$, *show that* $\frac{dy}{dx} = \frac{1}{2}\sqrt{1-\sin x}$

f) *If* $y = \frac{1+\sin^2 x}{1+\cos^2 x}$, *show that* $\frac{dy}{dx} = \frac{3\sin 2x}{(1+\cos^2 x)^2}$

g) *If* $y = \sin m\theta$, *show that* $\frac{d^2y}{d\theta^2} + m^2\theta = 0$

h) *If* $y = \cot^2\theta$, *show that* $\frac{d^2y}{d\theta^2} = 2(1+y)(1+3y)$

i) *If* $y = \tan x$, *show that* $\frac{d^2y}{dx^2} = 2y(1+y^2)$

j) *If* $y = \sqrt{4+3\sin x}$, *prove that* $2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = 4$

Solution

a) $y = (\tan x + \sec x)^m$

$\frac{dy}{dx} = m(\tan x + \sec x)^{m-1}.(\sec^2 x + \tan x \sec x)$

$= m \sec x\,(\tan x + \sec x)^{m-1}.(\tan x + \sec x)$

$= m \sec x\,(\tan x + \sec x)^m = m \sec x\,.\,y$

b) $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$, squaring both sides to simplify

$y^2 = \frac{1+\sin x}{1-\sin x}$, differentiating w.r.t x

$2y\frac{dy}{dx} = \frac{(1-\sin x).\cos x-(1+\sin x).(-\cos x)}{(1-\sin x)^2}$

$2y\frac{dy}{dx} = \frac{\cos x(1-\sin x+1+\sin x)}{(1-\sin x)^2} = \frac{2\cos x}{(1-\sin x)^2}$

$\therefore \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)^2}.\frac{1}{y} = \frac{\cos x}{(1-\sin x)^2.\sqrt{\frac{1+\sin x}{1-\sin x}}}$

$= \frac{\sqrt{1-\sin^2 x}.\sqrt{1-\sin x}}{(1-\sin x)^2.\sqrt{1+\sin x}} = \frac{\sqrt{(1+\sin x)(1-\sin x)}.\sqrt{1-\sin x}}{(1-\sin x)^2.\sqrt{1+\sin x}}$

$= \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x}$

c) Left as an exercise

d) Left as an exercise

e) Left as an exercise

f) Left as an exercise

g) Left as an exercise

h) Left as an exercise

i) Left as an exercise

j) $y = \sqrt{4 + 3\sin x}$, $2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = 4$

Squaring both sides to simplify

$y^2 = 4 + 3\sin x$ ..... (1)

Differentiating w.r.t x

$2y\frac{dy}{dx} = 3\cos x$

Differentiating again w.r.t x

$2y\frac{d^2y}{dx^2} + \frac{dy}{dx}.2\frac{dy}{dx} = -3\sin x$

From eqn. (1), $-3\sin x = 4 - y^2$

$\therefore 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 4 - y^2$

$\therefore 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = 4$

Differentiating $\sec^n[f(x)]$, $\operatorname{cosec}^n[f(x)]$ and $\cot^n[f(x)]$ w.r.t x

a) Let $y = \sec^n[f(x)] = [\sec[f(x)]]^n$

$u = \sec[f(x)] \Longrightarrow \frac{du}{dx} = f'(x).\sec[f(x)].\tan[f(x)]$

$\Longrightarrow y = u^n \Longrightarrow \frac{dy}{du} = nu^{n-1}$

By Chain Rule

$\therefore \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$= nu^{n-1}.f'(x).\sec[f(x)].\tan[f(x)]$

$= n\sec^{n-1}[f(x)].f'(x).\sec[f(x)].\tan[f(x)]$

$$\therefore \frac{d}{dx}\{\sec^n[f(x)]\} = n\sec^{n-1}[f(x)].f'(x).\sec[f(x)].\tan[f(x)]$$
$$= n\sec^n[f(x)].f'(x).\tan[f(x)]$$

b) Proceeding as in (a) above

$$\therefore \frac{d}{dx}\{\operatorname{cosec}^n[f(x)]\} = -n\operatorname{cosec}^{n-1}[f(x)].f'(x).\operatorname{cosec}[f(x)].\cot[f(x)]$$
$$= -n\operatorname{cosec}^n[f(x)].f'(x).\cot[f(x)]$$

c) Proceeding as above in a similar way

$$\therefore \frac{d}{dx}\{\cot^n[f(x)]\} = -n\cot^{n-1}[f(x)].f'(x).\operatorname{cosec}^2[f(x)]$$

Example

Differentiate w.r.t x

a) $\sqrt{\sec x}$ b) $\sin x.\sec^2 x$ c) $\sec x(1+\cot x)$

d) $\sec^3(2\tan 3x)$

Solution

a) Let $y = \sqrt{\sec x} = (\sec x)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(\sec x)^{\frac{1}{2}}.\frac{d}{dx}(\sec x)$

$= \frac{1}{2}(\sec x)^{\frac{1}{2}}.\sec x.\tan x = \frac{1}{2}\sqrt{\sec x}\tan x$

b) Let $y = \sin x.\sec^2 x$

$\frac{dy}{dx} = \sec^2 x.\cos x + \sin x.2\sec x.\sec x\tan x$

$= \sec^2 x(\cos x + 2\sin x.\tan x)$

c) Let $y = \sec x(1+\cot x)$

$\frac{dy}{dx} = \sec x.(-\operatorname{cosec}^2 x) + (1+\cot x).\sec x\tan x$

$= \sec x[-\operatorname{cosec}^2 x + \tan x + \cot x.\tan x] = \sec x[-\operatorname{cosec}^2 x + \tan x + 1]$

d) Let $y = \sec^3(2\tan 3x) = [\sec(2\tan 3x)]^3$

$\frac{dy}{dx} = 3[\sec(2\tan 3x)]^2.\frac{d}{dx}[\sec(2\tan 3x)]$

$= 3\sec^2(2\tan 3x) \cdot \frac{d}{dx}(2\tan 3x) \cdot \sec(2\tan 3x)\tan(2\tan 3x)$

$= 3\sec^2(2\tan 3x) \cdot (6\sec^2 3x) \cdot \sec(2\tan 3x)\tan(2\tan 3x)$

$= 18\sec^3(2\tan 3x) \cdot \sec^2 3x \cdot \tan(2\tan 3x)$

PROOFS INVOLVING THE ABOVE THEOREMS

a) *If* $y^2 = \sec 2x$, *prove that* $\frac{d^2y}{dx^2} + y = 3y^5$

b) *If* $y = \sec\theta$, *show that* $\frac{d^2y}{d\theta^2} = y(2y^2 - 1)$

c) *If* $y = \cot^2\theta$, *prove that* $\frac{d^2y}{d\theta^2} = 2(1 + y)(1 + 3y)$

Solution

a) $y^2 = \sec 2x$, differentiating w.r.t x

$2y\frac{dy}{dx} = 2\sec 2x \tan 2x$

$2y.\frac{dy}{dx} = 2y^2 \tan 2x \Longrightarrow \frac{dy}{dx} = y.\tan 2x$ ..... (1)

Differentiating again w.r.t

$\frac{dy}{dx} = y.2\sec^2 2x + \tan 2x \cdot \frac{dy}{dx}$

$= 2y.y^4 + \tan 2x \cdot y\tan 2x$ ..... [From eqn. (1)]

$= 2y^5 + y.\tan^2 2x = 2y^5 + y.(\sec^2 2x - 1)$

$= 2y^5 + y.(y^4 - 1) = 3y^5 - y$

$\therefore \frac{d^2y}{dx^2} + y = 3y^5$

b) Left as an exercise

c) $y = \cot^2\theta$, differentiating w.r.t θ

$\frac{dy}{d\theta} = 2\cot\theta \cdot (-\text{cosec}^2\theta)$

$= 2\cot\theta \cdot (1 + \cot^2\theta) = -2\cot\theta + 2\cot^3\theta$

Differentiating w.r.t θ

$\frac{dy}{d\theta} = -2(-\text{cosec}^2\theta) - 2.3\cot^2\theta \cdot (-\text{cosec}^2\theta)$

$= 2\,\text{cosec}^2\theta\,(1 + 3\cot^2\theta)$

$= 2(1 + \cot^2\theta)(1 + 3\cot^2\theta) = 2(1 + y)(1 + 3y)$

DIFFERENTIATING INVERSE OF TRIGONOMETRIC FUNCTIONS

a) *To differentiate* $\cos^{-1}[f(x)]$ *w.r.t* x

Let $y = \cos^{-1}[f(x)]$

$\cos y = f(x)$

Differentiate w.r.t x

$-\sin y . \frac{dy}{dx} = f'(x)$

$\frac{dy}{dx} = \frac{f'(x)}{-\sin y}$

But from $\cos^2 y + \sin^2 y = 1 \Longrightarrow \sin y = +\sqrt{1-\cos^2 y}$

$\frac{dy}{dx} = \frac{f'(x)}{-\sqrt{1-\cos^2 y}} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$

$$\therefore \frac{d}{dx}[\cos^{-1}[f(x)]] = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$$

b) *To differentiate* $\sin^{-1}[f(x)]$ *w.r.t* x *, proceed as above*

$$\therefore \frac{d}{dx}[\sin^{-1}[f(x)]] = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$$

c) *To differentiate* $\tan^{-1}[f(x)]$ *w.r.t* x *, proceed as above*

$$\therefore \frac{d}{dx}[\tan^{-1}[f(x)]] = \frac{f'(x)}{1+[f(x)]^2}$$

d) *To differentiate* $\sec^{-1}[f(x)]$ *w.r.t* x

Let $y = \sec^{-1}[f(x)]$

$\sec y = f(x)$

Differentiating w.r.t x

$\sec y \tan y \frac{dy}{dx} = f'(x)$

$\frac{dy}{dx} = \frac{f'(x)}{\sec y \tan y}$

But from $\sec^2 y = 1 + \tan^2 y = 1 \Longrightarrow \tan y = +\sqrt{\sec^2 y - 1} = \sqrt{[f(x)]^2 - 1}$

$\frac{dy}{dx} = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$

$$\frac{d}{dx}\{\sec^{-1}[f(x)]\} = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$$

e) *To differentiate* $\text{cosec}^{-1}[f(x)]$ *w.r.t* x, proceeding as in (f) above

$$\frac{d}{dx}\{\sec^{-1}[f(x)]\} = -\frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$$

f) *To differentiate* $\cot^{-1}[f(x)]$ *w.r.t* x, proceeding as in (f) above

$$\boxed{\frac{d}{dx}\{\cot^{-1}[f(x)]\} = -\frac{f'(x)}{1+[f(x)]^2}}$$

Example

Differentiate w.r.t x

a) $\sin^{-1}(2x)$ b) $\cos^{-1}\sqrt{x^3}$ c) $\tan^{-1}\left(5/\sqrt{x}\right)$

d) $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ e) $\sin^{-1}(\cos x)$ f) $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Solution

a) Let $y = \sin^{-1}(2x)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(2x)}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$$

b) Let $y = \cos^{-1}\sqrt{x^3} = \cos^{-1}x^{3/2}$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^{3/2})}{\sqrt{1-(x^{3/2})^2}} = \frac{-\frac{3}{2}x^{1/2}}{\sqrt{1-x^3}} = -\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

c) Let $y = \tan^{-1}\left(5/\sqrt{x}\right) = \tan^{-1}\left(5x^{-1/2}\right)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\left(5x^{-\frac{1}{2}}\right)}{1+\left(5x^{-\frac{1}{2}}\right)^2} = \frac{-\frac{5}{2}x^{-\frac{3}{2}}}{1+\left(5x^{-\frac{1}{2}}\right)^2}$$

$$= \frac{-\frac{5}{2}x^{-\frac{3}{2}}}{1+25x^{-1}} = \frac{-\frac{5}{2}x^{-\frac{3}{2}}.x}{x+25}$$

$$= \frac{-\frac{5}{2}x^{-\frac{1}{2}}}{x+25} = -\frac{5}{2(x+25).\sqrt{x}}$$

d) Let $y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \sin^{-1}\left(\frac{x}{(1+x^2)^{1/2}}\right)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\left(\frac{x}{(1+x^2)^{1/2}}\right)}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} = \frac{\frac{(1+x^2)^{1/2}.1-x.\frac{1}{2}(1+x^2)^{-1/2}.2x}{1+x^2}}{\sqrt{1-\frac{x^2}{1+x^2}}}$$

$$= \frac{(1+x^2)^{-1/2}(1+x^2-x^2)}{(1+x^2)\sqrt{\frac{1+x^2-x^2}{1+x^2}}} = \frac{(1+x^2)^{-1/2}}{(1+x^2).\frac{1}{\sqrt{1+x^2}}}$$

$$= \frac{(1+x^2)^{-1/2}.(1+x^2)^{1/2}}{1+x^2} = \frac{1}{1+x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

e) Let $y = \sin^{-1}(\cos x)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x)}{\sqrt{1-(\cos x)^2}} = \frac{-\sin x}{\sqrt{1-\cos^2 x}}$$

$$= -\frac{\sin x}{\sin x} = -1$$

$$\therefore \frac{dy}{dx} = -1$$

f) Left as an exercise. **Ans:** $\frac{1}{\sqrt{1-x^2}}$

Task

1. Differentiate w.r.t x

a) $\sin^{-1}\sqrt{x}$ b) $\tan^{-1}x^{5/2}$ c) $\cos^{-1}2/\sqrt{x}$ d) $\sin^{-1}(2x+3)/7$ e) $\cos^{-1}(3x-2)/4$

f) $\tan^{-1}(x-5)/2$ g) $\sin^{-1}(2x+7)/\sqrt{2}$ h) $\cos^{-1}(2x+7)/\sqrt{2}$ i) $\tan^{-1}(2x+7)/\sqrt{2}$

j) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ k) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ l) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Answers

a) $\frac{1}{2\sqrt{x(1-x)}}$ b) $\frac{5x^{3/2}}{2(1+x^5)}$ c) $\frac{1}{x\sqrt{x-4}}$ d) $\frac{1}{\sqrt{10-3x-x^2}}$

e) $-\frac{3}{\sqrt{12+12x-9x^2}}$ f) $\frac{2}{x^2-10x-29}$ g) $\frac{2}{\sqrt{-47-28x-4x^2}}$

h) $\frac{-2}{\sqrt{-47-28x-4x^2}}$ i) $\frac{2\sqrt{2}}{\sqrt{51+28x+4x^2}}$ j) $\frac{2}{1+x^2}$

k) $\frac{2}{1+x^2}$ l) $\frac{2}{1+x^2}$

2. a) *If* $\sqrt{y}=\tan^{-1}x$, *show that* $(1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}=2$

b) *If* $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, *show that* $\frac{dy}{dx}=-\frac{2}{1+x^2}$

c) *If* $y=\sin^{-1}(3x-4x^3)$, *show that* $\sqrt{1-x^2}\frac{dy}{dx}=3$

d) *If* $u=\tan^{-1}\theta$, *show that* $(1+\theta^2)\frac{d^2u}{d\theta^2}+2\theta\frac{du}{d\theta}=0$

e) *Show that:*

i) $\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$

ii) $\frac{d}{dx}\left(\cot^{-1}\left(\frac{x}{a}\right)\right)=-\frac{1}{\sqrt{a^2-x^2}}$

iii) $\frac{d}{dx}(\text{cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}$

iv) *If* $y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $\frac{dy}{dx}=\frac{2}{1+x^2}$

v) *If* $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, $(1-x^2)\frac{dy}{dx}=xy+1$

vi) *If* $y=(\tan^{-1}x)^2$, $\frac{d}{dx}\left\{(1+x^2).\frac{dy}{dx}\right\}=\frac{2}{1+x^2}$

vii) *If* $y=\cos^{-1}(\sin 2x)$, $\frac{dy}{dx}=-2$

viii) *If* $y=\tan(m\ \tan^{-1}x)$, $(1+x^2)\frac{d^2y}{dx^2}=2(my-x)\frac{dy}{dx}$

ix) *If* $y=\sin(m\ \sin^{-1}x)$, $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$

x) *If* $y=\sqrt{1-x^2}\ \sin^{-1}x$, $(1-x^2)\frac{dy}{dx}+xy=1-x^2$

Solution

a) $\sqrt{y}=\tan^{-1}x$; $(1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}=2$

Squaring both sides

$y=(\tan^{-1}x)^2$

Differentiating w.r.t x

$\frac{dy}{dx} = \frac{2\ \tan^{-1} x}{1+x^2}$

$(1+x^2).\frac{dy}{dx} = 2\ \tan^{-1} x$

Differentiating w.r.t x

$(1+x^2).\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{2}{1+x^2}$, multiplying through by $(1+x^2)$

$\therefore (1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$

b) Left as an exercise

c) $y = \sin^{-1}(3x - 4x^3)$; $\sqrt{1-x^2}\frac{dy}{dx} = 3$

$\frac{dy}{dx} = \frac{(3-12x^2)}{\sqrt{1-(3x-4x^3)^2}} = \frac{3(1-4x^2)}{\sqrt{(1+3x-4x^3)(1-3x+4x^3)}}$

$= \frac{3(1-2x)(1+2x)}{\sqrt{(x-1)(1+2x)(1+2x)[-(x+1)(2x-1)(2x-1)]}} = \frac{3(1-2x)(1+2x)}{(1-2x)(1+2x)\sqrt{(x-1)[-(x+1)]}}$

$= \frac{3}{1-x^2}$

$\therefore \sqrt{1-x^2}\frac{dy}{dx} = 3$

d) Left as an exercise

e) (i) up to (vii), (ix) , (x) are left as an exercise

viii) $y = \tan(m\ \tan^{-1} x)$, $(1+x^2)\frac{d^2y}{dx^2} = 2(my - x)\frac{dy}{dx}$

Differentiating w.r.t x

$\frac{dy}{dx} = \frac{d}{dx}(m\ \tan^{-1} x).\sec^2(m\ \tan^{-1} x)$

$= \frac{m}{1+x^2}.(1+\tan^2(m\ \tan^{-1} x)) = \frac{m}{1+x^2}.(1+y^2)$

$\Longrightarrow (1+x^2)\frac{dy}{dx} = m(1+y^2)$

Differentiating w.r.t x

$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = m.2y\frac{dy}{dx}$

$(1+x^2)\frac{d^2y}{dx^2} = m.2y\frac{dy}{dx} - 2x\frac{dy}{dx}$

$\therefore (1+x^2)\frac{d^2y}{dx^2} = 2(my - x)\frac{dy}{dx}$

DIFFERENTIATION INVOLVING PARAMETRIC EQUATIONS

Given two parametric equations

$$y = f(\theta) \text{ and } x = g(\theta)$$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Solution

By chain Rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

By chain Rule

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \times \frac{d\theta}{dx}$$

Examples

1. *Find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *in terms of the parameter when*

a) $y = \cos 4\theta \ , \ x = \sin 2\theta$

b) $y = \theta - \sin\theta \ , \ x = 2\cos\theta$

c) $x = a\cos^3\theta \ , \ y = a\sin^3\theta$

d) $x = a(\theta + \sin\theta) \ , \ y = a(1 - \cos\theta)$

e) $x = a(\theta\sin\theta + \cos\theta - 1) \ , \ y = a(\sin\theta - \theta\cos\theta)$

f) $x = 4\cos\theta + 3\sin\theta + 2 \ , y = 3\cos\theta - 4\sin\theta - 1$ *at the point where* $\theta = \pi/2$

g) $x = 3\cos\theta - \cos^3\theta \ , \ y = 3\ \sin\theta - \sin^3\theta$

h) $x = b\cos^3\theta \ , \ y = b\sin^3\theta$

i) $x = a(2\cos\theta + \cos 2\theta) \ , \ y = a(2\sin\theta - \sin 2\theta)$

j) $x = 4b\cos\theta - b\cos 4\theta \ , \ y = 4b\sin\theta - b\sin 4\theta$

Solution

a) $y = \cos 4\theta \Longrightarrow \frac{dy}{d\theta} = -4\sin 4\theta$

$x = \sin 2\theta \Longrightarrow \frac{dx}{d\theta} = 2\cos 2\theta$

By Chain Rule, $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= -4\sin 4\theta \times \frac{1}{2\cos\ 2\theta} = \frac{-4\sin\ 4\theta}{2\cos\ 2\theta}$

$= \frac{-2.2\sin\ 2\theta\cos\ 2\theta}{\cos\ 2\theta} = -4\sin 2\theta$

$\therefore \frac{dy}{dx} = -4\sin 2\theta$

By Chain Rule, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \times \frac{d\theta}{dx}$

$\frac{d^2y}{dx^2} = -8\cos 2\theta \,.\frac{d\theta}{dx} = -8\cos 2\theta \,.\frac{1}{2\cos\ 2\theta} = -4$

b) $y = \theta - \sin\theta \implies \frac{dy}{d\theta} = 1 - \cos\theta$

$x = 2\cos\theta \implies \frac{dx}{d\theta} = -2\sin\theta$

By Chain Rule, $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$\therefore \frac{dy}{dx} = -\frac{1-\cos\theta}{2\sin\theta}$

By Chain Rule, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) \times \frac{d\theta}{dx}$

$\frac{d^2y}{dx^2} = -\left[\frac{2\sin\theta.\sin\theta-(1-\cos\theta).2\cos\theta}{4\sin^2\theta}\right].\frac{1}{-2\sin\theta}$

$= \frac{\sin^2\theta-\cos\theta+\cos^2\theta}{4\sin^3\theta} = \frac{1-\cos\theta}{4\sin^3\theta}$

$= \frac{1}{4}(\text{cosec}^3\theta - \cot\theta\,\text{cosec}^2\theta)$

c) Left as an exercise. **Ans:** $\frac{dy}{dx} = -\tan\theta$, $\frac{d^2y}{dx^2} = \frac{\sec^4\theta\,\text{cosec}\,\theta}{3a}$

d) Left as an exercise. **Ans:** $\frac{dy}{dx} = \tan\frac{\theta}{2}$, $\frac{d^2y}{dx^2} = \frac{1}{4a}\sec^4\frac{\theta}{2}$

e) Left as an exercise. **Ans:** $\frac{dy}{dx} = \tan\theta$, $\frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta}$

f) Left as an exercise. **Ans:** $3/4$

g) Left as an exercise. **Ans:** $\frac{dy}{dx} = -\cot^3\theta$, $\frac{d^2y}{dx^2} = -\cot^2\theta\,\text{cosec}^5\theta$

h) Left as an exercise. **Ans:** $\frac{dy}{dx} = -\tan\theta$, $\frac{d^2y}{dx^2} = \frac{\sec^4\theta\,\text{cosec}\,\theta}{3b}$

i) Left as an exercise. **Ans:** $\frac{dy}{dx} = -\tan\frac{\theta}{2}$, $\frac{d^2y}{dx^2} = \frac{1}{8a}\text{cosec}\frac{3}{2}\theta\sec^3\frac{\theta}{2}$

j) Left as an exercise. **Ans:** $\frac{dy}{dx} = \frac{1}{b}\tan\frac{5\theta}{2}$, $\frac{d^2y}{dx^2} = \frac{5}{16b}\sec\frac{5}{2}\theta\,\text{cosec}\frac{3}{2}\theta$

2. a) *Prove that if* $\sin\theta = 2\sin\varphi$ *and* $x = \cos\theta - 2\cos\varphi$ *then* $\frac{dx}{d\theta} = x\tan\varphi$

b) *if* $x = a(2\cos\theta + \cos 2\theta)$, $y = a(2\sin\theta - \sin 2\theta)$, *prove that* $8a\frac{d^2y}{dx^2} = \text{cosec}\frac{3}{2}\theta.\sec^3\frac{\theta}{2}$

c) *Given that* $x = 4b\cos\theta - b\cos 4\theta$, $y = 4b\sin\theta - b\sin 4\theta$, *prove that* $\frac{d^2y}{dx^2} = \frac{5}{16b}\sec^3\frac{5\theta}{2}\text{cosec}\frac{3\theta}{2}$

d) *Given that b* $x = a(\theta - \sin\theta)$ *and* $y = a(1-\cos\theta)$, *prove that* $\frac{dy}{dx} = \cot\frac{\theta}{2}$ *and* $\frac{d^2y}{dx^2} = -\frac{1}{4a}\text{cosec}^3\frac{\theta}{2}$

e) *If* $y = \sin n\theta\,\text{cosec}\,\theta$ *and* $x = \cos\theta$, *prove that*

i) $(1-x^2)\frac{dy}{dx} - xy + n\cos n\theta = 0$

ii) $(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + (n^2-1)y = 0$

Solution

a) $\sin\theta = 2\sin\varphi \quad , \; x = \cos\theta - 2\cos\varphi$

Differentiating w.r.t θ

$\frac{dx}{d\theta} = -\sin\theta + 2\sin\varphi\frac{d\varphi}{d\theta}$

$= -2\sin\varphi + 2\sin\varphi\frac{d\varphi}{d\theta}$

Also $\cos\theta = 2\cos\varphi\frac{d\varphi}{d\theta} \Longrightarrow \frac{d\varphi}{d\theta} = \frac{\cos\theta}{2\cos\varphi}$

$\frac{dx}{d\theta} = -2\sin\varphi + 2\sin\varphi.\frac{\cos\theta}{2\cos\varphi} = -2\sin\varphi + \tan\varphi\cos\theta$

$= -2\sin\varphi + \tan\varphi\cos\theta = \tan\varphi\left[\cos\theta - \frac{2\sin\varphi}{\tan\varphi}\right]$

$= \tan\varphi\left[\cos\theta - \frac{2\sin\varphi\cos\varphi}{\sin\varphi}\right] = \tan\varphi\left[\cos\theta - 2\cos\varphi\right]$

$\therefore \frac{dx}{d\theta} = x\tan\varphi$

b) $x = a(2\cos\theta + \cos 2\theta)\,, \; y = a(2\sin\theta - \sin 2\theta)$

$\frac{dx}{d\theta} = a(-2\sin\theta - 2\sin 2\theta) = -2a(\sin\theta + \sin 2\theta)$

$\frac{dy}{d\theta} = a(2\cos\theta + 2\cos 2\theta) = 2a(\cos\theta - \cos 2\theta)$

By Chain Rule, $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$\frac{dy}{dx} = \frac{2a(\cos\theta - \cos 2\theta)}{-2a(\sin\theta + \sin 2\theta)} = \frac{-2\sin\frac{3}{2}\theta\sin\left(-\frac{1}{2}\theta\right)}{-2\sin\frac{3}{2}\theta\cos\left(-\frac{1}{2}\theta\right)}$

$= \frac{\sin\left(\frac{1}{2}\theta\right)}{-\cos\left(\frac{1}{2}\theta\right)} = -\tan\frac{\theta}{2}$

By Chain Rule, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \times \frac{d\theta}{dx}$

$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \times \frac{d\theta}{dx} = \frac{d}{d\theta}\left(-\tan\frac{\theta}{2}\right) \times \frac{1}{-2a(\sin\theta + \sin 2\theta)}$

$= -\frac{1}{2}\sec^2\frac{\theta}{2}.\frac{1}{-2a.\left[2\sin\frac{3}{2}\theta\cos\left(-\frac{1}{2}\theta\right)\right]} = -\frac{1}{8a}\sec^2\frac{\theta}{2}.\operatorname{cosec}\frac{3\theta}{2}.\sec\frac{\theta}{2}$

$\therefore 8a\frac{d^2y}{dx^2} = \operatorname{cosec}\frac{3}{2}\theta\,.\sec^3\frac{\theta}{2}$

c) Left as an exercise

d) Left as an exercise

LOGARITHMIC DIFFERENTIATION

Differentiating $\log_a x$ from first principles

Let $y = \log_a x$

$y + \delta y = \log_a(x + \delta x)$

$\delta y = \log_a(x + \delta x) - y = \log_a(x + \delta x) - \log_a x$

$= \log_a\left(\frac{x+\delta x}{x}\right) = \log_a\left(1 + \frac{\delta x}{x}\right)$

From Maclaurin's Expansion

$\ln(1 + x) = x - \frac{x^2}{2} + \cdots \ldots \ldots \ldots \ldots$

$\log_a\left(1 + \frac{\delta x}{x}\right) = \frac{\ln\left(1+\frac{\delta x}{x}\right)}{\ln a}$ [Changing base to e]

$\therefore \delta y = \frac{1}{\ln a}\left[\frac{\delta x}{x} - \frac{1}{2}\left(\frac{\delta x}{x}\right)^2\right]$ [Higher power terms neglected]

Dividing through by δx

$\frac{\delta y}{\delta x} = \frac{1}{\ln a}\left[\frac{1}{x} - \frac{\delta x}{2x}\right]$

Introducing limits

$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{1}{\ln a}\left[\frac{1}{x} - \frac{\delta x}{2x}\right]\right)$

$\frac{dy}{dx} = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$

$$\therefore \frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}$$

Corrary

If the base $a = e$ .i.e $\log_a x = \log_e x = \ln x$, the derivative becomes

$$\therefore \frac{\boldsymbol{d}}{\boldsymbol{dx}}\left(\boldsymbol{\log_e x}\right) = \frac{\boldsymbol{d}}{\boldsymbol{dx}}(\boldsymbol{\ln x}) = \frac{\boldsymbol{1}}{\boldsymbol{x}} \qquad \because \ln e = \log_e e = 1$$

Differentiating $\log_e[f(x)]$

Let $y = \ln[f(x)]$

$u = f(x) \Longrightarrow \frac{du}{dx} = f'(x)$

$y = \ln u \Longrightarrow \frac{dy}{du} = \frac{1}{u}$

By Chain Rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{u} \times f'(x) = \frac{f'(x)}{f(x)}$

$$\therefore \frac{d}{dx}\left(\ln[f(x)]\right) = \frac{f'(x)}{f(x)}$$

Example

1. *Find* $\frac{dy}{dx}$ *if*

a) $y = \ln 2x$

b) $y = \ln\sqrt{x^2 - 1}$

c) $y = x\ln x - x$

d) $y = \ln\left(\frac{1-x}{1+x}\right)$

e) $y = \frac{\ln x}{x}$

f) $y = \ln(\sec x + \tan x)$

g) $y = \ln\left(\frac{\cos x}{1+\sin x}\right)$

Solution

a) $y = \ln 2x \Longrightarrow \frac{dy}{dx} = \frac{\frac{d}{dx}(2x)}{2x} = \frac{1}{x}$

b) $y = \ln\sqrt{x^2 - 1} = \frac{1}{2}\ln(x^2 - 1)$

$\frac{dy}{dx} = \frac{1}{2}.\frac{2x}{x^2-1} = \frac{x}{x^2-1}$

c) $y = x\ln x - x$

$\frac{dy}{dx} = x.\frac{d}{dx}(\ln x) + \ln x\,.\frac{d}{dx}(x) - 1$

$= x.\frac{1}{x} + \ln x - 1 = \ln x$

d) $y = \ln\left(\frac{1-x}{1+x}\right) = \ln(1-x) - \ln(1+x)$

$\frac{dy}{dx} = -\frac{1}{1-x} - \frac{1}{1+x} = \frac{-1-x-1+x}{1-x^2} = -\frac{2}{1-x^2}$

e) $y = \frac{\ln x}{x} \Longrightarrow \frac{dy}{dx} = \frac{x.\frac{d}{dx}(\ln x) - \ln x.\frac{d}{dx}x}{x^2}$

$\frac{dy}{dx} = \frac{x.\frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$

f) $y = \ln(\sec x + \tan x)$

$\frac{dy}{dx} = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$

2. Logarithmic Differentiation of Complicated Products or Quotients or Functions of the form $[f(x)]^{g(x)}$

Differentiate w.r.t x

a) $y = \frac{x(1+x^2)^3}{\sqrt[3]{1+x^3}}$

b) $y = \sqrt{\left(\frac{a^2+ax+x^2}{a^2-ax+x^2}\right)}$

c) $y = (\cos x)^{\sqrt{\sin 2x}}$

d) $y = 2x.(\sin x)^x$

Solution

a) $y = \frac{x(1+x^2)^3}{\sqrt[3]{1+x^3}}$

Introducing ln on both sides

$\ln y = \ln x + 3\ln(1+x^2) - \frac{1}{3}\ln(1+x^3)$

Differentiating w.r.t x

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{6x}{1+x^2} - \frac{x^2}{1+x^3}$

$$\frac{dy}{dx} = \frac{(1+x^2)(1+x^3)+6x^2(1+x^3)-x^3(1+x^2)}{x(1+x^2)(1+x^3)}.y$$

$$\frac{dy}{dx} = \frac{1+x^2+x^3+x^5+6x^2+6x^5-x^3-x^5}{x(1+x^2)(1+x^3)}.y$$

$$= \frac{1+7x^2+6x^5}{x(1+x^2)(1+x^3)}.y$$

$$= \frac{(1+7x^2+6x^5).x.(1+x^2)^3}{x(1+x^2)(1+x^3).\sqrt[3]{1+x^3}} = \frac{(1+7x^2+6x^5)(1+x^2)^2}{(1+x^3)^{4/3}}$$

b) $y = \sqrt{\left(\frac{a^2+ax+x^2}{a^2-ax+x^2}\right)}$

Introducing ln on both sides

$\ln y = \frac{1}{2}\ln(a^2+ax+x^2) - \frac{1}{2}\ln(a^2-ax+x^2)$

Differentiating w.r.t x

$\frac{1}{y}\frac{dy}{dx} = \frac{a+2x}{2(a^2+ax+x^2)} - \frac{-a+2x}{2(a^2-ax+x^2)}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\frac{(a+2x)(a^2-ax+x^2)-(-a+2x)(a^2+ax+x^2)}{(a^2+ax+x^2)(a^2-ax+x^2)}\right].y$

$= \frac{1}{2}\left[\frac{2a(a^2+x^2)-4ax^2}{(a^2+ax+x^2)(a^2-ax+x^2)}\right].y = \left[\frac{a(a^2+x^2)-2ax^2}{(a^2+ax+x^2)(a^2-ax+x^2)}\right].y$

$= \frac{a(a^2-x^2)}{\sqrt{\{(a^2+ax+x^2)(a^2-ax+x^2)^3\}}}$

c) $y = (\cos x)^{\sqrt{\sin 2x}}$

Introducing ln on both sides

$\ln y = \sqrt{\sin 2x}.\ln\cos x$

Differentiating w.r.t x

$\frac{1}{y}\frac{dy}{dx} = \ln\cos x.\frac{1}{2}.(\sin 2x)^{-1/2}.2\cos 2x + \sqrt{\sin 2x}.\left(-\frac{\sin x}{\cos x}\right)$

$= \frac{\cos 2x \ln\cos x}{\sqrt{\sin 2x}} + \sqrt{\sin 2x}.(-\tan x)$

$= \frac{\cos 2x \ln\cos x - \sin 2x.\tan x}{\sqrt{\sin 2x}}$

$\therefore \frac{dy}{dx} = \left[\frac{\cos 2x \ln\cos x - \sin 2x.\tan x}{\sqrt{\sin 2x}}\right].(\cos x)^{\sqrt{\sin 2x}}$

d) $y = 2x.(\sin x)^x$

Introducing ln on both sides

$\ln y = \ln 2x + x\ln\sin x$

Differentiating w.r.t x

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + x.\frac{\cos x}{\sin x} + \ln\sin 2x$

$\frac{dy}{dx} = \frac{\sin x + x^2\cos x + x\sin x\ln\sin 2x}{x\sin x}.y$

$= \frac{\sin x + x^2\cos x + x\sin x\ln\sin 2x}{x\sin x}.2x.(\sin x)^x$

$= (\sin x + x^2\cos x + x\sin x\ln\sin 2x)(\sin x)^{x-1}$

Task

1. Differentiate w.r.t x

a) $\frac{e^{x^2}\sqrt{\sin x}}{(2x+1)^3}$ **Ans:** $\frac{e^{x^2}}{2(2x+1)^3\sqrt{\sin x}}\{(8x^2+4x-12)\sin x+(2x+1)\cos x\}$

b) $\sqrt[3]{\frac{x+1}{x-1}}$ **Ans:** $-\frac{2}{3}(x+1)^{-2/3}(x-1)^{-4/3}$

c) $\frac{\sqrt{x^2+1}}{(2x-1)^2}$ **Ans:** $-\frac{2x^2+x+4}{(2x-1)^3\sqrt{x^2+1}}$

d) $\frac{x^2e^x}{(x-1)^3}$ **Ans:** $\frac{xe^x(x^2-2x-2)}{(x-1)^4}$

e) $3x^x$ **Ans:** $(\ln x+1)(3x^x)$

f) $x^{\cos x}$ **Ans:** $(-2x\sin 2x\ln x+\cos 2x)(x^{\cos x-1})$

g) $2x^x+3x^{e^x}$ **Ans:** $(\ln x+1)(2x^x)+3x^{e^{2x-1}}(1+x\ln x)$

h) $(\sin 2x)^x+x^{\sin^{-1}2x}$ **Ans:** $\frac{\sqrt{1-4x^2}.\sin^{-1}2x+2x\ln x}{\sqrt{1-4x^2}}+(2x\cot 2x+\ln\sin 2x)(\sin 2x)^x$

i) $x.5^{1+\sin^2 x}$ **Ans:** $\left(\frac{1+x\sin 2x\ln 5}{x}\right)\left(x.5^{1+\sin^2 x}\right)$

j) $\sqrt{\left\{\frac{(x+2)^3}{1-5x}\right\}}$ **Ans:** $\frac{1}{2}(13-10x)\sqrt{\frac{x+2}{(1-5x)^3}}$

k) $4^{\sin^{-1}x}$ **Ans:** $\frac{4^{\sin^{-1}x}\ln 4}{\sqrt{1-x^2}}$

l) 4^{x+x^2} **Ans:** $4^{x+x^2}\ln 4\,(1+2x)$

m) $2^{\sin^{-1}x}+4^{\cos^{-1}x}$ **Ans:** $\frac{2^{\sin^{-1}x}\ln 2.-4^{\cos^{-1}x}\ln 4}{\sqrt{1-x^2}}$

n) 5^{e^x} **Ans:** $5^{e^x}.e^x\ln 5$

o) $4a^{x^2}+5b^{x^2+1}$ **Ans:** $8xa^{x^2}\ln a+10xb^{x^2+1}\ln b$

2. (i) If $y=\cos(\ln x)$, show that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

ii) If $y=\ln(x^2-5)$, show that $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=2e^{-y}$

iii) If $y=\sin 2x\,.\ln\tan x$, show that $\frac{d^2y}{dx^2}+4y=4\cot 2x$

3. If $y=a\cos(\ln bx)+b\sin(\ln ax)$, show that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

DIFFERENTIATING EXPONENTIAL FUNCTIONS

1. *Differential from first principles* $y = a^{bx}$

Solution

$y + \delta y = a^{b(x+\delta x)} = a^{bx}.a^{b\delta x}$

$\delta y = a^{bx}.a^{b\delta x} - y$

$= a^{bx}.a^{b\delta x} - a^{bx}$

$= a^{bx}\left(a^{b\delta x} - 1\right)$

Divide through by δx

$\frac{\delta y}{\delta x} = a^{bx}\left(\frac{a^{b\delta x}-1}{\delta x}\right)$

Introducing limits

$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left[a^{bx}\left(\frac{a^{b\delta x}-1}{\delta x}\right)\right]$

$\frac{dy}{dx} = a^{bx}\lim_{\delta x\to 0}\left(\frac{a^{b\delta x}-1}{\delta x}\right)$

$$\therefore \frac{d}{dx}\left(a^{bx}\right) = a^{bx}\lim_{\delta x\to 0}\left(\frac{a^{b\delta x}-1}{\delta x}\right)$$

Note

It can be shown that $\lim_{\delta x\to 0}\left(\frac{a^{b\delta x}-1}{\delta x}\right) = b\ln a$

$$\therefore \frac{d}{dx}\left(a^{bx}\right) = b.a^{bx}\ln a$$

2. *Differentiate from first principles*

i) $y = e^{ax}$

ii) $y = e^{x^2}$

Solution

i) $y = e^{ax}$

$y + \delta y = e^{a(x+\delta x)} = e^{ax}.e^{a\delta x}$

$\delta y = e^{ax}.e^{a\delta x} - y$

$= e^{ax}.e^{a\delta x} - e^{ax}$

$= e^{ax}\left(e^{a\delta x} - 1\right)$

Expanding $e^{a\delta x}$ using Maclaurin's Expansion

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$\therefore e^{a\delta x} = 1 + a\delta x$, taking only the first two terms and ignoring higher power terms

$\therefore \delta y = e^{ax}(1 + a\delta x - 1)$

$= a\delta x.e^{ax}$

Dividing through by δx

$\frac{\delta y}{\delta x} = ae^{ax}$

Introducing limits

$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}(ae^{ax})$

$\therefore \frac{dy}{dx} = ae^{ax}$

$\therefore \frac{d}{dx}(e^{ax}) = ae^{ax}$

NOTE

$$\frac{d}{dx}(e^{x}) = e^{x}$$

ii) $y = e^{x^2}$

$y + \delta y = e^{(x+\delta x)^2} = e^{x^2+2x\delta x+(\delta x)^2} = e^{x^2}.e^{2x\delta x+(\delta x)^2}$

$\delta y = e^{x^2}.e^{2x\delta x+(\delta x)^2} - y$

$= e^{x^2}.e^{2x\delta x+(\delta x)^2} - e^{x^2}$

$= e^{x^2}\left(e^{2x\delta x+(\delta x)^2} - 1\right)$

Expanding $e^{2x\delta x+(\delta x)^2}$ using Maclaurin's Expansion

$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$\therefore e^{2x\delta x+(\delta x)^2} = 1 + 2x\delta x + (\delta x)^2$, taking only the first two terms and ignoring higher power terms

$\therefore \delta y = e^{x^2}(1 + 2x\delta x + (\delta x)^2 - 1) = e^{x^2}(2x\delta x + (\delta x)^2)$

Dividing through by δx

$\frac{\delta y}{\delta x} = e^{x^2}(2x + \delta x)$

Introducing limits

$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(e^{x^2}(2x + \delta x)\right)$

$\therefore \frac{dy}{dx} = 2xe^{x^2}$

$\therefore \frac{d}{dx}\left(e^{x^2}\right) = 2xe^{x^2}$

Differentiating $e^{f(x)}$

Let $y = e^{f(x)}$

$u = f(x) \Longrightarrow \frac{du}{dx} = f'(x)$

$y = e^{u} \Longrightarrow \frac{dy}{du} = e^{u}$

By Chain Rule $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = e^{u}.f'(x) = e^{f(x)}.f'(x)$

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)}.f'(x)$$

Examples

Differentiate w.r.t x

a) $y = e^{x^3}$

b) $y = e^{ax^2.\sec bx}$

c) $y = 1/e^{x.\cos^{-1}x^2}$

d) $y = x/e^{\cos x}$

Solution

a) $y = e^{x^3} \Longrightarrow \frac{dy}{dx} = \frac{d}{dx}(x^3).e^{x^3} = 3x^2e^{x^3}$

b) $y = e^{ax^2.\sec bx} \Longrightarrow \frac{dy}{dx} = \frac{d}{dx}(ax^2.\sec bx).e^{ax^2.\sec bx}$

$\frac{dy}{dx} = (\sec bx\,.\,2ax + ax^2.b\sec bx\tan bx)e^{ax^2.\sec bx}$

$= a\sec bx\,(2x + bx^2\tan bx)e^{ax^2.\sec bx}$

c) $y = \frac{1}{e^{x.\cos^{-1}x^2}} = e^{-x.\cos^{-1}x^2}$

$\frac{dy}{dx} = \frac{d}{dx}(-x.\cos^{-1}x^2).e^{-x.\cos^{-1}x^2}$

$= \left[-x.\frac{-2x}{\sqrt{1-x^4}} + \cos^{-1}x^2\,.(-1)\right].e^{-x.\cos^{-1}x^2}$

$= \left[2x^2 - \sqrt{1-x^4}\cos^{-1}x^2\right].\frac{e^{-x.\cos^{-1}x^2}}{\sqrt{1-x^4}}$

$= \frac{e^{-x.\cos^{-1}x^2}}{\sqrt{1-x^4}}.\left[2x^2 - \sqrt{1-x^4}\cos^{-1}x^2\right]$

d) $y = \frac{x}{e^{\cos x}} = xe^{-\cos x}$

$\frac{dy}{dx} = x.\left[-\left(-\sin x\right)\right]e^{-\cos x} + e^{-\cos x}$

$= (x\sin x + 1).e^{-\cos x}$

NOTE

Logarithmic differentiation can also be used in handling such type of functions

PROOFS INVOLVING EXPONENTIAL FUNCTIONS

a) *If* $y = Ae^{-x}\cos(x+\alpha)$, *where A and α are constants prove that* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

b) *If* $y = e^{-2x}\cos 4x$, *prove that* $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 20y = 0$

c) *If* $y = e^x \sin x$, *show that* $\frac{dy}{dx} = \sqrt{2}e^x \sin(x+\pi/2)$ *and* $\frac{d^2y}{dx^2} = 2e^x \sin\left(x+\frac{\pi}{2}\right)$

d) *If* $y = e^{-t}\cos(t+\beta)$, *show that* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

e) *If* $y = e^{4x}\cos 3x$, *prove that* $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 25y = 0$

Solution

a) $y = Ae^{-x}\cos(x+\alpha)$, $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

$\frac{dy}{dx} = A[e^{-x}[-\sin(x+\alpha)] - e^{-x}\cos(x+\alpha)]$

$= -Ae^{-x}\sin(x+\alpha) - Ae^{-x}\cos(x+\alpha)$

$= -Ae^{-x}\sin(x+\alpha) - y$..... (1)

Differentiating w.r.t x

$\frac{d^2y}{dx^2} = -A[e^{-x}\cos(x+\alpha) - e^{-x}\sin(x+\alpha)] - \frac{dy}{dx}$

$= -Ae^{-x}\cos(x+\alpha) + Ae^{-x}\sin(x+\alpha) - \frac{dy}{dx}$

From eqn. (1), $Ae^{-x}\sin(x+\alpha) = -y - \frac{dy}{dx}$

$\therefore \frac{d^2y}{dx^2} = -y - y - \frac{dy}{dx} - \frac{dy}{dx}$

$\therefore \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

b) , (c) , (d) and (e) are left as an exercise

INTEGRATION

Definition and notation

The process which deals with finding the function when its derivative is given $.i.e.$.

If $\frac{d}{dx}\varphi(x) = f(x)$, then the in integral calculus we find $\varphi(x)$ when $f(x)$ is given.

In general, integration is the *inverse* of differentiation.

NOTE

When $\varphi(x)$ is found, it is called the *integral* of $f(x)$ with respect to x

Mathematically

If $\frac{d}{dx}\varphi(x) = f(x)$ then $\varphi(x) = \int f(x)\ dx$.

The symbol $\int f(x)\ dx$ is read as integral of $f(x)$ w.r.t x

NOTE

When the function $\varphi(x)$ is obtained, it is said that a differential equation is solved. $i.e.\ \varphi(x) = \int f(x)\ dx$

Now the process of determining an integral is called *Integration* and the function to be integrated is called the *Integrand.*

$\Rightarrow \frac{d}{dx}\int f(x)\ dx = \int \left[\frac{d}{dx}\ f(x)\right] dx = f(x)$

Addition of an arbitrary constant

$\frac{d}{dx}\varphi(x) = f(x)$

$\Rightarrow \int f(x)\,dx = \varphi(x) + c$, where c is an arbitrary constant

Proof

$\frac{d}{dx}[\varphi(x) + c] = \varphi(x) + 0 \ \left[\because \frac{d}{dx}(c) = 0\right]$

$= \frac{d}{dx}\varphi(x) = f(x)$, given.

$\therefore \int f(x)\ dx = \varphi(x) + c$

NOTE

$\varphi(x) + C$ is called an *indefinite integral of* $f(x)$, because the constant c is indefinite.

Table of Standard Integrals

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$

$\int \frac{1}{x}dx = \ln x + c$

$\int e^x\,dx = e^x + c$

$\int a^x\,dx = \frac{a^x}{\ln a} + c$

$\int \sec x \tan x\,dx = \sec x + c$

$\int \sin x\,dx = -\cos x + c$

$\int \cos x\,dx = \sin x + c$

$\int \sec^2 x\,dx = \tan x + c$

$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + c$ or $-\sin^{-1} x + c$

$\int \frac{1}{1+x^2}dx = \tan^{-1} x + c$ or $-\cot^{-1} x + c$

$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x + c$

Relate these with differentiation. *i.e.*

Differentiation Formula	**Integration Formula**
$\frac{d}{dx}(c) = 0$	$\int 0dx = c$
$\frac{d}{dx}kx = k$	$\int kdx = kx + c$

Integrating Functions of the Form x^n

Integrate

a) $\int (2x^3 + x^2)\,dx$

b) $\int \left(\sqrt{x}+1\right)\left(2\sqrt{x}-1\right)\,dx$

c) $\int (ax^2 + bx + c)dx$

d) $\int \frac{x^2+2x+3}{\sqrt{x}}dx$

Solution

a) $\int (2x^3 + x^2)\,dx = \frac{2x^{3+1}}{3+1} + C = \frac{x^4}{4} + \frac{x^3}{3} + c$

b) $\int \left(\sqrt{x}+1\right)\left(2\sqrt{x}-1\right)\,dx = \int \left(2x - \sqrt{x} + 2\sqrt{x} - 1\right)dx$

$= \int \left(2x + \sqrt{x} - 1\right)dx = \frac{x^2}{2} + \frac{x^{3/2}}{3/2} - x + c$

$= x^2 + \frac{2}{3}x^{3/2} - x + c$

c) $\int (ax^2 + bx + c)dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$

d) $\int \frac{x^2+2x+3}{\sqrt{x}}dx = \int \left(x^{2-1/2} + 2x^{1-1/2} + 3x^{-1/2}\right)dx = \int \left(x^{3/2} + 2x^{1/2} + 3x^{-1/2}\right)dx$

$= \frac{x^{5/2}}{5/2} + \frac{2x^{3/2}}{3/2} + \frac{3x^{1/2}}{1/2} + c = \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + 6x^{1/2} + c$

$= \frac{2}{5}\sqrt{x^5} + \frac{4}{3}\sqrt{x^3} + 6\sqrt{x} + c$

Task

a) $\int (2x^3 - 5x^2 - 3x + 4)\sqrt{x}\,dx$ **Ans:** $\frac{4}{9}x^{9/2} - \frac{10}{7}x^{7/2} - \frac{6}{5}x^{5/2} + \frac{8}{3}x^{3/2} + c$

b) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^3 dx$ **Ans:** $\frac{2}{5}x^{5/2} - 2x^{3/2} + 6x^{1/2} + 2x^{-1/2} + c$

Definite Integrals

Definition: If $\int f(x)\,dx = \varphi(x)$, then $\varphi(b) - \varphi(a)$ is called the *definite integral of* $f(x)$ *between the limits* a *and* b and is written as;

$\int_a^b f(x)\,dx$ [Read as integral from a to b, of $f(x)$ w.r.t x]

$\therefore \int_a^b f(x)\,dx = \varphi(b) - \varphi(a)$, this gives the Fundamental Theorem of Calculus.

Here a is called the *Lower Limit*, b is the *Upper Limit* of Integration. The interval $(a\,,b)$ is the *Range of Integration.*

Note

1) Since $\int_c^d f(x)\,dx$ has a definite value $\varphi(b) - \varphi(a)$ then the name "*Definite Integral*"
2) The difference $\varphi(b) - \varphi(a)$ is also written as $[\varphi(x)]_a^b = \varphi(b) - \varphi(a)$

Rule to write the Definite Integral $\int_a^b f(x)\,dx$

a) Integrate $\int f(x)\,dx$ and let the indefinite integral be $\varphi(x)$
b) Substitute in $\varphi(x)$ the first upper limit and then the lower limit
c) Subtract the second result from the first

NOTE

Definition of Two special Definite Integral

a) $\int_a^a f(x)\,dx = 0$
b) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$, where f integrable on $[a\,,b]$

Other properties include

1) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
2) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$, where f and g are integrable on $[a\,,b]$ and k is a constant

Example

Evaluate the definite integrals

a) $\int_3^0 (x+3)\,dx$ b) $\int_1^3 (-x^2+4x-3)\,dx$ c) $\int_{-1}^1 (2x^2-x^3)\,dx$

Solution

a) $\int_3^0 (x+3)\,dx = -\int_3^0 (x+3)\,dx$

$$= \left[\frac{x^2}{2} + 2x\right]_0^3 = -\left(\frac{9}{2} + 6 - 0\right) = -\frac{21}{2}$$

b) $\int_1^3 (-x^2+4x-3)\,dx = \left[\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$

$$= \left(\frac{-27}{2} + 18 - 9\right) - \left(\frac{1}{3} + 2 - 3\right) = \frac{4}{3}$$

c) $\int_{-1}^1 (2x^2-x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{-1}^1 = -\left(\frac{2}{3} - \frac{1}{4}\right) - \left(-\frac{2}{3} - \frac{1}{4}\right) = \frac{4}{3}$

INTEGRATION BY SUBSTITUTION

Introduction:

From the previous presentations, integrals can be obtained with the help of standard results of differentiation by mere inspection. However the integrand $f(x)$ may not be easily guessed by mere inspection, the function of which $f(x)$ is the derivative.

In such cases, a suitable substitution like $x = \varphi(z)$ may change $f(x)$ in to a function of z which can be integrated by mere inspection.

The procedure of evaluating integrals is called *Integration by Substitution.*

Review:

Differentiation formula	**Integration formula**
$\frac{d}{dx}[c] = 0$	$\int 0dx = c$
$\frac{d}{dx}[kx] = k$	$\int kdx = kx + c$
$\frac{d}{dx}[k\,f(x)] = k\,f'(x)$	$\int k\,f(x)\,dx = k\int k\,f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int[f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c\,, n \neq -1$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\ dx = \sin x + c$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\ dx = \cos x + c$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\ dx = \sec x + c$
$\frac{d}{dx}[\cot x] = -\operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
$\frac{d}{dx}[\operatorname{cosec} x] = -\operatorname{cosec} x \cot x$	$\int \operatorname{cosec} x \cot x\ \ dx = -\operatorname{cosec} x + c$

For easy understanding the discussion has been split in to two parts. *i.e.*

a) Pattern Recognition
b) Change of variable

Both techniques involve a $u-$*substitution*

Pattern Substitution

Here perform the substitution mentally, and with change of variables you write the substitution steps.

The role of substitution in integration is comparable to the role of chain rule in differentiation. *i.e.*

Given that $y = f[g(x)]$, then

Let $u = g(x) \Longrightarrow \frac{du}{dx} = g'(x)$

$y = f(u) \Longrightarrow \frac{dy}{du} = f'(u)$

By Chain Rule $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$\frac{dy}{dx} = f'(u).g'(x) = f'[g(x)].g'(x)$

$\therefore \frac{d}{dx}\{f[g(x)]\} = f'[g(x)].g'(x)$

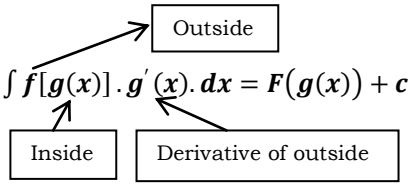
Now integrating both sides

$\int \frac{d}{dx}\{f[g(x)]\}\ dx = \int f'[g(x)].g'(x)\ dx$

$= f[g(x)] + c$

Or $\int f(u)\,.du = F(u) + c$

Generally



Recognizing the $f[g(x)].g'(x)$

Evaluate $\int 2x(x^2+1)^2\,dx$

Solution

Letting $g(x) = x^2 + 1$, $g'(x) = 2x$ and $f[g(x)] = [g(x)]^2$

From this you can recognize that the integrand follows $f[g(x)].g'(x)$ pattern

Using the Power Rule $.i.e$

$\int [g(x)]^n.g'(x)\ dx = \frac{[g(x)]^{n+1}}{n+1} + c\ ,\quad n \neq -1$

Equivalently if $u = g(x)$, then

$\int u^n\ du = \frac{u^{n+1}}{n+1} + c\ ,\quad n \neq -1$

$[g(x)]^2\ .\ g'(x)$

$\Rightarrow \int (x^2+1)^2.(2x)\,.dx = \frac{(x^2+1)^3}{3} + c$

NOTE

Many integrands contain the essential part (the Variable Part) of $g'(x)$ but are missing a constant multiple. In such cases you can multiply and divide by the necessary constant multiple.

Evaluate $\int x(x^2+1)^2\,dx$

Solution

$\int x(x^2+1)^2\,dx = \int (x^2+1)^2.(2x).\frac{1}{2}dx$ ……. Multiply and Divide by 2

$= \frac{1}{2}\int (x^2+1)^2.(2x)\,.dx \equiv \frac{1}{2}\int [g(x)]^2.g'(x)\,.dx$

$= \frac{1}{2}(x^2+1)^3 + c \qquad \equiv \frac{1}{2}[g(x)]^3 + c$

Change of Variable

Sometimes the change of the variable x in $\int f(x)\,dx$, by means of a substitution $x = \varphi(z)$ changes the integrand to a function of z which is either a standard form or a form easier to deal with.

Let $I = \int f(x)\,dx$

$\frac{dI}{dx} = f(x)$

But $x = \varphi(z)$ $\therefore \frac{dx}{dz} = \varphi'(z)$

Now $\frac{dI}{dz} = f(x).\varphi'(z)$

$\Rightarrow I = \int f\big(\varphi(z)\big).\varphi'(z)\,dz$, which is anew integral in z which is more simpler/easier to deal with.

General Approach

If $I = \int f(x)\,dx$,

let $x = \varphi(z)$, $dx = \varphi'(z).dz$ are the key substitutions

$\Rightarrow I = \int f(x)\,dx = \int f\big(\varphi(z)\big).\varphi'(z)\,dz$, which is as before

NOTE

Sometimes the substitution is of the form $F(x) = \varphi(z)$

$\Rightarrow F'(x)dx = \varphi'(x)dz$

, which eliminates the integrand, transforming to a new integrand in terms of z.

Example

Evaluate

a) $\int \sqrt{2x-1}\,dx$ b) $\int x\sqrt{2x-1}\,dx$

Solution

a) $\int \sqrt{2x-1}\,dx$, let $u = 2x-1 \Rightarrow du = 2dx$ $\therefore dx = \frac{du}{2}$

$$\int \sqrt{2x-1}\,dx = \int \sqrt{u}\,.\frac{du}{2}$$
$$= \frac{1}{2}\int u^{1/2}\,dx = \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + c$$
$$= \frac{1}{3}u^{3/2} + c = \frac{1}{3}(2x-1)^{3/2} + c$$

b) $\int x\sqrt{2x-1}\,dx$; let $2x-1 \Rightarrow du = 2dx$ $\therefore dx = \frac{du}{2}$

Now since the integrand has a factor of x, you must also solve for x in terms of u. $i.e.$

$x = \frac{u+1}{2}$

$$\Rightarrow \int x\sqrt{2x-1}\,dx = \int \left(\frac{u+1}{2}\right).\sqrt{u}.\frac{du}{2}$$
$$= \frac{1}{4}\int \left(u^{3/2} + u^{1/2}\right)du = \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + c$$
$$= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{5/2} + c$$

Other examples on Substitution and the general Power Rule

a) $\int 3.(3x-1)^4\,dx = \int \overbrace{(3x-1)^4}^{u}.\overbrace{(3).dx}^{du} = \overbrace{\frac{(3x-1)^5}{5}}^{u^5/5} + c$

(b) $\int (2x+1).(2x^2+x)\,dx = \int \overbrace{(2x+1)}^{u}.\overbrace{(2x^2+x).dx}^{du} = \overbrace{\frac{(x^2+x)^2}{2}}^{u^2/2} + c$

(c) $\int 3x^2.\sqrt{x^3-2}\,dx = \int \overbrace{(x^3-2)^{1/2}}^{u^{1/2}}.\overbrace{(3x^2)dx}^{du} = \overbrace{\frac{(x^3-2)^{3/2}}{2}}^{\frac{u^{3/2}}{3/2}} + c$

(d) $\int -\frac{4x}{(1-2x^2)^2}\,dx = \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}}.\overbrace{(-4x)dx}^{du} = \overbrace{\frac{(1-2x^2)^{-1}}{-1}}^{u^{-1}/-1} + c$

How to Identify the Proper Substitution

There is no general rule for finding a proper substitution, however the following suggestions will prove useful.

1. *If a part of the integrand, in any form whatsoever that its differential coefficient is a factor of the integrand, then*

Part $= u$

Example: $\int x.\cos x^2\ dx$

Here we note that the differential coefficient of x^2 is $2x$, which contains the factor x(Neglecting the constant factor). Hence proper substitution is

$x^2 = u$

$\Longrightarrow 2x\,dx = du$ or $x = \frac{1}{2}du$

$\therefore \int x.\cos x^2\ dx = \frac{1}{2}\cos u\,.du = \frac{1}{2}\sin u + c = \frac{1}{2}\sin x^2 + c$

2. *If the integrand contains "Trigonometrical ratio, a "Logarithmic function" or "an exponential" in which the index is a "a function of x" then put this*

Function$= u$

Example: $\int e^{\sqrt{x}}\,dx$

Put $u = \sqrt{x}$ or $u^2 = x \Longrightarrow 2udu = dx$

$\therefore \int e^{\sqrt{x}}\,dx = \int e^u.2u\,du = 2\int u.e^u\,du$, which can be integrated by parts

3. *If the integrand contains* $\log_e x$*, it may be helpful to use*

$u = \log_e x$ or $x = e^u$

4. *If the integrand contains an inverse trigonometrical function of* x *like* $\sin^{-1} x$*, this ratio may be put as*

Inverse of Trigonometrical Function $= \boldsymbol{u}$

Example: $\int \frac{\tan^{-1} x}{1+x^2}\,dx$, $\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}}\,dx$

If the trigonometrical ratio is that of a "function of x" then the trigonometrical ratio whole (and not merely the function) must be put as **Inverse of Trigonometrical Function** $= \boldsymbol{u}$**,** unless there is a better way to remove the inverseness.

Thus if $I = \int \sin^{-1}\sqrt{\left(\frac{x}{a+x}\right)}\,dx$, put $\sin^{-1}\sqrt{\left(\frac{x}{a+x}\right)} = u \Rightarrow \sin u = \sqrt{\left(\frac{x}{a+x}\right)}$

$\Rightarrow x = (a+x)\sin^2 u$ or $x = \frac{a\sin^2 u}{1-\sin^2 u}$

5. *If you integrate an* ***odd*** *power of* $\sin x$*, put* $\boldsymbol{u = \cos x}$

6. *If you integrate an* ***odd*** *power of* $\cos x$*, put* $\boldsymbol{u = \sin x}$

7. *If you integrate the product of functions of sine and cosine with power of* ***sine*** *being* ***odd****, put* $\boldsymbol{u = \cos x}$ *if the power of* ***cosine*** *is* ***even,*** *and put* $\boldsymbol{u = \sin x}$ *if the power of cosine is odd and that of sine is even.*

Example:

i) $\int \sin^2 x \cos^5 x\ dx$, let $u = \sin x$

ii) $\int \sin^3 x \cos^4 x\ dx$, let $u = \cos x$

8. *If the integral are in the form;*

a) $\int \sec^m x \tan^n x\,dx$, $m = even$, use $u = \tan x$, and if $n = odd$, use $u = \sec x$

b) $\int \sin^m x \cos^n x\,dx$, $m, n = odd$, use a function with the *highest power*

Now if $m = even, n = odd$, use $u = \sin x$

$m = odd, n = even$, use $u = \cos x$

c) $\int \tan^m x\,dx$, $m = even$, $u = \tan x$

9. *If the integrand is a fraction whose numerator is (constant of a function of* $\tan x$*) and the denominator a homogeneous function of second degree in* $\sin x$ *of* $\cos x$*, divide the numerator and denominator by* $\cos^2 x$ *and put* $u = \tan x$ *(or divide by* $\sin^2 x$*) and put* $u = \cot x$*.*

If the degree of denominator be fourth, divide by $\cos^4 x$ *or* $\sin^4 x$ *and make the same substitution.*

For some of the latter cases, a substitution of $\boldsymbol{u = \tan^2 x}$ *may be effective which is evidently better.*

Example

Let $I = \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} dx$

Divide the numerator and denominator by $\cos^2 x$

$I = \int \frac{\sec^2 x dx}{a^2 \tan^2 x + b^2} dx = \int \frac{du}{a^2 u^2 + b^2}$, after putting $u = \tan x$ and $du = \sec^2 x\, dx$

Again if the denominator be not homogeneous it may be possible to make it so

$\int \frac{dx}{1+3\sin^2 x} = \int \frac{dx}{(\cos^2 x + \sin^2 x) + 3\sin^2 x} = \int \frac{dx}{\cos^2 x + 4\sin^2 x}$

10. *When the integrand is a rational function of the trigonometrical ratios of x, the substitution $t = \tan\frac{x}{2}$ is often useful. This transforms the integrand to a rational function of t because;*

$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$

$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$, and $\frac{1}{2}\sec^2 x\, dx = dt \quad \therefore dx = \frac{2dt}{1+t^2}$

11. *If the integrand is a rational function of e^x put $u = e^x$ such that $e^x dx = du$ or $dx = \frac{du}{e^x} = \frac{du}{u}$*

12. *If the integrand contains a surd, it is better to first rationalize it by proper substitution.*

The following substitutions are very useful

Form of Surd	**Substitution**
$\sqrt{ax+b}$	$ax + b = u^2$
$(ax+b)^{r/s}$	$ax + b = u^s$
$\sqrt{x+b}$	$x + b = u^2$, or $x = \tan^2\theta$
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$, or $x = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$
$\sqrt{ax - x^2}$	$x = a\sin^2\theta$
$\sqrt{2ax - x^2}$	$x = 2a\sin^2\theta$ or $a(1-\cos\theta)$
$\sqrt{\left(\frac{a-x}{a+x}\right)}$	$x = a\cos 2\theta$
$\sqrt{\left(\frac{a^2-x^2}{a^2+x^2}\right)}$	$x^2 = a^2\cos 2\theta$

General standard formula for surds

- $\int (ax+b)^n . a . dx = \frac{(ax+b)^{n+1}}{(n+1)} + c$
- $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

8–Standard integrals to be used

1. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)$ or $-\cos^{-1}\left(\frac{x}{a}\right)$
2. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ or $-\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right)$
3. $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$ or $-\frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right)$
4. $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$
5. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \log_e\left(\frac{x+\sqrt{a^2+x^2}}{a}\right)$
6. $\int \sqrt{a^2+x^2}\, dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left(\frac{x+\sqrt{a^2+x^2}}{2}\right)$
7. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)$
8. $\int \sqrt{x^2-a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)$

Other Four Standard Integrals

Show that

a) $\int \cot x \, dx = \ln \sin x$

b) $\int \tan x \, dx = -\ln \cos x$ or $\ln \sec x$

c) $\int \operatorname{cosec} x \, dx = \ln \tan \frac{1}{2}x$ or $\ln(\operatorname{cosec} x - \cot x)$

d) $\int \sec x \, dx = \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$ or $\ln(\sec x + \tan x)$

Change of Variable for Definite Integrals

From $\int_a^b f\big(g(x)\big).g'(x)dx$, let $u = g(x) \Longrightarrow du = g'(x)dx$

$\Longrightarrow \int_a^b f\big(g(x)\big).g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du$

Example

a) *Evaluate* $\int_0^1 x(x^2+1)^3\,dx$

Solution

Let $u = x^2 + 1 \Longrightarrow du = 2x\,dx$

Before substituting, determine the new upper and lower limits of integration

Lower limit — When $x = 0\,, u = 0^2 + 1 = 1$

Upper limit — When $x = 1\,, u = 1^2 + 1 = 2$

Now substituting

$\int_0^1 x(x^2+1)^3\,dx = \frac{1}{2}\int_0^1 (x^2+1)^3\,.(2x)dx$

$= \frac{1}{2}\int_1^2 (u)^3\,du = \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2$

$= \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$

b) *Evaluate* $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

Solution

Let $u = \sqrt{2x-1} \Longrightarrow u^2 = 2x - 1 \;\; \therefore x = \frac{u^2+1}{2}$

$dx = u.du$

Before substituting, determine the new upper and lower limits of integration

Lower limit — When $x = 1\,, u = \sqrt{2-1} = 1$

Upper limit — When $x = 5\,, u = \sqrt{10-1} = 3$

Now substituting to obtain

$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \frac{1}{2}\int_1^3 \left(\frac{u^2+1}{u}\right)u.du$

$= \frac{1}{2}\int_1^3 (u^2+1)\,du = \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^3$

$= \frac{1}{2}\left(9 + 3 - \frac{1}{3} - 1\right) = \frac{16}{3}$

Task

1. *Evaluate*

a) $\int (1+2x)^4\,.2dx$ — **Ans:** $\frac{(1+2x)^5}{5} + c$

b) $\int \sqrt{9-x^2}\,.(-2x)dx$ — **Ans:** $\frac{2}{9}(9-x^2)^{3/2} + c$

c) $\int (5x)\sqrt[3]{\sqrt{1-x^2}}\,dx$ — **Ans:** $-\frac{15}{8}(1-x^2)^{4/3} + c$

d) $\int \frac{x^2}{(1+x^3)^2}dx$ — **Ans:** $-\frac{1}{3(1+x^3)} + c$

e) $\int \left(1+\frac{1}{t}\right)^3.\left(\frac{1}{t^2}\right)dt$ — **Ans:** $-\frac{1}{4}\left(1+\frac{1}{t}\right)^4 + c$

f) $\int \frac{1}{\theta^2}\cos\left(\frac{1}{\theta}\right)d\theta$ **Ans:** $-\sin\left(\frac{1}{\theta}\right)+c$

g) $\int \frac{\text{cosec}^{\,2} x}{\cot^3 x}dx$ **Ans:** $\frac{1}{2}\tan^2 x + c$

h) $\int \tan^4 x \sec^2 x\, dx$ **Ans:** $\frac{1}{5}\tan^5 x + c$

i) $\int \frac{x^2-1}{\sqrt{2x-1}}dx$ **Ans:** $\frac{\sqrt{2x-1}}{15}(3x^2+2x-13)+c$

j) $\int \frac{-x}{(x+1)-\sqrt{x+1}}dx$ **Ans:** $-x-1-2\sqrt{x+1}+c$

2. *Evaluate the definite integral*

a) $\int_{-1}^{1} x(x^2+1)^3\, dx$ **Ans:** 0

b) $\int_{1}^{9} \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\ dx$ **Ans:** $\frac{1}{2}$

c) $\int_{0}^{7} x\sqrt[3]{x+1}\,dx$ **Ans:** $\frac{1209}{28}$

d) $\int_{1}^{2}(x-1)\sqrt{2-x}\ dx$ **Ans:** $\frac{4}{15}$

Four important integrals

To integrate

i) $\int \frac{1}{x^2+a^2}dx$

ii) $\int \frac{1}{\sqrt{a^2-x^2}}dx$

iii) $\int \frac{1}{x\sqrt{x^2-a^2}}dx$

Solution

i) Put $x = a\tan\theta \Longrightarrow dx = a\sec^2\theta\, d\theta$

$$\int \frac{1}{x^2+a^2}dx = \int \frac{1}{a^2\tan^2\theta + a^2}.a\sec^2\theta\, d\theta$$

$$= \int \frac{a\sec^2\theta d\theta}{a^2(1+\tan^2\theta)} = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$\therefore \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

ii) Put $x = a\sin\theta \Longrightarrow dx = a\cos\theta\, d\theta$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}.a\cos\theta\, d\theta$$

$$= \int \frac{a\cos\theta}{a\cos\theta}d\theta = \theta + c$$

$$\therefore \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+c$$

iii) Put $x = a\sec\theta \Longrightarrow dx = a\sec\theta\tan\theta\, d\theta$

$$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \int \frac{1}{a\sec\theta\sqrt{a^2\sec^2\theta-a^2}}.a\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\tan\theta}{a\sqrt{\sec^2\theta-1}}d\theta = \theta + c$$

$$\therefore \int \frac{1}{x\sqrt{x^2-a^2}}dx = \sec^{-1}\left(\frac{x}{a}\right)+c$$

Illustrative examples

Evaluate

i) $\int \sin^3 x \cos x \, dx$

ii) $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

iii) $\int \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}} dx$

Solution

i) Let $u = \cos x \Longrightarrow du = -\sin x \; dx$

$$\int \sin^3 x \cos x \, dx = \int \sin^3 x \,.\, u.\frac{du}{-\sin x}$$
$$= -\int \sin^2 x \,.\, u.\, du = -\int (1-\cos^2 x).\, u.\, du$$
$$= -\int (u - u^3)\, du = -\frac{u^2}{2} + \frac{u^4}{4} + c$$
$$= -\frac{\cos^2 x}{2} + \frac{\cos^4 x}{4} + c$$

ii) The presence of $\frac{1}{1+x^2}.dx$ suggests the substitution of $u = \tan^{-1} x$

$$\Longrightarrow du = \frac{dx}{1+x^2}$$
$$\therefore \int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^u \, du = e^u + c$$
$$= e^{\tan^{-1} x} + c$$

iii) Let $t = 1 + \sin^2 x \Longrightarrow dt = 2 \sin x \cos x$

$$\int \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}} dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\int t^{-1/2}\, dt$$
$$= \sqrt{t} + c = \sqrt{1+\sin^2 x} + c$$

3. *Integrate*

a) $\frac{x^2 \tan^{-1}(x^3)}{1+x^6}$

b) $\frac{1}{x}\cos(\ln x)$

c) $\frac{\cos\sqrt{x}-3}{\sqrt{x}}$

d) $\frac{1}{x^{2/3}(1+x^{2/3})}$

Solution

a) Let $u = \tan^{-1}(x^3)$ because of $\frac{x^2}{1+x^6}dx$ being present

$$\Longrightarrow du = \frac{3x^2}{1+x^6}dx$$
$$\therefore \int \frac{x^2 \tan^{-1}(x^3)}{1+x^6} dx = \frac{1}{3}\int \frac{3x^2.\tan^{-1}(x^3)}{1+x^6}.dx$$
$$= \frac{1}{3}\int u \, du = \frac{1}{6}u^2 + c$$
$$= \frac{1}{6}(\tan^{-1}(x^3))^2 + c$$

b) The presence of $\frac{dx}{x}$ suggests that $u = \ln x \Longrightarrow du = \frac{dx}{x}$ or $dx = x\, du$

$$\therefore \int \frac{1}{x}\cos(\ln x)\, dx = \int \cos u \; du = \sin u + c$$
$$= \sin \ln x + c$$

c) $\int \frac{\cos\sqrt{x}-3}{\sqrt{x}} dx$ putting $t = \sqrt{x} \;\Longrightarrow dt = \frac{dx}{2\sqrt{x}}$ or $dx = 2\sqrt{x}dt$

$$\therefore \int \frac{\cos\sqrt{x}-3}{\sqrt{x}} dx = \int (\cos t - 3)\,.2dx$$
$$= 2(\sin t\, 3t) + c = 2\sin\sqrt{x} - 6\sqrt{x} + c$$

d) $\int \frac{1}{x^{2/3}(1+x^{2/3})}dx$, putting $t = x^{1/3} \implies dt = \frac{1}{3}x^{-2/3}dx$ or $dx = 3x^{2/3}dt$

$$\int \frac{1}{x^{2/3}(1+x^{2/3})}dx = \int \frac{3\,dt}{1+t^2}$$
$$= 3\tan^{-1} t + c = 3\tan^{-1}\left(x^{1/3}\right) + c$$

4. *Integrate*

a) $\int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$ b) $\int \frac{\sin^2 x \cos x}{1-\sin x}\,dx$

c) $\int \frac{1}{x\sqrt{1-x}}\,dx$

Solution

a) $\int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$, putting $t^2 = \tan x \implies 2t\,dt = \sec^2 x\,dx$ or $dx = \frac{2tdt}{\sec^2 x}$

$$\therefore \int \frac{\sqrt{\tan x}}{\sin x \cos x}dx = \int \frac{t}{\sin x \cos x}.\frac{2tdt}{\sec^2 x}$$
$$= \int \frac{2t^2}{\tan x}dt = \int \frac{2t^2}{t^2}dt$$
$$= 2t + c = 2\sqrt{\tan x} + c$$

b) $\int \frac{\sin^2 x \cos x}{1-\sin x}\,dx$, putting $t = \sin x \implies dt = \cos x\,dx$ or $dx = \frac{dt}{\cos x}$

$$\therefore \int \frac{\sin^2 x \cos x}{1-\sin x}\,dx = \int \frac{t^2}{1-t}dt = \int \frac{-t(1-t)+t}{1-t}dt$$
$$= \int \left(-t + \frac{t}{1-t}\right)dt = \int \left(-t + \frac{-(1-t)+1}{1-t}\right)dt \quad \text{(Apply partial fractions)}$$
$$= \int \left(-t - 1 + \frac{1}{1-t}\right)dt = -\frac{t^2}{2} - t - \ln(1-t) + c$$
$$= -\left(\sin x + \frac{\sin^2 x}{2} + \ln(1-\sin x)\right) + c$$

c) $\int \frac{1}{x\sqrt{1-x}}\,dx$, putting $t^2 = 1 - x \implies 2t\,dt = -1\,dx$ or $dx = -2t\,dt$

$$\therefore \int \frac{1}{x\sqrt{1-x}}\,dx = \int \frac{1}{(1-t^2).t}.-2t\,dt = \int \frac{2}{1-t^2}dt$$
$$= \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt \qquad \text{(Apply partial fractions)}$$
$$= \ln(t-1) - \ln(t+1) + c$$
$$= \ln\left(\frac{t-1}{t+1}\right) + c = \ln\left[\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right] + c$$

Integrals involving Inverse of Trigonometrical Functions: completing squares

The derivatives of six inverse trigonometrical functions occur in three pairs. In each pair the derivative of one function is the negative of the other. *e.g.*

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$

When listing the *anti derivative* that corresponds to each of the inverse of trigonometric functions, you need to use only one member of each pair. It is conventional to use $\arcsin x$ as the anti derivative of $\frac{1}{\sqrt{1-x^2}}$, rather than $-\arccos x$

Three important integrals

To integrate

i) $\int \frac{1}{x^2+a^2}dx$ iii) $\int \frac{1}{x\sqrt{x^2-a^2}}dx$

ii) $\int \frac{1}{\sqrt{a^2-x^2}}dx$, see the four important integrals presented above.

Solutions

i) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

ii) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+c$

iii) $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \sec^{-1}\left(\frac{x}{a}\right)+c$

NOTE

The above results are useful. The reader should commit them to memory.

Examples

Integration with Inverse Trigonometrical Functions

a) $\int \frac{1}{\sqrt{4-x^2}}dx$ c) $\int \frac{1}{x\sqrt{3x^2-1}}dx$

b) $\int \frac{1}{2+9x^2}dx$ d) $\int \frac{1}{\sqrt{2-x^2}}dx$

Solution

a) $\int \frac{1}{\sqrt{4-x^2}}dx = \int \frac{1}{\sqrt{2^2-x^2}}dx = \arcsin\frac{x}{2}+c$

b) $\int \frac{1}{2+9x^2}dx = \frac{1}{3}\int \frac{3\,dx}{(3x)^2+(\sqrt{2})^2}$, $u=3x$, $a=\sqrt{2}$

$= \frac{1}{3}.\frac{1}{\sqrt{2}}.\arctan\frac{3x}{\sqrt{2}}+c$

c) $\int \frac{1}{x\sqrt{3x^2-1}}dx = \int \frac{1}{x\sqrt{3(x^2-1/3)}}dx$

$= \frac{1}{\sqrt{3}}\int \frac{1}{x\sqrt{(x^2-1/3)}}dx = \frac{1}{\sqrt{3}}\int \frac{1}{x\sqrt{\left(x^2-(1/\sqrt{3})^2\right)}}dx$

$= \frac{1}{\sqrt{3}}.\frac{1}{1/\sqrt{3}}\operatorname{arcsec}\left(\frac{x}{1/\sqrt{3}}\right)+c$

$= \operatorname{arcsec}\left(\sqrt{3}\,x\right)+c$

Alternatively

$\int \frac{1}{x\sqrt{3x^2-1}}dx = \int \frac{1}{x\sqrt{(\sqrt{3}\,x)^2-(1)^2}}dx$

$= \int \frac{\sqrt{3}}{(\sqrt{3}\,x)\sqrt{(\sqrt{3}x)^2-(1)^2}}dx$, $u=\sqrt{3}\,x$, $a=1$

$= \operatorname{arcsec}\left(\frac{\sqrt{3}x}{1}\right)+c = \operatorname{arcsec}\left(\sqrt{3}\,x\right)+c$

NOTE

The above examples portray the application of the three important integration formulas above.

Disguised Inverse Trigonometric Integration Formulae
Integration by Substitution

Evaluate

a) $\int \frac{dx}{\sqrt{e^{2x}-1}}$ c) $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx$ e) $\int_0^{\pi/4} \frac{\cos x}{1+\sin^2 x}dx$

b) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ d) $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}dx$ f) $\int \frac{e^{2x}}{\sqrt{1-e^{2x}}}dx$

Solution

a) Let $u = e^x \implies du = e^x.dx$ or $dx = \frac{du}{e^x} = \frac{du}{u}$

$$\therefore \int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{du}{u\sqrt{(e^x)^2-1}}$$
$$= \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}\left(\frac{u}{1}\right) + c$$
$$= \text{arcsec}(e^x) + c$$

b) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx$

Let $u = e^x \implies du = e^x.dx$ or $dx = \frac{du}{e^x} = \frac{du}{u}$

$$\therefore \int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{u\sqrt{(e^x)^2-1}}$$
$$= \int \frac{u}{\sqrt{1-u^2}}.\frac{du}{u} = \sin^{-1}\left(\frac{u}{1}\right) + c$$
$$= \text{arcsin}(e^x) + c$$

c) $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx$, let $u = \sin x \implies du = \cos x\,.dx$ or $dx = \frac{du}{\cos x}$

$$\therefore \int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx = \int \frac{\cos x}{\sqrt{4-u^2}}.\frac{du}{\cos x} = \int \frac{1}{\sqrt{4-u^2}}du$$
$$= \int \frac{1}{\sqrt{2^2-u^2}}du\,, a = 2$$
$$= \text{arcsin}\left(\frac{u}{2}\right) + c = \text{arcsin}\left(\frac{\sin x}{2}\right) + c$$

d) $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}dx$, let $u = \cos x \implies du = -\sin x\,.dx$ or $dx = \frac{du}{-\sin x}$

Changing limits

When $x = \pi/2$, $u = \cos\pi/2 = 0$ and when $x = \pi\,, u = \cos\pi = -1$

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}dx = \int_0^{-1} \frac{\sin x}{1+u^2}.\frac{du}{-\sin x} = \int_0^{-1} \frac{1}{1+u^2}\,du$$
$$= [\tan^{-1} u]_{-1}^{0} = \tan^{-1}(0) - \tan^{-1}(-1) = \frac{\pi}{4}$$

e) Left as an exercise. **Ans:** $\tan^{-1}\left(1/\sqrt{2}\right)$

f) Left as an exercise **Ans:** $\frac{1}{4}\text{arctan}\frac{e^{2x}}{2} + c$

Rewriting as the Sum of Two Quotients

Evaluate

a) $\int \frac{x+2}{\sqrt{4-x^2}}dx$ b) $\int \frac{x+1}{x^2+1}dx$

Solution

a) $\int \frac{x+2}{\sqrt{4-x^2}}dx = \int \frac{x}{\sqrt{4-x^2}}dx + \int \frac{2}{\sqrt{4-x^2}}dx$

$= -\frac{1}{2}\int(-2x).(4-x^2)^{-1/2}\,dx + 2\int \frac{1}{\sqrt{(4)^2-x^2}}dx$

$= -\frac{1}{2}\left[\frac{(4-x^2)^{-1/2}}{1/2}\right] + 2\arcsin\left(\frac{x}{2}\right) + c$

$= -\sqrt{4-x^2} + 2\arcsin\left(\frac{x}{2}\right) + c$

b) $\int \frac{x+1}{x^2+1}dx = \int \frac{x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$

$= \frac{1}{2}\ln(x^2+1) + \arctan x + c$

NOTE

The above integrals when the integrand is split in to two, one yields an integral to be evaluated by the Power Rule and the second part yields an Inverse Trigonometric Function

Integrating an Improper Rational Function

Evaluate

a) $\int \frac{3x^3-2}{x^2+4}dx$ b) $\int \frac{x^3}{x^2+1}dx$

Solution

a) The integrand is an improper rational function because the degree of the numerator is greater than the degree of the denominator.

Use long division to rewrite the integrand.

$$\begin{array}{r} 3x \\ x^2+4\overline{)\,3x^3 \quad -2} \\ \underline{3x^3+12x} \\ -12x-2 \end{array}$$

$\Rightarrow \int \frac{3x^3-2}{x^2+4}dx = \int\left(3x - \frac{12x+2}{x^2+4}\right)dx$

$= \int 3x\ dx - 6\int \frac{2x}{x^2+4}\ dx - 2\int \frac{dx}{x^2+4}\ dx$

$= 3x^2 - 6\ln(x^2+4) - \arctan\frac{x}{2} + c$

b) Left as an exercise. **Ans:** $\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + c$

Completing squares

Evaluate

a) $\int \frac{dx}{x^2-4x+7}$ b) $\int \frac{dx}{x^2+x+1}$

Solution

a) $x^2 - 4x + 7 = (x^2 - 4x + 4) - 4 + 7$

$$= (x-2)^2 + 3 \equiv u^2 + a^2$$

$$\int \frac{dx}{x^2-4x+7} = \int \frac{dx}{(x-2)^2+(\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right) + c$$

b) $x^2 + x + 1 \equiv \left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} + 1$

$$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \equiv u^2 + a^2$$

$$\therefore \int \frac{dx}{x^2+x+1} = \frac{1}{\sqrt{3}/2} \arctan\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + c$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

NOTE

If the leading coefficient is not 1, it helps to factor before completing squares. *e. g*

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$$

Completing the squares (Leading coefficient not 1)

Evaluate

a) $\int \frac{dx}{2x^2-8x+10}$ b) $\int \frac{dx}{2x^2-3x+2}$

Solution

$$2x^2 - 8x + 10 = 2(x^2 - 4x + 5)$$

$$= 2(x^2 - 4x + 4 - 4 + 5)$$

$$= 2[(x-2)^2 + 1]$$

$$\int \frac{dx}{2x^2-8x+10} = \frac{1}{2}\int \frac{dx}{(x-2)^2+1}$$

$$= \frac{1}{2}\arctan(x-1) + c$$

Completing the squares (Negative Leading Coefficient)

Evaluate

a) $\int \frac{dx}{\sqrt{3x-x^2}}$ c) $\int \frac{dx}{\sqrt{2+3x-2x^2}}$

b) $\int \frac{dx}{\sqrt{-x^2-4x}}$ d) $\int \frac{3x-1}{\sqrt{1-x-2x^2}}$

Solution

a) $3x - x^2 = -(x^2 - 3x)$

$$= -\left[x^2 - 3x + \left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2\right]$$

$$= -\left[\left(x - \frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2\right]$$

$$= \left(\frac{3}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{dx}{\sqrt{3x-x^2}} = \int \frac{dx}{\sqrt{(3/2)^2-(x-3/2)^2}}$$
$$= \arcsin\frac{x-3}{3/2} + c$$
$$= \arcsin\frac{2x-3}{3} + c$$

b) Left as an exercise **Ans:** $\arcsin\frac{x+2}{2} + c$

c) $2 + 3x - 2x^2 = -2\left(x^2 - \frac{3}{2}x - 1\right)$
$$= -2\left[x^2 - \frac{3}{2}x + \left(-\frac{3}{4}\right)^2 - \left(-\frac{3}{4}\right)^2 - 1\right]$$
$$= -2\left[\left(x - \frac{3}{4}\right)^2 - \frac{25}{16}\right]$$
$$= -2\left[\left(x - \frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right]$$
$$= 2\left[\left(\frac{5}{4}\right)^2 - \left(x - \frac{3}{4}\right)^2\right]$$
$$\int \frac{dx}{\sqrt{2+3x-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(5/4)^2-(x-3/4)^2}}$$
$$= \frac{1}{\sqrt{2}}\arcsin\left(\frac{x-3/4}{5/4}\right) + c$$
$$= \frac{1}{\sqrt{2}}\arcsin\left(\frac{4x-3}{5}\right) + c$$

Task

a) $\int \frac{1}{(x-1)\sqrt{x^2-2x}}dx$

b) $\int \frac{x+2}{\sqrt{-x^2-4x}}dx$ **Ans:** $-\sqrt{x^2-4x} + c$

c) $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx$ **Ans:** $4 - 2\sqrt{3} + \pi/6 \simeq 1.059$

More Exploration of the Basic Rules of Integration

a) $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + c$

b) $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + c$

c) $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + c$

These rules involve the sum or difference of two squares:
$a^2 - [f(x)]^2$, $a^2 + [f(x)]^2$, $[f(x)]^2 - a^2$
With such an expression, consider the substitution $\boldsymbol{u = f(x)}$

Example

Evaluate

a) $\int \frac{x^2}{\sqrt{16-x^6}}dx$ b) $\int \frac{dx}{x^2+2x+5}$

Solution

a) Because the radical in the denominator can be written in the form $\sqrt{a^2-u^2} = \sqrt{4^2-(x^3)^2}$, try the substitution $u = x^3 \Rightarrow du = 3x^2dx$
$$\int \frac{x^2}{\sqrt{16-x^6}}dx = \frac{1}{3}\int \frac{x^2}{\sqrt{4^2-(x^3)^2}}dx = \frac{1}{3}\int \frac{du}{\sqrt{4^2-(x^3)^2}}$$

$$= \frac{1}{3}\arcsin\frac{u}{4} + c$$
$$= \frac{1}{3}\arcsin\frac{x^3}{4} + c$$

b) $\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+2^2}$
$$= \frac{1}{2}\arctan\frac{x+1}{2} + c$$

Two Important Forms of Integrals

A) $\int \frac{f'(x)}{f(x)}dx = \ln[f(x)] + c$

B) $\int [f(x)]^n . f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c \;; n \neq 1$ (Power Rule)

For (A) put $u = f(x) \implies du = f'(x)\,dx$

$\therefore \int \frac{f'(x)}{f(x)}dx = \int \frac{du}{u} = \ln u + c = \ln[f(x)] + c$

Thus $\int \frac{f'(x)}{f(x)}dx = \ln[f(x)] + c$

Thus the integral of a fraction whose numerator is the derivative of the denominator is equal to the logarithm of the denominator.

For (B) put $u = f(x) \implies du = f'(x)\,dx$

$\therefore \int [f(x)]^n . f'(x)\,dx = \int u^n\,du = \frac{u^{n+1}}{n+1} + c = \frac{[f(x)]^{n+1}}{n+1} + c \;; n \neq 1$

Thus $\int [f(x)]^n . f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c \;; n \neq 1$

Thus if the integrand is the product of a power of a function $f(x)$ and its derivative v, then the integral is obtained by increasing the index by unity and dividing the result by the increased index.

Example

1. *Integrate w.r.t x*

a) $\frac{2x+1}{x^2+x+1}$ b) $\frac{1}{(1+x^2)\tan^{-1}x}$ c) $\frac{x^{e-1}-e^{x-1}}{x^e-e^x}$

Solution

a) Since the numerator is the derivative of the denominator
$\int \frac{2x+1}{x^2+x+1}dx = \ln(x^2+x+1) + c$

b) $\int \frac{1}{(1+x^2)\tan^{-1}x}dx = \int \frac{\frac{1}{1+x^2}}{\tan^{-1}x}dx = \ln\tan^{-1}x + c$

c) $\int \frac{x^{e-1}-e^{x-1}}{x^e-e^x}dx = \frac{1}{e}\int \frac{e.x^{e-1}-e^x}{x^e-e^x}dx = \frac{1}{e}\ln(x^e - e^x) + c$

2. *Evaluate*

a) $\int \frac{x+1}{\sqrt{x^2+2x+3}}dx$ c) $\int \frac{1}{x(1+\ln x)^3}dx$

b) $\int \sqrt{\sin x}.\cos x\,dx$ d) $\int \cot^2 x\,\mathrm{cosec}^4 x\;dx$

Solution

a) $\int \frac{x+1}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int (x^2+2x+3)^{-1/2}(2x+2)\,dx$

Here $f(x) = x^2 + 2x + 3$, $f'(x) = 2x + 2$, $n = 1/2$

$\Rightarrow \frac{1}{2}\int (x^2 + 2x + 3)^{-1/2}(2x + 2)\,dx = \frac{1}{2}\frac{(x^2+2x+3)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$\therefore \int \frac{x+1}{\sqrt{x^2+2x+3}}dx = \sqrt{x^2 + 2x + 3} + c$

b) $\int \sqrt{\sin x}\,.\cos x\,dx = \int (\sin x)^{\frac{1}{2}}.\cos x\,dx$

$= \frac{(\sin x)^{1/2+1}}{1/2} + c = \frac{2}{3}(\sin x)^{\frac{3}{2}} + c$

c) $\int \frac{1}{x(1+\ln x)^3}dx = \int (1 + \ln x)^{-3}.\frac{1}{x}\,dx$

$= \frac{(1+\ln x)^{-3+1}}{-3+1} + c = -\frac{1}{2(1+\ln x)^2} + c$

d) $\int \cot^2 x \operatorname{cosec}^4 x\ dx = \int \cot^2 x\,.\operatorname{cosec}^2 x\ .\operatorname{cosec}^2 x\,dx$

$= \int \cot^2 x\,.(1 + \cot^2 x)\,.\operatorname{cosec}^2 x\,dx$

$= \int \cot^2 x\,.\operatorname{cosec}^2 x + \int \cot^4 x \operatorname{cosec}^2 x\,dx$

$= -\int \cot^2 x\,.\frac{d}{dx}\cot x\ dx + \int \cot^4 x\,.\frac{d}{dx}\cot x\ dx$

$= -\frac{\cot^3 x}{3} - \frac{\cot^5 x}{5} + c$

Disguised Integrals

a) $\int \frac{1}{1+e^x}\,dx$ b) $\int \cot x\,[\ln \sin x]dx$

Solution

a) This integral does not appear to fit any of the basic rules. However the quotient form suggests the logarithm rule.

Let $u = 1 + e^x \Rightarrow du = e^x.dx$

To get the required du, add and subtract e^x in the numerator

$\int \frac{1}{1+e^x}\,dx = \int \frac{1+e^x-e^x}{1+e^x}\,dx$

$= \left(\int \frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$

$= x - \ln(1 + e^x) + c$

Note

This integral can still be handled by multiplying numerator /denominator by e^{-x}

b) $\int \cot x\,[\ln \sin x]dx$, let $\ln \sin x \Rightarrow du = \frac{\cos x}{\sin x}\,dx$

$\int \cot x\,[\ln \sin x]dx = \int [\ln \sin x]\,.\cot x\,dx$

$= \int u\ du = \frac{u^2}{2} + c$

$\therefore \int \cot x\,[\ln \sin x]dx = \frac{1}{2}[\ln \sin x]^2 + c$

Procedures for Fitting Integrands to Basic Rules

Technique	**Example**
Expand (Numerator)	$(1+e^x)^2 = 1+2e^x+e^{2x}$
Separate Numerator	$\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$
Complete the square	$\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$
Divide improper fraction	$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$
Add and subtract terms In the numerator	$\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$
Multiply and divide by the Pythagorean conjugate	$\frac{1}{1+\sin x} = \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right)$ $= \frac{1-\sin x}{1-\sin^2 x} = \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$

Further Application of Trigonometric Substitutions

Integrals involving radicals,

$\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, $\sqrt{u^2-u^2}$

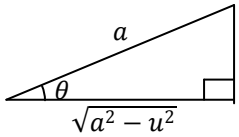
, can be evaluated using *Trigonometric Substitution.*

The objective with trigonometric substitution is to eliminate the radical in the integrand using Pythagorean identities. i.e.

$\cos^2\theta = 1-\sin^2\theta$, $\sec^2\theta = 1+\tan^2\theta$, $\tan^2\theta = \sec^2\theta - 1$

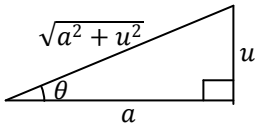
Trigonometric Substitution $(a > 0)$

1. *For integrals involving* $\sqrt{a^2-u^2}$, let $u = a\sin\theta$



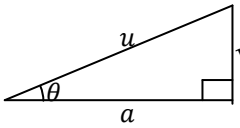
Then $\sqrt{a^2-u^2} = a\cos\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

2. *For integrals involving* $\sqrt{a^2+u^2}$, let $u = a\tan\theta$



Then $\sqrt{a^2+u^2} = a\sec\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

3. *For integrals involving* $\sqrt{u^2-u^2}$, let $u = a\sec\theta$



Then$\sqrt{u^2-u^2} = \pm a\tan\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

If $u > a$, use the positive value.

Examples

Evaluate

a) $\int \frac{dx}{x^2\sqrt{9-x^2}}$

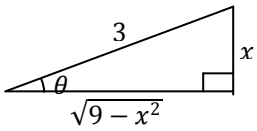
b) $\int \frac{dx}{\sqrt{4x^2+1}}$

Solution

a) Since $\sqrt{9-x^2}$ is of the form $\sqrt{a^2-u^2}$, use the substitution

$x = a\sin\theta = 3\sin\theta$

$\Longrightarrow dx = 3\cos\theta\, d\theta$, $\sqrt{9-x^2} = 3\cos\theta$ and $x^2 = 9\sin^2\theta$



$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta\, d\theta}{(9\sin^2\theta)(3\cos\theta)}$$

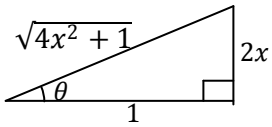
$$= \frac{1}{9}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{9}\int \operatorname{cosec}^2\theta\, d\theta$$

$$\therefore \int \frac{dx}{x^2\sqrt{9-x^2}} = -\frac{1}{9}\cot\theta + c$$

$$= -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + c = -\frac{\sqrt{9-x^2}}{9x} + c$$

b) $\int \frac{dx}{\sqrt{4x^2+1}} = \int \frac{dx}{\sqrt{(2x)^2+1}}$, let $2x = \tan\theta$

$\Longrightarrow 2dx = \sec^2\theta\, d\theta$ or $dx = \frac{1}{2}\sec^2\theta$, and $\sqrt{4x^2+1} = \sec\theta$



$$\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\int \frac{\sec^2\theta\, d\theta}{\sec\theta}$$

$$= \frac{1}{9}\int \sec\theta\, d\theta$$

$$\therefore \int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\ln|\sec\theta + \tan\theta| + c$$

$$= -\frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + c$$

NOTE

Extension of the use of trigonometric substitution to cover integrals involving expressions such as

$$(a^2-u^2)^{n/2}$$

, by writing the expression

$$\boxed{(a^2-u^2)^{n/2} = \left(\sqrt{a^2-u^2}\right)^n}$$

Example

Evaluate

a) $\int \frac{1}{(25-x^2)^{3/2}}dx$

b) $\int \frac{\sqrt{25-x^2}}{x}dx$

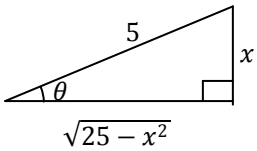
c) $\int \frac{1}{x^2\sqrt{25-x^2}}dx$

d) $\int \frac{x^2}{\sqrt{25-x^2}}dx$

Solution

a) $\int \frac{1}{(25-x^2)^{3/2}}dx = \int \frac{1}{\sqrt{(25-x^2)^3}}dx = \int \frac{1}{\sqrt{(5^2-x^2)^3}}dx$

$x = 5\sin\theta \Longrightarrow dx = 5\cos\theta\, d\theta$, $\sqrt{25-x^2} = 5\cos\theta$



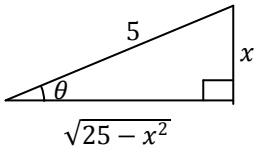
$$\therefore \int \frac{1}{(25-x^2)^{3/2}}dx = \int \frac{5\cos\theta\, d\theta}{125\cos^3\theta}$$

$$= \frac{1}{25}\int \sec^2\theta\, d\theta = \frac{1}{25}\tan\theta + c$$

$$= \frac{1}{25}\cdot\frac{x}{\sqrt{25-x^2}} + c$$

b) $\int \frac{\sqrt{25-x^2}}{x}dx$, let $x = 5\sin\theta \Longrightarrow dx = 5\cos\theta\, d\theta$, $\sqrt{25-x^2} = 5\cos\theta$

$$\int \frac{\sqrt{25-x^2}}{x}dx = \int \frac{5\cos\theta .5\cos\theta\, d\theta}{\sin\theta}$$
$$= 5\int \frac{\cos^2\theta}{\sin\theta}d\theta = 5\int \frac{1-\sin^2\theta}{\sin\theta}d\theta$$
$$= 5\int (\operatorname{cosec}\theta - \sin\theta)\, d\theta$$
$$= 5[\ln(\operatorname{cosec}\theta - \cot\theta) + \cos\theta] + c$$
$$= 5\left[\ln\left|\frac{5}{x} - \frac{\sqrt{25-x^2}}{x}\right|\right] + \sqrt{25-x^2} + c$$
$$= 5\left[\ln\left|\frac{5-\sqrt{25-x^2}}{x}\right|\right] + \sqrt{25-x^2} + c$$



c) Left as an exercise

d) Left as an exercise

Trigonometric Substitution: Rational Powers

Evaluate

a) $\int \frac{dx}{(x^2+1)^{3/2}}$

b) $\int \frac{dx}{(x^2+1)^2}$

c) $\int \frac{dx}{(x^2+3)^{3/2}}$

d) $\int \frac{dx}{(x^2-4)^{3/2}}$

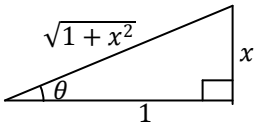
Solution

a) $\int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{dx}{\left(\sqrt{x^2+1}\right)^3}$

Let $x = \tan\theta \implies dx = \sec^2\theta\, d\theta$, $\sqrt{1+x^2} = \sec\theta$

$\implies \int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{dx}{\left(\sqrt{x^2+1}\right)^3} = \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta}$

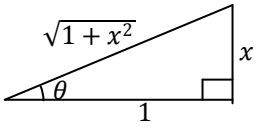
$$= \int \frac{d\theta}{\sec\theta} = \int \cos\theta\, d\theta$$
$$= \sin\theta + c = \frac{x}{\sqrt{1+x^2}} + c$$



b) $\int \frac{dx}{(x^2+1)^2}$, let $x = \tan\theta \implies dx = \sec^2\theta\, d\theta$, $\sqrt{1+x^2} = \sec\theta$, $\sec^2\theta = 1 + x^2$

$\therefore \int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \cos^2\theta\, d\theta$

$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + c$$
$$= \frac{1}{2}[\theta + \sin\theta\cos\theta] + c$$
$$= \frac{1}{2}\left[\theta + \frac{x}{\sqrt{1+x^2}}.\frac{1}{\sqrt{1+x^2}}\right] + c = \frac{1}{2}\left[\arctan x + \frac{x}{1+x^2}\right] + c$$



c) Left as an exercise

d) Left as an exercise **Ans:** $-\frac{x}{4\sqrt{x^2-4}}$

Completing Squares

Evaluate

a) $\int \frac{dx}{(x^2-4x)^{3/2}}$, $x > 4$

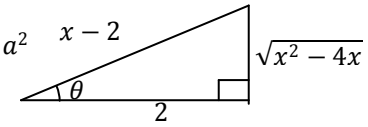
b) $\int \frac{x^2}{\sqrt{2x-x^2}}\, dx$

c) $\int \frac{x^2}{\sqrt{x^2+10x+9}}\, dx$

Solution

a) Completing squares

$x^2 - 4x = x^2 - 4x + 4 - 4 = (x-2)^2 - 2^2 \equiv u^2 - a^2$



Let $a = 2$ and $u = x - 2 = 2\sec\theta$

$\Rightarrow dx = \sec\theta\tan\theta\, d\theta$

, and $\sqrt{(x-2)^2 - 2^2} = 2\tan\theta$

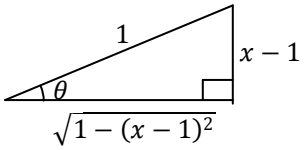
$\therefore \int \frac{dx}{(x^2-4x)^{3/2}} = \int \frac{\sec\theta\tan\theta\, d\theta}{(2\tan\theta)^3}$

$= \frac{2}{8}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{4}\int (\sin\theta)^{-2}(\cos\theta)\, d\theta$

$= -\frac{1}{4}\operatorname{cosec}\theta + c = -\frac{x-2}{4\sqrt{x^2-4x}} + c$

b) Completing squares

$2x - x^2 = -(x^2 - 2x + 1 - 1) = 1 - (x-1)^2$



$\int \frac{x^2}{\sqrt{2x-x^2}}\, dx = \int \frac{x^2}{\sqrt{1-(x-1)^2}}\, dx$

Let $x - 1 = \sin\theta \implies dx = \cos\theta\, d\theta$

, and $x^2 = (1 + \sin\theta)^2$

$\therefore \int \frac{x^2}{\sqrt{2x-x^2}}\, dx = \int \frac{(1+\sin\theta)^2}{\cos\theta}\, .\cos\theta\, d\theta$

$= \int (1 + 2\sin\theta + \sin^2\theta)\, d\theta$

$= \theta - 2\cos\theta + \frac{1}{2}\int (1 - \cos 2\theta)\, d\theta$

$= \theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + c$

$= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{2}\sin\theta\cos\theta + c$

$= \frac{3}{2}\sin^{-1}(x-1) - 2\sqrt{2x-x^2} - \frac{1}{2}(x-1)\sqrt{2x-x^2} + c$

$= \frac{3}{2}\sin^{-1}(x-1) - 2\sqrt{2x-x^2}.\left(1 + \frac{1}{4}(x-1)\right) + c$

$= \frac{3}{2}\arcsin(x-1) - \frac{1}{2}(x+3)\sqrt{2x-x^2} + c$

Integrals of Trigonometric Functions

Integrals of $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$

These are handled using the logarithmic rule

a) $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$

Knowing that $\frac{d}{dx}\cos x = -\sin x$, let $u = \cos x$

$\Rightarrow \int \tan x\, dx = -\int \frac{-\sin x}{\cos x}\, dx \equiv -\int \frac{u'}{u} dx$

$= -\ln|u| + c$

$= -\ln|\cos x| + c = \ln|\sec x| + c$

$\therefore \int \tan x\, dx = \ln|\sec x| + c$

b) $\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$, let $u = \sin x \Rightarrow du = \cos x\, dx$

$= \int \frac{du}{u} = \ln|u| + c$

$\therefore \int \cot x \ dx = \ln|\sin x| + c$

c) $\int \sec x \ dx$

Rewriting $\sec x$ as:

$\sec x = \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) = \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$

$\int \sec x \ dx = \int \left(\frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\right) dx$

, let $u = \sec x + \tan x \Longrightarrow du = (\sec^2 x + \sec x \tan x)dx$

$\Longrightarrow \int \sec x \ dx = \int \frac{du}{u} = \ln|u| + c$

$\therefore \int \sec x \ dx = \ln|\sec x + \tan x| + c$

d) $\int co \sec x \ dx$

Rewriting $\text{cosec}\, x$ as:

$\text{cosec}\, x = \text{cosec}\, x \cdot \left(\frac{\text{cosec}\ x - \cot x}{\text{cosec}\ x - \cot x}\right) = \frac{\text{co}\sec^2 x - \text{cosec}\ x \cot x}{\text{cosec}\ x - \cot x}$

$\int \text{cosec}\, x \ dx = \int \left(\frac{\text{co}\sec^2 x - \text{cosec}\ x \cot x}{\text{cosec}\ x - \cot x}\right) dx$

, let $u = \text{cosec}\, x - \cot x \Longrightarrow du = (-\text{cosec}\, x \cot x + \text{cosec}^2 x)dx$

$\Longrightarrow \int \text{cosec}\, x \ dx = \int \frac{du}{u} = \ln|u| + c$

$\therefore \int \text{cosec}\, x \ dx = \ln|\text{cosec}\, x - \cot x| + c$

NOTE

$\text{cosec}\, x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1 - \cos x}{\sin x}$

$= \frac{2\sin^2 \frac{x}{2}}{2\sin \frac{x}{2}\cos \frac{x}{2}} = \tan \frac{x}{2}$

Also $\int \text{cosec}\, x \ dx = \ln\left|\tan \frac{x}{2}\right| + c$

Integration of Positive Even Integral powers of $\sec x$ *or* $\text{cosec}\, x$

Rule to integrate $\int \sec^n x \ dx$ or $\int \text{cosec}^n x \ dx$, *where* n *is a positive even integer*

i) If the index of $\sec x$ is a positive even integer, put the substitution

$\boxed{u = \tan x}$

ii) If the index of $\text{cosec}\, x$ is a positive even integer, put the substitution

$\boxed{u = \cot x}$

NOTE

When n an odd positive integer, Reduction formula is is used.

Example

i) $\int \sec^4 2x \, dx$ ii) $\int \text{cosec}^4 2x \, dx$

Solution

i) Let $u = \tan 2x \ \Longrightarrow du = 2\sec^2 2x \, dx$ or $dx = \frac{du}{2\sec^2 2x}$

$\int \sec^4 2x \, dx = \int \sec^4 2x \cdot \frac{du}{2\sec^2 2x}$

$= \frac{1}{2}\int \sec^2 2x \, du = \frac{1}{2}\int (1 + \tan^2 2x) \, du$

$$= \frac{1}{2}\int (1+u^2)\,du = \frac{1}{2}\left[u + \frac{1}{3}u^3\right] + c$$

$$\therefore \int \sec^4 2x\,dx = \frac{1}{2}\tan 2x + \frac{1}{6}\tan^3 2x + c$$

ii) Let $u = \cot 2x \implies du = -2\,\text{cosec}^2\, 2x\,dx$ or $dx = -\frac{du}{2\,\text{cosec}^{\,2}\,2x}$

$$\int \text{cosec}^4\, 2x\,dx = \int \text{cosec}^4\, 2x \,.\, \frac{du}{-2\,\text{cosec}^{\,2}\,2x}$$

$$= -\frac{1}{2}\int \text{cosec}^2\, 2x\,du = \frac{1}{2}\int (1+\cot^2 2x)\,du$$

$$= -\frac{1}{2}\int (1+u^2)\,du = \frac{1}{2}\left[u + \frac{1}{3}u^3\right] + c$$

$$\therefore \int \sec^4 2x\,dx = -\frac{1}{2}\cot 2x - \frac{1}{6}\cot^3 2x + c$$

Integration of positive Even Integral Powers of $\cot x$ and $\tan x$

Rule to integrate $\int \cot^n x\,dx$ or $\int \tan^n x\,dx$, where n is a positive even integer

i) If the index of $\cot x$ is a positive even integer, put the substitution

$\boxed{u = \cot x}$, then apply $1 + \cot^2 x = \text{cosec}^2\, x$

ii) If the index of $\tan x$ is a positive even integer, put the substitution

$\boxed{u = \tan x}$, then apply $1 + \tan^2 x = \sec^2 x$

Example

i) $\int \tan^4 x\,dx$ ii) $\int \cot^4 x$

Solution

i) Let $u = \tan x \implies u = \tan^{-1} x$

$\implies du = \sec^2 x\,dx$ or $dx = \frac{du}{\sec^{\,2} x} = \frac{du}{1+\tan^{\,2} x} = \frac{du}{1+u^2}$

$\int \tan^4 x\,dx = \int u^4 \,.\, \frac{du}{1+u^2}$

By long division

$$\begin{array}{r} u^2 - 1 \\ u^2+1\,\overline{)\;u^4} \\ \underline{u^4 + u^2} \\ -u^2 \\ \underline{-u^2 - 1} \\ 1 \end{array}$$

$$\therefore \int \tan^4 x\,dx = \int \frac{u^4 du}{1+u^2} = \int \left(u^2 - 1 + \frac{1}{u^2+1}\right) du$$

$$= \frac{1}{3}u^3 - u + \tan^{-1} u + c$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + c$$

ii) Let $u = \cot x \implies u = \cot^{-1} x$

$\implies du = -\,\text{cosec}^2\, x\,dx$ or $dx = -\frac{du}{\text{cosec}^{\,2}\,x} = -\frac{du}{1+\cot^{\,2} x} = -\frac{du}{1+u^2}$

$\int \cot^4 x\,dx = -\int u^4 \,.\, \frac{du}{1+u^2}$

By long division

$$\begin{array}{r|l} & u^2 - 1 \\ \hline u^2+1 & u^4 \\ & u^4 + u^2 \\ \hline & -u^2 \\ & -u^2 - 1 \\ \hline & 1 \end{array}$$

$$\therefore \int \cot^4 x\, dx = -\int \frac{u^4 du}{1+u^2} = -\int \left(u^2 - 1 + \frac{1}{u^2+1}\right) du$$
$$= -\left[\frac{1}{3}u^3 - u + \cot^{-1} u\right] + c$$
$$= -\frac{1}{3}\cot^3 x + \cot x - x + c$$

NOTE

If the index in integrals $\int \cot^n x\ dx$ or $\int \tan^n x\ dx$ is *odd* use the substitutions

a) $\int \cot^n x\ dx$, $u = \operatorname{cosec} x$

b) $\int \tan^n x\ dx$, $u = \sec x$

Example

a) $\int \cot^3 2x\, dx$ b) $\int_0^{\pi/4} \tan^3 x\ dx$

Solution

a) Let $u = \operatorname{cosec} x \implies du = -2\operatorname{cosec} 2x \cot 2x\, dx$ or $dx = -\frac{du}{2\operatorname{cosec}\ 2x \cot\ 2x}$

$$\int \cot^3 2x\, dx = \int -\frac{\cot^3 2x du}{2\operatorname{cosec}\ 2x \cot\ 2x}.$$
$$= -\frac{1}{2}\int \frac{\cot^2 2x}{\operatorname{cosec}\ 2x}.du = -\frac{1}{2}\int \frac{\operatorname{cosec}^2 2x - 1}{\operatorname{cosec}\ 2x}.du$$
$$= -\frac{1}{2}\int \frac{u^2-1}{u} du = -\frac{1}{2}\int \left(u - \frac{1}{u}\right) du$$
$$= -\frac{1}{2}\left[\frac{u^2}{2} - \ln u\right] + c$$
$$= \frac{1}{2}\left[\ln u - \frac{u^2}{2}\right] + c = \frac{1}{4}[2\ln u - u^2] + c$$
$$= \frac{1}{4}[\ln|\operatorname{cosec}^2 x| - \operatorname{cosec}^2 x] + c$$
$$= \frac{1}{4}[\ln|u|^2 - u^2] + c$$

b) Let $u = \sec x \implies du = \sec x \tan x\, dx$ or $dx = \frac{du}{\sec\ x \tan\ x}$

<u>Changing limits</u>

When $x = 0$, $u = 1$ and when $x = \frac{\pi}{4}$, $u = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$

$$\int_0^{\frac{\pi}{4}} \tan^3 x\ dx = \int_0^{\sqrt{2}} \tan^3 x . \frac{du}{\sec\ x \tan\ x}$$
$$= \int_0^{\sqrt{2}} \tan^2 x . \frac{du}{\sec\ x} = \int_0^{\sqrt{2}} (\sec^2 x - 1) . \frac{du}{\sec\ x}$$
$$= \int_0^{\sqrt{2}} (u^2 - 1) . \frac{du}{u} = \int_0^{\sqrt{2}} \left(u - \frac{1}{u}\right) du$$
$$= \left[\frac{u^2}{2} - \ln|u|\right]_1^{\sqrt{2}} = \left(\frac{(\sqrt{2})^2}{2} - \ln\sqrt{2}\right) - \left(\frac{1^2}{2} - \ln 1\right)$$
$$= 1 - \frac{1}{2}\ln 2 - \frac{1}{2} = \frac{1}{2}(1 - \ln 2)$$

Integrals of the form $\int \cos^n x\ dx$ and $\int \sin^n x\, dx$

Case I: n is positive and odd

a) $\int \cos^n x\ dx$, use the substitution $\boldsymbol{u = \sin x}$

b) $\int \sin^n x\ dx$, use the substitution $\boldsymbol{u = \cos x}$

, then apply the identity $\boldsymbol{\cos^2 x + \sin^2 x = 1}$

Example

a) $\int_0^{\pi/2} \cos^3 x\, dx$ b) $\int \sin^5 2x\, dx$

Solution

a) Let $u = \sin x \implies du = \cos x\, dx$

$\int_0^{\pi/2} \cos^3 x\, dx = \int_0^{\frac{\pi}{2}} \cos^2 x \,.\cos x\, dx$

$= \int_0^{\frac{\pi}{2}} (1 - \sin^2 x)\,.\cos x\, dx$

$= \int_0^1 (1 - u^2)\,.\, du$

$= \left[u - \frac{u^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$

Changing limits

When $x = 0$, $u = \sin 0 = 0$

When $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

b) Let $u = \cos 2x \implies du = -2\sin 2x\, dx$

$\int \sin^5 2x\, dx = -\frac{1}{2}\int \sin^4 2x\,.(-2\sin 2x\, dx)$

$= -\frac{1}{2}\int (\sin^2 2x)^2\,.\, du = -\frac{1}{2}\int (1 - \cos^2 2x)^2\,.\, du$

$= -\frac{1}{2}\int (1 - u^2)^2\,.\, du = -\frac{1}{2}\int (1 - 2u^2 + u^4)\, du$

$= \frac{1}{2}\left[u - \frac{2}{3}u^3 + \frac{u^5}{5}\right] + c = \frac{1}{30}[15u - 20u^3 + 3u^5] + c$

$= \frac{1}{30}[3\cos^5 2x - 20\cos^3 2x + 15\cos 2x\ ] + c$

Case II: n is positive and even

a) $\int \cos^n x\ dx$, use the substitution $\boldsymbol{\cos^2 x = \frac{1+\cos 2x}{2}}$

b) $\int \sin^n x\ dx$, use the substitution $\boldsymbol{\sin^2 x = \frac{1+\cos 2x}{2}}$

NOTE: The above identities are for double angles

Example

a) $\int \cos^4 x\, dx$ b) $\int_0^{\pi/2} \sin^4 x\, dx$

Solution

a) $\cos^4 x = [\cos^2 x]^2 = \left[\frac{1+\cos\ 2x}{2}\right]^2$

$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$

$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1+\cos\ 4x}{2}\right) = \frac{1}{8}(\cos 4x + 4\cos 2x + 3)$

$\therefore \int \cos^4 x\, dx = \frac{1}{8}\int (\cos 4x + 4\cos 2x + 3)\, dx$

$= \frac{1}{8}\left[\frac{1}{4}\sin 4x + 4.\frac{1}{2}\sin 2x + 3x\ \right] + c$

$$= \frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$$

b) $\sin^4 x = [\sin^2 x]^2 = \left[\frac{1-\cos 2x}{2}\right]^2$

$$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1-\cos 4x}{2}\right) = \frac{1}{8}(2 - 4\cos 2x + 1 + \cos 4x)$$

$$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$

$$\therefore \int_0^{\pi/2} \sin^4 x\, dx = \frac{1}{8}\int_0^{\pi/2}(3 - 4\cos 2x + \cos 4x)\, dx$$

$$= \frac{1}{8}\left[3x - 2\sin 2x + \frac{1}{4}\sin 4x\right]_0^{\pi/2}$$

$$= \frac{1}{8}\left[\frac{3\pi}{2} - 2\sin\pi + \frac{1}{4}\sin 2\pi\right] - 0 = \frac{3\pi}{16}$$

Integrals involving Sine-Cosine Products with same/different angle

These integrals are evaluated using the product-to-sum identities (Factor Formula).

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$
$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$
$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

Example

a) $\int \sin 5x \cos 4x\, dx$ b) $\int_0^{\pi/4} \sin 2\theta \sin 3\theta\, d\theta$

Solution

a) $\int \sin 5x \cos 4x\, dx = \frac{1}{2}\int 2\sin 5x \cos 4x\, dx$

$$= \frac{1}{2}\int(\sin 9x + \sin x)\, dx$$

$$= \frac{1}{2}\left[-\frac{\cos 9x}{9} - \cos x\right] + c = -\frac{\cos 9x}{18} - \frac{\cos x}{2} + c$$

b) $\int_0^{\pi/4} \sin 2\theta \sin 3\theta\, d\theta = \frac{1}{2}\int_0^{\pi/4} 2\sin 2\theta \sin 3\theta\, d\theta$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}(\cos(-\theta) - \cos 5\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}(\cos\theta - \cos 5\theta)\, d\theta$$

$$= \frac{1}{2}\left[\sin\theta - \frac{\sin 5\theta}{5}\right]_0^{\pi/4} = \frac{1}{2}\left[\sin\frac{\pi}{4} - \frac{\sin 5\pi/4}{5}\right] - 0$$

$$= \frac{1}{2}\left[\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{10}\right)\right] = \frac{1}{2}\left(\frac{5\sqrt{2}+\sqrt{2}}{10}\right)$$

$$= \frac{3\sqrt{2}}{10}$$

Integration of $\sin^m x \cos^n x$ (at least one index is a positive integer)

The integral $\int \sin^m x \cos^n x\, dx$ can easily be evaluated when either m or n is a positive *odd integer,* for all values of the other index *(*including zero)

Different cases which arise are:

i) $\int \sin^n x\ dx$ and $\int \cos^n x\ dx$ where n is a positive **odd** integer.

ii) $\int \sin^m x \cos^n x\, dx$ where either m or n or both are positive *odd* integers

The rule to integrate these is as follows.

a) If the index of $\sin x$ is **odd**, put $u = \cos x$

b) If the index of $\cos x$ is **odd**, put $u = \sin x$

Example

a) $\int \sin^5 x \cos^4 x\, dx$

b) $\int \sin^2 x \cos^3 x\, dx$

c) $\int \sin^5 x \cos^3 x\, dx$

Solution

a) The index of $\sin x$ being odd, let $t = \cos x \Longrightarrow dt = -\sin x\ dx$

$$\therefore \int \sin^5 x \cos^4 x\, dx = \int \sin^4 x \cos^4 x \,.\sin x\, dx$$
$$= \int \cos^4 x \,.(1-\cos^2 x)^2 .\sin x\, dx$$
$$= -\int t^4 .(1-t^2)^2 \,.dt = -\int (t^4 - 2t^6 + t^8)\,.dt$$
$$= -\frac{t^5}{5} + \frac{2t^7}{7} - \frac{t^9}{9} + c$$
$$= \frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + c$$

b) The index of $\cos x$ being odd, let $t = \sin x \ \Longrightarrow dt = \cos x\ dx$

$$\therefore \int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \,.\cos x\, dx$$
$$= \int \sin^2 x\,(1-\sin^2 x).\cos x\, dx$$
$$= \int t^2(1-t^2).dt = \int (t^2 - t^4)\ dt$$
$$= \frac{t^3}{3} - \frac{t^5}{5} + c = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

c) Since both the indices are odd on sine and cosine we put either $u = \sin x$ or $u = \cos x$

In practice it is convenient to take the smaller index as odd and use the corresponding substitution.

Since both indices are odd but the index of $\cos x$ is smaller, we put

$u = \sin x \ \Longrightarrow du = \cos x\ dx$

$$\therefore \int \sin^5 x \cos^3 x\, dx = \int \sin^5 x \cos^2 x \,.\cos x\, dx$$
$$= \int \sin^5 x\,(1-\sin^2 x)\,.\cos x\, dx$$
$$= \int u^5(1-u^2)\,.du = \frac{u^6}{6} - \frac{u^8}{8} + c$$
$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c$$

NOTE

If the integral $\int \sin^m x \cos^n x\, dx$ where m and n are even, express each of the functions using double angle identities as:

$\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$

Example

Evaluate $\int \sin^2 x \cos^4 x\, dx$

Solution

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^4 x = \frac{(1+\cos 2x)^2}{4} = \frac{1+2\cos 2x+\cos^2 2x}{4} = \frac{1+2\cos 2x+\frac{1+\cos 4x}{2}}{4} = \frac{3+4\cos 2x+\cos 4x}{8}$

Now $\sin^2 x \cos^4 x = \left(\frac{1-\cos 2x}{2}\right)\left(\frac{3+4\cos 2x+\cos 4x}{8}\right)$

$= \frac{3+4\cos 2x+\cos 4x-3\cos 2x-4\cos^2 2x-\cos 4x\cos 2x}{16}$

$= \frac{3+\cos 2x+\cos 4x-4\left(\frac{1+\cos 2x}{2}\right)-\cos 4x\cos 2x}{16}$

$= \frac{1+\cos 2x+\cos 4x-\cos 4x\cos 2x}{16}$

$\therefore \int \sin^2 x \cos^4 x\, dx = \frac{1}{16}\int(1+\cos 2x+\cos 4x-\cos 4x\cos 2x)\, dx$

$= \frac{1}{16}\int\left(1+\cos 2x+\cos 4x-\frac{1}{2}(\cos 6x+\cos 2x)\right)dx$

$= \frac{1}{16}\left[x+\frac{\sin 2x}{2}-\frac{\sin 4x}{4}-\frac{\sin 6x}{12}-\frac{\sin 2x}{4}\right]+c$

Integrals Involving Powers of Secant and Tangent

Integrals of the form $\int \sec^m x \tan^n x\, dx$

Case I: Power of the secant is even. i.e. $m = 2k$

Rewrite the integrand as:

Save a secant squared factor

$\int \sec^{2k} x \tan^n x\, dx = \int(\sec^2 x)^{k-1}.\sec^2 x\,.\tan^n x\, dx$

$= \int(1+\tan^2 x)^{k-1}.\sec^2 x\,.\tan^n x\, dx$

Rest of the terms converted to tangents

Now let $\boldsymbol{u = \tan x}$

Case II: power of the tangent is odd and positive. i.e. $n = 2k+1$

Rewrite the integrand as:

Save a secant squared factor

$\int \sec^m x \tan^{2k+1} x\, dx = \int \sec^{m-1} x\,.\sec x\,.(\tan^2 x)^k.\tan x\, dx$

$= \int \sec^{m-1} x\,.(\tan^2 x)^k.(\sec x\tan x)\, dx$

$= \int \sec^{m-1} x\,.(\sec^2 x-1)^k.(\sec x\tan x)\, dx$

Rest of the terms converted to secants

Now let $\boldsymbol{u = \sec x}$

Example

a) $\int \sec^4 3x \tan^3 3x\, dx$ b) $\int \sec^3 x \tan x\, dx$

Solution

a) Since the index on secant is even, let $u = \tan 3x \implies du = 3\sec^2 3x\, dx$

$\therefore \int \sec^4 3x \tan^3 3x\, dx = \frac{1}{3}\int \sec^2 3x\,.\tan^3 3x\,.(3\sec^2 3x\, dx)$

$= \frac{1}{3}\int(1+\tan^2 3x).\tan^3 3x\,.(3\sec^2 3x\, dx)$

$$= \frac{1}{3}\int(\tan^3 3x + \tan^5 3x).(3\sec^2 3x\,dx)$$
$$= \frac{1}{3}\int(u^3 + u^5)\,du = \frac{1}{3}\left[\frac{u^4}{4} + \frac{u^6}{6}\right]$$
$$= \frac{\tan^4 x}{12} + \frac{\tan^6 x}{18} + c$$

b) Since the index on tangent is odd, let $u = \sec x \Longrightarrow du = \sec x \tan x\,dx$
$\therefore \int \sec^3 x \tan x\,dx = \int \sec^3 x\,.(\sec x \tan x)\,dx$
$$= \int u^2\,du = \frac{u^3}{3} + c = \frac{\sec^3 x}{3} + c$$

INTEGRATION OF ALGEBRAIC RATIONAL FRACTIONS

General Method for Integrating a Rational Fraction $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$

1) If the degree of $f(x)$ be equal to or greater than that of $g(x)$, divide $f(x)$ until the degree of remainder is less than that of $g(x)$ *.i.e.*
$f(x) = Q(x) + \frac{R(x)}{g(x)}$
, where $Q(x)$ is the quotient of the polynomial, $R(x)$ is the remainder whose degree is less than that of $g(x)$.
The part $Q(x)$ can be integrated easily by the formula $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$
Now how do we integrate $\frac{R(x)}{g(x)}$ *where the degree of* $R(x)$ *is smaller than that of* $g(x)$*?*

Cases to consider

a) By algebra, if $g(x)$ can be resolved in to factors of first and second degree. i.e.
$g(x) = A(a_1x + b_1)^{p_1}(a_1x + b_1)^{p_2}$ …… or
$g(x) = (A_1x + B_1x + C_1)^{q_1}(A_2x + B_2x + C_2)^{q_2}$ ……
, where $p_1, p_2,$ …… and $q_1, q_2,$ …… are positive integers denoting the number of times each factor is repeated.
Since some/all of the integers $p_1, p_2, \ldots\ldots, q_1, q_2,$ …… may be repeated, then $g(x)$ has four major factors;

i) Linear Non Repeated
ii) Linear Repeated
iii) Quadratic Non Repeated
iv) Quadratic Repeated

Hence the fraction $\frac{R(x)}{g(x)}$ may have partial fractions of the form, where n is positive

i) $\frac{1}{ax+b}$
ii) $\frac{1}{(ax+b)^n}$
iii) $\frac{lx+m}{Ax^2+Bx+C}$
iv) $\frac{lx+m}{(Ax^2+Bx+C)^2}$

, respectively.

Integration of (i) and (ii) is quite easy

i) $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$

ii) $\int \frac{1}{(ax+b)^n}dx = \int (ax+b)^{-n}\,dx = \frac{1}{a(-n+1)} \cdot \frac{1}{(ax+b)^{n-1}} + c$

Now let us discuss the integration of (iii) and (iv).i.e.

Integration of $\int \frac{lx+m}{Ax^2+Bx+C}dx$, where Ax^2+Bx+C cannot be resolved in to real linear factors. i.e. Ax^2+Bx+C is non factorisable.

Two cases arise

Case I: $l = 0$ .i.e. $\int \frac{m}{Ax^2+Bx+C}dx$

First express Ax^2+Bx+C in terms of completing squares. i.e.

$$Ax^2+Bx+C = A\left(x^2+\frac{B}{A}x+\frac{C}{A}\right)$$
$$= A\left[x^2+\frac{B}{A}x+\frac{B^2}{4A^2}+\left(\frac{C}{A}-\frac{B^2}{4A^2}\right)\right]$$
$$= A\left[\left(x+\frac{B}{2A}\right)^2+\left(\frac{\sqrt{4AC-B^2}}{2A}\right)^2\right]$$
$$= A[(x+h)^2+k^2], \text{ where } h=\frac{B}{2A} \text{ and } k=\frac{\sqrt{4AC-B^2}}{2A}$$

$\therefore \int \frac{m}{Ax^2+Bx+C}dx = \frac{m}{A}\int \frac{dx}{(x+h)^2+k^2} = \frac{m}{Ak}\tan^{-1}\left(\frac{x+h}{h}\right) + c_1$

Case II: $l \neq 0$ .i.e. $\int \frac{lx+m}{Ax^2+Bx+C}dx$

Here split the numerator

Let $lx+m \equiv \lambda\frac{d}{dx}(Ax^2+Bx+C)+\mu$

$$\equiv \lambda(2Ax+B)+\mu$$

From which values of λ and μ can be obtained.

$\therefore \int \frac{lx+m}{Ax^2+Bx+C}dx = \lambda\int \frac{2Ax+B}{Ax^2+Bx+C}dx + \mu\int \frac{1}{Ax^2+Bx+C}dx$

$\therefore \int \frac{lx+m}{Ax^2+Bx+C}dx = \lambda\ln|Ax^2+Bx+C| + \frac{\mu}{Ak}\tan^{-1}\left(\frac{x+h}{h}\right) + c_2$

NOTE

1) In Ax^2+Bx+C, if B^2-4AC is positive, then Ax^2+Bx+C can be resolved in to two integrals of the form $\int \frac{l}{ax+b}dx$, already discussed.
2) In Ax^2+Bx+C, if B^2-4AC is negative or $4AC-B^2$ is positive, then Ax^2+Bx+C cannot be resolved in to linear factors. Hence complete squares.

Integration of $\int \frac{lx+m}{(Ax^2+Bx+C)^n}dx$ where n is a positive integer greater than 1

Put $lx + m \equiv \lambda(2Ax + B) + \mu$, so that

$\frac{l}{2A} = \lambda$ and $\mu = \frac{2Am-Bl}{2A}$

$\therefore \int \frac{lx+m}{(Ax^2+Bx+C)^n}dx = \lambda \int \frac{2Ax+B}{(Ax^2+Bx+C)^n}dx + \mu \int \frac{1}{(Ax^2+Bx+C)^n}dx$

$= \frac{\lambda}{-n+1} \cdot \frac{1}{(Ax^2+Bx+C)^{n-1}} + \mu \int \frac{dx}{(Ax^2+Bx+C)^n}$

Now to evaluate the integral on R.H.S,

Let $Ax^2 + Bx + C = A[(x + h)^2 + k^2]$, where $h = \frac{B}{2A}$, $k = \frac{\sqrt{4AC-B^2}}{2A}$

$\Longrightarrow \int \frac{dx}{(Ax^2+Bx+C)^n} = \int \frac{dx}{[(x+h)^2-k^2]^n}$

, which can be evaluated in to integrals

i) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$; Substitution is $x = a\sec\theta$

ii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$; Resolve in to partial fractions

Example

a) $\int \frac{x+1}{x^2+4x+8}dx$ b) $\int \frac{2-5x}{\sqrt{3-4x-4x^2}}dx$

Solution

a) Since $B^2 - 4AC = 16 - 32 = -16 < 0$, $x^2 + 4x + 8$ has no real linear factors

By completing squares

$x^2 + 4x + 8 = x^2 + 4x + 4 + 4$

$= (x + 2)^2 + 4$

Also splitting the numerator

$x + 1 = A(2x + 4) + B$

$1 = 2A$, $A = \frac{1}{2}$

$1 = 4A + B$, $B = 1 - 2 = -1$

$\therefore \int \frac{x+1}{x^2+4x+8}dx = \int \frac{\frac{1}{2}(2x+4)}{x^2+4x+8}dx - \int \frac{dx}{x^2+4x+8}$

$= \frac{1}{2}\ln|x^2 + 4x + 8| - \int \frac{dx}{(x+2)^2+2^2}$

$= \frac{1}{2}\ln|x^2 + 4x + 8| - \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)$

b) Let $2 - 5x = A(-4 - 8x) + B$

$-5 = -8A$, $A = 5/8$

$2 = -4A + B$, $B = 2 + \frac{5}{2} = \frac{9}{2}$

$\int \frac{2-5x}{\sqrt{3-4x-4x^2}}dx = \int \frac{A(-4-8x)}{\sqrt{3-4x-4x^2}}dx + B\int \frac{dx}{\sqrt{3-4x-4x^2}}$

$= A\int(-4 - 8x).(3 - 4x - 4x^2)^{-1/2} + B\int \frac{dx}{\sqrt{3-4x-4x^2}}$

Now dealing with the R.H.S integral

$-4x^2 - 4x + 3 = -4\left(x^2 + x - \frac{3}{4}\right)$

Completing squares

$$= -4\left[x^2 + x + \frac{1}{4} - \frac{1}{4} - \frac{3}{4}\right]$$
$$= -4[(x+1/2)^2 - 1] = 4[1 - (x+1/2)^2]$$

$$\int \frac{dx}{\sqrt{3-4x-4x^2}} = \frac{1}{4}\int \frac{dx}{\sqrt{1-(x+1/2)^2}}$$
$$= \frac{1}{4}\sin^{-1}\left(\frac{x+1/2}{1}\right) + c_1$$
$$= \frac{1}{4}\sin^{-1}\left(\frac{2x+1}{2}\right) + c_1$$

$$\therefore \int \frac{2-5x}{\sqrt{3-4x-4x^2}}dx = A\int(-4-8x).(3-4x-4x^2)^{-1/2} + B\int \frac{dx}{\sqrt{3-4x-4x^2}}$$
$$= \frac{5}{4\sqrt{3-4x-4x^2}} + \frac{9}{4}\sin^{-1}\left(\frac{2x+1}{2}\right) + c$$

Task

a) $\int \frac{3x-1}{\sqrt{5x^2-2x-3}}dx$ **Ans:** $\frac{3}{5}\sqrt{5x^2-2x-3} - \frac{2}{5\sqrt{5}}\ln\left|x-5+\sqrt{x^2-\frac{2}{5}x-\frac{3}{5}}\right| + c$

b) $\int \frac{3x+5}{x^2+6x+11}$ **Ans:** $\frac{3}{2}\ln|x^2+6x+11| - 2\sqrt{2}\tan^{-1}\frac{x+3}{\sqrt{2}} + c$

Integration of Irrational Algebraic Functions

a) **Integration of rational functions of x^n and $(ax+b)^{\frac{1}{n}}$, n is a positive integer.**

i) $\int x^n.(ax+b)^{\frac{1}{n}}\,dx$

ii) $\int \frac{x^n}{(ax+b)^{\frac{1}{n}}}dx$

These are solved using the substitution

$$u = (ax+b)^{\frac{1}{n}} \quad \text{or} \quad u^n = ax+b$$

Rational functions of x^n, $(ax+b)^{\frac{1}{n}}$ and $(ax+b)^{\frac{1}{m}}$ can similarly be integrated by the substitution

$$u^p = ax+b, \text{ where } p \text{ is the L.C.M of } m \text{ and } n$$

Example

a) $\int \frac{x^2}{\sqrt[3]{x+2}}dx$ b) $\int \frac{1+x^{1/2}}{1+x^{1/3}}dx$ c) $\int \frac{dx}{(1+x)^{1/3}+(1+x)^{1/2}}dx$

Solution

a) Let $x+2 = u^3 \Longrightarrow dx = 3u^2\,du$

$$\int \frac{x^2}{\sqrt[3]{x+2}}dx = \int \frac{(u^3-2)^2}{u}.3u^2\,du$$
$$= 3\int(u^7 - 4u^4 + 4u)\,du = 3\left[\frac{u^8}{8} - \frac{4}{5}u^5 + 2u^2\right] + c$$
$$= \frac{3}{40}u^2[5u^6 - 32u^3 + 80] + c$$
$$= \frac{3}{40}(x+2)^{\frac{2}{3}}[5(x+2)^2 - 32(x+2) + 80] + c$$

$$= \frac{3}{40}(x+2)^{\frac{2}{3}}(5x^2 - 12x + 36) + c$$

b) $\int \frac{1+x^{1/2}}{1+x^{1/3}} dx$, L.C.M of 2 and 3 is 6

Let $x = u^6 \implies dx = 6u^5\, du$

$\int \frac{1+x^{1/2}}{1+x^{1/3}} dx = \int \frac{1+u^3}{1+u^2}.6u^5\, du = 6\int \frac{u^5+u^8}{1+u^2} du$

By long division

$$\begin{array}{r|l}
 & u^6 - u^4 + u^3 + u^2 - u - 1 \\
\hline
u^2+1 & u^8 + u^5 \\
 & \underline{u^8 + u^5} \\
 & -u^6 + u^5 \\
 & \underline{-u^6 - u^4} \\
 & u^5 + u^4 \\
 & \underline{u^5 + u^3} \\
 & u^4 - u^3 \\
 & \underline{u^4 + u^2} \\
 & -u^3 - u^2 \\
 & \underline{-u^3 - u} \\
 & -u^2 - u \\
 & \underline{-u^2 - 1} \\
 & -u + 1
\end{array}$$

$6\int \frac{u^5+u^8}{1+u^2} du$

$= \frac{6}{7}u^7 - \frac{6}{5}u^5 + \frac{3}{2}u^4 + 2u^2 - 3u^2 - 6u + 3\ln|u^2+1| + 6\tan^{-1} u + c$

$= \frac{6}{7}x^{\frac{7}{6}} - \frac{6}{5}x^{\frac{5}{6}} + \frac{3}{2}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{\frac{1}{2}} - 6x^{\frac{1}{6}} + 3\ln\left|x^{\frac{1}{3}} + 1\right| + 6\tan^{-1} x^{\frac{1}{6}} + c$

c) L.C.M of 3 and 2 being 6

Let $1 + x = u^6 \implies dx = 6u^5\, du$

$\int \frac{dx}{(1+x)^{1/3}+(1+x)^{1/2}} dx = \int \frac{6u^5\, du}{u^2+u^3} = 6\int \frac{u^3}{1+u} du$

$= 6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right) du$

$= 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right] + c$

$= 2(1+x)^{\frac{1}{2}} - 3(1+x)^{\frac{1}{3}} + 6(1+x)^{\frac{1}{6}} - 6\ln\left|(1+x)^{\frac{1}{6}} + 1\right| + c$

NOTE

A rational function of $\left(\frac{ax+b}{cx+d}\right)^{1/n}$ can also be integrated by the substitution

$$\boxed{u^n = \frac{ax+b}{cx+d}}$$

Example

a) $\int \frac{1}{(1+x)^{3/2}+(1+x)^{1/2}} dx$ b) $\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}} dx$

Solution

a) Let $u = (1+x)^{1/2} \implies u^2 = 1 + x$ or $dx = 2u\, du$

$$\therefore \int \frac{1}{(1+x)^{3/2}+(1+x)^{1/2}}dx = \int \frac{2u\,du}{u^3+u} = 2\int \frac{du}{u^2+1}$$
$$= 2\tan^{-1}u + c = 2\tan^{-1}(1+x)^{1/2} + c$$

b) Let $y^2 = \frac{1-x}{1+x} \Longrightarrow x = \frac{1-y^2}{1+y^2}$, $dx = \frac{4y\,du}{(1+y^2)^2}$

$$\therefore \int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx = -4\int \frac{y^2}{(1-y^2)(1+y^2)}dy$$
$$= -2\int \left(\frac{1}{1-y^2} - \frac{1}{1+y^2}\right)dy = -2\int \left[\frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right) - \frac{1}{1+y^2}\right]dy$$
$$= -2\left[\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| - \tan^{-1}y\right] + c$$
$$= 2\tan^{-1}\sqrt{\frac{1-x}{1+x}} - \ln\left|\frac{1+\sqrt{\frac{1-x}{1+x}}}{1-\sqrt{\frac{1-x}{1+x}}}\right| + c$$
$$= 2\tan^{-1}\sqrt{\frac{1-x}{1+x}} - \ln\left|\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right| + c$$

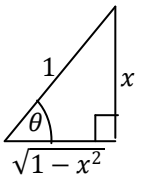
NOTE

1) For $\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx$, $x = \cos 2\theta$ or $x = \cos\theta$ can evaluate the integral

2) Also $\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx$ can be handled by Rationalization of the numerator. i.e.

$\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx = \int \frac{1-x}{x\sqrt{1-x^2}}dx$, let $x = \sin\theta$ or $dx = \cos\theta\ d\theta$

$$\therefore \int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx = \int \frac{1-\sin\theta}{\sin\theta\sqrt{1-\sin^2\theta}}\cos\theta\ d\theta$$
$$= \int \frac{1-\sin\theta}{\sin\theta}d\theta = \int(\operatorname{cosec}\theta - 1)\ d\theta$$
$$= \ln|\operatorname{cosec}\theta - \cot\theta| - \theta + c$$



But $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{x}$, $\cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{1-x^2}}{x}$

$$\therefore \int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx = \ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| - \sin^{-1}x + c$$
$$= \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| - \sin^{-1}x + c$$

Example

Integrate

a) $\int \sqrt{\frac{1+x}{1-x}}dx$

b) $\int \sqrt{\frac{x-1}{2-x}}dx$

c) $\int \sqrt{\frac{4-3x}{5x+2}}dx$ **Ans:** $\tan^{-1}\sqrt{\frac{3}{5}}\sqrt{\frac{4-3x}{5x+2}} + c$ use $t^2 = \frac{4-3x}{5x+2}$

d) $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$ **Ans:** $\sqrt{x-x^2} - 2\sqrt{1-x} - \sin^{-1}\sqrt{x} + c$

e) $\int \sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}dx$ **Hint:** use $\sqrt{x} = \cos\theta$

f) $\int \sqrt{\frac{1-x}{1+x}}dx$ **Ans:** $\sin^{-1}x + \sqrt{1-x^2} + c$

g) $\int \sqrt{\frac{x}{a-x}}dx$

h) $\int \frac{\sqrt{x}}{\sqrt{x}+2}\,dx$ **Ans:** $x - 4\sqrt{x} + 8\ln\left|\sqrt{x}+2\right| + c$

i) $\int_0^2 \sqrt{\frac{2+x}{2-x}}dx$ **Ans:** $+2$

j) $\int \sqrt{\frac{5-x}{x-2}}dx$ **Ans:** $\sqrt{7x-x^2-10} + 3\sin^{-1}x + \sqrt{1-x^2} + c$;

Use $t = \sqrt{x-2}$

k) $\int \sqrt{\frac{x-2}{5-x}}dx$ **Ans:** $-\left[\sqrt{7x-x^2-10} + 3\sin^{-1}x + \sqrt{1-x^2}\right] + c$;

Use $t = \sqrt{5-x}$

l) $\int \frac{dx}{\sqrt{1+x}+\sqrt{x}}$ **Ans:** $\frac{2}{3}\sqrt{(1+x)^3} - \frac{2}{3}\sqrt{x^3} + c$; Rationalize

m) $\int_0^1 x^3\sqrt{\frac{1+x^2}{1-x^2}}\,dx$ **Ans:** $\frac{\pi}{8}\ln 2$; use $x^2 = \cos 2\theta$

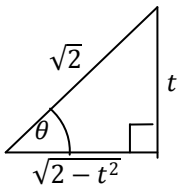
n) Using the substitution $x = 3\sin\theta$, evaluate $\int_0^3 \sqrt{\frac{3+x}{3-x}}\,dx$ (**Uneb 2001 16(c)**)

More techniques of handling $\int \sqrt{\frac{1-x}{1+x}}dx$

Alternative I

Let $t^2 = 1+x \implies 1-x = 2-t^2$ and $dx = 2t\,dt$

$$\int \sqrt{\frac{1-x}{1+x}}dx = \int \sqrt{\frac{2-t^2}{t^2}}.2t\,dt = 2\int \sqrt{2-t^2}\,dt$$



Let $t = \sqrt{2}\sin\theta \implies dt = \sqrt{2}\cos\theta\ d\theta$

$$2\int \sqrt{2-t^2}\,dt = 2\int \sqrt{2-2\sin^2\theta}\,.\sqrt{2}\cos\theta\,d\theta$$

$$= 4\int \cos^2\theta\ d\theta = 2\int(1+\cos 2\theta)\ d\theta$$

$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right] + c = 2\theta + \sin 2\theta + c$$

$$= 2\theta + 2\sin\theta\cos\theta + c$$

$$= 2\sin^{-1}\left(\frac{t}{\sqrt{2}}\right) + \frac{2t}{\sqrt{2}}.\frac{\sqrt{2-t^2}}{\sqrt{2}} + c = 2\sin^{-1}\left(\frac{t}{\sqrt{2}}\right) + t\sqrt{2-t^2} + c$$

But $t = \sqrt{1+x}$, $1-x = 2-t^2$

$$\therefore \int \sqrt{\frac{1-x}{1+x}}dx = 2\sin^{-1}\left(\frac{\sqrt{1+x}}{\sqrt{2}}\right) + \sqrt{1+x}.\sqrt{1-x} + c$$

$$= 2\sin^{-1}\left(\sqrt{\frac{1+x}{2}}\right) + \sqrt{1-x^2} + c$$

Simplifying using trigonometry

Let $y = 2\sin^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$, $\theta = \sin^{-1}\left(\sqrt{\frac{1+x}{2}}\right) \implies y = 2\theta$

$$\sin y = \sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta}$$

$$= 2\sqrt{\frac{1+x}{2}}.\sqrt{1-\frac{1+x}{2}} = 2\sqrt{\frac{1-x^2}{4}} = \sqrt{1-x^2}$$

$$\therefore \int \sqrt{\frac{1-x}{1+x}}\,dx = \sin^{-1}\sqrt{1-x^2} + \sqrt{1-x^2} + c$$

Alternative II

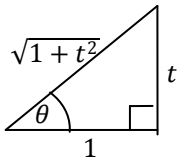
Let $t^2 = \frac{1-x}{1+x} \Longrightarrow t^2 + t^2x = 1 - x \quad \therefore x = \frac{1-t^2}{1+t^2}$

Now $2t\,dt = \frac{-(1+x)-(1-x)}{(1+x)^2}dx = -\frac{2}{(1+x)^2} \quad \therefore dx = -t\,(1+x)^2 = -t\left(1+\frac{1-t^2}{1+t^2}\right)$

$$\therefore \int \sqrt{\frac{1-x}{1+x}}\,dx = -\int t.t\left(1+\frac{1-t^2}{1+t^2}\right)^2 dt$$

$$= -\int \frac{4t^2}{(1+t^2)^2}\,dt$$

Let $t = \tan\theta \quad \Longrightarrow dt = \sec^2\theta\ d\theta$



$$\therefore \int \sqrt{\frac{1-x}{1+x}}\,dx = -\int \frac{4t^2}{(1+t^2)^2}\,dt = \int \frac{-4\tan^2\theta}{(1+\tan^2\theta)^2}.\sec^2\theta\ d\theta$$

$$= \int \frac{-4\tan^2\theta}{\sec^2\theta}d\theta = \int \frac{-4\sin^2\theta}{\cos^2\theta}.\cos^2\theta\,d\theta$$

$$= -4\int \sin^2\theta\,d\theta = -4\int \frac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= -2\left[\theta - \frac{1}{2}\sin 2\theta\right] + c = -2\theta + \sin 2\theta + c$$

$$= -2\theta + 2\sin\theta\cos\theta + c$$

$$= -2\tan^{-1}t + \frac{2t}{\sqrt{1+t^2}}.\frac{1}{\sqrt{1+t^2}} + c$$

$$= -2\tan^{-1}t + \frac{2t}{1+t^2} + c$$

$$= -2\tan^{-1}\sqrt{\frac{1-x}{1+x}} + \frac{2\sqrt{\frac{1-x}{1+x}}}{1+\frac{1-x}{1+x}} + c$$

$$= -2\tan^{-1}\sqrt{\frac{1-x}{1+x}} + \frac{2\sqrt{1-x}.\sqrt{1+x}}{2} + c$$

$$= -2\tan^{-1}\sqrt{\frac{1-x}{1+x}} + \sqrt{1-x^2} + c$$

$$\therefore \int \sqrt{\frac{1-x}{1+x}}\,dx = -\sin^{-1}\sqrt{1-x^2} + \sqrt{1-x^2} + c$$

<u>By Trigonometry</u>

Let $y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$, $z = 2\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

$\tan y = \sqrt{\frac{1-x}{1+x}}$

$z = 2y \Longrightarrow \sin z = \sin 2y = \frac{2\tan y}{1+\tan^2 y}$

$\sin z = 2\frac{\sqrt{\frac{1-x}{1+x}}}{1+\frac{1-x}{1+x}} = \sqrt{1-x^2}$

Alternative III

By Rationalization

$$\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}x + \frac{1}{2}\int(-2x).(1-x^2)^{-1/2}\,dx$$

$$= \sin^{-1}x + \frac{1}{2}.\frac{(1-x^2)^{1/2}}{1/2} + c$$

$$\therefore \int \sqrt{\frac{1-x}{1+x}}\,dx = \sin^{-1}\sqrt{1-x^2} + \sqrt{1-x^2} + c$$

Alternative IV

$\int \sqrt{\frac{1-x}{1+x}}dx$, let $x = \cos 2\theta \Longrightarrow dx = -2\sin 2\theta \ d\theta$

$$\int \sqrt{\frac{1-x}{1+x}}dx = \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}.-2\sin 2\theta \ d\theta$$

$$= -\int \sqrt{\tan^2\theta}\,.-2.(2\sin\theta\cos\theta)d\theta = -\int \sqrt{\tan^2\theta}\,.-2.(2\sin\theta\cos\theta)d\theta$$

$$= -\int \frac{\sin\theta}{\cos\theta}.-2.(2\sin\theta\cos\theta)d\theta = -2\int 2\sin^2\theta\, d\theta$$

$$= -2\int(1-\cos 2\theta)d\theta$$

$$= -2\left[\theta - \frac{1}{2}\sin 2\theta\right] + c = -2\theta + \sin 2\theta + c$$

$$= -2\left[\frac{\cos^{-1}x}{2} - \frac{1}{2}\sqrt{1-\cos^2 2\theta}\right] + c$$

$$= -\cos^{-1}x + \sqrt{1-x^2} + c$$

From trigonometry

Let $y = \cos^{-1}x \Longrightarrow \cos y = x$ and $z = -\cos^{-1}x \quad \therefore z = -y$

$\sin z = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$

$\therefore \int \sqrt{\frac{1-x}{1+x}}dx = -\sin^{-1}\sqrt{1-x^2} + \sqrt{1-x^2} + c$

Comparing the above approaches, Rationalization leads us to the answer easily.

Integrals of the type $\int \frac{dx}{X\sqrt{Y}}$, where X and Y are Linear or Quadratic Functions of x

Four cases are to be considered

Case I: Rule to integrate $\int \frac{1}{Linear\ .\sqrt{Linear}}dx$

Put $\sqrt{Linear} = t$ or $Linear = t^2$

Case II: Rule to integrate $\int \frac{1}{Quadratic\ .\sqrt{Linear}}dx$

Put $\sqrt{Linear} = t$ or $Linear = t^2$

Case III: Rule to integrate $\int \frac{1}{Linear\ .\sqrt{Quadratic}}dx$

Put $Linear = \frac{1}{t}$

NOTE

In all the above cases if instead of *unity* in the numerator there is a polynomial in x whose degree is equal or greater than that of the rational factor of the denominator, first divide the numerator by this rational factor, and then use the rule, where necessary.

Case IV: Rule to integrate $\int \frac{1}{Pure\ Quadratic\ .\sqrt{Pure\ Quadratic}}dx$

Put $x = \frac{1}{t}$ and then in the resulting integral put $u = \sqrt{Quadratic}$

Example

a) $\int \frac{dx}{(x+2)\sqrt{x+3}}$ b) $\int \frac{x^2-x+1}{(x+1)\sqrt{x-2}}dx$

Solution

a) Put $x+3=t^2 \implies x=t^2-3,\ dx=2t\,dt$

$$\int \frac{dx}{(x+2)\sqrt{x+3}}=\int \frac{2t\,dt}{(t^2-1).t}=\int \frac{2}{(t+1)(t-1)}dt$$

$$=\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt=\ln|t-1|-\ln|t+1|+c$$

$$=\ln\left|\frac{t-1}{t+1}\right|+c=\ln\left|\frac{\sqrt{x+3}-1}{\sqrt{x+3}+1}\right|+c$$

b) Here the degree of the numerator x^2-x+1 is greater than that of the rational factor $(x+1)$ of the denominator. Thus we divide x^2-x+1 by $x+1$

$$\frac{x^2-x+1}{x+1}=x-2+\frac{3}{x+1}$$

$$\therefore \int \frac{x^2-x+1}{(x+1)\sqrt{x-2}}dx=\int\left(x-2+\frac{3}{x+1}\right).\frac{1}{\sqrt{x-2}}dx$$

$$=\int\sqrt{x-2}\ dx+3\int\frac{1}{(x+1)\sqrt{x-2}}dx=\frac{(x-2)^{\frac{1}{2}+1}}{\frac{1}{2}+1}+3\int\frac{1}{(x+1)\sqrt{x-2}}dx$$

For the integral $\int \frac{1}{(x+1)\sqrt{x-2}}dx$, put $x-2=t^2 \implies dx=2t\,dt$

$$\therefore \int \frac{1}{(x+1)\sqrt{x-2}}dx=\int \frac{1}{(t^2+3).t}.2t\,dt=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right)$$

$$\therefore \int \frac{x^2-x+1}{(x+1)\sqrt{x-2}}dx=\frac{2}{3}(x-2)^{3/2}+3.\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right)+c$$

$$\therefore \int \frac{x^2-x+1}{(x+1)\sqrt{x-2}}dx=\frac{2}{3}(x-2)^{3/2}+2\sqrt{3}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right)+c$$

Task

a) $\int \frac{1}{x\sqrt{1-x}}\ dx$ **Ans:** $\ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right|+c$

b) $\int \frac{\sqrt{x}}{1+x}\ dx$ **Ans:** $2\left(\sqrt{x}-\tan^{-1}\sqrt{x}\right)+c$

c) $\int \frac{x+3}{\sqrt{x-3}}\ dx$ **Ans:**

d) $\int_8^{15}\int \frac{1}{(x-3)\sqrt{x+1}}\ dx$ **Ans:** $\frac{\ln 5-\ln 3}{2}$

e) $\int_3^8\int \frac{2-3x}{x\sqrt{x+1}}\ dx$ **Ans:** $2\ln(3/2e^3)$

f) $\int \frac{dx}{(x-3)\sqrt{x+1}}$ **Ans:** $\frac{1}{2}\ln\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+c$

g) $\int_0^1 \frac{dx}{(x+1)\sqrt{x^2+2x}}$ **Ans:** $\frac{\pi}{3}$

h) $\int \frac{dx}{(x+1)\sqrt{x^2+1}}$ **Ans:** $\sqrt{\frac{x-1}{x+1}}+c$

Integration of Trigonometric Functions

Integration of $\int \frac{dx}{a\sin x+b\cos x}$

Let $a\sin x + b\cos x = R\sin(x+\alpha)$

$= R\sin x\cos\alpha + R\cos x\sin\alpha$

Comparing

For $\sin x$: $a = R\cos\alpha$ ..... (1)

$\cos x$: $a = R\sin\alpha$ ..... (2)

Solving for R and α

Squaring the equations and adding gives $R = \sqrt{a^2+b^2}$

Dividing eqn. (2) by eqn. (1) gives $\tan\alpha = \frac{b}{a}$ or $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

$\therefore \int \frac{dx}{a\sin x+b\cos x} = \frac{1}{R}\int \frac{dx}{\sin(x+\alpha)} = \frac{1}{R}\int \operatorname{cosec}(x+\alpha)\,dx$

$= \frac{1}{R}\ln\left|\tan\frac{x+\alpha}{2}\right| + c = \frac{1}{R}\ln\left|\tan\frac{1}{2}\left(x+\tan^{-1}\left(\frac{b}{a}\right)\right)\right| + c$

$\therefore \int \frac{dx}{a\sin x+b\cos x} = \frac{1}{R}\ln\left|\tan\frac{1}{2}\left(x+\tan^{-1}\left(\frac{b}{a}\right)\right)\right| + \mathrm{c}$

Example

a) $\int \frac{dx}{2\sin x-\cos x}$ b) $\int \frac{dx}{5\sin x+12\cos x}$

Solution

a) Let $2\sin x - \cos x = R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

Comparing

$2 = R\cos\alpha$, $-1 = R\sin\alpha$

Solving $R = \sqrt{5}$, $\tan\alpha = -\frac{1}{2}$ or $\alpha = \tan^{-1}(-1/2) = -\tan^{-1}(1/2)$

$\therefore \int \frac{dx}{2\sin x-\cos x} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sin(x+\tan^{-1}(-1/2))}$

$= \frac{1}{\sqrt{5}}\int \operatorname{cosec}(x+\tan^{-1}(-1/2))\,dx = \frac{1}{\sqrt{5}}\ln\left|\tan\left(\frac{x-\tan^{-1}(1/2)}{2}\right)\right| + c$

b) Left as an exercise. **Ans:** $\frac{1}{13}\ln\left|\tan\left(\frac{x+\tan^{-1}(12/5)}{2}\right)\right| + c$

Alternative Approach

Using the $t-$ substitution $t = \tan\frac{x}{2}$

$\int \frac{dx}{5\sin x+12\cos x} = \int \frac{dx}{10\sin\frac{x}{2}\cos\frac{x}{2}+12\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$

Divide the numerator and denominator by $\cos^2\frac{x}{2}$

$\int \frac{dx}{5\sin x+12\cos x} = \int \frac{\sec^2\frac{x}{2}dx}{10\tan\frac{x}{2}+12-\tan^2\frac{x}{2}}$, let $t = \tan\frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}dx$

$\therefore \int \frac{dx}{5\sin x+12\cos x} = \int \frac{2\,dt}{10t+12-12t^2} = \int \frac{dt}{6+5t-6t^2}$

Since $B^2-4AC < 0$, completing squares

$6+5t-6t^2 \equiv -6\left(t^2-\frac{5}{6}t-1\right) = -6\left(\left(t-\frac{5}{12}\right)^2-\frac{25}{144}-1\right) = 6\left(\frac{169}{144}-\left(t-\frac{5}{12}\right)^2\right)$

$\int \frac{dx}{5\sin x+12\cos x} = \frac{1}{6}\int \frac{dt}{\frac{169}{144}-\left(t-\frac{5}{12}\right)^2}$, taking $a^2 = \frac{169}{144}$, $u^2 = \left(t-\frac{5}{12}\right)^2$

$\therefore \int \frac{dx}{5\sin x+12\cos x} = \frac{1}{6}\int \frac{du}{a^2-u^2} = \frac{1}{6}\int \frac{du}{(a+u)(a-u)}$

By partial fractions

$\frac{1}{(a+u)(a-u)} = \frac{A}{a+u} + \frac{B}{a-u}$

$1 = A(a-u) + B(a+u)$, solving for A and B

$1 = Aa + Ba$..... (1)

$0 = -A + B$..... (2)

Adding the equations gives $\frac{1}{a} = 2B$, $B = \frac{1}{2a}$ also $A = B = \frac{1}{2a}$

$\therefore \int \frac{du}{a^2-u^2} = \int \frac{du}{(a+u)(a-u)} = \frac{1}{2a}\int \left(\frac{A}{a+u} + \frac{B}{a-u}\right) du$

$= \frac{1}{2a}\ln|a+u| - \ln|a-u| = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + c$

$\therefore \int \frac{dx}{5\sin x+12\cos x} = \frac{1}{6}.\frac{1}{2(13/12)}\ln\left|\frac{\frac{13}{12}+t-\frac{5}{12}}{\frac{13}{12}-t+\frac{5}{12}}\right| + c$

$= \frac{1}{13}\ln\left|\frac{8+12t}{18-12t}\right| = \frac{1}{13}\ln\left|\frac{4+6t}{9-6t}\right| + c$

$\therefore \int \frac{dx}{5\sin x+12\cos x} = \frac{1}{13}\ln\left|\frac{2/3+t}{3/2-t}\right| + c = \frac{1}{13}\ln\left|\tan\left(\frac{x+\tan^{-1}(12/5)}{2}\right)\right| + c$

Integrals of Rational Functions of $\sin x$ and $\cos x$

To integrate $\int \frac{dx}{a+b\cos x}$

$\int \frac{dx}{a+b\cos x} = \int \frac{dx}{a+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)} = \int \frac{\sec^2\frac{x}{2}dx}{a\sec^2\frac{x}{2}+b\left(1-\tan^2\frac{x}{2}\right)} \int \frac{\sec^2\frac{x}{2}dx}{a+a\tan^2\frac{x}{2}+b\tan^2\frac{x}{2}}$

Let $t = \tan\frac{x}{2} \Longrightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}dx$

$\therefore \int \frac{dx}{a+b\cos x} = \int \frac{2\,dt}{(a+b)+(a-b)t^2}$

$= \frac{2}{a-b}\int \frac{dt}{\frac{a+b}{a-b}+t^2}$

Case I: $a > b$

In this case $\frac{a+b}{a-b}$ is positive

$\therefore \frac{2}{a-b}\int \frac{dt}{\frac{a+b}{a-b}+t^2} = \frac{2}{a-b}.\frac{1}{\sqrt{\frac{a+b}{a-b}}}\tan^{-1}\left(\frac{t}{\sqrt{\frac{a+b}{a-b}}}\right) + c$

$\therefore \int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}.\tan^{-1}\left(\sqrt{\frac{a+b}{a-b}}.\tan\frac{x}{2}\right) + c$

NOTE

$\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}.\left|\tan^{-1}\left(\sqrt{\frac{a+b}{a-b}}.\tan\frac{x}{2}\right)\right|_0^{\pi}$

$= \frac{2}{\sqrt{a^2-b^2}}[\tan^{-1}\infty - \tan^{-1}0] = \frac{2}{\sqrt{a^2-b^2}}.\frac{\pi}{2}$

$= \frac{\pi}{\sqrt{a^2-b^2}}$

Case II: $a < b$

In this case $\frac{a+b}{a-b}$ is negative

$$\therefore \int \frac{dx}{a+b\cos x} = \int \frac{2\,dt}{(a+b)-(b-a)t^2}$$

$$= \frac{2}{b-a}\int \frac{dt}{\frac{a+b}{b-a} - t^2}$$

$$= \frac{2}{b-a}.\frac{1}{2.\sqrt{\frac{a+b}{b-a}}} \ln\left|\frac{\sqrt{\frac{a+b}{b-a}}+t}{\sqrt{\frac{a+b}{b-a}}-t}\right| + c$$

$$= \frac{1}{\sqrt{b^2-a^2}} \ln\left|\frac{\sqrt{a+b}+\sqrt{b-a}.t}{\sqrt{a+b}-\sqrt{b-a}.t}\right| + c$$

$$= \frac{1}{\sqrt{b^2-a^2}} \ln\left|\frac{\sqrt{a+b}+\sqrt{b-a}.\tan\frac{x}{2}}{\sqrt{a+b}-\sqrt{b-a}.\tan\frac{x}{2}}\right| + c$$

Thus we get $\int \frac{dx}{a+b\cos x}$

$$\therefore \int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}.\tan^{-1}\left(\sqrt{\frac{a+b}{a-b}}.\tan\frac{x}{2}\right) + c \quad (\text{where a} > b)$$

$$= \frac{1}{\sqrt{b^2-a^2}} \ln\left|\frac{\sqrt{a+b}+\sqrt{b-a}.\tan\frac{x}{2}}{\sqrt{a+b}-\sqrt{b-a}.\tan\frac{x}{2}}\right| + c \quad (\text{where a} > b)$$

To integrate $\int \frac{dx}{a+b\sin x}$

Putting $t = \tan\frac{x}{2} \Longrightarrow dt = \frac{1}{2}\sec^2\frac{x}{2}dx$, $\sin x = \frac{2t}{1+t^2}$

$$\int \frac{dx}{a+b\sin x} = \int \frac{1}{a+b\left(\frac{2t}{1+t^2}\right)}.\frac{2\,dt}{1+t^2}$$

$$= 2\int \frac{dt}{at^2+2bt+a} = \frac{2}{a}\int \frac{dt}{t^2+\frac{2b}{a}t+1}$$

$$= \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2+\frac{a^2-b^2}{a^2}}$$

Case I: $a^2 > b^2$

$$\int \frac{dx}{a+b\sin x} = \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2+\left(\frac{\sqrt{a^2-b^2}}{a}\right)^2}$$

$$= \frac{2}{a}.\frac{1}{\sqrt{a^2-b^2}/a}\tan^{-1}\left(\frac{t+b/a}{\sqrt{a^2-b^2}/a}\right) + c$$

$$\therefore \int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}.\tan^{-1}\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}} + c$$

Case II: $a^2 < b^2$

$$\int \frac{dx}{a+b\sin x} = \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2+\left(\frac{a^2-b^2}{a^2}\right)}$$

$$= \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2-\left(\frac{b^2-a^2}{a^2}\right)}$$

$$= \frac{2}{a}\int \frac{dt}{\left(t+\frac{b}{a}\right)^2-\left(\frac{\sqrt{b^2-a^2}}{a}\right)^2}$$

$$= \frac{2}{a}.\frac{1}{2\frac{\sqrt{b^2-a^2}}{a}}.\ln\left|\frac{t+\frac{b}{a}-\frac{\sqrt{b^2-a^2}}{a}}{t+\frac{b}{a}+\frac{\sqrt{b^2-a^2}}{a}}\right|+c$$

$$= \frac{1}{\sqrt{b^2-a^2}}.\ln\left|\frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right|+c$$

Evaluate

Evaluate

a) $\int \frac{dx}{5+4\cos x}$ b) $\int_0^{\pi/2} \frac{dx}{4+5\cos x}$ c) $\int \frac{dx}{5+4\sin x}$

Solution

a) $\int \frac{dx}{5+4\cos x}$, put $t=\tan\frac{x}{2} \Longrightarrow dt=\frac{1}{2}\sec^2\frac{x}{2}dx$, $\cos x=\frac{1-t^2}{1+t^2}$

$$\Longrightarrow \int \frac{dx}{5+4\cos x} = \int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)}.\frac{2\,dt}{1+t^2}$$

$$= 2\int \frac{dt}{9+t^2} = 2.\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)+c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan x/2}{3}\right)+c$$

NOTE:

$$\int_0^{\pi/2} \frac{dx}{5+4\cos x} = \left|\frac{2}{3}\tan^{-1}\left(\frac{\tan x/2}{3}\right)\right|_0^{\pi/2} = \left(\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)-\tan^{-1}(0)\right) = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$$

b) $\int_0^{\pi/2} \frac{dx}{4+5\cos x} = \int_0^{\pi/2} \frac{dx}{4+5\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$

$$= \int_0^{\pi/2} \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{4+4\tan^2\frac{x}{2}+5-5\tan^2\frac{x}{2}} = \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}dx}{9-\tan^2\frac{x}{2}}$$

Let $t=\tan\frac{x}{2} \Longrightarrow dt=\frac{1}{2}\sec^2\frac{x}{2}dx \;\therefore 2\,dt=\sec^2\frac{x}{2}dx$

Changing limits

When $x=0$, $t=0$ and when $x=\frac{\pi}{2}$, $t=1$

$$\therefore \int_0^{\pi/2} \frac{dx}{4+5\cos x} = \int_0^1 \frac{2\,dt}{9-t^2} = \int_0^1 \frac{2\,dt}{(3+t)(3-t)}$$

$$= 2.\frac{1}{3.2}\ln\left|\frac{3+t}{3-t}\right|_0^1 = \frac{1}{3}\left(\log\frac{4}{2}-\log 1\right)$$

$$= \frac{1}{3}\ln 2$$

c) Let $t=\tan\frac{x}{2} \Longrightarrow dt=\frac{1}{2}\sec^2\frac{x}{2}dx \;\therefore \frac{2\,dt}{1+t^2}=dx$ and $\sin x=\frac{2t}{1+t^2}$

$$\int \frac{dx}{5+4\sin x} = \int \frac{1}{5+4\left(\frac{2t}{1+t^2}\right)}.\frac{2\,dt}{1+t^2}$$

$$= 2\int \frac{dt}{5t^2+8t+5} = \frac{2}{5}\int \frac{dt}{(t+4/5)^2+(3/5)^2}$$

$$= \frac{2}{5}.\frac{1}{3/5}\tan^{-1}\left[\frac{5\tan x/2+4}{3}\right]$$

Task

$\int_{\pi/3}^{\pi/2} \frac{dx}{4-5\cos x}$ **Ans:** $\frac{1}{3}\ln\left|\frac{\sqrt{3}+1}{2(\sqrt{3}-1)}\right|$

Example

Evaluate

a) $\int \frac{dx}{3\cos x-4\sin x+5}$ b) $\int \frac{dx}{3\cos x+4\sin x+6}$

Solution

a) Putting $t=\tan\frac{x}{2} \Longrightarrow dt=\frac{1}{2}\sec^2\frac{x}{2}dx \ \therefore dx=\frac{2\,dt}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$

$$\int \frac{dx}{3\cos x-4\sin x+5}=\int \frac{1}{3\left(\frac{1-t^2}{1+t^2}\right)-4\left(\frac{2t}{1+t^2}\right)+5}\cdot\frac{2\,dt}{1+t^2}$$

$$=\int \frac{dt}{t^2-t+4}=\int \frac{dt}{(t-2)^2}$$

$$=\int (t-2)^{-2}\,dt=\frac{(t-2)^{-2+1}}{-2+1}=-\frac{1}{t-2}$$

$$\therefore \int \frac{dx}{3\cos x-4\sin x+5}=-\frac{1}{\tan\frac{x}{2}-2}+c=\frac{1}{2-\tan\frac{x}{2}}+c$$

b) $3\cos x+4\sin x+6=3\left(\frac{1-t^2}{1+t^2}\right)+\frac{8t}{1+t^2}+6$

$$=\frac{3-3+8t+6+6t^2}{1+t^2}=\frac{3t^2+8t+9}{1+t^2}$$

Putting $t=\tan\frac{x}{2} \Longrightarrow dt=\frac{1}{2}\sec^2\frac{x}{2}dx \ \therefore dx=\frac{2\,dt}{1+t^2}$

$$\int \frac{dx}{3\cos x+4\sin x+6}=\int \frac{2\,dt}{3t^2+8t+9}$$

$$=\frac{2}{3}\int \frac{dt}{t^2+(8/3)t+3}=\frac{2}{3}\int \frac{dt}{(t+4/3)^2+11/9}$$

$$=\frac{2}{3}\times\frac{1}{\sqrt{11}/3}\tan^{-1}\left(\frac{t+4/3}{\sqrt{11}}\right)+c$$

$$=\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{t+4/3}{\sqrt{11}}\right)+c$$

$$\therefore \int \frac{dx}{3\cos x+4\sin x+6}=\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{3\tan(x/2)+4}{3\sqrt{11}}\right)+c$$

To integrate $\int \frac{a\cos x+b\sin x}{c\cos x+d\sin x}dx$

Here the Numerator $= l$ (Denominator) + m(Diff. coeff. of Denominator) where l and m are the constants to be found

$$\Longrightarrow a\cos x+b\sin x=l(c\cos x+d\sin x)+\mathrm{m}(-c\sin x+d\cos x)$$

$$=(cl+dm)\cos x+(dl-cm)\sin x$$

Equating coefficients

For $\cos x$: $a=cl+dm$..... (1)

$\sin x$: $b=dl-cm$..... (2)

Solving eqn. (1) and eqn. (2)

$l=\frac{ac+bd}{c^2+d^2}$, $m=\frac{ad-bc}{c^2+d^2}$

$$\therefore \int \frac{a\cos x+b\sin x}{c\cos x+d\sin x}dx=\int \frac{l(c\cos x+d\sin x)+\mathrm{m}(-c\sin x+d\cos x)}{c\cos x+d\sin x}dx$$

$$=\int\left(l+m.\frac{-c\sin x+d\cos x}{c\cos x+d\sin x}\right)dx$$

$$\therefore \int \frac{a\cos x+b\sin x}{c\cos x+d\sin x}dx=lx+m\ln|c\cos x+d\sin x|+c$$

Example

Evaluate $\int \frac{26\sin x-13\cos x}{4\sin x-7\cos x}dx$

Solution

Let $26\sin x - 13\cos x = l(4\sin x - 7\cos x) + m(4\cos x + 7\sin x)$

$= (4l + 7m)\sin x + (4m - 7l)\cos x$

Equating coefficients

For $\sin x$: $26 = 4l + 7m$

$\cos x$: $-13 = -7 + 4m$

Solving equations: $l = 3\ ,\ m = 2$

$\therefore \int \frac{26\sin x-13\cos x}{4\sin x-7\cos x}dx = \int \frac{l(4\sin x-7\cos x)+m(4\cos x+7\sin x)}{4\sin x-7\cos x}dx$

$= \int \left(l + m.\frac{4\cos x+7\sin x}{4\sin x-7\cos x}\right)dx$

$\therefore \int \frac{26\sin x-13\cos x}{4\sin x-7\cos x}dx = lx + m\ln|4\sin x - 7\cos x| + c$

To integrate $\int \frac{1}{a^2\cos^2 x+b\sin^2 x}dx$

Divide the Numerator and Denominator by $\cos^2 x$

$\int \frac{1}{a^2\cos^2 x+b^2\sin^2 x}dx = \int \frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x}$

Now put $t = \tan x \implies dt = \sec^2 x\,dx$

$\int \frac{1}{a^2\cos^2 x+b\sin^2 x}dx = \int \frac{dt}{a^2+b^2t^2} = \frac{1}{b^2}\int \frac{dt}{t^2+\frac{a^2}{b^2}}$

$= \frac{1}{b^2}.\frac{1}{a/b}\tan^{-1}\left(\frac{t}{a/b}\right) + c = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + c$

$\therefore \int \frac{1}{a^2\cos^2 x+b\sin^2 x}dx = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + c$

Example

a) $\int \frac{dx}{1+3\sin^2 x}$

b) $\int \frac{dx}{1+\cos^2 x}$

c) $\int \frac{dx}{9\sin^2 x+4\cos^2 x}$

d) $\int_0^{\pi/2} \frac{dx}{9+16\cos^2 x}$

Solution

a) $\int \frac{dx}{1+3\sin^2 x} = \int \frac{\sec^2 x\,dx}{\sec^2 x+3\tan^2 x} = \int \frac{\sec^2 x\,dx}{1+4\tan^2 x}$

Let $t = \tan x \implies dt = \sec^2 x\,dx$

$\int \frac{dx}{1+3\sin^2 x} = \int \frac{dt}{1+4t^2}$

$= \frac{1}{4}\int \frac{dt}{t^2+1/4} = \frac{1}{4}\int \frac{dt}{t^2+(1/2)^2}$

$= \frac{1}{4}.\frac{1}{1/2}\tan^{-1}\left(\frac{t}{1/2}\right) + c$

$\therefore \int \frac{dx}{1+3\sin^2 x} = \frac{1}{2}\tan^{-1}(2\tan x) + c$

b) Left as an exercise **Ans:** $\frac{1}{2}\tan^{-1}\left[\frac{1}{\sqrt{2}}\tan x\right]$

c) Left as an exercise **Ans:** $\frac{1}{6}\tan^{-1}\left[\frac{3}{2}\tan x\right]$

d) Left as an exercise **Ans:** $\frac{\pi}{30}$

Integration by Parts

This technique applies to a wide variety of functions and is particularly useful for integrands involving a *product* of algebraic and functions involving trigonometry. E.g. $\int x^n \ln x\, dx$, $\int x^n e^{ax}\, dx$ and $\int x^n \sin mx\, dx$

Statement

If u and v are functions of x, then

$\int u.v\, dx = u\int v\, dx - \int \frac{du}{dx}.(\int v\, dx)\, dx$ or $\int u\, dv = uv - \int v\, dx$

Proof

Integration by parts is based on the formula for the derivative of a product.

$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$

Integrating both sides

$\int \frac{d}{dx}(u.v).dx = \int u\frac{dv}{dx}.dx + \int v\frac{du}{dx}.dx$

$\therefore u.v = \int u.dv + \int v.du$

Rewriting

$$\int u.dv = u.v - \int v.du$$

NOTE

From $\int \frac{d}{dx}(u.v).dx = \int u\frac{dv}{dx}.dx + \int v\frac{du}{dx}.dx$,

Let $w = \int \frac{dv}{dx} \Longrightarrow v = \int w\, dx$

$\therefore \int u.w.dx = u\int w\, dx - \int(\int w\, dx).\frac{du}{dx}.dx$

Guidelines for Integration by Parts

The choices of u and dv in the integration by parts, because it easens the evaluation of second integral than the original one.

1) Try letting dv be the most complicated portion of the integrand that fits the basic integration rule. Then u be the remaining factors of the integrand.
2) Try letting u be the portion of the integrand whose derivative is a simpler function than dv then dv will be the remaining factors of the integrand.

Example

To integrate $\int x^n e^{ax}\, dx$, $\int x^n \sin ax\, dx$ or $\int x^n \cos ax\, dx$

Let $u = x^n$ and $dv = e^{ax}\, dx, \sin ax\ ,\ \cos ax$

Integrate

a) $\int x^2.e^{2x}\, dx$ b) $\int x^2.\sin x\, dx$ c) $\int_0^{\pi/2} x\cos x\, dx$

Solution

a) Let $u = x^2 \Longrightarrow \frac{du}{dx} = 2x$, $\frac{dv}{dx} = e^{2x}$ or $v = \int e^{2x}\, dx = \frac{e^{2x}}{2}$

$$\therefore \int x^2.e^{2x}\,dx = \frac{x^2e^{2x}}{2} - \int x.e^{2x}\,dx$$
$$= \frac{x^2e^{2x}}{2} - \left[\frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx\right]$$
$$\therefore \int x^2.e^{2x}\,dx = \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + c = \frac{e^{2x}}{4}(2x^2 - 2x + 1) + c$$

b) Let $u = x^2 \implies \frac{du}{dx} = 2x$, $\frac{dv}{dx} = \sin x$ or $v = \int \sin x\,dx = -\cos x$

$$\therefore \int x^2.\sin x\,dx = -x^2\cos x - \int -\cos x\,.2x\,dx$$
$$= -x^2\cos x + 2\int x.\cos x\,dx$$
$$= -x^2\cos x + 2[x\sin x - \int \sin\, x\,dx]$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

c) Let $u = x \implies \frac{du}{dx} = 1$, $\frac{dv}{dx} = \cos x$ or $v = \int \cos x\,dx = \sin x$

$$\int_0^{\pi/2} x\cos x\,dx = [x\sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x\,dx$$
$$= \left[\frac{\pi}{2}\sin\frac{\pi}{2} - 0\right] + [\cos x]_0^{\pi/2}$$
$$= \frac{\pi}{2} + \cos\frac{\pi}{2} - \cos 0 = \frac{\pi}{2} - 1$$

NOTE

Tabular Method

For problems involving repeated application of integration by parts, tabular method can help organize work.

It works best on integrals of the form

$$\int x^n \sin ax\,dx\,, \int x^n \cos ax\,dx \text{ and } \int x^n e^{ax}\,dx$$

Example

Evaluate $\int x^2 \sin 4x\,dx$

Solution

Begin as usual by letting $u = x^2$ and $\frac{dv}{dx} = \sin 4x$

Next create a table consisting of three columns

Alternate signs	u **and its derivatives**	$\frac{dv}{dx}$ **and its anti derivatives**
+	x^2	$\sin 4x$
−	$2x$	$-\frac{\cos 4x}{4}$
+	2	$-\frac{\sin 4x}{16}$
−	0	$\frac{\cos 4x}{64}$

Differentiate until you obtain 0 as a derivative

Finally the solution is given by adding the *signed* products of the diagonal entries to obtain

$\therefore \int x^2 \sin 4x\, dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32}\cos 4x + c$

To integrate $\int x^n \ln x\, dx$, $\int x^n \arcsin ax\, dx$ or $\int x^n \arctan ax\, dx$

Let $u = \ln x$, $\arcsin ax$ or $\arctan ax$ and $dv = x^n\, dx$

Evaluate

a) $\int x^2 \ln x\, dx$ b) $\int_0^1 \arcsin x\, dx$ c) $\int x^2 \tan^{-1} x\, dx$

Solution

a) In this case x^2 is more easily integrated than $\ln x$.
Furthermore, the derivative of $\ln x$ is simpler than $\ln x$

$dv = x^2 dx \Longrightarrow v = \int x^2\, dx = \frac{1}{3}x^3$, $u = \ln x \Longrightarrow \frac{du}{dx} = \frac{1}{x}$ or $du = \frac{dx}{x}$

$\therefore \int x^2 \ln x\, dx = \frac{x^3 \ln x}{3} - \int \frac{x^3}{3}.\frac{1}{x}.dx$

$= \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2\, dx = \frac{x^3 \ln x}{3} - \frac{1}{9}x^3 + c$

b) Let $dv = dx \Longrightarrow v = \int dx = x$, $u = \arcsin x \Longrightarrow du = \frac{1}{\sqrt{1-x^2}}dx$

$\therefore \int \arcsin x\, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx$

$= x \arcsin x - \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\, dx$

$= x \arcsin x - \sqrt{1-x^2} + c$

Now evaluating a definite integral

$\int_0^1 \arcsin x\, dx = \left[x \arcsin x - \sqrt{1-x^2} + c\right]_0^1$

$= \frac{\pi}{2} - 1 \simeq 0.571$

c) Let $dv = x^2 \Longrightarrow v = \int x^2 dx = \frac{1}{3}x^3$, $u = \tan^{-1} x \Longrightarrow du = \frac{1}{1+x^2}dx$

$\int x^2 \tan^{-1} x\, dx = \frac{x^3 \tan^{-1} x}{3} - \int \frac{x^3}{3}.\frac{1}{1+x^2}dx$

$= \frac{x^3 \tan^{-1} x}{3} - \int \frac{x(1+x^2)-x}{1+x^2}dx$

$= \frac{x^3 \tan^{-1} x}{3} - \int \left(x - \frac{x}{1+x^2}\right)dx = \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}\int \frac{2x}{1+x^2}dx\right]$

$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{6}x^2 + \frac{1}{6}\ln|x^2+1| + c$

To integrate $\int e^{ax} \sin bx\, dx$ or $\int e^{ax} \cos bx\, dx$

Let $u = \sin bx$ or $\cos bx$ and $dv = e^{ax}\, dx$

These require repeated use of integration by parts

1. *Evaluate*

   a) $\int e^x \cos 2x\, dx$ b) $\int e^{2x} \cos x\, dx$

Solution

a) Let $dv = e^x.dx \implies v = \int e^x.dx = e^x$, $u = \cos 2x \implies du = -2\sin 2x\, dx$

$\therefore \int e^x \cos 2x\, dx = e^x \cos 2x + 2\int e^x \sin 2x\, dx$ ..... (1)

Dealing with $\int e^x \sin 2x\, dx$

Let $dv = e^x.dx \implies v = \int e^x.dx = e^x$, $u = \sin 2x \implies du = 2\cos 2x\, dx$

$\therefore \int e^x \sin 2x\, dx = e^x \sin 2x - 2\int e^x \cos 2x\, dx$ ..... (2)

Eqn. (2) in to eqn. (1)

$\therefore \int e^x \cos 2x\, dx = e^x \cos 2x + 2[e^x \sin 2x - 2\int e^x \cos 2x\, dx]$

$= e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x\, dx$

Because the R.H.S integral is a constant multiple of original integral, you can add it to both sides of the equation

$\therefore 5\int e^x \cos 2x\, dx = e^x \cos 2x + 2e^x \sin 2x$

$\therefore \int e^x \cos 2x\, dx = \frac{1}{5}e^x \cos 2x + \frac{2}{5}e^x \sin 2x + c$

2. *Show that*

   a) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

   b) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

Solution

a) Let $dv = e^{ax}.dx \implies v = \int e^{ax}.dx = \frac{1}{a}e^{ax}$, $u = \sin bx \implies du = b\cos bx\, dx$

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}\sin bx}{a} - \frac{1}{a}\int b\, e^{ax} \cos bx\, dx$

$= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax} \cos bx\, dx$

Integrating again

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left[\frac{e^{ax}\cos bx}{a} - \frac{1}{a}\int -b\, e^{ax} \sin bx\, dx\right]$

$= \frac{e^{ax}\sin bx}{a} - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax} \sin bx\, dx$

Transposing the integral on the R.H.S to the L.H.S

$\left(1 + \frac{b^2}{a^2}\right)e^{ax} \sin bx = \frac{e^{ax}}{a^2}[a\sin bx - b\cos bx]$

$\therefore \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

b) Left as an exercise

NOTE

From integration of trigonometric functions of secants . i. e. $\int \sec^n(mx)\, dx$, when n is *odd*, use integration by parts and for n = *even*, use $t = \tan mx$

Example

$\int \sec^3 x \ dx$

Since the power on secant is odd, use $u = \sec x \Longrightarrow du = \sec x \tan x \, dx$ and $dv = \sec^2 x \Longrightarrow v = \int \sec^2 x \, dx = \tan x$

$$\int \sec^3 x \ dx = \int \sec^2 x \sec x \ dx = \sec x \ \tan x - \int \tan x \, . \sec x \tan x \, dx$$
$$= \sec x \ \tan x - \int \tan^2 x \, . \sec x \, dx$$
$$= \sec x \ \tan x - \int (\sec^2 x - 1) \, . \sec x \, dx$$
$$= \sec x \ \tan x - \int \sec^3 x \, dx - \int \sec x \, dx$$

$2 \int \sec^3 x \ dx = \sec x \tan x + \ln|\sec x + \ \tan x|$

$\therefore \int \sec^3 x \ dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \ \tan x| + c$

Task

Show that

a) $\int_0^1 x^2 \tan^{-1} x \, dx = \frac{\pi}{12} - \frac{1}{16}(1 - \ln 2)$

b) $\int_0^1 x \sin^{-1} x \, dx = \frac{\pi}{8}$

c) $\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + c$

d) $\int x^2 \sin^{-1} x \, dx = \frac{x^3}{3} \sin^{-1} x - \frac{1}{9}(1 - x^2)^{3/2} + \frac{1}{4}\sqrt{1 - x^2} + c$

e) $\int x^{-n} \ln x \, dx = \frac{x^{1-n}}{1-n}\left[\ln x - \frac{1}{1-n}\right] + c$

f) $\int \ln(x^2 + 2) \, dx = x \ln|x^2 + 2| - 2x + 2\sqrt{2} \tan^{-1} \frac{x}{\sqrt{2}} + c$

g) $\int (\ln x)^2 \, dx = x(\ln x)^2 - 2x \ln x + 2x + c$

h) $\int_1^2 x \ln(x^2 + 1) \, dx = \frac{5}{2} \ln 5 - \ln 2 - \frac{3}{2}$

Disguised Integration by Parts

Evaluate

a) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ **Ans:** $-\sin^{-1} x \sqrt{1 - x^2} + x + c$ use $[u = \sin^{-1} x]$

b) $\int \frac{x^2 \tan^{-1} x}{1+x^2} dx$ **Ans:** $x \tan^{-1} x - \frac{1}{2} \ln|x^2 + 1| - \frac{1}{2}(\tan^{-1} x)^2 + c$ Use $[u = \tan^{-1} x]$

c) $\int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx$ **Ans:** $\frac{x - \tan^{-1} x}{\sqrt{1+x^2}}$

d) $\int \frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx$ **Ans:**

e) $\int \cos \sqrt{x} \, dx$ **Ans:**

f) $\int \frac{x^3 \sin^{-1} x}{\sqrt{1-x^2}} dx$ **Ans:**

g) $\int_0^{\pi/2} \frac{x}{x + \sin x} \ dx$ **Ans:**

h) $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} \ dx$ **Ans:** $\frac{\pi}{4} - \frac{1}{2} \ln 2$

Other numbers on Integration by Parts

a) $\int \sqrt{9-x^2}\,dx$ b) $\int \frac{x\cos x}{\sin^2 x}dx$

Solution

a) This can be evaluated using $x = 3\sin\theta$

By integration by parts

$$\int \sqrt{9-x^2}\,dx = \int 1\times\sqrt{9-x^2}\,dx$$

$$= \sqrt{9-x^2}.\int 1\,dx - \int \frac{d}{dx}\sqrt{9-x^2}\,.\int 1\,dx$$

$$= x\sqrt{9-x^2} - \int -\frac{x^2}{\sqrt{9-x^2}}dx$$

$$= x\sqrt{9-x^2} - \int \frac{9-x^2-9}{\sqrt{9-x^2}}dx$$

$$= x\sqrt{9-x^2} - \int \frac{9-x^2}{\sqrt{9-x^2}}dx + 9\int \frac{dx}{\sqrt{9-x^2}}$$

$$= x\sqrt{9-x^2} - \int \sqrt{9-x^2}\,dx + 9\sin^{-1}\frac{x}{3}$$

$$2\int \sqrt{9-x^2}\,dx = x\sqrt{9-x^2} + 9\sin^{-1}\frac{x}{3}$$

$$\therefore \int \sqrt{9-x^2}\,dx = \frac{1}{2}x\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + c$$

Integrals of Exponential Functions

a) **Integrating an exponential function to another base other than e**

$\int a^x\,dx$

To evaluate this integral, differentiate the same function a^x

Let $u = a^x \Longrightarrow \ln u = x\ln a$, differentiating w.r.t x

$$\frac{1}{u}\frac{du}{dx} = \ln a\,.\frac{d}{dx}(x) = \ln a$$

$$\therefore \frac{d}{dx}(a^x) = a^x\ln a \Longrightarrow a^x = \frac{1}{\ln a}.\frac{d}{dx}(a^x)$$

Now integrating both sides

$$\Longrightarrow \int a^x\,dx = \frac{1}{\ln a}\int \frac{d}{dx}(a^x)\,.dx = \frac{a^x}{\ln a}$$

$$\boxed{\therefore \int a^x\,dx = \frac{a^x}{\ln a} + c}$$

Corrary: If $a = e$, then

$$\int e^x\,dx = \frac{e^x}{\ln e} + c = e^x + c$$

$$\boxed{\therefore \int e^x\,dx = e^x + c}$$

Example

Evaluate

a) $\int 2^x\,dx$ b) $\int 5^{x+3}\,dx$ c) $\int 4^{-x}\,dx$

Solution

a) Let $u = 2^x \implies \ln u = x\ln 2$

$\frac{1}{u}\frac{du}{dx} = \ln 2\,dx \implies du = u\ln 2\,dx = 2^x \ln 2\;dx \;\therefore \frac{du}{\ln 2} = 2^x\,dx$

$\therefore \int 2^x\,dx = \int \frac{du}{\ln 2} = \frac{u}{\ln 2} + c$

$= \frac{2^x}{\ln 2} + c$

b) Let $u = 5^{x+3} \implies \ln u = (x+3)\ln 5$

$\frac{1}{u}\frac{du}{dx} = \ln 5\,dx \implies du = u\ln 5 = 5^{x+3}\ln 5\;dx \;\therefore \frac{du}{\ln 5} = 5^{x+3}\,dx$

$\therefore \int 5^{x+3}\,dx = \int \frac{du}{\ln 5} = \frac{u}{\ln 5} + c$

$= \frac{5^{x+3}}{\ln 5} + c$

c) Left as an exercise. **Ans:** $-\frac{4^{-x}}{\ln 4} + c$

b) **Integrating an Exponential Function of the form $\int f'(x).a^{f(x)}\,dx$**

Here let $\boldsymbol{u = f(x)}$

$\implies du = f'(x)dx$

$\therefore \int f'(x).a^{f(x)}\,dx = \int a^u\,du = \frac{a^u}{\ln a} + c$

$$\therefore \int f'(x).a^{f(x)}\,dx + c = \frac{a^{f(x)}}{\ln a} + c$$

Corrary: If $a = e$, then

$\therefore \int f'(x).e^{f(x)}\,dx + c = \frac{e^{f(x)}}{\ln e} + c = e^{f(x)} + c$

$$\therefore \int f'(x).e^{f(x)}\,dx = e^{f(x)} + c$$

Example

Evaluate

a) $\int 5xe^{-x^2}\,dx$

b) $\int \frac{e^{\frac{1}{x}}}{x^2}dx$

c) $\int \cos \pi x\, e^{\sin \pi x}\,dx$

d) $\int x.5^{-x^2}\,dx$

e) $\int (3-x).7^{(3-x)^2}$

Solution

a) $\int 5xe^{-x^2}\,dx = -\frac{5}{2}\int(-2x).e^{-x^2}\,dx$

Let $u = -x^2 \Longrightarrow du = -2x\,dx$

$\therefore \int 5xe^{-x^2}\,dx = -\frac{5}{2}\int e^u\,du = -\frac{5}{2}e^u + c$

$= -\frac{5}{2}e^{-x^2} + c$

b) $\int \frac{e^{\frac{1}{x}}}{x^2}dx = -\int e^{\frac{1}{x}}.\left(-\frac{1}{x^2}\right)dx$

Let $u = \frac{1}{x} \Longrightarrow du = -\frac{1}{x^2}\,dx$

$\therefore \int \frac{e^{\frac{1}{x}}}{x^2}dx = -\int e^u\,du = -e^{\frac{1}{x}} + c$

c) $\int \cos \pi x\, e^{\sin \pi x}\,dx$

Let $u = \sin \pi x \Longrightarrow du = \pi \cos \pi x\,dx$

$\int \cos \pi x\, e^{\sin \pi x}\,dx = \frac{1}{\pi}\int e^u\,du = \frac{e^u}{\pi} + c$

$= \frac{e^{\sin \pi x}}{\pi} + c$

d) $\int x.5^{-x^2}\,dx$

Let $u = -x^2 \Longrightarrow du = -2x\,dx$

$\int x.5^{-x^2}\,dx = -\frac{1}{2}\int\left(5^{-x^2}\right)(-2x)dx$

$= -\frac{1}{2}\int 5^u\,du = -\frac{1}{2}.\frac{5^u}{\ln 5} + c$

$= -\frac{5^{-x^2}}{2\ln 5} + c$

e) $\int (3-x).7^{(3-x)^2}$

Let $u = (3-x)^2 \Longrightarrow du = -2(3-x)\,dx$

$\int (3-x).7^{(3-x)^2} = -\frac{1}{2}\int[-2(3-x)].7^{(3-x)^2}\,dx$

$= -\frac{1}{2}\int 7^u\,du = -\frac{1}{2}.\frac{7^u}{\ln 7} + c$

$= -\frac{7^{(3-x)^2}}{\ln 7} + c$

Integration by Partial Fractions

A function of the form $\frac{F(x)}{g(x)}$ where $F(x)$ and $g(x)$ are polynomials is called a Rational Function.

Decomposition of $\frac{F(x)}{g(x)}$ into Partial Fractions

1. Divide if improper

When the degree of the numerator is less than that of the denominator, the fraction is known as *proper* otherwise improper.

i) If the fraction is proper, we resolve it into simple partial fractions and integrate each partial fraction

ii) In case the fraction is improper $.i.e.$ degree of the numerator is not less than that of the denominator then we divide the numerator by the denominator till the remainder is of degree less than that of the denominator. Thus an improper fraction can be expressed in terms of a polynomial and a proper fraction. E.g.

$\frac{2x^3-3x^2+5}{x^2-3x+2} = 2x + 3 + \frac{5x-1}{x^2-3x+2}$

It is easy to integrate a polynomial. To integrate a proper fraction we resolve in to partial fractions.

For resolving proper fractions into partial fractions, we break the denominator in to prime factors which may be linear/quadratic.

2. Factor Denominator

Factorizing the denominator can give factors of the form $(px+q)^n$ or $(ax^2+bx+c)^n$, where $(ax^2+bx+c)^n$ is irreducible.

3. Linear Factors

To each linear factor (non repeated) factor $(px+q)$ there is a partial fraction of the form $\frac{A}{px+q}$ where A is a constant.

Further more for each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions

$\frac{A_1}{px+q} + \frac{A_1}{(px+q)^2} + \cdots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{A_m}{(px+q)^m}$

4. Quadratic Factors

To each non repeated quadratic factor (ax^2+bx+c) there is a partial fraction of the from $\frac{Ax+B}{ax^2+bx+c}$

Further more for each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of $n-$ factors

$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$

Four major cases are illustrated below

Examples

a) **Linear Factors (Non Repeated) only, in the Denominator**

1. *Resolve* $f(x) = \frac{x+1}{x^2-5x+6}$ *into partial fractions*

Solution

$f(x) = \frac{x+1}{x^2-5x+6} = \frac{x+1}{(x-2)(x-3)}$

$\frac{x+1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$

Multiply this equation by the L.C.M of denominator $(x-2)(x-3)$ leads to a *basic equation*

$x + 1 = A(x-3) + B(x-2)$ (Basic equation)

Because this equation is to be true for all x you substitute any *convenient* values for x to obtain equation in A and B. the most convenient values are the ones that make particular factors equal to zero.

To solve for B, let $x = 3$

$4 = A(3-3) + B(3-2) \quad \therefore B = 4$

To solve for A, let $x = 2$

$3 = A(2-3) + B(2-2) \quad \therefore A = -3$

$\therefore f(x) = \frac{x+1}{x^2-5x+6} = -\frac{3}{x-2} + \frac{4}{x-3}$

2. *Integrate*

a) $\frac{3x}{(x-1)(x-2)(x-3)}$ b) $\frac{x^2}{x^2+7x+10}$

Solution

a) Let $\frac{3x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$3x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

Put $x = 1 \Longrightarrow 3 = A(-1)(-2) \quad \therefore A = 3/2$

Put $x = 2 \Longrightarrow 6 = B(1)(-1) \quad \therefore B = -6$

Put $x = 3 \Longrightarrow 9 = C(2)(1) \quad \therefore C = 9/2$

$\therefore \frac{3x}{(x-1)(x-2)(x-3)} = \frac{3}{2(x-1)} - \frac{6}{x-2} + \frac{9}{2(x-3)}$

$\therefore \int \frac{3x}{(x-1)(x-2)(x-3)} = \frac{3}{2}\int \frac{1}{(x-1)}dx - 6\int \frac{1}{x-2}dx + \frac{9}{2}\int \frac{1}{(x-3)}dx$

$= \frac{3}{2}\ln|x-1| - 6\ln|x-2| + \frac{9}{2}\ln|x-3| + c$

b) Since the degree of the numerator is the same as that of the denominator, we divide x^2 by $x^2 + 7x + 10$

$\frac{x^2}{x^2+7x+10} = 1 - \frac{7x+10}{x^2+7x+10}$

Now let $\frac{7x+10}{x^2+7x+10} = \frac{7x+10}{(x+2)(x+5)} = \frac{A}{x+2} + \frac{B}{x+5}$

$\therefore 7x + 10 = A(x+5) + B(x+2)$

Put $x = -2 \Longrightarrow 7(-2) + 10 = A(-2+5) \quad \therefore A = -4/3$

Put $x = -5 \Longrightarrow 7(-5) + 10 = B(-5+2) \quad \therefore B = 25/3$

$\therefore \frac{x^2}{x^2+7x+10} = 1 + \frac{4}{3(x+2)} - \frac{25}{3(x+5)}$

$$\therefore \int \frac{x^2}{x^2+7x+10}dx = \int \left[1 + \frac{4}{3(x+2)} - \frac{25}{3(x+5)}\right]dx$$

$$= x + \frac{4}{3}\ln|x+2| + \frac{25}{3}\ln|x+5| + c$$

Task

Evaluate

a) $\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}dx$ **Ans:** $\boldsymbol{x + \ln\left|\frac{(x-3)^3(x-6)^{30}}{(x-5)^{24}}\right| + c}$

b) $\int \frac{x^3}{x(x^2-1)}dx$ **Ans:** $\boldsymbol{x - \ln|x| + \ln|x-1| + c}$

3. *Two Important Integrals*

i) $\int \frac{dx}{x^2-a^2}$ $(x > a)$ ii) $\int \frac{dx}{a^2-x^2}$ $(x < a)$

Solution

i) $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$

Here $A = \frac{1}{2a}$, $B = -\frac{1}{2a}$

$$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\left[\int \frac{1}{x-a}dx + \int \frac{1}{x+a}dx\right]$$

$$= \frac{1}{2a}[\ln|x-a| - \ln|x+a|] + c$$

Hence $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$ $(x > a)$

ii) $\frac{1}{a^2-x^2} = \frac{1}{(a-x)(a+x)} = \frac{A}{x-a} + \frac{B}{x+a}$

Here $A = \frac{1}{2a}$, $B = \frac{1}{2a}$

$$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\left[\int \frac{1}{a+x}dx + \int \frac{1}{a-x}dx\right]$$

$$= \frac{1}{2a}[\ln|a+x| - \ln|a-x|] + c$$

Hence $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$ $(x < a)$

4. *Evaluate*

a) $\int \frac{dx}{2x^2-3}$ b) $\int \frac{dx}{x^2-25}$

Solution

a) $\int \frac{dx}{2x^2-3} = \frac{1}{2}\int \frac{dx}{x^2-3/2} = \frac{1}{2}\int \frac{dx}{x^2-\left(\sqrt{3/2}\right)^2}$

$$= \frac{1}{2}.\frac{1}{2.\sqrt{3/2}}\ln\left|\frac{x-\sqrt{3/2}}{x+\sqrt{3/2}}\right| + c = \frac{1}{4}\sqrt{\frac{2}{3}}\ln\left|\frac{x-\sqrt{3/2}}{x+\sqrt{3/2}}\right| + c$$

b) $\int \frac{dx}{x^2-25} = \int \frac{dx}{x^2-5^2}$

$$= \frac{1}{2(5)}\ln\left|\frac{x-5}{x+5}\right| + c = \frac{1}{10}\ln\left|\frac{x-5}{x+5}\right| + c$$

b) **Linear Repeated and Non Repeated Factors in the Denominator**

Evaluate

a) $\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$

b) $\int \frac{3x+2}{(x+1)^2(x-2)}dx$

c) $\int \frac{x^2}{(x-1)^3(x+1)}dx$

d) $\int \frac{x^2}{(x+2)^2(x+1)}dx$

Solution

a) $\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx = \int \frac{5x^2+20x+6}{x(x^2+2x^2+1)}dx = \int \frac{5x^2+20x+6}{x(x+1)^2}dx$

Let $\frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

Multiplying through by the L.C.M $x(x+1)^2$

$5x^2 + 20x + 6 = A(x+1)^2 + B(x+1) + Cx$

Put $x = 0$, $\Longrightarrow 6 = A + 0 + 0 \ \therefore A = 6$

Put $x = -1$,$\Longrightarrow 5 - 20 + 6 = 0 + 0 - C = A + 0 + 0 \ \therefore C = 9$

The most convenient choices of values of x have been used, so to find the value of B you can use *any other value* of x along with the calculated values of A and C.

Using $x = 1$, $A = 6$ and $C = 9$

$5 + 20 + 6 = A(4) + B(2) + C$

$31 = 6(4) + B(2) + 9 \ \therefore B = -1$

$\therefore \int \frac{5x^2+20x+6}{x^3+2x^2+x}dx = \int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right)dx = \int \left(\frac{6}{x} - \frac{1}{x+1} + 9(x+1)^{-2}\right)dx$

$= 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + c$

b) Left as an exercise. **Ans:** $-\frac{1}{3(x+1)} + \frac{8}{9}\ln\left|\frac{x-2}{x+1}\right| + c$

c) Left as an exercise. **Ans:** $-\frac{1}{4(x-1)^2} - \frac{3}{4(x-1)} + \frac{1}{8}\ln\left|\frac{x-1}{x+1}\right| + c$

d) Left as an exercise. **Ans:** $\ln|x+1| + \frac{4}{x+2} + c$

c) **Linear and Quadratic Factors (Non Repeated) in the Denominator**

Evaluate

a) $\int \frac{x}{(x-1)(x^2+4)}\,dx$ b) $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx$

Solution

a) Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$

Multiplying both sides by $(x-1)(x^2+4)$

$x = A(x^2+4) + (Bx+C)(x-1)$

Put $x = 1$, $\Longrightarrow 1 = A(1^2+4) + 0 \ \therefore A = 1/5$

Put $x = 0$,$\Longrightarrow 0 = 4A - C$, but $A = 1/5 \ \therefore C = 4/5$

Using the other convenient value of x to obtain B

Putting $x = -1$, then using $A = 1/5$ and $C = 4/5$

$\Longrightarrow -1 = A(1+4) + (-B+C)(-1-1)$

$\Longrightarrow -1 = \frac{1}{5}(5) - 2\left(-B + \frac{4}{5}\right) \ \therefore -1 = 1 + 2B - \frac{8}{5} \ \Longrightarrow B = -\frac{1}{5}$

$\therefore \int \frac{x}{(x-1)(x^2+4)}\,dx = \frac{1}{5}\int \frac{1}{x-1}dx + \int \frac{-\frac{1}{5}x+\frac{4}{5}}{x^2+4}dx$

$= \frac{1}{5}\ln|x-1| - \frac{1}{5}\int \frac{x-4}{x^2+4}dx$

$= \frac{1}{5}\ln|x-1| - \frac{1}{5}\int \frac{x}{x^2+4}dx + \frac{1}{5}\int \frac{4}{x^2+4}dx$

$= \frac{1}{5}\ln|x-1| - \frac{1}{5(2)}\int \frac{2x}{x^2+4}dx + \frac{4}{5}\int \frac{1}{x^2+2^2}dx$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|x^2+4| \, dx + \frac{4}{5}.\frac{1}{2}\tan^{-1}\frac{x}{2} + c$$
$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|x^2+4| \, dx + \frac{2}{5}\tan^{-1}\frac{x}{2} + c$$

b) Because $(x^2-x)(x^2+4) = x(x-1)(x^2+4)$

Let $\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$

Multiplying through by the least common denominator $x(x-1)(x^2+4)$ yields the *basic equation*

$2x^3 - 4x - 8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1)$

Putting $x = 0 \implies -8 = -4A + 0 + 0 \quad \therefore A = 2$

Putting $x = 1 \implies -10 = 0 + 5B + 0 \quad \therefore B = -2$

At this point, C and D are yet to be determined. You can find these remaining constants by choosing two other values of x and solving the resulting system of linear equations.

If $x = -1$, then using $A = 2$ and $B = -2$

$-6 = (2)(-2)(5) + (-2)(-1)(5) + (-C+D)(-1)(-2) \quad \therefore 2 = -C + D$

If $x = 2$, then using $A = 2$ and $B = -2$

$0 = (2)(1)(8) + (-2)(2)(8) + (2C+D)(2)(1) \quad \therefore 8 = 2C + D$

Solving the linear system of equations

$\therefore C = 2 \ , \ D = 4$

$$\therefore \int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)} \, dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right) dx$$
$$= 2\ln|x| - 2\ln|x-1| + \ln|x^2+4| - 2\tan^{-1}\frac{x}{2} + c$$

Repeated Quadratic Factors

Evaluate

a) $\int \frac{8x^3+13x}{(x^2+2)^2} dx$ b) $\int \frac{x^2}{(x^2+1)^2} dx$

Solution

a) Let $\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$

Multiplying through by the least common denominator $(x^2+2)^2$ yields the *basic equation.*

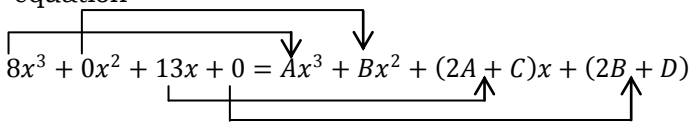
$8x^3 + 13x = (Ax+B)(x^2+2) + Cx + D$

Expanding the equation and collecting the like terms

$8x^3 + 13x = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$

$= Ax^3 + Bx^2 + (2A+C)x + (2B+D)$

Now you can equate coefficients of like terms on opposite sides of the equation



Using the known values $A = 8$ and $B = 0$

$13 = 2A + C = 2(8) + C \quad \therefore C = -3$

$0 = 2B + D = 2(0) + D \quad \therefore D = 0$

$\therefore \int \frac{8x^3+13x}{(x^2+2)^2}dx = \int \left(\frac{8x}{x^2+2} + \frac{-3x}{(x^2+2)^2}\right)dx = 4\ln|x^2+2| + \frac{3}{2(x^2+2)} + c$

b) Let $\frac{x^2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+2)^2}$

Multiplying through by the least common denominator $(x^2+1)^2$ yields the *basic equation.*

$x^2 = (Ax+B)(x^2+1) + Cx + D$

$= Ax^3 + Bx^2 + (A+C)x + (B+D)$

Equating coefficients you obtain

$A = 0, B = 1, C = 0, D = -1$

$\therefore \int \frac{x^2}{(x^2+1)^2}dx = \int \left[\frac{1}{x^2+1} - \frac{1}{(x^2+2)^2}\right]dx$

$= \arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + c$

$= \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + c$

Task

Evaluate

a) $\int \frac{x^4-x^3-x-1}{x^3-x^2}dx$ **Ans:** $\frac{x^2}{2} + 2\ln|x| - \frac{1}{x} - 2\ln|x-1| + c$

b) $\int \frac{x}{(1+x)(1+x^2)}dx$ **Ans:** $-\frac{1}{2}\ln|1+x| + \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x + c$

c) $\int \frac{x^2}{x^4-2x^2-8}dx$ **Ans:** $\frac{1}{6}\left[\ln\left|\frac{x-2}{x+2}\right| + \sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right)\right] + c$

d) $\int \frac{x}{16x^4-1}dx$ **Ans:** $\frac{1}{16}\ln\left|\frac{4x^2-1}{4x^2+1}\right| + c$

Guidelines for Solving the Basic Equation

Linear factors

1. Substitute the roots of the distinct linear factors in to the basic equation
2. For repeated linear factors, use the coefficients determined in part (1) to rewrite the basic equation. Then substitute *other* convenient values of x and solve for the remaining coefficients

Quadratic factors

1. Expand the basic equation
2. Collect terms according to powers of x
3. Equate the coefficient of like terms to obtain a system of linear equations involving A, B,C etc
4. Solve the system of linear equations

NOTE

When integrating rational expressions, keep in mind that for *improper* rational expressions such as;

$\frac{F(x)}{g(x)} = \frac{2x^3+x^2-7x+7}{x^2+x-2}$. You must first divide to obtain;

$\frac{F(x)}{g(x)} = 2x - 1 + \frac{-2x+5}{x^2+x-2}$

The proper rational expression is then decomposed in to its partial fractions.

AREA OF PLANE CURVES

DISC METHOD

***a)* Areas of plane curves. Rectangular coordinates**

Consider the curve $y = f(x)$

The area between the curve, the $x-$axis and the ordinates at $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b y\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The value of y in terms of x is substituted in to the above equation. It is obtained from the equation of the curve.

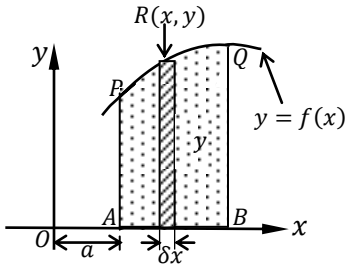
Proof

Considering a vertical rectangle element of base δx and altitude y

Area of the rectangle, $\delta A = y.\delta x$

Total area is obtained as the limit of sum of all areas of these elements between the ordinates AP and BQ

$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y.\delta x = \int_a^b y\,dx$



Similarly the area bounded by the curve $x = \varphi(y)$, the $y-$axis and the horizontal lines $y = c$ and $y = d$ is given by

$$\text{Area} = \int_c^d x\,dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The value of x in terms of y is substituted from the equation of the curve. It is obtained from the equation of the curve.

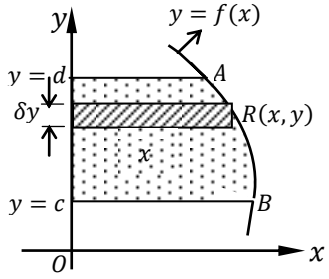
Proof

Considering a horizontal rectangle element of base δy and altitude x

Area of the rectangle, $\delta A = x.\delta y$

Total area is obtained as the limit of sum of all areas of these elements between the ordinates $y = c$ and $y = d$

$A = \lim_{\delta x \to 0} \sum_{y=c}^{y=d} x.\delta y = \int_c^d x.dy$



NOTE

1. It is always necessary to trace the curve with reference to which the region whose area is to be determined is given. The shape of the curve will enable the area to be expressed as a definite integral or a suitable combination of integrals.

2. If the area to be determined lies *above* the $x-$ axis, the area formula $\int_a^b y\,dx$ gives a *positive value* as in fig (i) and a *negative value* if the area is *below* the $x-$axis as in fig (ii).
   If however, the curve AEC crosses the $x-$axis at say, at E, the area formula $\int_a^b y\,dx$ *does not* give the sum of the area of the two portions AEC and EDC but their *difference.*

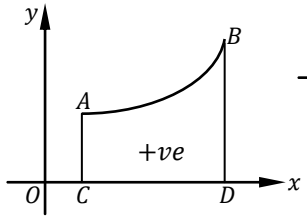


Fig. (i)

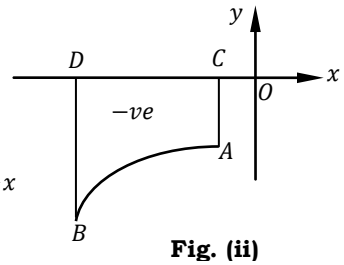


Fig. (ii)

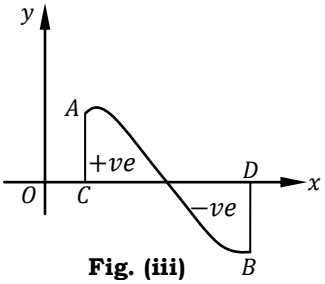


Fig. (iii)

Area = Area ACE − Area BDE , when the formula of area $\int_a^b y\,dx$ is used

NOTE

This area can be obtained by taking separate numerical values of ACE and BDE, then adding. i.e

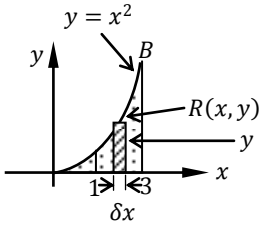
Area = Area ACE (Numerically) + Area BDE (Numerically)

Example

1. *Find the area bounded by the curve $y = x^2$, the $x-$axis and the ordinates $x = 1$ and $x = 3$*

Solution

$$\text{Required area} = \int_1^3 y\,dx$$
$$= \int_1^3 x^2\,dx$$
$$= \left[\frac{x^3}{3}\right]_1^3 = 9 - \frac{1}{3}$$
$$= \frac{26}{3}\ sq.units$$

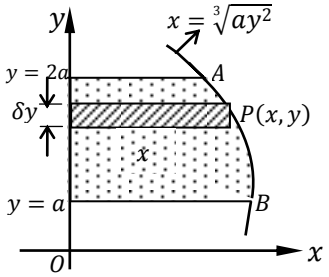


P

2. *Find the area bounded by the curve $ay^2 = x^3$, the $y-$ axis and the lines.*

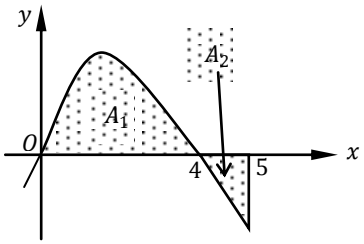
Solution

$$\text{Area required} = \int_a^{2a} x.dy$$
$$= \int_a^{2a} a^{1/3}.y^{2/3}dy$$
$$= \frac{3}{5}a^{1/3}\left[y^{5/3}\right]_0^{2a}$$
$$= \frac{3}{5}a^2\left(\sqrt[3]{32} - 1\right)$$
$$= 1.304a^2$$



3. *Find the area between the curve $y = x(4 - x)$ and the $x-$ axis from $x = 0$ to $x = 5$*

Solution



First, make a sketch of the curve $y = x(4 - x)$. The sketch shows that the required area is in two parts. One part lies above the x –axis and therefore its area is $+ve$ and the other part lies below the $x -$ axis and has a $-ve$ area.

Using $\int_a^b y\,dx$, calculating the areas separately

$$A_1 = \int_0^4 x(4-x)\,dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = \left(32 - \frac{64}{3}\right) - 0 = +\frac{32}{3} = +10\frac{2}{3}$$

$$A_2 = \int_4^5 x(4-x)\,dx = \left[2x^2 - \frac{x^3}{3}\right]_4^5 = \left(50 - \frac{125}{3}\right) - \left(32 - \frac{64}{3}\right) = -2\frac{1}{3}$$

Thus the total area under the curve between $x = 0$ and $x = 5$ is given by the sum of the *numerical* values of these two parts.

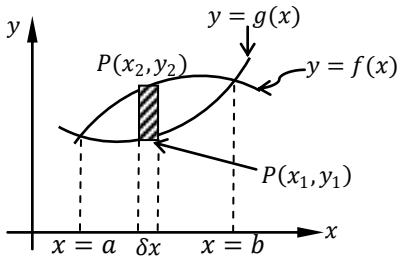
Required area $= 10\frac{2}{3} + 2\frac{1}{3} = 13\ sq.units$

NOTE

1. Integrating $\int_0^5 x(4-x)\,dx$ doe not give that required area.
2. Required area $= A_1 + A_2$ (Taking areas with their signs)
   $= 10\frac{2}{3} - 2\frac{1}{3} = 13\ sq.units$

Area of a Region between Two Curves

Let $y = f(x)$ and $y = g(x)$ be the two equations of the two curves. Solve these two equations simultaneously for x by eliminating y and thus find the $x -$ coordinates of the point of intersection.



Let these curves intersect at $x = a$ and $x = b$.

Taking the elementary strip of base/width δx and altitude/height *[value of y at the upper boundary] – [value of y at the lower boundary]*

i.e. $[f(x) - g(x)]$

Area of the strip $= [f(x) - g(x)]\delta x$

$$\therefore \text{Required Area} = \int_a^b [f(x) - g(x)]\,dx$$

Thus area of the region between $y = f(x)$ and $y = g(x)$ is:

(Area of Region under $y = f(x)$) – (Area of region under $y = g(x)$) or

$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

NOTE

The above formula Area $= \int_a^b [f(x) - g(x)]\ dx$ on the interval $[a\,,b]$ is used provided $g(x) \leq f(x)$.

Vertical elements (of width δx) implies integration w.r.t x, where as a horizontal element (of width δy) implies integration w.r.t y.

Example

1. *Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$*

Solution

Obtaining $x-$ coordinates of points of intersection

$4x = (2x-4)^2 \Rightarrow 4x = 4x^2 - 16x + 1 \Rightarrow 6$

$\Rightarrow 4x^2 - 20x + 16 = 0$ or $x^2 - 5x + 4 = 0$

$\Rightarrow (x-1)(x-4) = 0 \quad \therefore x = 1\,,4$

For $x = 1\,, y = 2(1) - 4 = -2;\ \ x = 4\,, y = 2(4) - 4 = 4$

$\therefore (1\,,-2)\ ,\ (4\,,4)$

Sketching curves

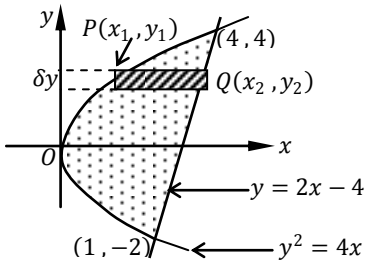
Considering a horizontal strip of width δy and height $(x_2 - x_1)$, area of strip is

$\delta A = (x_2 - x_1)\delta y$

$= \left[\frac{y+4}{2} - \frac{y^2}{4}\right]\delta y$

Area of the region $= \int_{-2}^{4} \left(2 + \frac{y}{2} - \frac{y^2}{4}\right) dy$

$= \left[2y + \frac{y^2}{4} - \frac{y^3}{12}\right]_{-2}^{4} = 9\ sq.units$



NOTE

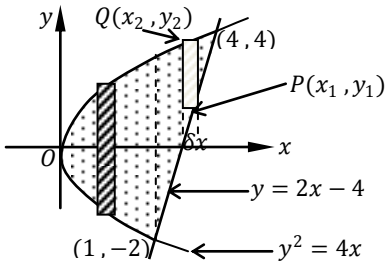
When a horizontal strip is drawn, this strip runs from the parabola to the line at any point drawn, hence one definite integral is formulated.

Alternatively

Considering a vertical strip, one strip can run from the line to the parabola and another runs from one branch of parabola to the other branch.

Dividing the area by the line $x = 1$.

For the approximating rectangle to the right, the width is δx, the height is:



$2\sqrt{x} - (2x-4) = 2\sqrt{x} - 2x + 4$, and area is:

$\delta A = \left(2\sqrt{x} - 2x + 4\right)\delta x$

For the approximating rectangle to the left of this line the width is, δx , the height (Making use of symmetry) is $2y = 4.\sqrt{x}$, and the area is $4\sqrt{x}.\delta x$

Required area $= \int_0^1 4\sqrt{x}\,dx + \int_1^4 \left(2\sqrt{x} - 2x + 4\right) dx$

$$= \left[\frac{8}{3}x^{3/2}\right]_0^1 + \left[\frac{4}{3}x^{3/2} - x^2 + 4x\right]_1^4$$
$$= \frac{8}{3} + \frac{19}{3} = 9\ sq.units$$

NOTE

When finding area between two curves. i.e. a curve and a line, choose a suitable element which gives only one *definite integral*. For the above case, a vertical element gives two integrals where as a horizontal element gives one definite integral.

2. *Find the area of the region bounded by the graphs*

a) $y = x^2 + 2\,, y = -x\,, x = 0$ *and* $x = 1$

b) $y = 2 - x^2\,,\ y = x$

c) $x = 3 - y^2$ *and* $y = x - 1$

Solution

a) Points of intersection

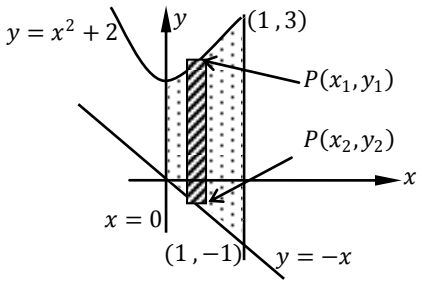
$y = x^2 + 2\,, y = -x \Longrightarrow -x = x^2 + 2$ or $x^2 + x + 2 = 0$

This equation has no real values of x, hence no points of intersection

$y = x^2 + 2\,,\ x = 0 \Longrightarrow (0\,, 2)$ $\quad$ $y = -x\,, x = 0 \Longrightarrow (0\,, 0)$

$y = x^2 + 2\,,\ x = 1 \Longrightarrow (1\,, 3)$ $\quad$ $y = -x\,, x = 1 \Longrightarrow (1\,, -1)$

Sketching the graphs



Taking a vertical element of width δx and a height $(y_2 - y_1)$

Its area, $\delta A = (y_2 - y_1)\delta x$

Required area $= \int_0^1 \left(x^2 + 2 - (-x)\right) dx$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1 = \frac{1}{3} + \frac{1}{2} + 2$$
$$= \frac{17}{6}\ sq.units$$

b) See example (a) above. **Ans:** $\frac{9}{2}$

c) See example (a) above. **Ans:** $\frac{9}{2}$

3. *Find the area of the region between the graphs of* $f(x) = 3x^3 - x^2 - 10x$ *and* $g(x) = -x^2 + 2x$

Solution

Obtaining the points of intersection (x –coordinates)

$3x^3 - x^2 - 10x = -x^2 + 2x$

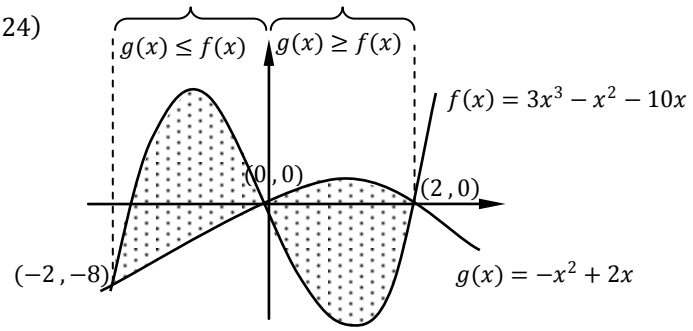
$3x^3 - 12x = 0 \Longrightarrow 3x(x^2 - 4) = 0 \quad \therefore x = 0\ , \pm 2$

Sketch the curves to get the curve below, from which,

Required area, $A = \int_{-2}^{0} [f(x) - g(x)]\, dx + \int_0^2 [g(x) - f(x)]\, dx$

$$= \int_{-2}^{0} (3x^3 - 12x)\, dx + \int_0^2 (-3x^3 + 12x)\, dx$$
$$= \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[-\frac{3x^4}{4} + 6x^2\right]_0^2$$

$= -(12 - 24) + (-12 + 24)$
$= 24\ sq.units$



NOTE

In general, to detyermine the area between two curves you can use

$A = \int_{x_1}^{x_2}[(Top\ Curve) - (Bottom\ Curve)]\,dx$ [Vertical rectanglar strip used]

$A = \int_{y_1}^{y_2}[(Right\ Curve) - (Left\ Curve)]\,dx$ [Horizantal rectanglar strip used]

, where $(x_1\ , y_1)$ and $(x_2\ , y_2)$ are either adjacent points of intersection of the two curves involved or points on the specified boundary lines.

4. *Sketch the region bounded by the graphs and find the area of the region*

a) $y = \sin x$ *and* $y = \cos x$ *for* $0 \le x \le 2\pi$

b) $y = 2\sin x$ *and* $y = \tan x$ *for* $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

c) $y = 2\sin x + \sin 2x$ *and* $y = 0$ *for* $0 \le x \le \pi$

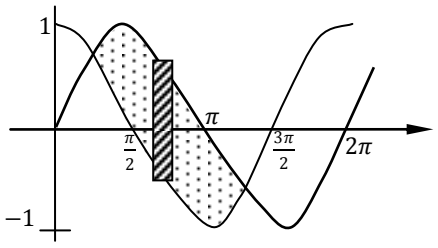
Solution

a) Points of intersection

$\sin x = \cos x \Longrightarrow \frac{\sin x}{\cos x} = 1$ or $\tan x = 1 \Longrightarrow x = \frac{\pi}{4}\ , \frac{5\pi}{4}$

Thus $a = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

Sketching the curves



Since $\sin x \ge \cos x$ on the interval $\left[\frac{\pi}{4}\ , \frac{5\pi}{4}\right]$, the area is given by

$A = \int_{\pi/4}^{5\pi/4}(\sin x - \cos x)\,dx$

$= [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = 2\sqrt{2}$

b) Left as an exercise. **Ans:** $A = 2(1 - \ln 2) \simeq 0.614$

c) Left as an exercise. **Ans:** $A = 2(1 - \ln 2) \simeq 0.614$

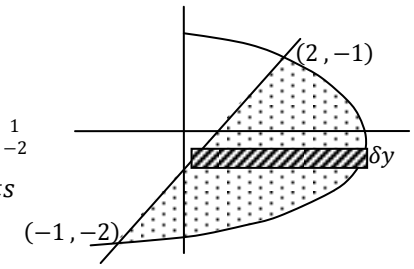
Task

a) Find the area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2$.
Ans: $\frac{1}{6}(3\pi + 2)$

b) Find the area common to the parabolas $y^2 = 5x + 6$ and the circle $x^2 = y$.
Ans: $\frac{27}{5}$

c) Find the area common to the parabolas $x^2 = 9y$ and the circle $y^2 = 9x$.
Ans: 27

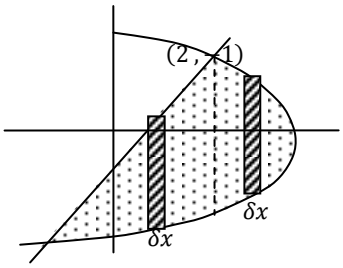
5. *Find the area of the region bounded by graphs* $x = 3 - y^2$ *and* $y = x - 1$
Solution
Points of intersection
$y + 1 = 3 - y^2 \Rightarrow y^2 + y - 2 = 0 \quad \therefore y = 1, -2$ points are: $(-1, -2), (2, 1)$
Let $f(y) = y + 1, g(y) = 3 - y^2$; *since* $f(y) \leq g(y)$ on the interval $[-2, 1]$ then



$\delta A = [g(y) - f(y)]\delta y = \left[\left(3 - y^2 - (y + 1)\right)\right]\delta y$
Hence area , $A = \int_{-2}^{1}\left(3 - y^2 - (y + 1)\right) dy$
$= \int_{-2}^{1}(-y^2 - y + 2)\, dy = \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1}$
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}\ sq.units$

Alternaively

Let $f(x) = x - 1, g(x) = \pm\sqrt{3 - x}$. For the interval $x = -1$ to $x = 2, f(x) \geq -g(x)$ and for the interval $x = -1$ to $x = 2$, $g(x) \leq -g(x)$
Total area, $A = \int_{-1}^{2}\left[f(x) - g(x)\right) dx +$
$\int_{1}^{2}\left[g(x) - \left(-g(x)\right)\right) dx$



Area, $A = \int_{-1}^{2}\left[(x - 1) + \sqrt{3 - x}\right] dx + \int_{-1}^{2}\left[\sqrt{3 - x} - \left(-\sqrt{3 - x}\right)\right] dx$
$= \left[\frac{x^2}{2} - x - \frac{(3-x)^{3/2}}{3/2}\right]_{-1}^{2} + 2.\left[\frac{(3-x)^{3/2}}{3/2}\right]_{2}^{3}$
$= \left(\frac{2^2}{2} - 2 - \frac{(3-2)^{3/2}}{3/2} - \left(\frac{(-1)^2}{2} + 1 - \frac{(3+1)^{3/2}}{3/2}\right)\right) + 2\left(\frac{(3-2)^{3/2}}{3/2} - \frac{(3-3)^{3/2}}{3/2}\right)$
$= -\frac{2}{3} - \frac{3}{2} + \frac{16}{3} + \frac{4}{3} = \frac{9}{2}$

VOLUMES AND SURFACES OF SOLIDS OF REVOLUTION

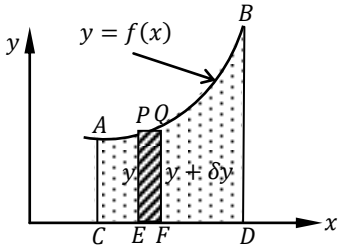
Solids of Revolution

A solid generated by revolving a plane area about a line in the plane is called a *solid of revolution.* The line is called the *axis of revolution*

Revolution about $x-$ axis

The area bounded by the curve $y=f(x)$, the ordinates $x=a$ and $x=b$ and the $x-$ axis, revolves about the $x-$ axis. Find the volume of the solid generated.

Solution



Let AB be the curve $f=f(x)$, CA and DB be ordinates $x=a$ and $x=b$ respectively.
Let $P(x,y)$ be any point on the curve and $Q(x+\delta x, y+\delta y)$ be the neighboring point

PE and QF be ordinates of P and Q so that $EP=y$ and $FQ=y+\delta y$ and $EF=\delta x$

When that area ACDB revolves about the x −axis, the strip PEQF (rectangle) forms a disc of volume $\delta V=\pi y^2\delta x$ (Approximately)

Summing up the volumes of all such discs generated by the different strips from $x=a$ and $x=b$, the required volume of revolution, V is given by

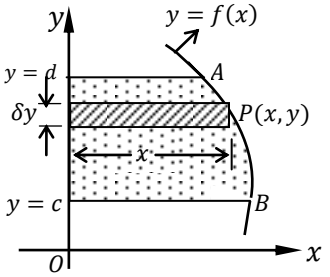
$V=\lim_{\delta x\to 0}\sum_{x=a}^{x=b}\pi y^2\delta x=\pi\int_a^b y^2dx$

$$\therefore V=\pi\int_a^b y^2dx$$

Revolution about $y-$ axis

Proceeding as above, the volume of solid generated by revolution of the area bounded by the curve $x=\varphi(y)$, the line $y=c$ and $y=d$ and $y-$ axis, about $y-$ axis is given by:

$$\therefore V=\pi\int_c^d x^2dy$$



NOTE

1. The above method of finding volume of a solid of revolution about the axis is known as *Disc Method*. Its major application involves a plane bounded by the graph $y=f(x)$ or $x=\varphi(y)$ and the x −axis or $y-$ axis respectively.
   If the axis of revolution is x −axis, then the radius is simply $f(x)$ and if the radius is y −axis, the radius is $\varphi(y)$

2. In general to find the volume of a solid of revolution with the disc method, use the formula

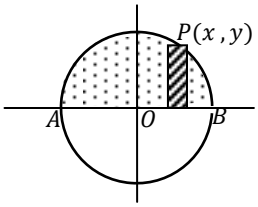
Horizontal axis of revolution	*Vertical axis of revolution*
Volume $= \pi \int_a^b [R(x)]^2 dx$	$\therefore V = \pi \int_c^d [R(y)]^2 dy$

Example

1. *Find the volume of a sphere of radius a.*

Solution

$\therefore V = \pi \int_c^d x^2 dy$Let the sphere be formed by the revolution of the circle $x^2 + y^2 = a^2$ about the x −axis



Consider a vertical element of width δx and height y.

Revolving the strip gives the volume of a disc

$\delta V = \pi y^2 \delta x$

Required volume $= \int_{-a}^{a} \pi y^2 \, dx = \int_{-a}^{a} \pi(a^2 - x^2) \, dx$

$= 2\pi \int_0^a \pi(a^2 - x^2) \, dx$ ($\because a^2 - x^2$ is an even function)

$= 2\pi \left[a^2 x - \frac{x^3}{3}\right] a\,0 = 2\pi \left(a^3 - \frac{a^3}{3}\right) = \frac{4}{3}\pi a^3$

NOTE

1. The sphere is formed by revolution of a semi circular area

2. The same sphere is formed by revolution of the above circle about y −axis

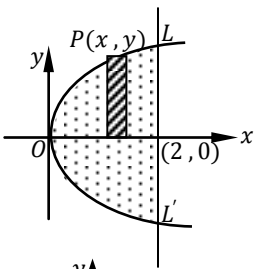
2. *Find the volume generated by revolving*

   i) The area in the first quadrant bounded by the parabola $y^2 = 8x$ and its latus rectum about x − axis

   ii) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x −axis

Solution

i) The latus rectum of a parabola $y^2 = 8x$ is $x = 2$



Required is the volume generated by revolving the area in the first quadrant bounded by the parabola $y^2 = 8x$, the line $x = 2$ and x −axis.

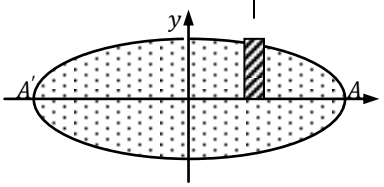
Using $V = \pi \int_0^2 y^2 \, dx = \pi \int_0^2 8x \, dx$

$= 8\pi \left[\frac{x^2}{2}\right] 2\,0 = 16\pi \; cubic \; units$

ii) $V = \int_{-a}^{a} \pi y^2 \, dx = 2\pi \int_0^a b^2 \left(1 - \frac{x^2}{a^2}\right) dx$

$\left(\because \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}\right)$

$= 2\pi \left[x - \frac{x^3}{3a^2}\right] a\,0 = 2\pi b^2 \left(a - \frac{a^3}{3a^2}\right) = \frac{4\pi b^2 a}{3}$



3. *Find the volume of the solid generated by revolving the region bounded by*

a) $y = 2 - x^2$ *and* $y = 1$ *about the line* $y = 1$

b) $y^2 = 8x$ *and its latus rectum about the latus rectum*

Solution

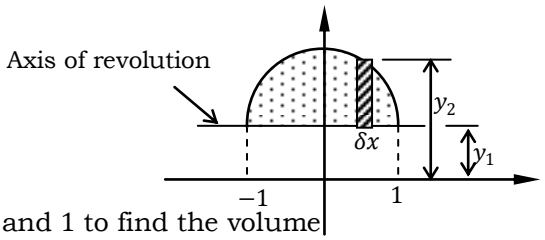
a) Points of intersection.

$1 = 2 - x^2 \Rightarrow x^2 \therefore x = \pm 1$

The radius is $y_2 - y_1$

$\therefore R(x) = 2 - x^2 - 1$

$= 1 - x^2$



Finally, integrate between -1 and 1 to find the volume

$V = \pi \int_a^b [R(x)]^2\, dx = \pi \int_{-1}^{1} (1 - x^2)^2\, dx = \pi \int_{-1}^{1} [1 - 2x^2 + x^4]\, dx$

$= \pi \left[x - \frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^{1} = \frac{16\pi}{15}\ cubic\ units$

b) Taking a strip to be horizontal

$R(x) = 2 - x$

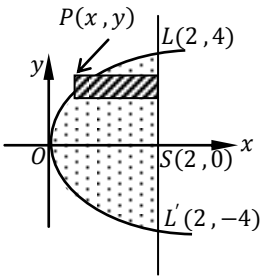
$\therefore$Required volume$= \pi \int_{-4}^{4} (2 - x)^2\, dy$

$= \pi \int_{-4}^{4} \left(2 - \frac{y^2}{8}\right)^2 dy \qquad \left(\because y^2 = 8x, x = \frac{y^2}{8}\right)$

$= 2\pi \int_0^4 \left(4 + \frac{y^4}{64} - \frac{y^2}{2}\right) dy \qquad$ (Integrand is even)

$= 2\pi \left[4y + \frac{1}{64}.\frac{y^5}{5} - \frac{1}{2}.\frac{y^3}{3}\right]_0^4$

$= 2\pi \left[4(4) + \frac{1}{64}.\frac{(4)^5}{5} - \frac{1}{2}.\frac{(4)^3}{3}\right] = \frac{256\pi}{15}$



NOTE

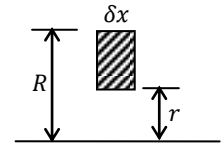
The disc method is very useful if the axis of revolution is one of the boundaries of the curve to be rotated.

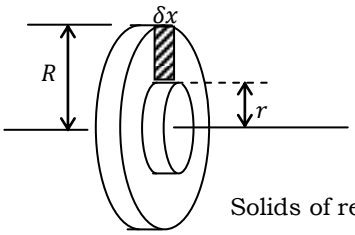
THE WASHER METHOD

The washer is formed by revolving a rectangle about an axis.

If r and R are the inner and outer radii of the washer and δx is the width of the washer, then the volume is given by

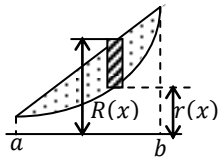
Volume of washer$= \pi(R^2 - r^2).\delta x$

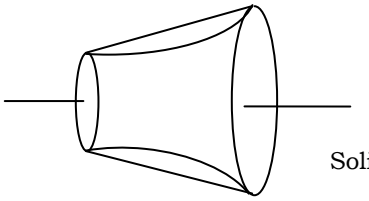




Solids of revolution

Consider a region bounded by an *outer radius* $R(x)$ and an *inner radius* $r(x)$ as below.





Solid of revolution with a hole

If the region is revolved about its axis of revolution, then the volume of the resulting solid is given by;

$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)\, dx$

$$\boxed{V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)\, dx}$$

NOTE

The axis of revolution is not the boundary of the area being rotated.

Example

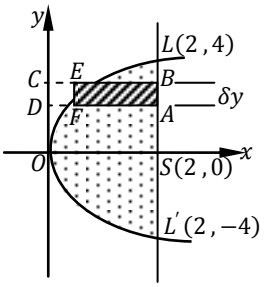
1. a) *Find the volume generated by revolving the area bounded by the parabola* $y^2 = 8x$ *and its latus rectum about the* $y-$*axis*

b) *Find the volume generated by revolving about the* $x-$*axis the area cut off from the parabola* $9y = 4(9 - x^2)$ *by the line* $4x + 3y = 12$

c) *Find the volume of the solid generated by revolving the region bounded by the graphs of* $y = \sqrt{x}$ *and* $y = x^2$ *about the* $x-$*axis*

Solution

a)



The revolution of the strip about the $y-$ axis generates a washer whose volume

=(Volume generated by revolving Rectangle ABCD)

Minus

(Volume generated by revolving Rectangle DPEC

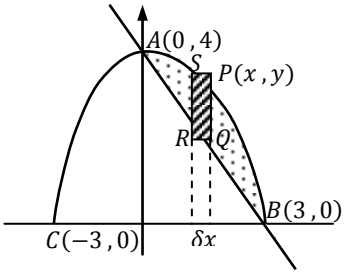
$= \pi(2)^2 \delta y - \pi x^2 \delta y$

$= \pi(4 - x^2)\delta y$

Required volume $= \int_{-4}^{4} \pi(4 - x^2) dy = 2\pi \int_{-4}^{4} \left(4 - \frac{y^4}{64}\right) dy$

$= 2\pi \left[4y - \frac{y^5}{320}\right] = \frac{128\pi}{5}$

b) Parabola is symmetrical about $y-$ axis and its intercepts are: $C(-3\,,0)$ and $B(3\,,0)$



Points of intersection

$9y = 4(9-x^2)$, $4x+3y=12$ or $y=\left(\frac{12-4x}{3}\right)$

$\Rightarrow 9\left(\frac{12-4x}{3}\right)=4(9-x^2)$, solving $x=0\,,3$

$\therefore$ Points are: $A(0\,,4)$ $B(3\,,0)$

When the strip PQRS revolves about $x-$ axis it generates a washer with volume $\delta V=\pi({y_1}^2-{y_2}^2)\delta x$

Hence Required volume $=\pi\int_0^3({y_1}^2-{y_2}^2)\,dx$

$$=\pi\int_0^3\left[\left(\frac{4(9-x^2)}{9}\right)^2-\left(\frac{12-4x}{3}\right)^2\right]dx$$

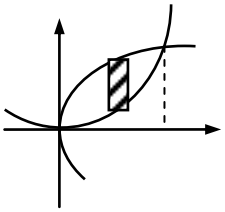
$$=\pi\int_0^3\left[\left(\frac{16}{81}(81+x^4-18x^2)\right)-\frac{16}{81}(9+x^2-6x)\right]dx$$

$$=\frac{16\pi}{81}\int_0^3(x^4-27x^2+54x)\,dx$$

$$=\frac{16\pi}{81}\left[\frac{x^5}{5}-9x^3+27x^2\right]_0^3$$

$$=\frac{16\pi}{81}\left[\frac{243}{5}-243+243\right]=\frac{48\pi}{5}\ cubic\ units$$

c) Points of intersection



$y=\sqrt{x}$, $y=x^2\Rightarrow x^4=x$

$\Rightarrow x(x^3-1)=0\quad\therefore x=0\,,1$

Points area; $(0\,,0)$, $(1\,,1)$

$\therefore V=\int_a^b([R(x)]^2-[r(x)]^2)\,dx$

$=\pi\int_0^1(x-x^4)\,dx=\pi\left[\frac{x^2}{2}-\frac{x^5}{5}\right]_0^1$

$=\frac{3\pi}{10}$ cubic units

NOTE

For all the above formulae on solids of revolution the angle of rotation is $360^0=2\pi$. However if the angle is θ, the formulae are multiplied by the constant $\frac{\theta}{360^0}$ if θ is in degrees or $\frac{\theta}{2\pi}$ if θ is in radians.

Example

If rotation is through 90^0 about the $x-$ axis, the formula will be:

$V=\frac{\pi}{4}\int_a^b y^2\,dx$ or $V=\frac{\pi}{4}\int_a^b([R(x)]^2-[r(x)]^2)\,dx$

VOLUME

Using the disc method, the volume of a solid having a circular section whose area is $A = \pi R^2$ can be obtained.

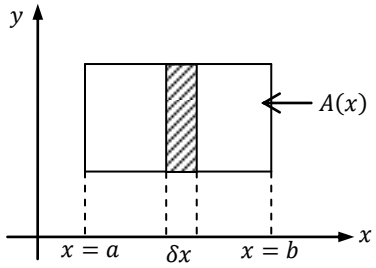
In general, solids of any shape, as long as you know a formula for the area of an arbitrary cross section, volume can be obtained.

Some of the common cross sections are:

- Squares
- Rectangles
- Triangles
- Semi circles
- Trapezoids

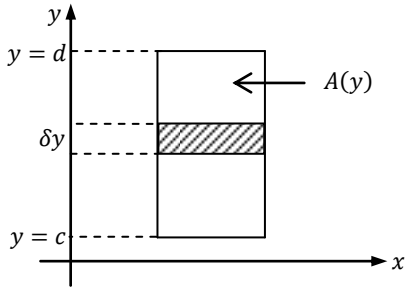
Formula used

1. For cross sections of area $A(x)$ taken perpendicular to the $x-$ axis



$$\text{Volume} = \int_a^b A(x)\,dx$$

2. For cross sections of area $A(y)$ taken perpendicular to the $y-$ aixs



$$\text{Volume} = \int_c^d A(y)\,dx$$

Example

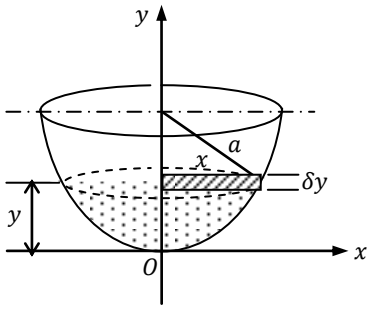
1. *Water is poured into a hemispherical bowl whose axis is vertical and radius a. Show that at the instant when the depth measured from the water's surface to the lowest point of bowl is x, the area of the water's surface is given by:*

$$S(x) = \pi x(2a - x)$$

Find by integration the volume of water at this instant

Solution

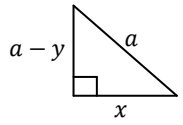
Consider a cross section of area πx^2 at a height y from its horizontal base.



$x^2 = a^2 - (a-y)^2$

$= a^2 - (a^2 - 2ay + y^2)$

$= 2ay - y^2$



$\therefore S(y) = \pi(2ay - y^2)$

But $x = y$ as per the question

$\therefore S(x) = \pi(2ax - x^2)$

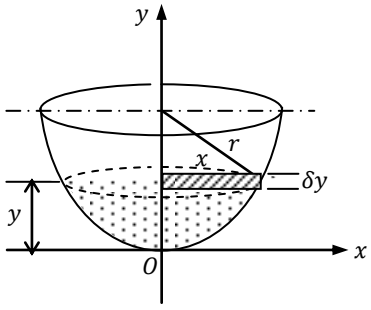
$\therefore$ Volume $= \int_0^x S(y)\,dy$

$= \int_0^x \pi(2ay - y^2)\,dy$

$= \pi\left[ay^2 - \frac{y^3}{3}\right]_0^x = \pi\left(ax^2 - \frac{x^3}{3}\right)$

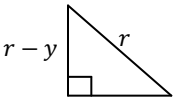
2. *A hemispherical bowl of internal radius r is fixed with its rim horizontal and contains liquid to a depth h, show by integration that the volume of the liquid in the bowl is $\frac{1}{3}\pi h^2(3r-h)$*

Solution



Consider a cross section $A(y)$, y units above $x-$ axis

$A(y) = \pi x^2$



$r^2 = x^2 + (r-y)^2$

$= x^2 + (r^2 - 2ry + y^2)$

$x^2 = 2ry - y^2$

$\therefore A(y) = \pi(2ry - y^2)$

Volume $= \int_0^h \pi(2ay - y^2)\,dy$

$= \pi\left[ry^2 - \frac{y^3}{3}\right]_0^h = \pi\left(rh^2 - \frac{h^3}{3}\right) = \frac{1}{3}\pi h^2(3r-h)$

3. *A hemispherical bowl of internal radius 13 cm contains water to a maximum depth of 8 cm. Find the volume of the water.*

Solution

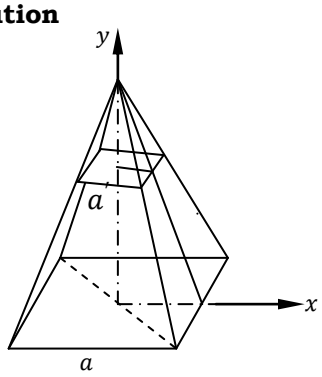
Drawing the above diagrams in question (2) above, $r = 13$, $h = 8$

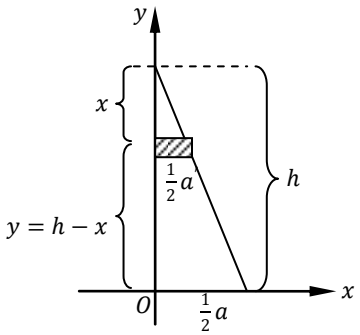
$\therefore$ Volume $= \frac{1}{3}\pi h^2(3r-h) = \frac{1}{3}\pi(8)^2(3(13)-8) = 661\frac{1}{3}\pi\ cm^3$

An Application to Geometry

4. *A regular pyramid has a square base of side a and its height is h. Show that the area of a plane section at a depth x below the vertex is $\left(\frac{ax}{h}\right)^2$ Prove also that the volume of the pyramid of the pyramid is $a^2h/3$*

Solution





Using similar triangles

$\frac{a'}{a} = \frac{x}{h} \Rightarrow a' = \left(\frac{ax}{h}\right)$

Area of the cross section $A(y) = a'^2$

$\Rightarrow A(y) = \left(\frac{ax}{h}\right)^2$

Volume $= \int_0^h A(y)\,dy = \int_0^h \left(\frac{ax}{h}\right)^2 dy$, but $x = y - h$

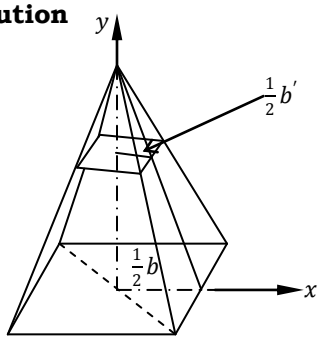
$= \int_0^h \frac{a^2}{h^2}(y-h)^2\,dy$

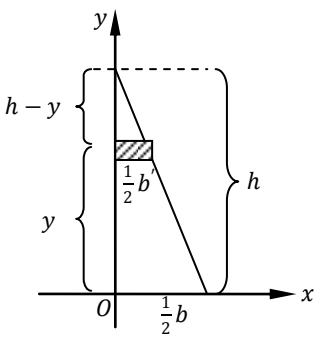
$= \frac{a^2}{h^2}\int_0^h (y-h)^2\,dy$

$= \frac{a^2}{h^2}\left[\frac{(y-h)^3}{3}\right]_0^h = \frac{a^2}{h^2}\left[0 - \frac{(-h)^3}{3}\right] = \frac{a^2h}{3}$

5. *Prove that the volume of a pyramid with a square base is $V = \frac{1}{3}h.B$ where h is the height of the pyramid, and B is the area of the base.*

Solution





Using similar triangles

$\frac{b'}{b} = \frac{h-y}{h} \Rightarrow b' = \frac{b}{h}(h-y)$, where b is the length of the sides of the base of the pyramid

Area of the cross section $A(y) = b'^2$

$\Rightarrow A(y) = \frac{b^2}{h^2}(h-y)^2$

Volume $= \int_0^h A(y)\,dy = \int_0^h \frac{b^2}{h^2}(h-y)^2\,dy$

$= \frac{b^2}{h^2}\int_0^h (h-y)^2\,dy$

$= \frac{b^2}{h^2}\left[\frac{(h-y)^3}{-3}\right]_0^h = -\frac{b^2}{h^2}\left[0 - \frac{(h)^3}{3}\right]$

$= \frac{b^2h^3}{3h^2} = \frac{1}{3}b^2h$

$\therefore V = \frac{1}{3}h.B$

Task

1. A gold fish bowl is a glass sphere of inside diameter 20 cm. Calculate the volume of the water it contains when the maximum depth is 18 cm.
   Ans: 1296π
2. A wall vessel has a plane face and its volume is equivalent to that generated by rotating area enclosed by $x = \frac{y^3}{64} + 1$, the $y-$ axis and $y = 8$ through two right angles about $y-$ axis, and units being cm. calculate the volume. **Ans:** $57\frac{6}{7}cm^3$

FIRST ORDER DIFFERENTIAL EQUATIONS

Definition:

A differential equation is an equation showing the relationship between an independent variable x , a dependednt variable, y and more or one differential coefficients of y w.r.t x.

e.g. $x^2\frac{dy}{dx} + y\sin x = 0$, $xy\frac{d^2y}{dx^2} + y\frac{dy}{dx} = e^{-3x}$

if an equation has only *one* independent variable, the equation is called an *Ordinary Differential Equation.*

e.g. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 2y$

Order of a Differential Equation

The order of a differential equation is given by the highest derivative involved in the equation.

Classifying Differential Equation

Equation	Order
a) $\frac{d^3y}{dx^3} + 4y = 2$	3
b) $\frac{d^2s}{dt^2} = -32$	2
c) $\left(\frac{dy}{dx}\right)^2 - 3y = e^x$	1
d) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2$	2

Formation of a Differential Equation

Differential equations may be formed from:

a) Considering the physical problems to which they refer
b) Mathematically eliminating arbitrary constants

Forming a Differential Equation from Eliminating Arbitrary Constants

F*orm a differential equation, and state the order if:*

a) $y = A\sin x + B\cos x$
b) $y = x + \frac{A}{x}$
c) $y = Ax^2 + Bx$
d) $y = e^{3x}(A\cos 3x + B\sin 3x)$

Solution

NOTE

When forming a differential equation from eliminating arbitrary constants,

1) An nth order differential equation is derived from the function having $n-$ arbitrary constants
2) The number of arbitrary constants in an equation determines the number of times you are to differentiate the equation

a) $y = A\sin x + B\cos x$..... (1)

$\frac{dy}{dx} = A\cos x - B\sin x$ ..... (2)

$\frac{d^2y}{dx^2} = -A\sin x - B\cos x = -(A\sin x + B\cos x)$ ..... (3)

Solving the three equations simultaneously

Eqn. (1) in to eqn. (3)

$\Rightarrow \frac{d^2y}{dx^2} = -y \quad \therefore \frac{d^2y}{dx^2} + y = 0$ is an equation of order 2

b) $y = x + \frac{A}{x}$..... (1)

$\frac{dy}{dx} = 1 - \frac{A}{x^2}$ ..... (2)

Eliminating A from eqn. (1) and eqn. (2)

$\text{eqn. (1)} \times \frac{1}{x} + \text{ eqn. (2)}$

$\Rightarrow \frac{y}{x} = 1 + \frac{A}{x^2}$ ..... (3)

$\text{eqn. (2)} + \text{eqn. (3)}$

$\frac{dy}{dx} + \frac{y}{x} = 2 \Rightarrow x\frac{dy}{dx} + y = 2x$, which is an equation of first order

c) $y = Ax^2 + Bx$..... (1)

$\frac{dy}{dx} = 2Ax + B$ ..... (2)

$\frac{d^2y}{dx^2} = 2A$ ..... (3)

Since eqn. (3) does not contain, eliminating B from eqn. (1) and eqn. (2)

Eqn. (2) $\times\, x$ − eqn. (1)

$x\frac{dy}{dx} - y = 2Ax^2 + Bx - (2Ax^2 + Bx) = Ax^2$

$\therefore x\frac{dy}{dx} - y = Ax^2$ ..... (4)

Now eqn. (4) ÷ eqn. (3) to eliminate A

$\Rightarrow \frac{x\frac{dy}{dx} - y}{\frac{d^2y}{dx^2}} = \frac{Ax^2}{2A} = \frac{x^2}{2}$

$\therefore x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$, which is an equation of order 2

Methods of Solving Differential Equation

1. **BY INTEGRATION**

An equation of the form $\frac{dy}{dx} = f(x)$ can be solved by Direct Integration

Example

Solve the differential equation

a) $\frac{dy}{dx} = 3x^2 - 6x + 4$

b) $\cos x\frac{dy}{dx} = 1$

c) $e^{2x}\frac{dy}{dx} = 4$

Solution

a) $\frac{dy}{dx} = 3x^2 - 6x + 4$, integrating w.r.t x

$\int \frac{dy}{dx}.dx = \int (3x^2 - 6x + 4)\,dx$

$\therefore y = x^3 - 3x^2 + 4x + c$

b) $\cos x \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos x} = \sec x$, integrating w.r.t x

$\int \frac{dy}{dx}.dx = \int \sec x\,dx$

$\therefore y = \ln|\sec x + \tan x| + c$

c) Left as an exercise. **Ans:** $y = 2e^{-2x} + c$

NOTE

1. An arbitrary constant c is included in the solution which we always get when solving a 1st order differential equation.
2. The arbitrary constant's value can only be got when further information about the function is given
3. The solution containing an arbitrary constant is called *a General Solution/Primitive Solution* of the equation. e.g $y = \ln|\sec x + \tan x| + c$ is a general solution.
4. If we are told the value of y for a given value of x, then c can be evaluated, and the resultant solution is known as a *Particular Solution.*

Example

Find the particular solution of the equation $e^x \frac{dy}{dx} = 4$ *given that when* $x = 0\,, y = 3$

Solution

$e^x \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = 4e^{-x}$

$\Rightarrow \int \frac{dy}{dx}.dx = \int 4e^{-x}\ dx$

$y = -4e^{-x} + c$

Put $x = 0\,, y = 3$ in to the equation

$\Rightarrow 3 = -4e^0 + c \ \ \therefore c = 7$

$y = -4e^{-x} + 7$

2. **BY SEPARATION OF VARIABLES**

An equation of the form

- $\frac{dy}{dx} = M(x).N(y)$ or
- $\frac{dy}{dx} = \frac{M(x)}{N(y)}$ or
- $M(x) + N(y).\frac{dy}{dx} = 0$

, can be solved by this method.

For this equation, all x –terms can be collected with dx and y – terms with dy, and the solution can be obtained by integration.

Example

Solve:

a) $(x^2+4)\frac{dy}{dx} = xy$

b) $\frac{dy}{dx} = \frac{y^2+xy^2}{x^2y-x^2}$

c) $y\tan x\frac{dy}{dx} = (4+y^2)\sec^2 x$

, and simplify your answer.

Solution

a) $(x^2+4)\frac{dy}{dx} = xy$

Separating variables

$\Rightarrow \frac{dy}{y} = \frac{x}{x^2+4}$, integrating

$\int \frac{dy}{y}dy = \int \frac{x}{x^2+4}dx \Rightarrow \ln|y| = \frac{1}{2}\ln|x^2+4| + c_1$

Simplifying using logarithms

$\ln|y| - \frac{1}{2}\ln\left|\sqrt{x^2+4}\right| = c_1 \Rightarrow \ln\left|\frac{y}{\sqrt{x^2+4}}\right| = c_1$

$e^{c_1} = \frac{y}{\sqrt{x^2+4}} \therefore y = e^{c_1}\sqrt{x^2+4}$

Since e^{c_1} is a constant, replacing it with a general constant c

$\therefore y = c\sqrt{x^2+4}$

b) $\frac{dy}{dx} = \frac{y^2+xy^2}{x^2y-x^2} = \frac{y^2}{x^2}\left(\frac{1+x}{y-1}\right)$

Separating variables

$\Rightarrow \frac{y-1}{y^2} = \frac{x+1}{x^2}$, integrating

$\int\left(\frac{y}{y^2} - y^{-2}\right)dy = \int\left(\frac{1}{x^2} + \frac{x}{x^2}\right)dx$

$\Rightarrow \int\left(\frac{1}{y} - y^{-2}\right)dy = \int\left(x^{-2} + \frac{1}{x}\right)dx$

$\Rightarrow \ln|y| + \frac{1}{y} = -\frac{1}{x} + \ln|x| + c$

c) $y\tan x\frac{dy}{dx} = (4+y^2)\sec^2 x$

Separating variables

$\Rightarrow \frac{y.dy}{4+y^2} = \frac{\sec^2 x}{\tan x}\,dx$, integrating

$\int \frac{y}{4+y^2}dy = \int \frac{\sec^2 x}{\tan x}dx \Rightarrow \frac{1}{2}\ln|4+y^2| = \ln\tan x + c_1$

$\therefore \ln|4+y^2| = 2\ln\tan x + 2c_1$ let $2c_1 = \ln c$

$\Rightarrow \ln|4+y^2| = 2\ln\tan x + \ln c$

$\Rightarrow \ln|4+y^2| = \ln|\tan^2 x| + \ln|c| = \ln|c\tan^2 x|$

$\Rightarrow 4+y^2 = c\ \tan^2 x \ \ \therefore y^2 = c\ \tan^2 x - 4$

Task

Find the general solution to the equation

a) $\frac{dy}{dx} = \frac{y}{x}$ **Ans:** $y = A.x$

b) $\cos^2 x\frac{dy}{dx} = y+3$ **Ans:** $\ln y = \frac{x^2}{2} - x + c$

c) $\frac{\sin x}{1+y}\ \frac{dy}{dx} = \cos x$ **Ans:** $y = A\sin x - 1$

3. **LINEAR EQUATIONS: Use Integrating Factor**

This solves the equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = f(x)$, $Q = g(x)$ or P , Q are constants

To solve this equation, we multiply the equation by an integrating factor (I.F) which is always

$$\text{I.F} = e^{\int p\,dx}$$

This converts the L.H.S in to a complete differential coefficient.

NOTE

When evaluating $\text{I.F} = e^{\int p\,dx}$, we donot include a constant of integration. This omission is for convenience because if the constant is added on both sides of the equation, they would subsequently cancel.

Rule to Solve a Linear Differential Equation

$\boldsymbol{\frac{dy}{dx} + Py = Q}$**, where** $\boldsymbol{P = f(x)}$, $\boldsymbol{Q = g(x)}$ **or P , Q are constants**

Multiply both sides by an Integrating Factor.

The general solution to Linear Differential equation is

$$\text{y.I.F} = \int Q.\text{I.F}\,dx$$

Example

Solve the differential equation

a) $\frac{dy}{dx} - y = x$

b) $x\frac{dy}{dx} + y = x^3$

c) $(1 - x^2)\frac{dy}{dx} = xy + 1$

Solution

a) Comparing $\frac{dy}{dx} - y = x$ with $\frac{dy}{dx} + Py = Q$

$\Rightarrow P = -1$, $Q = x$

$\Rightarrow \text{I.F} = e^{\int p\,dx} = e^{\int -dx} = e^{-x}$

Multiplying through by e^{-x}

$e^{-x}.\frac{dy}{dx} - (y.e^{-x}) = x.e^{-x}$

$\Rightarrow \frac{d}{dx}(y.e^{-x}) = x.e^{-x}$, integrating

$\Rightarrow \int \frac{d}{dx}(y.e^{-x})\,dx = \int x.e^{-x}\,dx$

Integrating $\int x.e^{-x}\,dx$

Let $u = x \Rightarrow du = dx$, $dv = e^{-x}\,dx \Rightarrow v = -e^{-x}$

$\therefore \int x.e^{-x}\,dx = -x.e^{-x} - \int -e^{-x}\,dx$

$= -x.e^{-x} - e^{-x} + c$

$\therefore y.e^{-x} = -x.e^{-x} - e^{-x} + c$

$\therefore y = -x - 1 + ce^{x}$

b) $x\frac{dy}{dx} + y = x^3$, rewriting, divide through by x

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^2$

Comparing $\frac{dy}{dx} + \frac{y}{x} = x^2$ with $\frac{dy}{dx} + Py = Q$

$\Rightarrow P = \frac{1}{x}, Q = x^2$

$\Rightarrow \text{I.F} = e^{\int p\,dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

Multiplying through by x

$x.\frac{dy}{dx} - \left(\frac{y}{x}.x\right) = x^3$

$\Rightarrow \frac{d}{dx}(y.x) = x^3$, integrating

$\Rightarrow \int \frac{d}{dx}(y.x)\,dx = \int x^3\ dx$

$\Rightarrow y.x = \frac{x^4}{4} + c$ or $y = \frac{x^3}{4} + \frac{c}{x}$

c) $(1 - x^2)\frac{dy}{dx} = xy + 1$

Re arranging

$(1 - x^2)\frac{dy}{dx} - xy = 1$, dividing through by $(1 - x^2)$

$\Rightarrow \frac{dy}{dx} - y.\frac{x}{1-x^2} = \frac{1}{1-x^2}$

Comparing $\frac{dy}{dx} - y.\frac{x}{1-x^2} = \frac{1}{1-x^2}$ with $\frac{dy}{dx} + Py = Q$

$\Rightarrow P = -\frac{x}{1-x^2}, Q = \frac{1}{1-x^2}$

$\Rightarrow \text{I.F} = e^{\int p\,dx} = e^{\int \frac{-x}{1-x^2} dx} = e^{\frac{1}{2}\ln|1-x^2|} = \sqrt{1-x^2}$

Using $\text{y.I.F} = \int Q.\text{I.F}\,dx$

$\Rightarrow y.\sqrt{1-x^2} = \int \frac{\sqrt{1-x^2}}{1-x^2} dx = \int \frac{1}{\sqrt{1-x^2}} dx$

$\Rightarrow y.\sqrt{1-x^2} = \sin^{-1} x + c$

$\therefore y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} + \frac{c}{\sqrt{1-x^2}}$

Task

Solve the differential equation

a) $\frac{dy}{dx} + 3y = e^{4x}$ **Ans:** $y = \frac{e^{4x}}{7} + ce^{-3x}$; $I.F = e^{3x}$

b) $x\frac{dy}{dx} + y = x\sin x$ **Ans:** $xy = \sin x - x\cos x + c$ $I.F = \sin x$

c) $(x^2 - 1)\frac{dy}{dx} + 2xy = x$ **Ans:** $y(x^2 - 1) = \frac{x^2}{2} + c$

d) $x\frac{dy}{dx} - 2y = x^3\cos x$ **Ans:** $y = x^2(\sin x + c)$

e) $(y^2 - x)dx + 2ydy = 0$ **Ans:** $y^2 - x + 1 = ce^{-x}$

HOMOGENEOUS EQUATIONS

By substitution $y = vx$

A homogeneous differential equation is an equation of the form

$M(x, y)dx + N(x, y)dy = 0$, where M and N are homogeneous functions of x and y of same degree

NOTE

Example

i) $(x^2 + xy)dx + y^2dy = 0$ or $y^2\frac{dy}{dx} = x^2 + xy$..... Degree 2^0

ii) $\frac{dy}{dx} = \frac{x+2y}{2x}$ ..... Degree 1^0

iii) $x^3 + y^3 = 3xy^2\frac{dy}{dx}$..... Degree 3^0

To solve the equation use:

a) $y = vx$..... (1)

b) Differentiating eqn. (1) w.r.t x

$\frac{dy}{dx} = x\frac{dv}{dx} + v$ ..... (2)

Example

1. *Solve the differential equation*

a) $\frac{dy}{dx} = \frac{x+3y}{2x}$ b) $(x^2 + y^2)dx = xy.dy$

Solution

a) $\frac{dy}{dx} = \frac{x+2y}{2x}$

Let $y = vx$..... (1)

$\Rightarrow \frac{dy}{dx} = \frac{x+3.vx}{2x} = \frac{1+3v}{2}$

Differentiating eqn. (1) w.r.t x

$\frac{dy}{dx} = x\frac{dv}{dx} + v$ ..... (2)

$\therefore x\frac{dv}{dx} + v = \frac{1+3v}{2}$

$\Rightarrow x\frac{dv}{dx} = \frac{1+3v}{2} - v = \frac{1+v}{2}$ $\therefore x\frac{dv}{dx} = \frac{1+v}{2}$

Separating variables

$\frac{2}{1+v}\,dv = -\frac{1}{x}\,dx$, integrating

$\int \frac{2}{1+v}\,dv = \int -\frac{1}{x}\,dx$

$2\ln|1+v| = \ln|x| + c_1 \Rightarrow \ln\left|\frac{(1+v)^2}{x}\right| = c_1$

$\Rightarrow e^{c_1} = \frac{(1+v)^2}{x}$

$\Rightarrow xe^{c_1} = (1+v)^2$, but $v = \frac{y}{x}$

$\Rightarrow xe^{c_1} = \left(1 + \frac{y}{x}\right)^2 = \frac{(x+y)^2}{x^2}$

$\Rightarrow x^3e^{c_1} = (x+y)^2$ $\therefore x^3A = (x+y)^2$; $A = e^{c_1}$

b) $(x^2 + y^2)dx = xy.dy$

Let $y = vx$..... (1)

$\Rightarrow \frac{dy}{dx} = \frac{x^2(1+v^2)}{x^2 v} = \frac{1+v^2}{v}$ ..... (*)

Differentiating eqn. (1) w.r.t x

$\frac{dy}{dx} = x\frac{dv}{dx} + v$ ..... (2)

Eqn. (2) in to eqn. (*)

$\therefore x\frac{dv}{dx} + v = \frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{v} - v = \frac{1}{v}$, separating variables

$v\,dv = \frac{dx}{x}$, integrating

$\int v\,dv = \int \frac{dx}{x} \Rightarrow \frac{v^2}{2} = \ln|x| + c$, but $v = \frac{y}{x}$

$\Rightarrow \frac{y^2}{2x^2} = \ln|x| + c \quad \therefore y^2 = 2x^2\ln|x| + 2x^2 c$

Solution

Solve the differential equation

a) $(x-y)\frac{dy}{dx} = x + y$ **Ans:** $\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\ln\left|\frac{x^2+y^2}{x^2}\right| + \ln|x| + \ln|A|$

b) $2x^2 dy = (x^2 + y^2)dx$ **Ans:** $\frac{2x}{x-y} = \ln|x| + c$

c) $x^3 + y^3 = 3xy^2\frac{dy}{dx}$ **Ans:** $x^3 - 2y^3 = 2Ax$

d) $(2y - x)\frac{dy}{dx} = 2x + y$ given that when $x = 2, y = 3$ **Ans:** $y^2 - xy - x^2 + 1 = 0$

e) $xy + y^2 + (x^2 - xy)\frac{dy}{dx} = 0$ **Ans:** $xy = Ae^{y/x}$

Transformations

Using the substitutions given, solve the differential equation

a) $(3x + 3y - 4)\frac{dy}{dx} = -(x + y)$, use $v = x + y$ **Ans:** $x + 3y + 2\ln|x + y - 2| = A$

b) $(x - y - 1) + (4y + x - 1)\frac{dy}{dx} = 0$, use $v = x - 1$

Ans: $\ln|4y^2 + (x-1)^2| + \tan^{-1}\left(\frac{2y}{x-1}\right) = A$

APPLICATION OF DIFFERENTIAL EQUATIONS

Some of the applications of differential equations are:

- Radioactivity Decay
- Population Growth
- Chemical Reactions
- Newton's Law of Cooling
- Gravitational Force
- Tangents

a) **Newton's Law of Cooling**

This states that "Rate of decrease of the difference in temperature of a body is proportional to the difference between the temperature of the body and that of the medium"

If θ is the temperature of the body at time t and θ_0 the constant temperature of the medium, then

$\frac{d\theta}{dt} \propto (\theta - \theta_0)$ or $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

$\Rightarrow \frac{d\theta}{\theta-\theta_0} = -k\,dt$, integrating

$\int \frac{d\theta}{\theta-\theta_0} = -\int k\,dt \Rightarrow \ln|\theta - \theta_0| = -k\,t + c$

If initially θ_1 is the temperature of the body at $t = 0$, then

$\ln|\theta - \theta_0| = c$

$\therefore \ln|\theta - \theta_0| - \ln|\theta_1 - \theta_0| = -k\,t$

$\ln\left|\frac{\theta-\theta_0}{\theta_1-\theta_0}\right| = -k\,t \Rightarrow e^{-kt} = \frac{\theta-\theta_0}{\theta_1-\theta_0}$

$$\therefore \theta = \theta_0 + (\theta_1 - \theta_0)e^{-kt}$$

Example

1. *Water at a temperature of* 100^0C *cools in 10 minutes to* 88^0C *in a room of temperature* 25^0C*. Find the temperature of water after 20 minutes.*

Solution

Using the above result

$\theta = \theta_0 + (\theta_1 - \theta_0)e^{-kt}$

$\Rightarrow \theta_0 = 25\,, t = 10\,, \theta_1 = 100\,, \theta = 88$

$\therefore 88 = 25 + (100 - 25)e^{-kt} \quad \Rightarrow \frac{21}{25} = e^{-10t}$ ..... (1)

We want to find θ after 20 minutes

$\therefore \theta = 25 + (100 - 25)e^{-20t}$

$\theta = 25 + 75e^{-20t}$ ..... (2)

Eqn. (1) in to eqn. (2)

$\therefore \theta = 25 + 75\left(\frac{21}{25}\right)^2 = 77.92^0C$

2. *The rate at which the body cools is proportional to the difference between the temperature of the body and that of the surrounding air.*
If a body in air at 25^0C *will cool from* 100^0C *to* 75^0C *in 1 minute, find its temperature at the end of 3 minutes.*

Solution

Let the temperature of the body at any time t be θ.

$\frac{d\theta}{dt} = -k(\theta - 25) \Rightarrow \frac{d\theta}{\theta-25} = -k\,dt$, integrating

$\int \frac{d\theta}{\theta-\theta_0} = -\int k\,dt \Rightarrow \ln|\theta - 25| = -k\,t + c$ ..... (1)

When $t = 0\,,\ \theta = 100$

$\therefore c = \ln(100 - 25) = \ln 75$

$\Rightarrow \ln|\theta - 25| = -k\,t + \ln 75$ ..... (2)

When $t = 1\,,\ \theta = 75$

$\Rightarrow \ln|75 - 25| = -k\ + \ln 75 \quad \therefore -k = \ln\left(\frac{50}{75}\right) = \ln\left(\frac{2}{3}\right)$

$\Rightarrow e^{-k} = \frac{2}{3}$

From eqn. (2)

$\ln|\theta - 25| = -k\,t + \ln 75$

$\Rightarrow \ln\left|\frac{\theta-25}{75}\right| = -k\,t \qquad \therefore \frac{\theta-25}{75} = e^{-kt}$

$\therefore \theta = 25 + 75e^{-kt}$

When $t = 3$, then θ =?

$\therefore \theta = 25 + 75e^{-3k} = 25 + 75\left(\frac{2}{3}\right)^3 = 47.22^0C$

3. *The rate at which ice melts is proportional to the amount of ice at that instant. Find the amount of ice left after 2 hours if half the quantity melts in 30 minutes.*

Solution

Let m = amount of ice at any instant time t

$\frac{dm}{dt} \propto m \Rightarrow \frac{dm}{dt} = k$

$\Rightarrow \frac{dm}{m} = k\,dt$, integrating

$\ln m = k\,t + c$ ..... (1)

At $t = 0\,, m = M$

$\therefore \ln M = c$

$\ln m = k\,t + \ln M$ ..... (2)

Now when $t = \frac{1}{2} hrs = 30\ minutes, m = \frac{M}{2}$

$\Rightarrow \ln\left(\frac{M}{2}\right) = \frac{k}{2} + \ln M$

$\Rightarrow \frac{k}{2} = \ln\left(\frac{M}{2}\right) - \ln M = \ln\left(\frac{1}{2}\right) = -\ln 2$

$\therefore k = -2\ln 2$

From eqn. (2)

$\ln m = -2\ln 2\,t + \ln M$ ..... (3)

On putting $t = 2\ hours$ into eqn. (3)

$\ln\left(\frac{m}{M}\right) = -4\ln 2 = \ln\left(\frac{1}{16}\right)$

$\frac{m}{M} = \frac{1}{16}\ \therefore m = \frac{M}{16}$

After 2 hours, amount of ice left is $\frac{1}{16}$ of the amount of ice at the beginning.

4. *According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air.*

a) *If the temperature of the air is* $300k$ *and the substance cools from* $370k$ *to* $340k$ *in 15 minutes, find when the temperature will be* $310k$

b) *Suppose a room is kept at a constant temperature of* 70^0C, *and an object cools from* 350^0C *to* 150^0C *in 45 minutes. How long will it take for the object to cool to a temperature of* 80^0C?

Solution

let T be the temperature of the substance at time t min

$\Rightarrow \frac{dT}{dt} = -k\,(T - 300)$ or $\frac{dT}{T-300} = -k\,dt$

NOTE

The use of $-k$ here is optional. It will be found that k is $+ve$, but if $+k$ is used it will be found that k is equally negative.

Integrating between limits

$t = 0\,, T = 370$ and $t = 15\,, T = 340$

$\int_{370}^{340} \frac{dT}{T-300} = \int_0^{15} -k\,dt$

$\Rightarrow \ln 40 - \ln 70 = -15\,k = \ln\left(\frac{4}{7}\right)$ and $15k = \ln\frac{7}{4} = 0.56$

Integrating between limits

$t = 0\,, T = 370$ and $t = t\,, T = 310$

$\int_{370}^{310} \frac{dT}{T-300} = \int_0^{t} -k\,dt$

$\Rightarrow \ln 10 - \ln 170 = -kt = \ln\left(\frac{4}{7}\right)$ and $15kt = 15\ln 7$

$\therefore t = \frac{15\ln 7}{0.56} = 52\,min$

$t = 119.7\,min$

Task

1. *A hot body at a temperature of* 100^0C *is placed in a hot room of temperature of* 20^0C. *10 minutes later its temperature is* 60^0C.

i) *Write down a differential equation to represent the rate of change of temperature* θ *of the body with time* t.

ii) *Determine the temperature of the body after another 10 minutes* (**Uneb 2005**)

Answers

i) $\frac{d\theta}{dt} = -k(\theta - R)\,, R$ =room temperature= 20^0C

ii) $\theta = 40^0C$ at $t = 10 + 10 = 20\,min$

2. *At* 3.00 *pm, the temperature of a hot metal was* 80^0C *and that of the surroundings* 20^0C. *At 3:03 pm the temperature of the metal had dropped to* 42^0C. *The rate of cooling of the metal was directly proportional to the difference between its temperature* θ *and that of the surroundings.*

a) (i) *write a differential equation to represent rate of cooling of the metal.*

b) *Find the temperature of the metal at 3:05 pm* (**Uneb 2012**)

3. *It is observed that the rate at which at which a body cools is proportional to the amount by which its temperature exceeds that of its surroundings. A body at* 78^0C *is placed in a room at* 20^0C *and after 5 minutes the body*

*has cooled to 65^0C, what will be the temperature after a further 5 minutes? (***Uneb 2011)** **Ans:**

4. *According to Newton's law of cooling, the rate of cooling of a body in air is proportional to the difference between the temperature T of the body and the temperature T_0 of the air. If the air temperature is kept constant at 20^0C and the body cools from 100^0C to 60^0C in 20 minutes, in what further time will the body cool to 30^0C? **Ans:** 40 minute*
5. *A body is placed in a room which is kept at a constant temperature. The temperature of the body falls at a rate $k\theta$ C per minute; hwrer k is a constant and θ is the difference between the temperature of thje body and that of the room at time t.*
   Express this information in form of a differential equation and hence show that $\theta = \theta_0 e^{-kt}$, where θ_0 is the temperature difference at time $t = 0$. The temperature of the body falls 5^0 C in the first minute and 4^0 C in the second minute. Show that the fall of temperature in the third minute is 3.2^0 C. **Ans:** $\frac{d\theta}{dt} = -k\theta$
6. *The rate at which a heated body cools in air is proportional to the difference between the temperature of the body and that of the surrounding air. A body originally at 80^0 C cools down to 60^0 C in 20 minutes, the temperature of air being 40^0 C. What will be the temperature of the body after 40 minutes from the original?*

Population Growth

1. *If the population of a country doubles in 50 years, in how many years will it treble, assuming that the rate of increase is proportional to the number on inhabitants?*

Solution

Let t = time in years

y = Population after t − years

p = Original population (when $t = 0$)

Rate of increase $\propto$ populationn

$\Rightarrow \frac{dy}{dt} = k\,y$, where t = constant

$\frac{dy}{y} = \int k\,dt \Rightarrow \ln|y| = kt + c$

When $t = 0$, $y = p$

When $t = 50$, $y = 2p$

$\Rightarrow \ln|p| = c \therefore \ln|y| = kt + \ln|p|$

$\Rightarrow \ln|2p| = 50k + \ln|p|$

$\Rightarrow \ln\left|\frac{2p}{p}\right| = 50k \quad \therefore k = \frac{1}{50}\ln 2$

Now when $y = 3p$, t =?

$\Rightarrow \ln|y| = \frac{t}{50}\ln 2 + \ln|p|$

$\Rightarrow \ln\left|\frac{y}{p}\right| = \frac{t}{50}\ln 2$

$\therefore \ln\left|\frac{3p}{p}\right| = \frac{t}{50}\ln 2 \Rightarrow \frac{50\ln 3}{\ln 2} = t \quad \therefore t = 79\ years$

2. *The rate of growth of a population of fruit flies is proportional to the size of the population at any instant. If there are 180 flies after the* 2nd *day of the experiment and 300 after the* 4th *day, how many flies were there in the original population?*

Solution

Let the original population $= p_0$ at any time and $p =$ population at any time.

$\frac{dp}{dt} = k\,p$

$\int \frac{dp}{p} = \int k\,dt \Rightarrow \ln p = kt + c$

At $t = 0\,, p = p_0 \Rightarrow \ln p_0 = c$

$\therefore \ln p = kt + \ln p_0 \quad$ or $\quad \ln\left|\frac{p}{p_0}\right| = kt$

Now when $t = 2\,, p = 180$

$t = 4\,, p = 300$

$\Rightarrow \ln\left|\frac{180}{p_0}\right| = 2k$ ..... (1)

$\Rightarrow \ln\left|\frac{300}{p_0}\right| = 4k$ ..... (4)

Eqn. (2)÷eqn. (1)

$\Rightarrow \frac{\ln\left|\frac{300}{p_0}\right|}{\ln\left|\frac{180}{p_0}\right|} = \frac{4k}{2k} = 2$

$\Rightarrow \ln\left|\frac{300}{p_0}\right| = 2\ln\left|\frac{180}{p_0}\right| = \ln\left|\frac{32400}{{p_0}^2}\right|$

$\Rightarrow \frac{300}{p_0} = \frac{32400}{{p_0}^2} \qquad \therefore p_0 = \frac{32400}{300} = 108$

$\therefore$ Original population = 108

3. *In a certain culture of bacteria the rate of increase is proportional to the number present.*

a) *If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours?*

b) *If there are* 10^4 *at the end of 3 hours and* 4×10^4 *at the end of 5 hours, how many were there at the beginning?*

Solution

Let x denote the number of bacteria at time t hours

$\Rightarrow \frac{dx}{dt} = k\,x \quad$ or $\quad \frac{dx}{x} = k\,dt$

a) Integrating

$\ln|x| = kt + c$

Assuming that $x = x_0$ at time $t = 0, \quad c = \ln x_0$

$\therefore \ln|x| = kt + \ln|x_0|$ ..... (1)

At $t = 4\,,\ x = 2x_0$

$\ln|2x_0| = 4k + \ln x_0$

$4k = \ln\left|\frac{2x_0}{x_0}\right| = \ln 2 \qquad \therefore k = \frac{\ln 2}{4}$

$\therefore \ln|x| = \frac{\ln 2}{4}t + \ln x_0$

Now at $t = 12$, x =?

$\ln|x| = \ln|x_0| + \frac{12\ln 2}{4} \Rightarrow \ln|x| = \ln|x_0| + \ln 8$

$\ln x = \ln 8x_0 \qquad \therefore x = 8x_0$

$\therefore$ There are 8 times the original numbers

b) When $t = 3$, $x = 10^4$

From eqn. (1)

$\ln|x| = \ln|x_0| + k\,t \quad \Rightarrow \ln\left|\frac{x}{x_0}\right| = k\,t$

$\frac{x}{x_0} = e^{kt} \qquad \Rightarrow x = x_0 e^{kt}$

$\Rightarrow 10^4 = x_0 e^{3k}$ ..... (1)

When $t = 5$, $x = 4 \times 10^4$

$\Rightarrow 4 \times 10^4 = x_0 e^{5k}$ ..... (2)

Eqn. (2)÷eqn. (1)

$\Rightarrow \frac{4\times 10^4}{10^4} = \frac{x_0 e^{5k}}{x_0 e^{3k}} = e^{2k}$

$\therefore e^{2k} = 4$

From eqn. (1)

$\Rightarrow 10^4 = x_0 e^{3k} = x_0 (e^k)^3$

$\Rightarrow 10^4 = x_0 (4)^3 \qquad \therefore x_0 = \frac{10^4}{8}$ bacteria.

Radioactive Decay

1. *Uranium disintegrates at a rate proportional to the amount present at any instant. If M_1 and M_2 grams of uranium are present at a time T_1 and T_2 respectively, show that the half life of uranium is*

$$\frac{(T_2 - T_1)\ln 2}{\ln(M_1/M_2)}$$

Solution

Let m = Amount present at any time t

$\frac{dm}{dt} \propto m \quad \Rightarrow \frac{dm}{dt} = -k\,m$

$\Rightarrow \int \frac{dm}{dt} = \int -k\,dt$

$\Rightarrow \ln|m| = -kt + c$

For $m = M_1$, $t = T_1$; $\quad m = M_2$, $t = T_2$

$\Rightarrow \ln|M_1| = -kT_1 + c$ ..... (1)

$\Rightarrow \ln|M_2| = -kT_2 + c$ ..... (2)

Eqn. (1)− eqn. (2)

$\Rightarrow \ln\left|\frac{M_1}{M_2}\right| = -k(T_1 - T_2) = k(T_2 - T_1)$ ..... (3)

From $\ln|m| = -kt + c$

If the initial mass is M_0 at $t = 0$

$\ln|M_0| = c$

$\therefore \ln|m| = -kt + \ln|M_0|$

$\ln\left|\frac{m}{M_0}\right| = -kt$

Now for $m = \frac{M_0}{2}$, $t = T$ =?

$\Rightarrow \ln\left|\frac{M_0}{2M_0}\right| = -kT$

$\Rightarrow \ln\frac{1}{2} = -kT$, from eqn. (3) $\ln\left|\frac{M_1}{M_2}\right| = -k(T_1 - T_2) = k(T_2 - T_1) \quad \Rightarrow k = \frac{\ln\left|\frac{M_1}{M_2}\right|}{(T_2 - T_1)}$

$\therefore T = \frac{\ln 2}{k} = \frac{\ln 2}{\frac{\ln\left|\frac{M_1}{M_2}\right|}{(T_2 - T_1)}} = \frac{(T_2 - T_1)\ln 2}{\ln\left|\frac{M_1}{M_2}\right|}$

$\therefore T = \frac{(T_2 - T_1)\ln 2}{\ln\left|\frac{M_1}{M_2}\right|}$

2. *The rate of decomposition of radioactive radium is proportional to the amount present at a given instant. Find the percentage of a present amount that remains after 25 years if the half life of radioactive radium is 1600 years.*

Solution

Let the amounts present at any instant be m

$\Rightarrow \frac{dm}{dt} \propto m \Rightarrow \frac{dm}{dt} = -k\,m$

$\Rightarrow \int \frac{dm}{m} = \int -k\,dt$ or $\ln|m| = -kt + c$

Let the present amount be m_0 at $t = 0$

$\Rightarrow \ln|m_0| = c$

$\ln|m| = -kt + \ln|m_0|$ ..... (1)

Now for $m = \frac{m_0}{2}$, $t = 1600$

$\Rightarrow \ln\left|\frac{m_0}{2}\right| = -1600k + \ln|m_0| \quad \therefore k = \frac{\ln 2}{1600}$ ..... (2)

Now for $t = 25, m = M$

$\ln|M| = -25k + \ln|m_0| \Rightarrow \ln\left|\frac{m_0}{M}\right| = 25k = \frac{25\ln 2}{1600}$

$\ln\left|\frac{m_0}{M}\right| = \frac{\ln 2}{64} \Rightarrow \frac{m_0}{M} = 2^{1/64} = 1.010889$

$\therefore \frac{M}{m_0} = 0.9892$

$\%ge$ of original amount = 98.92%

Task

A substance looses mass s at a rate proportional to the amount M present at any time t.

a) Form a differential equation connecting M, t and the constant of proportionality k.
b) If initially the mass of the substance is M_0 show that $M = M_0 e^{-kt}$

c) Given that half life of the substance is lost in 1600 years, determine the number of years $15g$ of the substance would reduce to $13.6g$

Chemical Reactions

1) *Under certain conditions, cane sugar is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If out of 75grams of sugar at $t = 0$, 8 grams are converted during the first 30 minutes, find the amount converted in $1\frac{1}{2}$ hours.*

Solution

Let M be the amount of cane sugar initially,

m be the amount of cane sugar converted in to dextrose

$\Rightarrow \frac{dm}{dt} = k(M - m)$ or $\frac{dm}{dt} + k\,m = k\,M$

$\text{I.F} = e^{\int k\,dt} = e^{kt}$

Using $I.F.m = \int k\,Me^{kt}\,dt$

$\Rightarrow e^{kt}.m = \int k\,Me^{kt}\,dt = Me^{kt} + c$

$\therefore m = M + c\,e^{-kt}$ ..... (1)

i) At $t = 0\,, m = 0\,, M = 75$

Eqn. (1) becomes

$0 = 75 + c\,e^{-k(0)} \Rightarrow c = -75$

$\therefore m = 75 - 75e^{-kt}$

ii) At $t = 30\,, m = 8\,, M = 75$

$\therefore 8 = 75 - 75e^{-30t} \Rightarrow 67 = 75e^{-30t}$

Now at $t = 1\frac{1}{2}\ hours = 30\ minutes\,,\ m = ?$

$m = 75 - 75e^{-90k}$

$= 75 - 75\left(\frac{67}{75}\right)^3 = 21.55$

Alternatively

Let m be the amount unconverted

M be the initial amount

<u>*Given:*</u> $\frac{dm}{dt} \propto m \Rightarrow \frac{dm}{dt} = -k\,m$

[$-ve$ is used because when cane sugar is converted, the amount unconverted decrease]

$\Rightarrow \int \frac{dm}{m} = \int -k\,dt$ or $\ln|m| = -kt + c$

Now at $t = 0\,,\ m = 75 \quad \therefore c = \ln 75$

$\Rightarrow \ln|m| = -kt + \ln 75$ ..... (1)

Now at $t = 30\,,\ m = 75 - 8 = 67 \quad \therefore c = \ln 75$

$\Rightarrow \ln 67 = -30k + \ln 75$

$\Rightarrow \ln\left|\frac{67}{75}\right| = -30k$ or $e^{-30k} = \frac{67}{75}$ ..... (2)

<u>Required:</u> when $t = 1\frac{1}{2} hrs = 90\ minutes,\ m = ?$

$m = 75 - M$, $M =$ Amount converted

$\Rightarrow \ln|75 - M| = -90k + \ln 75$

$\Rightarrow e^{-90k} = \frac{75-M}{75}$

$\therefore 75e^{-90k} = 75 - M \quad \Rightarrow M = 75 - 75e^{-90k}$

$M = 75 - 75(e^{-90k})^3$

$= 75 - 75\left(\frac{67}{75}\right)^3 = 21.55$

2) *A research to investigate the effect of a certain chemical on a virus infection on crops revealed that the rate at which the virus population is destroyed is proportional to the population at that time. Initially the population was p_0 at t months later it was found to be p.*

a) *Form a differential equation connecting p and t*

b) *Given that the virus population reduced to $\frac{1}{3}$ of the initial population in 4 months solve the equation in (a) above*

c) *Find:*

i) *how long it will take for only 5% of the original population*

ii) *What percentage of the original virus population will be left after 2½ months?* **(Uneb 1999)**

Solution

a) $\frac{dp}{dt} \propto p$ or $\frac{dp}{dt} = -kp$

b) When $t = 4$, $p = \frac{1}{3}p_0$; $t = 0$, $p = p_0$

$\Rightarrow \frac{dp}{p} = -k\,dt \quad \Rightarrow \ln|p| = -kt + c$

For $t = 4$, $p = \frac{1}{3}p_0$, $c = \ln|p_0|$

$\therefore \ln\left|\frac{1}{3}\,p_0\right| = -4k + \ln|p_0|$

$\Rightarrow 4k = \ln|p_0| - \ln\left|\frac{1}{3}\,p_0\right| \quad \therefore k = \frac{1}{4}\ln 3$

$\therefore \ln|p| = -\frac{t}{4}\ln 3 + \ln|p_0|$

$\Rightarrow \ln\left|\frac{p}{p_0}\right| = -\frac{t}{4}\ln 3 = \ln\left|3^{-t/4}\right|$

$\therefore p = p_0.3^{-t/4}$

c) (i) For $p = \frac{5p_0}{100} = \frac{p_0}{20}$, $t =$?

$\Rightarrow \frac{p_0}{20} = p_0.3^{-t/4}$

$\Rightarrow 20 = 3^{\frac{t}{4}}$ or $\ln 20 = \frac{t}{4}\ln 3$

$\therefore t = \frac{4\ln 20}{\ln 3} = 10.907hrs$

ii) Now for $t = 2\frac{1}{2} = \frac{5}{2}$ months, $p =$?

$\Rightarrow p = p_0.3^{-\frac{5}{2}\times\frac{1}{4}} \quad \therefore \frac{p}{p_0} = 3^{-5/8} = 0.503268$

$\therefore \frac{p}{p_0} = 50.3268\%$

Wakissha 2009

A boy starts to sip a 900ml soda from a bottle at a rate of 10cm³/min. Given that the rate of consumption is inversely proportional to the square root of the volume of the soda remaining at any time, find the time it takes to empty the bottle.

Solution

Let volume of the soda remaining at any time t be v

Given: $\frac{dv}{dt} \propto \frac{1}{\sqrt{v}} \Rightarrow \frac{dv}{dt} = -\frac{k}{\sqrt{v}}$

$\Rightarrow \sqrt{v}\, dv = -k\, dt$, integrating

$\int \sqrt{v}\, dv = \int -k\, dt \quad \therefore \frac{2}{3}v^{3/2} = -k\, t + c$

$[k = -ve$ because volume of the soda remaining keeps on decreasing]

Now at $t = 0\,, v = 900$

$\Rightarrow \frac{2}{3}(900)^{3/2} = c \quad \therefore c = 18000$

$\therefore \frac{2}{3}\sqrt{v^3} = -k\, t + 18000$

Now when $v = 0\,, t = T$

$\frac{2}{3}(0) = -k\, T + 18000 \Rightarrow T = \frac{18000}{k}$ ..... (1)

But $\frac{dv}{dt} = -10 = -\frac{k}{\sqrt{900}} \Rightarrow k = 300$

From eqn. (1), $T = \frac{18000}{300} = 60\ minutes$

Thus the bottle will be finished in 60 minutes or 1 hour.

Alternatively

Let $v =$ volume of soda sipped at any time t

Given: $\frac{dv}{dt} \propto \frac{1}{\sqrt{900-v}} \Rightarrow \frac{dv}{dt} = +\frac{k}{\sqrt{900-v}}$

$\Rightarrow \sqrt{900-v}\, dv = k\, dt$, integrating

$\int \sqrt{900-v}\, dv = \int k\, dt \quad \therefore \frac{(900-v)^{3/2}}{-3/2} = kt + c_1$

$\Rightarrow \frac{2}{3}(900-v)^{3/2} = -kt + c\,; c = -c_1$

At $t = 0\,, v = 0 \quad \therefore c = \frac{2}{3}(900)^{3/2} = 18000$

$\Rightarrow \frac{2}{3}(900-v)^{3/2} = -kt + 18000$ ..... (1)

Now $\frac{dv}{dt} = +10 = +\frac{k}{\sqrt{900-v}}$, taking at $t = 0\,, v = 0$

$\Rightarrow k = 10\sqrt{900} = 300$

Eqn. (1) becomes

$\frac{2}{3}(900-v)^{3/2} = -300t + 18000$

Now to empty the bottle, $v = 900\,, t = T$

$\Rightarrow \frac{2}{3}(0) = -300T + 18000 \quad \therefore T = 60\ minutes$

Thus the bottle will be finished within 60 minutes or 1 hour.

Uneb 2004 no. 16

An athlete runs at a speed proportional to the square root of the distance he has to cover. If the athlete starts running at 10 m/s and he has a distance of 1600m to cover, find how long he will take to cover the distance.

Solution

Let x = distance he still has to cover

$\frac{dx}{dt} \propto \sqrt{x}$ or $\frac{dx}{dt} = -k\sqrt{x}$ $\Rightarrow \frac{dx}{\sqrt{x}} = -k\,dt$, integrating

$\int \frac{dx}{\sqrt{x}} = \int -k\,dt \Rightarrow \int x^{-1/2}\,dx = \int -k\,dt$

$\therefore 2\sqrt{x} = -k\,t + c$

At $t = 0\,, x = 1600\,, c = 80$

$\Rightarrow 2\sqrt{x} = -k\,t + 80$..... (1)

Now $\frac{dx}{dt} = -k\sqrt{x} = -10 \;\Rightarrow\; k\sqrt{x} = 10$

When he starts to cover the distance, $x = 1600$

$\Rightarrow k\sqrt{1600} = 10 \;\; \therefore k = \frac{1}{4}$

Eqn. (1) becomes

$2\sqrt{x} = -\frac{t}{4} + 80$

To cover the whole distance 1600m, $x = 0\,, t = T$

$0 = -\frac{T}{4} + 80 \;\; \therefore T = 320$

He takes 320 seconds to cover the whole distance.

Mixture Problems

1) *A tank of volume* $0.5m^3$ *is filled with brine containing 30kg of dissolved salt. Water runs in to the tank at a rate of* $15 \times 10^{-5} m^3/s$ *and the mixture, kept uniform by stirring, runs out at the same rate. How much salt is in the tank after 1 hour?*

Solution

Let x =no. of kg of salt in the tank after t (seconds)

Concentration of salt/proportion of salt in brine at any time

$\frac{x}{0.5} = 2x\ kg/m^3$

Considering a small change in t .i.e. dt for water to run in and out

Quantity of salt out of tank in (m^3) $= (15 \times 10^{-5})dt$

Now quantity of salt flowing out in dt

$(15 \times 10^{-5})dt \times 2x = 3x \times 10^{-4}\,dt$

Thus, the change dx of the amount of salt in the tank is

$dx = 3x \times 10^{-4}\,dt$

$\Rightarrow \frac{1}{3\times10^{-4}} . \frac{dx}{x} = -dt$, integrating

$\frac{1}{3\times10^{-4}} \ln|x| = -t + c_1$

$\Rightarrow \ln|x| = -3 \times 10^{-4} t + c_1(3 \times 10^{-4})$

$\Rightarrow x = e^{-3\times10^{-4}t+c_1(3\times10^{-4})} = c\,e^{-3\times10^{-4}t}\ ,\ c = e^{c_1(-3\times10^{-4})}$

Now when $t = 0\ , x = 30\ \Rightarrow c = 30$

$\therefore x = 30\,e^{-3\times10^{-4}t}$

Now when $t = 1hr = 3600s\ ,\ x =?$

$\therefore x = 30\,e^{-3\times10^{-4}\times3600} = 10kg$

2) *A tank being cleaned initially contains 3900litres of water and 100kg of dissolved dye. The mixture is kept uniform by stirring.*

Pure water is run in at the rate of 130litres per minute and the mixture is removed at the same rate. If the mass of dissolved dye remaining after t minutes is x kg, show that $\frac{dx}{dt} = -\frac{x}{30}$.

How long will it be before half the dye has been removed? Determine how much dye remains after 30 minutes.

Solution

Let $x =$ mass of dissolved dye remaining after t (seconds)

Concentration of dye in solution $= \frac{x}{3900} = kg/l$

Let the mixture run in/out in dt minutes

Volume rate $= 130l/min$

Quantity of dye running out $= (130\,dt)$ litres

Mass of dye running out $= \frac{x}{3900}\times 130\,dt = \frac{x}{03}dt$ kg

Thus the change dx of the dye in mixture

$dx = -\frac{x}{30}dt \quad \therefore \quad \frac{dx}{dt} = -\frac{x}{30}$

Solving the differential equation

$\frac{dx}{x} = -\frac{dt}{30} \Rightarrow \ln|x| = -\frac{t}{30} + c_1$

$x = e^{-\frac{t}{30}+c_1} \quad \Rightarrow x = ce^{-\frac{t}{30}}\ ,\ c = e^{c_1}$

$\therefore x = ce^{-\frac{t}{30}}$

Now when $x = 100\,kg\ ,\ t = 0\ \ \therefore c = 100$

$\Rightarrow x = 100e^{-\frac{t}{30}}$

Now when $x = \frac{1}{2}\times 100 = 50\,kg\ ,\ t = T$

$\Rightarrow 50 = 100e^{-\frac{T}{30}}$

$\Rightarrow \frac{1}{2} = e^{-\frac{T}{30}} \quad \text{or} \quad \ln\left|\frac{1}{2}\right| = -\frac{T}{30}$

$\therefore T = 30\ln 2\ minutes$

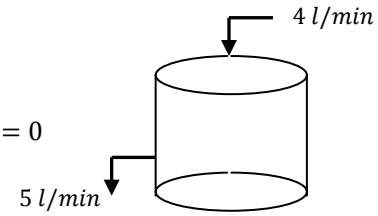
Also when $t = 30\min\ ,\ x =?$

$\Rightarrow x = 100e^{-1}\ kg$

3) *A tank contains 50litres of solution composed of 90% water and 10% alcohol*

A second solution containing 50% water and 50% alcohol is added to the tank at the rate of 4litres/min. as the second solution is being added, the tank is being drained at the rate of 5litres/min.

Assuming the solution in the tank is stirred constantly, how much alcohol is in the tank after 10 minutes?



Solution

Let $y =$ no. of litres of alcohol in the tank

At any time t

Amount of alcohol in tank initially i.e. at $t = 0$

$$= \frac{10}{100} \times 50 = 5$$

At $t = 0$, $y = 5$

No. of litres of solution in tank at any time t is $(50 - t)$

No. of litres lost/drained from tank = 5

$\therefore$ Litres of alcohol lost per minute $= \left(\frac{5}{50-t}\right).y$

No. of litres gained of alcohol per minute $= \frac{50}{100} \times 4 = 2$

Rate of change of alcohol in the tank

$\frac{dy}{dt} = 2 - \left(\frac{5}{50-t}\right).y$ or $\frac{dy}{dt} + \left(\frac{5}{50-t}\right).y = 2$

$\text{I.F} = e^{\int\left(\frac{5}{50-t}\right)dt} = e^{-5\ln|50-t|} = \frac{1}{(50-t)^5}$

Using $\text{I.F.y} = \int \text{I.F.Q}\, dx$

$\frac{y}{(50-t)^5} = \int \frac{2}{(50-t)^5} dt = \frac{1}{2(50-t)^4} + c$

$\therefore y = \frac{50-t}{2} - c(50-t)^5$

Now when $t = 0, y = 5 \quad \therefore c = -\frac{20}{50^5}$

$\therefore y = \frac{50-t}{2} - 20\left(\frac{50-t}{50}\right)^5$

Now when $t = 10, y =$?

$\therefore y = \frac{50-10}{2} - 20\left(\frac{50-10}{50}\right)^5 = 13.45\, l$, which represents a solution containing 33.6% alcohol.

4) *In a certain process the rate of production of yeast is k x grammes per minute, where x = amount produced and* $k = 0.003$.

a) *Show that the amount of yeast is doubled in about 230 minutes*

b) *If in addition, yeast is removed at a constant rate m grammes per minute, find the:*

i) Amount of yeast at time t, given that when $t = 0, x = p$ grammes

ii) Value of m if $p = 20000$grammes and the supply of yeast is exhausted in 100 minutes. **(Uneb 2009 no.16)**

Solution

a) $\frac{dx}{dt} = k\,x \Rightarrow \frac{dx}{x} = k\,dt$, integrating

$\ln|x| = k\,t + c_1$ or $x = e^{k\,t+c_1}$

$\therefore x = ce^{k\,t};\ c = e^{c_1}$

Now when $t = 0, x = x_0$ and $t = T,\ x = 2x_0$

$\therefore x_0 = ce^0 = c$

$\Rightarrow x = x_0 e^{k\,t}$

Now for $x = 2x_0\,, t = T$

$\Rightarrow 2x_0 = x_0 e^{0.003T} \quad \therefore 2 = e^{0.003T}$

$\Rightarrow \ln 2 = \ln e^{0.003T} = 0.003T$

$\therefore T = \frac{\ln 2}{0.003} \simeq 231\ min$

b) Rate of production of yeast = $0.003x$ grammes/min

Rate of removal of yeast = m grammes/min

Rate of change of production in yeast = $0.003x - m$

$\therefore \frac{dx}{dt} = 0.003x - m$ or $\frac{dx}{dt} - 0.003x = -m$

$\text{I.F} = e^{\int -0.003\,dt} = e^{-0.003t}$

$\therefore x.e^{-0.003t} = -\int m.e^{-0.003t}\,dt = \frac{m}{0.003}e^{-0.003t} + c$

Now at $t = 0\,, x = p$

$\therefore pe^0 = \frac{m}{0.003} + c$

$\Rightarrow c = p - \frac{m}{0.003} = p - \frac{1000\,m}{3} = \frac{3p - 1000\,m}{3}$

$\therefore x.e^{-0.003t} = \frac{m}{0.003}e^{-0.003t} + \frac{3p-1000\,m}{3}$

Now when $p = 20000$, $t = 100\,\text{min}$, $x = 0$

$\Rightarrow 0 = \frac{m}{0.003}e^{-0.03} + \frac{60000 - 1000\,m}{3}$

$\Rightarrow 0 = \frac{1000\,m}{3}e^{-0.03} + \frac{60000 - 1000\,m}{3}$

$\Rightarrow 0 = \frac{1000\,m\left(e^{-0.03} - 1\right) + 60000}{3}$

$\therefore m = -\frac{60000}{1000\,(e^{-0.03} - 1)} = \frac{60}{(1 - e^{-0.03})} = 2030.15$

Task

1. Glucose is added intravenously to the blood stream at a rate of $q\ units$ per minute, and the body removes glucose from the blood stream at a rate proportional to the amount present.

Assume $Q(t)$ is the amount of glucose in the blood stream at time t,

a) Determine the differential equation describing the rate of change w.r.t time of glucose in the blood stream.

b) Taking $Q = Q_0$ when $t = 0$, solve the differential equation

Answers

a) $\frac{dQ}{dt} = q - k\,Q$ b) $Q = \frac{q}{k} + \left(Q_0 - \frac{q}{k}\right)e^{-kt}$

2. A 200 m^3 tank is full of a solution containing 25 kg of concentrate. Starting at $t = 0$, distilled water is admitted to the tank at a rate of 10 m^3/min, and the well stirred solution is withdrawn at the same rate.

a) Find the amount of concentrate in the solution as a function of t

b) Find the time at which the concentrate in the tank reaches 15kg

c) Assuming that the solution entering the tank contains 0.05 kg of concentrate per m^3. Repeat (a) and (b) above.

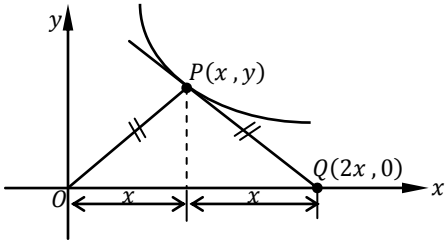
Answer (ii)

a) $q = 10 + 15e^{-0.005t}$ b) $t = -20\ln\frac{1}{3} \simeq 21.97\ min$

Tangent to the Curve

1. *The tangent at any point P on a curve in the first quadrant cuts the $x-axis$ at Q. Given that $OP = PQ$, where O is the origin and that the point $(1,4)$ lies on the curve, find the equation of the curve.*

Solution



Let $P(x,y)$ be the point
ΔOPQ is isosceles
$\Rightarrow Q(2x,0)$

Now Gradient of a tangent at P $= \frac{dy}{dx}$. i.e. Grad $\overline{PQ}$

$\Rightarrow \frac{dy}{dx} = \frac{0-y}{2x-x} = -\frac{y}{x}$

Separating variables

$\frac{dy}{y} = -\frac{1}{x}\,dx$, integrating

$\int \frac{dy}{y} = \int -\frac{1}{x}\,dx \ \Rightarrow \ln|y| = -\ln|x| + c$

$\therefore \ln|x.y| = c \ \Rightarrow e^c = x.y$

Now $(1,4)$ is on this curve

$\Rightarrow e^c = 1 \times 4 = 4$

$\therefore xy = 4$

Task

1. The tangent at any point P on a curve cuts the x −axis a t the point Q. given that $< OPQ = 90^0$ where O is the origin and that the point $(1,2)$ lies on the curve, find the equation of the curve.
Ans: $x^2 + y^2 = 5$

2. The tangent at any point P on the curve cuts the $x-axis$ at A and the y −axis at B. given that $2\overrightarrow{AP} = \overrightarrow{PB}$ and the curve passes through the point $(1,1)$, find the equation of the curve. **Ans:** $y = 1/x^2$

3. The tangent of a curve at the point (x,y) cuts the $x-$ axis at Q. if $OQ = kx$(where O is the origin of the coordinate system) show that the points on the curve satisfy the differential equation. **Ans:** $\frac{dy}{dx} = \frac{y}{x-kx}$. By solving the differential equation show that the equation of the curve has the form $y = Ax^{1/(1-k)}$ where A is an arbitrary constant.

CIRCLES

Definition:

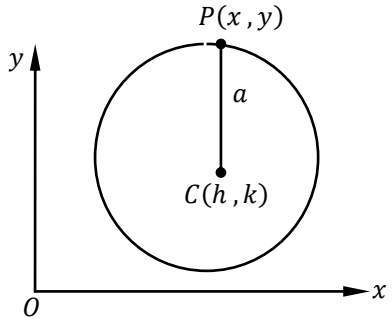
The locus of a point which moves in such a way that its distance from a fixed point is always a constant

The fixed point is called the *centre* and the fixed/constant distance is the *radius*.

Equation of a Circle
(Cartesian Equation of Locus of a Point)

A. To find the Equation of the Circle given the Centre (h ,k) and Radius, a

Consider the figure below



The point $P(x\,,y)$ whose locus is required lies on the circle.

From geometry,

Distance $\overline{CP} = a \Rightarrow \sqrt{(x-h)^2+(y-k)^2} = a$

Squaring both sides gives the equation of the circle.

$$\boxed{(x-h)^2+(y-k)^2 = a^2}$$

NOTE:

If the centre $C(h\,,k)$ is $(0\,,0)$ $.i.e.$ at the origin the equation of the circle is $\boxed{x^2+y^2=a^2}$, which is the simplest equation of a circle, called the *standard equation* of the circle.

Examples

1) *Find the equation of a circle having*

a) *Radius* 2 *units* and centre $(-1\,,2)$

b) *Radius* $\sqrt{3}$ *units* and centre $(-2\,,-3)$

Solution

Using the result $(x-h)^2+(y-k)^2=a^2$

a) $(h\,,k)=(-1\,,2)\ \therefore h=-1\,,k=2$

$Radius = a = 2$

$\Rightarrow (x--1)^2+(y-2)^2=2^2$

$\therefore\ (x+1)^2+(y-2)^2=4$

b) $(h\,,k)=(-2\,,-3)\ \therefore h=-2\,,k=-3$

$Radius=a=\sqrt{3}$

$\Longrightarrow (x--2)^2+(y--3)^2=\left(\sqrt{3}\right)^2$

$\therefore\ (x+2)^2+(y+3)^2=3$

Task

Find the equation of a circle with

a) *Radius* 3 *units* and centre *origin* **Ans:** $x^2+y^2=9$

b) *Radius* $\frac{1}{2}\sqrt{5}\,units$ and centre $\left(\frac{1}{2}\,,\frac{1}{2}\right)$ **Ans:** $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{5}{4}$

Stating the Centre and Radius of the circle in the form

$(\boldsymbol{x-h})^{\boldsymbol{2}}+(\boldsymbol{y-k})^{\boldsymbol{2}}=\boldsymbol{a^2}$

Example

State the centre and radius of the circle;

i) $(x-1)^2+(y-3)^2=4$

ii) $(x+2)^2+(y+5)^2=5$

iii) $(x+5)^2+(y-3)^2-8=0$

Solution

The above equations must be written in the form

$(x-h)^2+(y-k)^2=a^2$

i) $(x-1)^2+(y-3)^2=4$

Re writing

$\Longrightarrow (x-1)^2+(y-3)^2=2^2$ [4 is written in terms of power 2]

$\therefore Centre\,(1\,,3)\,,Radius=2\,units$

ii) $(x+2)^2+(y+5)^2=5$

Re writing

$\Longrightarrow [x-(-2)]^2+[y-(-5)]^2=\left(\sqrt{5}\right)^2$ [the $-ve$ sign in the squared brackets has been introduced, 5 is written in terms of power 2]

$\therefore Centre\,(-2\,,-5)\,,Radius=\sqrt{5}\,units$

Task

State the centre and radius of the circle;

i) $(x+2)^2+\left(y-\frac{1}{4}\right)^2=10$ **Ans:** $Centre\,\left(-2\,,\frac{1}{4}\right),Radius=\sqrt{10}\,units$

ii) $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=20$ **Ans:** $Centre\left(-\frac{1}{2}\,,\frac{1}{2}\right),Radius=\sqrt{20}\,units$

iii) $(x+5)^2+(y+4)^2-40=0$ **Ans:** $Centre\,(-5\,,-4),Radius=\sqrt{40}\,units$

General Equation of the Circle

From the equation of the circle $(x-h)^2+(y-k)^2=a^2$,

Expanding the brackets

$x^2-2hx+h^2+y^2-2ky+k^2=a^2$

$\boxed{x^2+y^2-2hx-2ky+h^2+k^2-a^2=0}$, which is the general equation of a circle.

Rewriting this equation of the circle

Let $-h=g$; $-k=f$ and $h^2+k^2-a^2=c$

Thus the general equation of a circle can be stated as;

$\boxed{x^2+y^2+2gx+2fy+c=0}$

Deduction:

For the general equation to represent a circle,

- The coefficient of x^2 and y^2 must be equal
- No term involving xy
- It's a second degree equation

Examples

State with reasons whether the equations represent a circle

a) $x^2+y^2-2x+4y+2=0$
b) $2x^2+2y^2-4x+2y-3=0$
c) $4x^2-4y^2+4x+2y-3=0$
d) $3x^2+2y^2-5x+y-1=0$
e) $3x^2+3y^2-5xy+x+y=0$

Solution

a) $x^2+y^2-2x+4y+2=0$ is a circle because the coefficients of x^2, $y^2$are the same, no product of xy term
b) $2x^2+2y^2-4x+2y-3=0$ is a circle because the coefficients of x^2, $y^2$are the same, no product of xy term
c) $4x^2-4y^2+4x+2y-3=0$ is Not a circle because the coefficients of x^2, $y^2$are not the same.
d) $3x^2+2y^2-5x+y-1=0$ is Not a circle because the coefficients of x^2, $y^2$are not the same.
e) $3x^2+3y^2-5xy+x+y=0$ is Not a circle because of the product xy

STATING THE CENTRE AND RADIUS OF THE GENERAL EQUATION OF THE CIRCLE $x^2 + y^2 + 2gx + 2fy + c = 0$

Firs rewriting the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ in to the form
$(x-h)^2 + (y-k)^2 = a^2$
$(x^2 + 2gx) + (y^2 + 2fy) = -c$
Completing squares
$\Rightarrow (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = -c + g^2 + f^2$
$\Rightarrow (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$

$$\therefore\ [x-(-g)]^2 + [y-(-f)]^2 = \left[\sqrt{g^2+f^2-c}\right]^2$$

$\therefore$ Centre $(-g, -f)$, Radius is $\sqrt{g^2+f^2-c}$

$\therefore$ $\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0}$ has **centre** $\boldsymbol{(-g, -f)}$ and **Radius** $\boldsymbol{\sqrt{g^2+f^2-c}}$

NOTE:
In general, a circle expressed in the form $x^2 + y^2 + 2gx + 2fy + c = 0$ has:
Centre $(-g, -f) = \left[-\frac{1}{2} \times coeff.of\ x\ ,\ -\frac{1}{2} \times coeff.of\ y\right]$ and
Radius
$= \sqrt{g^2+f^2-c} = \sqrt{(sum\ of\ squares\ of\ centre\ coodinatres) - \ constant}$

Example

1) *State the coordinates of centre and radius of the circle*

i) $x^2 + y^2 + 2x - 4y - 4 = 0$

ii) $x^2 + y^2 - 3x + 4y = 0$

Solution

i) $x^2 + y^2 + 2x - 4y - 4 = 0$
Rewriting the equation
$(x^2 + 2x) + (y^2 - 4y) = 4$
Completing squares
$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 4 + 1 + 4$
$(x+1)^2 + (y-2)^2 = 2^2$
$\therefore Centre\ (-1, 2)\ Radius = 2\ units$

ii) $x^2 + y^2 - 3x + 4y = 0$
Rewriting the equation
$(x^2 - 3x) + (y^2 + 4y) = 0$
Completing squares
$\left(x^2 - 3x + \frac{9}{4}\right) + (y^2 + 4y + 4) = \frac{9}{4} + 4 = \frac{25}{4}$
$\left(x - \frac{3}{2}\right)^2 + (y+2)^2 = \left(\frac{5}{4}\right)^2$

$\therefore Centre\ \left(\frac{3}{2}, 2\right)\ Radius = \frac{5}{2}\ units$

2) *State the coordinates of centre and radius of the circle*

i) $2x^2 + 2y^2 - 3x + 4y + 2 = 0$

ii) $3x^2 + 3y^2 - 4x + 6y + 3 = 0$

Solution

i) $2x^2 + 2y^2 - 3x + 4y + 2 = 0$

Making coefficients of $x^2,\ y^2$unity by dividing through by 2

$x^2 + y^2 - \frac{3}{2}x + 2y + 1 = 0$

Rewriting the equation

$\left(x^2 - \frac{3}{2}x\right) + (y^2 + 2y) = -1$

Completing squares

$\left(x^2 - 3x + \frac{9}{16}\right) + (y^2 + 2y + 1) = -1 + \frac{9}{16} + 1$

$\left(x - \frac{3}{4}\right)^2 + (y + 1)^2 = \left(\frac{3}{4}\right)^2$

$\therefore Centre\ \left(-\frac{3}{4}, -1\right)\ Radius = \frac{3}{4}\ units$

ii) $3x^2 + 3y^2 - 4x + 6y + 3 = 0$

Making coefficients of $x^2,\ y^2$unity by dividing through by 3

$x^2 + y^2 - \frac{4}{3}x + 2y + 1 = 0$

Rewriting the equation

$\left(x^2 - \frac{4}{3}x\right) + (y^2 + 2y) = -1$

Completing squares

$\left(x^2 - 3x + \frac{4}{9}\right) + (y^2 + 2y + 1) = -1 + \frac{4}{9} + 1$

$\left(x - \frac{2}{3}\right)^2 + (y + 1)^2 = \left(\frac{2}{3}\right)^2$

$\therefore Centre\ \left(\frac{2}{3}, -1\right)\ Radius = \frac{2}{3}\ un殴ts$

Task

State the coordinates of centre and radius of the circle

i) $4x^2 + 4y^2 - 5x + 3y - 8 = 0$

ii) $x^2 + y^2 - 5x + 3y - 1 = 0$

iii) $Ax^2 + Ay^2 + Bx + Cy + D = 0$

Ans: (i) $Centre$: $\left(\frac{5}{8}, -\frac{3}{8}\right), Radius = \frac{9}{4}\sqrt{2}\ units$

Ans: (ii) $Centre$: $\left(\frac{5}{2}, -\frac{3}{2}\right), Radius = \frac{\sqrt{19}}{2}\ units$

Ans: (i) $Centre$: $\left(-\frac{D}{2A}, -\frac{F}{2A}\right), Radius = \frac{1}{2A}\sqrt{D^2 + F^2 - 4AC}\ units$

APPLICATION OF GENERAL EQUATION OF A CIRCLE $x^2 + y^2 + 2gx + 2fy + c = 0$

To find the equation of a circle passing through <u>THREE</u> points

Example

Find the Cartesian equation of a circle through points

i) *Origin , $A(1\,,2)$ and $B(-2\,,-1)$*

ii) $A(1\,,1)\,,B(2\,,-1)$ and $C(3\,,2)$

Solution

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

i) For $O(0\,,0)$; $0 + 0 + 0 + 0 + c = 0 \quad \therefore c = 0$

For $A(1\,,2)$; $1 + 4 + 2g + 4f + c = 0$

$\therefore 2g + 4f = -5$ ..... (1)

For $B(-2\,,-1)$; $4 + 1 - 4g - 2f + c = 0$

$\therefore -4g - 2f = -5$ ..... (2)

Solving eqn. (1) and eqn. (2) simultaneously

Eqn. (1)$\times 2$ + eqn. (2) yields

$f = -\frac{5}{2}$ ..... (3)

Eqn. (3) in to eqn. (2)

$\Rightarrow 2g + 4\left(-\frac{5}{2}\right) = -5$, from which $g = \frac{5}{2}$

$\Rightarrow x^2 + y^2 + 2\left(\frac{5}{2}\right)x + 2\left(-\frac{5}{2}\right)y = 0$

$\therefore x^2 + y^2 + 5x - 5y = 0$

ii) For $A(1\,,\ 1)$; $1 + 1 + 2g + 2f + c = 0$

$\therefore 2g + 2f + c = 0$ ..... (1)

For $B(2\,,-1)$; $4 + 1 + 4g - 2f + c = 0$

$\therefore 4g - 2f + c = -5$ ..... (2)

For $C(3\,,\ 2)$; $9 + 4 + 6g + 4f + c = 0$

$\therefore 6g + 4f + c = -13$ ..... (3)

Eliminating c from eqn. (1) and eqn. (2)

Eqn. (1) $-$ eqn. (2) yields

$-2g + 4f = 3$ ..... (4)

Eliminating c from eqn. (1) and eqn. (3)

Eqn. (3) $-$ eqn. (2) yields

$-4g - 2f = 11$ ..... (5)

Now solving eqn. (4) and eqn. (5) simultaneously

Eqn. (4)$\times 2$ $-$eqn. (5) to eliminate gives

$f = -\frac{1}{2}$ ..... (6)

Putting eqn. (6) in to eqn. (5) or eqn. (4) yields

$g = -\frac{5}{2}$ ..... (7)

Putting eqn. (7) in to eqn. (1) or eqn. (2) or eqn. (3) yields

$c = 4$

Equation of the circle is

$x^2 + y^2 + 2\left(-\frac{5}{2}\right)x + 2\left(-\frac{1}{2}\right)y + 4 = 0$

$\therefore x^2 + y^2 - 5x - y + 4 = 0$

Task

a) *Find the equation of a circle passing through points* $A(-2\,,-1)\,, B(4\,,-2)$ *and* $C(2\,,1)$

Ans: $8x^2 + 8y^2 - 15x + 50y - 20 = 0$

b) *Form an equation of a circle which passes through the points* $A(-1\,,4)\,, B(2\,,5)$ *and* $C(0\,,1)$ (**Uneb 2006**)

Ans: $x^2 + y^2 - 7x - 9y + 24 = 0$

c) *Find the equation of a circle which passes through the points* $(5\,,7)\,,(1\,,3)$ *and* $(2\,,2)$. (**Uneb 2005**)

Ans: $x^2 + y^2 - 7x - 9y + 24 = 0$

d) *Find the equation of a circle circumscribing the triangle whose vertices are* $A(1\,,3)\,, B(4\,,-5)$ *and* $C(9\,,-1)$. *Find also the centre and radius.*

Ans: $13x^2 + 13y^2 - 113x + 8y - 41 = 0$, centre: $\left(\frac{113}{26}\,, -\frac{8}{26}\right)$, Radius: 4.71

To find an Equation of the Circle passing through <u>TWO</u> given points and satisfying another condition

Example

Find an equation of a circle which passes through points

a) $(4\,,1)$ *and* $(6\,,5)$ *and having its centre n the line* $4x + y - 16 = 0$

b) $(0\,,0)$ *and* $(1\,,1)$ *and having its radius as 4 units*

Solution

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

a) For $(4\,,1)$; $16 + 1 + 8g + 2f + c = 0$

$\therefore 8g + 2f + c = -17$ ..... (1)

For $(6\,,5)$; $36 + 25 + 12g + 10f + c = 0$

$\therefore 12g + 10f + c = -61$ ..... (2)

Centre of the circle $(-g\,,-f)$ lies on the line $4x + y - 16 = 0$

$\therefore -4g - f = 16$ ..... (3)

Solving eqn. (1) and eqn. (2) simultaneously, eliminating c

Eqn. (1) − eqn. (2) yields

$-4g - 8f = 44$ ..... (4)

Eliminating c from Eqn. (3) and eqn. (4)

Eqn. (3) − eqn. (4) yields

$\Rightarrow 7f = -28$, from which $f = -4$..... (5)

Putting eqn. (5) in to eqn. (3) or eqn. (4)

$g = -3$ ..... (6)

Putting eqn. (6) in to eqn. (1) or eqn. (2)

$c = 15$

$\Rightarrow x^2 + y^2 - 6x - 8y + 15 = 0$

b) For $(0\,,0)$; $0 + 0 + 0 + 0 + c = 0 \quad \Rightarrow c = 0$

For $(1,1)$; $1+1+2g+2f+c=0$

$\therefore g+f=-1$ ..... (1)

Radius $=\sqrt{g^2+f^2-c}=4$

$\Rightarrow g^2+f^2=16$ ..... (2)

From eqn. (1), $g=-(f+1)$

Now eqn. (2) becomes

$[-(f+1)]^2+f^2=16$

$\Rightarrow f^2+2f+1+f^2=16$ *or* $2f^2+2f-15=0$

Solving the equation yields

$\therefore f=\frac{-1\pm\sqrt{31}}{2}$ *or* $f=\frac{-1+\sqrt{31}}{2}$, $\frac{-1-\sqrt{31}}{2}$

From eqn. (1),

$g=-(f+1)$

For $f=\frac{-1+\sqrt{31}}{2}$, $g=-\left(\frac{-1+\sqrt{31}}{2}+1\right)=\frac{\sqrt{31}-1}{2}$

For $f=\frac{-1-\sqrt{31}}{2}$, $g=-\left(\frac{-1-\sqrt{31}}{2}+1\right)=-\left(\frac{\sqrt{31}+1}{2}\right)$

Circle are:

$x^2+y^2+2\left(\frac{\sqrt{31}-1}{2}\right)x+2\left(\frac{-1+\sqrt{31}}{2}\right)y=0$

$\therefore x^2+y^2+\left(\sqrt{31}-1\right)x-\left(\sqrt{31}+1\right)=0$

Also

$x^2+y^2+2\left(\frac{-\sqrt{31}-1}{2}\right)x-2\left(\frac{1+\sqrt{31}}{2}\right)y=0$

$\therefore x^2+y^2-\left(\sqrt{31}+1\right)x-\left(\sqrt{31}-1\right)y=0$

To find the equation of a circle passing through <u>ONE</u> given point and with a stated centre coordinates

Example

Find the equation of a circle with

a) *Centre* $(-1,2)$ *and passing through point* $(-3,1)$

b) *Centre* $(4,-1)$ *and passing through point* $(1,3)$

Solution

Let the circle be $x^2+y^2+2gx+2fy+c=0$

Centre $(-g,-f)$

a) $\Rightarrow(-g,-f)=(-1,2)$ $\therefore g=1, f=-2$..... (1)

For $(-3,1)$, $9+1-6g+2f+c=0$

$\therefore -6g+2f+c=-10$ ..... (2)

Putting eqn. (1) in to eqn. (2)

$\Rightarrow -6-4+c=-10$ $\therefore c=0$

The equation of the circle is

$\therefore x^2+y^2+2x-4y=0$

b) $\Rightarrow(-g,-f)=(4,-1)$ $\therefore g=-4$, $f=1$..... (1)

For $(1,3)$, $1+9+2g+6f+c=0$

$\therefore 2g + 6f + c = -10$ .... (2)

Putting eqn. (1) in to eqn. (2)

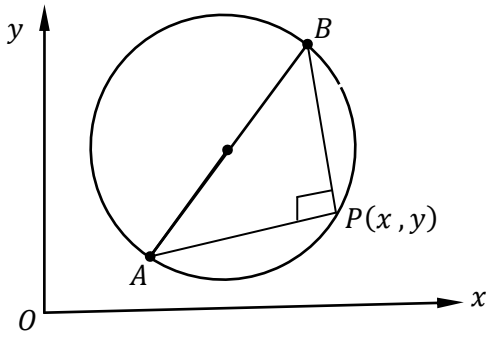
$\Rightarrow -8 + 2 + c = -4 \quad \therefore c = -4$

The equation of the circle is

$\therefore x^2 + y^2 - 8x + 2y - 4 = 0$

DIAMETER FORM OF EQUATION OF A CIRCLE

To find the equation of a circle having $A(x_1\,, y_1)$ and $B(x_2\,, y_2)$ as ends of the diameter



Considering point $P(x\,, y)$ to be the general point on the circle

From circle properties; $< A\hat{P}B = 90^0$

From geometry, $slope\ \overline{AP} \times slope\ \overline{PB} = -1$

Slope $\overline{AP} = \frac{y-y_1}{x-x_1}$

Slope $\overline{PB} = \frac{y-y_2}{x-x_2}$

$\therefore \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$

$$\therefore (\boldsymbol{y-y_1})(\boldsymbol{y-y_2}) + (\boldsymbol{x-x_1})(\boldsymbol{x-x_2}) = \boldsymbol{0}$$

, which is the equation of the circle in diameter form.

Examples

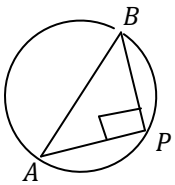
Find the equation of a circle having extremities points given as diameter

a) $A(1\,, 2)$ and $B(-5\,, 6)$

b) $A(3\,, 0)$ and $B(7\,, -2)$

Solution

a) $A(1\,, 2)$ and $B(-5\,, 6)$



Taking a general point $P(x\,, y)$ on the circle,

Grad $\overline{AP} = m_1 = \frac{y-2}{x-1}$

Grad $\overline{PB} = m_2 = \frac{y-6}{x+5}$

By geometry $m_1 \times m_2 = -1$

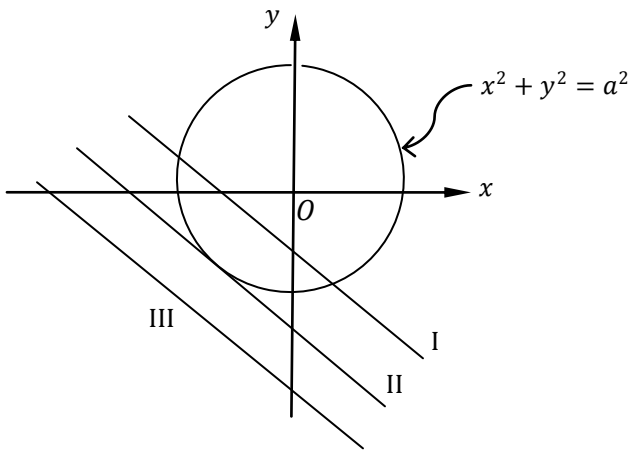
$\therefore \left(\frac{y-2}{x-1}\right) \times \left(\frac{y-6}{x+5}\right) = -1 \Rightarrow (x-1)(x+5) + (y-2)(y-6) = 0$

$\Rightarrow x^2 + 4x - 5 + y^2 - 8y + 12 = 0$
$\therefore x^2 + y^2 + 4x - 8y + 7 = 0$
b) Proceeding as in (a) above, the equation of the circle is;
$(x-3)(x-7) + y(y+2) = 0$ or $x^2 + y^2 - 10x + 2y + 21 = 0$

NOTE:
The question on diameter equation of a circle can also be transformed in to "To find the equation of a circle passing through **one** point and having a stated centre $i.e.$ midpoint of diameter extremity points.

Example
With reference to (a) above,
$A(1\,,2)$ and $B(-5\,,6)$
Centre of $AB = \left(\frac{1+-5}{2}\,,\frac{2+6}{2}\right) = (-2\,,6)$
Using $x^2 + y^2 + 2gx + 2fy + c = 0$
For $A(1\,,2)$; $1 + 4 + 2g + 4f + c = 0$
$\Rightarrow 2g + 4f + c = -5$ ..... (1)
Centre: $(-g\,,-f) = (-2\,,6)$ $\therefore g = 2\,, f = -6$ ..... (2)
Eqn. (2) in to eqn. (1) to obtain c
$\Rightarrow 4 - 16 + c = -5$ $\therefore c = 7$
$\therefore x^2 + y^2 + 4x - 8y + 7 = 0$

A LINE (TANGENT) AND A CIRCLE
A tangent is a line *touching* the circle at *only one point.* i.e. one point of contact.
Consider a circle $x^2 + y^2 = a^2$ and the line (tangent) $y = mx + c$



The *three* key questions asked about a line and tangents are:

To find the point(s) of intersection of a line and a circle

$x^2 + y^2 = a^2$ ..... (1)

$y = mx + c$ ..... (2)

Solving simultaneously

$x^2 + (mx + c)^2 = a^2$

$\Rightarrow x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$

$$x^2(1 + m^2) + (2mc)x + (c^2 - a^2) = 0$$

This equation is a quadratic in x which can be solved for values of x.

Three cases arise when solved.

Case I: *Two distinct points of contact*

For two distinct and real roots of a quadratic equation, $B^2 - 4AC > 0$

$\Rightarrow (2mc)^2 - 4(1 + m^2)(c^2 - a^2) > 0$

$\Rightarrow 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - m^2a^2) > 0$

$\Rightarrow -c^2 + a^2 + m^2a^2 > 0$

$$a^2(1 + m^2) > c^2$$

Case II: *One coincident root .i.e. one point of contact*

This is a case of showing that a line is a tangent to the circle

For the quadratic equation $B^2 - 4AC = 0$

Condition for a line to be a tangent is

$$a^2(1 + m^2) = c^2 \text{ or } c = \pm a\sqrt{1 + m^2}$$

To find the point of contact of tangent and a circle

Putting $c = \pm a\sqrt{1 + m^2}$ in to $x^2(1 + m^2) + (2mc)x + (c^2 - a^2) = 0$

$\Rightarrow x^2(1 + m^2) \pm \left(2ma\sqrt{1 + m^2}\right)x + (a^2(1 + m^2) - a^2) = 0$

$\Rightarrow x^2(1 + m^2) \pm \left(2ma\sqrt{1 + m^2}\right)x + a^2m^2 = 0$ this should be factorizable to a perfect square

$\Rightarrow \left(x\sqrt{1 + m^2} \pm am\right)^2 = 0$ from which $x = \pm \frac{am}{\sqrt{1+m^2}}$

Obtaining the corresponding value of y,

From $y = mx + c, x = \pm \frac{am}{\sqrt{1+m^2}}$ and $c = \pm a\sqrt{1 + m^2}$

$\Rightarrow y = \pm \frac{am^2}{\sqrt{1+m^2}} \pm a\sqrt{1 + m^2} = \pm\left(am^2 + a(1 + m^2)\right) = \pm a(2m^2 + 1)$

$\therefore$ *Point of contact is*

$$\left[\pm \frac{am}{\sqrt{1+m^2}}, \pm a(2m^2 + 1)\right]$$

Case III: N*on real roots. i.e. No point of contact*

A line and a circle do not intersect if $B^2 - 4AC < 0$

$$a^2(1 + m^2) < c^2$$

Examples

Prove that the line

a) $3x + 4y - 16 = 0$ *touches the circle* $x^2 + y^2 - 14x - 10y + 49 = 0$

b) $3x - 4y + 1 = 0$ *touches the circles* $x^2 + y^2 - 4x - 16y + \frac{187}{9} = 0$ *and* $(x-2)^2 + (y-8)^2 = 25$

c) $12x + 5y - 115 = 0$ *touches the circle* $x^2 + y^2 - 14x - 2y + 46 = 0$

d) $3x - 4y - 7 = 0$ *touches the circle* $x^2 + y^2 - 14x - 2y + 46 = 0$. *Hence find the point of contact in each of the above questions*

Solution

a) $3x + 4y - 16 = 0$, $x^2 + y^2 - 14x - 10y + 49 = 0$

$\Rightarrow x = \left(\frac{16-4y}{3}\right)$ ..... (1)

$x^2 + y^2 - 14x - 10y + 49 = 0$ ..... (2)

Eqn. (1) in to eqn. (2)

$\Rightarrow \left(\frac{16-4y}{3}\right)^2 + y^2 - 14\left(\frac{16-4y}{3}\right) - 10y + 49 = 0$

Simplifying

$25y^2 - 50y + 25 = 0$ or $y^2 - 2y + 1 = 0$

$\Rightarrow (y-1)^2 = 0 \ \therefore y = 1, 1$

Since there is only one value of $y = 1$ then there is only one point of contact. Hence the line is a tangent.

Point of contact

For $y = 1$, $x = \frac{16-4\times1}{3} = 4$

$\therefore$ $(4, 1)$ is the point of contact.

b) Left as an exercise. Point of contact is $\left(\frac{7}{3}, 2\right)$

c) Left as an exercise

d) Left as an exercise

EQUATION OF A TANGENT TO A GIVEN CIRCLE

To find the equation of a tangent to a circle at a point $P(x_1, y_1)$ on the circle $x^2 + y^2 = a^2$

The general equation of a line passing through one point is:

$y - y_1 = m(x - x_1)$ ..... (1)

Finding slope, m of the tangent

Since a tangent is passing through $P(x_1, y_1)$ which is on the circle, from differentiation, the derivative of a curve at a point is the slope of the tangent.

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(a^2)$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \ \therefore \frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow m = \frac{dy}{dx}\Big|_{at\ (x_1, y_1)} = -\frac{x_1}{y_1}$ ..... (2)

Eqn. (2) in to eqn. (1) yields

$y - y_1 = \left(-\frac{x_1}{y_1}\right)(x - x_1)$

$\Rightarrow yy_1 - {y_1}^2 = -xx_1 + {x_1}^2$

$$\boxed{xx_1 + yy_1 = {x_1}^2 + {y_1}^2 = a^2}$$

, which is the equation of the tangent to the standard equation of a circle.

Example

Find the equation of a tangent to the circle at the given points;

a) Circle: $x^2 + y^2 = 25$, point: $(-3\,, 4)$
b) Circle: $x^2 + y^2 + 4x - 4y + 4 = 0$, point $(0\,, 2)$

Solution

a) $x^2 + y^2 = 25$, point $(-3\,, 4)$ lies on the circle. i.e. $(-3)^2 + (4)^2 = 25$

Tangent equation

$y - 4 = m(x + 3)$ ..... (1)

Finding the slope m

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow m = \frac{dy}{dx}\Big|_{at\ (-3\,,4)} = -\frac{4}{-3} = \frac{4}{3}$

$\therefore y - 4 = \frac{4}{3}(x + 3)$, simplifying gives the tangent equation as

$\therefore 4x - 3y + 24 = 0$

b) $x^2 + y^2 + 4x - 4y + 4 = 0$, point $(0\,, 2)$ lies on the circle. i.e. $(0)^2 + (2)^2 + 4(0) - 4(2) + 4 = 0$

Tangent equation

$y - 2 = m(x - 0)$ ..... (1)

Finding the slope m

$\frac{d}{dx}(x^2 + y^2 + 4x - 4y + 4) = \frac{d}{dx}(0)$

$\Rightarrow 2x + 2y\frac{dy}{dx} + 4 - 4\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{4-2x}{2y-4}$

$\Rightarrow m = \frac{dy}{dx}\Big|_{at\ (0\,,2)} = -\frac{4-2(0)}{2(2)-4} = -\frac{4}{0}$

From eqn. (1),

$\frac{y-2}{x} = m = -\frac{4}{0}$

$\therefore x = 0$ is the tangent equation

Task

Find the equtaion of the tangent to the circle

i) $x^2 + y^2 + 4x - 4y + 4 = 0$ at the point $(-2\,, 0)$
ii) $x^2 + y^2 - 4x + 2y + 1 = 0$ at the point $(2\,, -1)$
iii) $x^2 + y^2 - 4x + 2y = 25$ at the point $(2\,, -3)$

Ans:

To find the equation of a circle touching a known line and satisfying atleast one other given condition.

Example

1) *Find the equation of a circle whose centre is* $\left(\frac{5}{2}, -3\right)$ *and touches the line* $3x + 2y + 5 = 0$

2) *Find the equation(s) of the circle(s) whhich touch the* $y-axis$ *and pass through points* $(1, -2)$ *and* $(3, -4)$

3) *Find the equation of the circle with radius* 5 *units and touching the line* $3x + 4y - 16 = 0$ *at the point* $(4, 1)$

4) *Find the equation of the circle which is inscribed in a triangle formrd by the lines* $3x + 4y - 15 = 0, 3x - 4y - 7 = 0$ *and* $12x + 5y - 115 = 0$

5) *Find the equation of the circle touchhing* $x-axis$ *and* $y-axis$ *at points* $(0, 4)$ *and* $(6, 0)$

6) (i) If $x = 0$ and $y = 0$ are tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, show that $c = g^2 = f^2$

ii) Given that the line $3x - 4y + 6 = 0$ is also a tangent to the circle in (i) above, determine the equation of the circle lying in the first quadrant. (**Uneb 2005**)

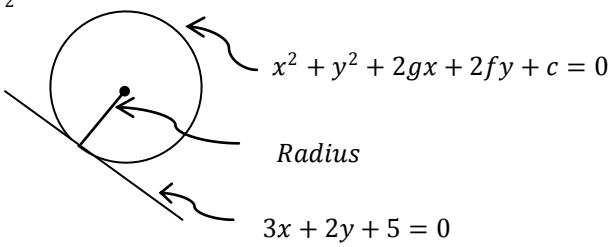
Solution

1) Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Line $3x + 2y + 5 = 0$, centre $\left(\frac{5}{2}, -3\right)$

$\left(\frac{5}{2}, -3\right) = (-g, -f)$

$\therefore g = -\frac{5}{2}, f = 3$ ..... (1)



Radius = perpendicular distance of centre from tangent

$= \frac{3(5/2)+2(-3)+5}{\sqrt{3^2+2^2}} = \frac{13}{2\sqrt{13}}$

But radius $= \sqrt{g^2 + f^2 - c}$

$\therefore \frac{13}{2\sqrt{13}} = \sqrt{g^2 + f^2 - c}$, squaring both sides

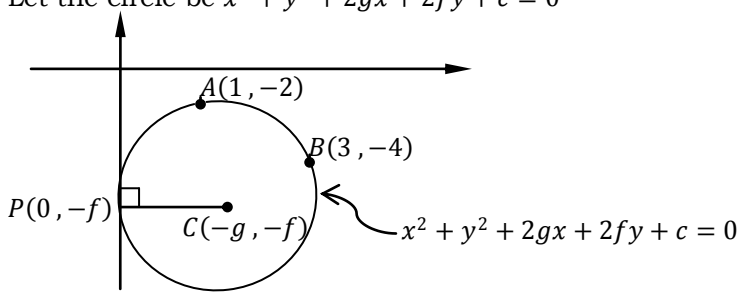
$\Rightarrow \frac{13}{4} = \frac{25}{4} + 9 - c \quad \therefore c = 12$

Required equation of a circle is:

$x^2 + y^2 + 2\left(-\frac{5}{2}\right)x + 2(3)y + 12 = 0$

$\therefore x^2 + y^2 - 5x + 6y + 12 = 0$

2) Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$



For $A(1\,,-2)$: $1 + 4 + 2g - 4f + c = 0$

$\therefore 2g - 4f + c = -5$ ..... (1)

For $B(3\,,-4)$: $9 + 16 + 6g - 8f + c = 0$

$\therefore 6g - 8f + c = -25$ ..... (2)

Let the circle touch $y - axis$ at P

$\Rightarrow$ Coordinates of P are $(0\,,-f)$. This point lies on the circle

$\Rightarrow 0 + f^2 + 0 - 2f^2 + c = 0$

$\therefore c = f^2$ ..... (3)

Eliminating g from eqn. (1) and eqn. (2)

Eqn. (1)$\times 3 -$ eqn. (2)

$\Rightarrow 3(2S - 4f + c) - (6g - 8f + c) = -15 - -25$

$\therefore -2f + c = 0$ ..... (4)

Eqn. (3) in to eqn. (4)

$\Rightarrow -2f + f^2 = 0 \;\; \therefore f = 0\,, 2$

From eqn. (1): $2g - 4f + c = -5$

For $f = 0\,, c = 0\,, g = -\frac{5}{2}$

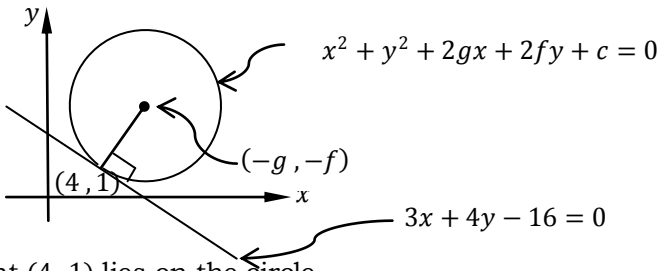
For $f = 2\,, c = 4\,, g = 2$

Thus the equation(s) of a circle are:

$\therefore x^2 + y^2 - 5x = 0$

$\therefore x^2 + y^2 + 4x + 4y + 4 = 0$

3)



Point $(4\,,1)$ lies on the circle

$\Rightarrow 16 + 1 + 8g + 2f + c = 0$

$\therefore 8g + 2f + c = -17$ ..... (1)

Radius $= \sqrt{g^2 + f^2 - c} = 5$

$\Rightarrow 25 = g^2 + f^2 - c$ ..... (2)

Also perpendicular distance of centre $(-g\,,-f)$, from the line $3x+4y-16=0$ is the radius

$\therefore 5=\left|\frac{3(-g)+4(-f)-16}{\sqrt{3^2+4^2}}\right|=\frac{-3g-4f-16}{5}$

$\Rightarrow -3g-4f-16=25$ or $-3g-4f=41$..... (3)

Eliminating c from eqn. (1) and eqn. (2)

Eqn. (1) + eqn. (2) yields

$\therefore g^2+f^2+8g+2f=8$ ..... (4)

From eqn. (3), $g=\left(\frac{41+4f}{-3}\right)$

Substituting this in to eqn. (4)

$\Rightarrow \left(\frac{41+4f}{-3}\right)^2+f^2+8\left(\frac{41+4f}{-3}\right)+2f=8$

Simplifying the equation we get

$f^2+10f+25=0\ \ or\ \ (f+5)^2=0\ \ \therefore f=-5$

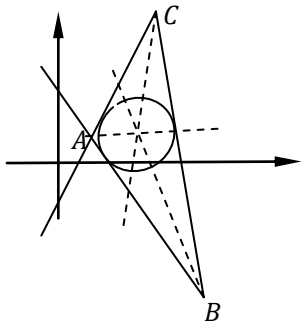
Now from $g=\left(\frac{41+4f}{-3}\right)$, $f=-5$ gives

$\therefore g=-7$

From eqn. (1) $8g+2f+c=-17$, $g=-7$, $f=-5\ \ \therefore c=49$

$\therefore x^2+y^2-14x-10y+49=0$

4) Sketching the lines



$\overline{AC}$ is $3x-4y-7=0$,
$\overline{AB}$ is $3x+4y-15=0$
$\overline{BC}$ is $12x+5y-115=0$
Dotted lines are Angle Bisectors

Finding the equations of Angle bisectors

Finding equations of angle bisectors between lines

$3x+4y-15=0$ and $3x-4y-7=0$

$\frac{3x+4y-15}{\sqrt{3^2+4^2}}=\pm\frac{3x-4y-7}{\sqrt{3^2+4^2}}$

Taking the $+ve$ sign since the slope of the bisector is $+ve$

$\frac{3x+4y-15}{5}=\frac{3x-4y-7}{5}\ \ \therefore y=1$ ..... (1)

Finding equations of angle bisectors between lines

$3x+4y-15=0$ and $12x+5y-115=0$

$$\frac{3x+4y-15}{\sqrt{3^2+4^2}}=\pm\frac{12x+5y-115}{\sqrt{12^2+5^2}}$$

Taking the $-ve$ sign since the slope of the bisector is $-ve$

$\frac{3x+4y-15}{5}=-\frac{12x+5y-115}{13}\ \ \therefore 9x+7y-70=0$ ..... (2)

Solving eqn. (1) and eqn. (2) simultaneously, $x = 7\,, y = 1$
Thus in centre of the circle is $(7\,, 1)$
Radius = perpendicular distance of centre from any tangent
Using $3x + 4y - 15 = 0$
Radius = $\frac{21+4-15}{5} = 2$
Thus equation of the circle is:
$\therefore (x-7)^2 + (y-1)^2 = 2^2 \;\; or\; x^2 + y^2 - 14x - 2y + 46 = 0$

Alternatively

Taking the centre of the circle to be $(-g\,, -f)$ for the circle
$x^2 + y^2 + 2gx + 2fy + c = 0$
Perpendicular distance of the centre from each line is the radius.
This gives two equations which can be solved for values of the centre.

5) Left as an exercise

6) (i) Solving $x = 0$ and $x^2 + y^2 + 2gx + 2fy + c = 0$ simultaneously
$\Rightarrow y^2 + 2fy + c = 0$
For tangency, $B^2 - 4AC = 0$
$\Rightarrow (2f)^2 = 4c \;\; \therefore f^2 = c$
Solving $y = 0$ and $x^2 + y^2 + 2gx + 2fy + c = 0$ simultaneously
$\Rightarrow x^2 + 2gx + c = 0$
For tangency, $B^2 - 4AC = 0$
$\Rightarrow (2g)^2 = 4c \;\; \therefore g^2 = c$
$\therefore c = g^2 = f^2$

(ii) Since $\therefore c = g^2 = f^2$, the equation of the circle becomes
$\Rightarrow\; x^2 + y^2 + 2fx + 2fy + f^2 = 0$
Now $3x - 4y + 6 = 0 \;\; \Rightarrow x = \left(\frac{4y-6}{3}\right)$is a tangent to this circle
Solving the equation of the circle and this line simultaneously
$\Rightarrow \left(\frac{4y-6}{3}\right)^2 + y^2 + 2\left(\frac{4y-6}{3}\right) + 2fy + f^2 = 0$
Simplifying the equation gives
$\therefore 25y^2 + y(-48 + 42f) + (36 - 36f + 9f^2) = 0$
Now for tangency $B^2 - 4AC = 0$
$\Rightarrow (-48 + 42f)^2 = 4(25)(36 - 36f + 9f^2)$
$\Rightarrow 4(-24 + 21f)^2 = 4(25)(36 - 36f + 9f^2)$
$\Rightarrow 576 - 1008f + 441f^2 = 900 - 900f + 225f^2$
Simplifying yields
$\Rightarrow 2f^2 - f - 3 = 0 \;\; or \;\; (f+1)(2f-3) = 0$ ..... (*)
$\therefore f = 1\,, -\frac{3}{2}$
Since the centre of the circle is $(-g\,, -f)$
$\Rightarrow -f = 1\,, -\frac{3}{2}$

Taking $f = -1$ to bring the circle to the first quadrant, from $c = g^2 = f^2 = (-1)^2$

$\therefore (1, 1)$ is the centre of the circle.

Hence the equation of the circle in the first quadrant is

$\Rightarrow x^2 + y^2 + 2(-1)x + 2(-1)y + 1 = 0$

$\therefore x^2 + y^2 - 2x - 2y + 1 = 0$

Note

Equation (*) can also be obtained by considering the perpendicular distance of the centre $(-g, -f)$ from the tangent $3x - 4y + 6 = 0$

TANGENT IN SLOPE FORM

Referring to the standard equation of the circle $x^2 + y^2 = a^2$ and the tangent $y = mx + c$, it was shown that if this line is a tangent to the circle then $a^2(1 + m^2) = c^2$

$\therefore c = \pm a\sqrt{(1 + m^2)}$

Equation of tangent to the circle in slope form is

$$y = mx = \pm a\sqrt{(1 + m^2)}$$

Note

The lines $y = mx + a\sqrt{(1 + m^2)}$ and $y = mx - a\sqrt{(1 + m^2)}$ have the same gradient, hence two tangents can be drawn to the circle parallel to any given line.

Example

1) *Find the equations of the tangents to the circle* $x^2 + y^2 = 4$ *which are parallel to the line* $x + 2y + 3 = 0$*.hence find their points of contact.*

Solution

Let the required tangent be $y = mx + c$

Here we are to obtain the values of c and m

Obtaining slope m

If $y = mx + c$ and $x + 2y + 3 = 0$ are parallel, then their gradients are the same.

$\Rightarrow m = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}x + c$ is the required tangent

Now obtaining c, apply condition for tangency

Solving $y = -\frac{1}{2}x + c$ and $x^2 + y^2 = 4$ simultaneously

$\Rightarrow x^2 + \left(-\frac{1}{2}x + c\right)^2 = 4$

Simplifying the equation yields

$5x^2 - 4cx + (4c^2 - 16) = 0$

For tangency $B^2 - 4AC = 0$

$\Rightarrow (-4c)^2 - 4(5)(4c^2 - 16) = 0$
$\Rightarrow 16c^2 = 80(c^2 - 4)$
$\Rightarrow c^2 = 5c^2 - 20 \ \ or \ \ c^2 - 5 = 0$
$\therefore c = \pm\sqrt{5}$
Tangents are: $y = -\frac{1}{2}x \pm \sqrt{5}$
Points of contact
From $5x^2 - 4cx + (4c^2 - 16) = 0$, $c = \pm\sqrt{5}$ .Taking $c = +\sqrt{5}$
$\Rightarrow 5x^2 - 4\sqrt{5}x + 4 = 0 \ \Rightarrow \left(\sqrt{5}x - 2\right)^2 = 0 \ \ \therefore x = \frac{2\sqrt{5}}{5}$
$\Rightarrow y = -\frac{1}{2}\left(\frac{2\sqrt{5}}{5}\right) + \sqrt{5} = \frac{4\sqrt{5}}{5}$. Point of contact $\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right)$
Taking $c = -\sqrt{5}$,
$\Rightarrow 5x^2 + 4\sqrt{5}x + 4 = 0 \ \Rightarrow \left(\sqrt{5}x + 2\right)^2 = 0 \ \ \therefore x = -\frac{2\sqrt{5}}{5}$
$\Rightarrow \left(\sqrt{5}x + 2\right)^2 = 0 \ \ \therefore x = -2/\sqrt{5}$
$\Rightarrow y = -\frac{1}{2}\left(-\frac{2\sqrt{5}}{5}\right) + \sqrt{5} = \frac{6\sqrt{5}}{5}$. Point of contact $\left(-\frac{2\sqrt{5}}{5}, \frac{6\sqrt{5}}{5}\right)$

2) *Given that the line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$, show that $a^2(1 + m^2) = c^2$*
Hence find the equations of tangents to the circle $x^2 + y^2 = 4$ that are perpendicular to the line $3x + 4y + 5 = 0$
Solution
For the required proof see notes. i.e. left as an exercise
From $y = mx + c$ and $a^2(1 + m^2) = c^2$
Required tangent becomes
$\therefore y = mx \pm a\sqrt{(1 + m^2)}$
Obtaining m
If $y = mx + c$ is perpendicular to $3x + 4y + 5 = 0$ then from $m_1 \times m_2 = -1$,
slope of the tangent is $m = \frac{4}{3}$
Also comparing $x^2 + y^2 = a^2$ with $x^2 + y^2 = 4, \Rightarrow a = 2$
Thus equation of tangent becomes
$\therefore y = mx \pm a\sqrt{(1 + m^2)}$ where $m = \frac{4}{3}$, $a = 2$
$\Rightarrow \ y = \frac{4}{3}x \pm 2\sqrt{\left(1 + \left(\frac{4}{3}\right)^2\right)}$
$\Rightarrow y = \frac{4}{3}x \pm 2 \times \frac{5}{3}$
$\therefore 3y = 4x \pm 10$
3) *Find the condition that the line $ax + by + c = 0$ is a tangent to the circle $x^2 + y^2 + 2gx + 2fy + k = 0$. Hence or otherwise find the equations of tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ that are parallel to the line $x - y + 5 = 0$*

Solution

For a line to be a tangent to the circle,

Radius = perpendicular distance of the centre from the tangent line

Radius $= \sqrt{g^2 + f^2 - k} = \left|\frac{a(-g)+b(-f)+c}{\sqrt{a^2+b^2}}\right|$

Squaring both sides

$\Rightarrow g^2 + f^2 - k = \frac{(-ag-bf+c)^2}{a^2+b^2}$

$\therefore (g^2 + f^2 - k)(a^2 + b^2) = (-ag - bf + c)^2$

Let the required tangent be $y = mx + \alpha$

This tangent is parallel to $x - y + 5 = 0$, hence $m = 1$

$\therefore y = x + \alpha$ is the tangent.

Rewriting this equation gives $x - y + \alpha = 0$

By comparison of;

$x^2 + y^2 + 2gx + 2fy + k = 0$ with $x^2 + y^2 - 2x - 4y - 4 = 0$,

$\Rightarrow g = -1\,, f = -2$ and $k = -4$

Also comparing $ax + by + c = 0$ with $x - y + \alpha = 0$

$\Rightarrow a = 1\,, b = -1\,, \alpha = c$

Now from $(g^2 + f^2 - k)(a^2 + b^2) = (-ag - bf + c)^2$

$\Rightarrow \big((-1)^2 + (-2)^2 - (-4)\big)(1^2 + (-1)^2) = (-(1)(-1) - (-1)(-2) + \alpha)^2$

$\Rightarrow 2 \times 9 = (\alpha + 1 - 2)^2$

$\Rightarrow (\alpha - 1)^2 = 18 \;\; or \; \alpha - 1 = \pm\sqrt{18}$

$\therefore \alpha = 1 \pm 3\sqrt{2}$

The required tangents are: $y = x + \left(1 \pm 3\sqrt{2}\right)$

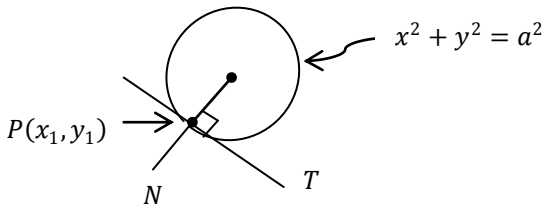
Note

If $x^2 + y^2 - 2x - 4y - 4 = 0$ and $y = x + \alpha$ are solved simultaneously, the applying condition for tangency yields the values of $\alpha = 1 \pm 3\sqrt{2}$, as before.

EQUATION OF THE NORMAL TO THE CIRCLE

To find the equation of the normal at any point $P(x_1, y_1)$ on the circle $x^2 + y^2 = a^2$

Consider the figure below



$\overline{PT}$ is a tangent, $\overline{NP}$ is a normal, both lines are at point P on the circle

Required equation of normal

$y - y_1 = m'(x - x_1)$

Finding m'

From $m \times m' = -1$, m =slope of the tangent at P

$\Rightarrow m' = -\frac{1}{m}$

First obtaining $m = \frac{dy}{dx}$ at point P

Differentiating the equation of the circle $x^2 + y^2 = a^2$ $w.r.x.$

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(a^2)$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \ \therefore \frac{dy}{dx} = -\frac{x}{y}$

$\therefore m = -\frac{x_1}{y_1}$

Slope of the normal $m' = -\frac{1}{m} = \frac{1}{\left(-\frac{x_1}{y_1}\right)} = \frac{y_1}{x_1}$

Equation of the normal is:

$y - y_1 = m'(x - x_1) \Rightarrow y - y_1 = \frac{y_1}{x_1}(x - x_1)$

$$\therefore x_1 y - x y_1 = 0$$

Examples

Find the equation of the normal at the given points

a) $x^2 + y^2 = 25$ at $(-3, 4)$

b) $x^2 + y^2 + 4x + 3y - 15 = 0$ at $(1, 2)$

Solution

a) Required equation f the normal

$y - 4 = m'(x + 3)$ ..... (1)

Finding m'

Let slope of the tangent be $m = \frac{dy}{dx}$ at $(-3, 4)$

Differentiating the equation of the circle $x^2 + y^2 = 25$ $w.r.x.$

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \ \therefore \frac{dy}{dx} = -\frac{x}{y}$

$\therefore m = -\frac{-3}{4} = \frac{3}{4}$

$\Rightarrow m' = -\frac{4}{3}$ ..... (2)

Eqn. (2) in to eqn. (1)

$\Rightarrow y - 4 = -\frac{4}{3}(x + 3)$ *or* $4x + 3y = 0$

b) Required equation f the normal

$y - 2 = m'(x - 1)$ ..... (1)

Finding m'

Let slope of the tangent be $m = \frac{dy}{dx}$ at $(1, 2)$

Differentiating the equation of the circle $x^2 + y^2 + 4x + 3y - 15 = 0$ w.r.t

$\frac{d}{dx}(x^2 + y^2 + 4x + 3y - 15) = \frac{d}{dx}(20)$

$\Rightarrow 2x + 2y\frac{dy}{dx} + 4 + 3\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{2x+4}{2y+3}$

$\therefore m = -\frac{2+4}{4+3} = -\frac{6}{7}$

$\Rightarrow m' = \frac{7}{6}$ ..... (2)

Eqn. (2) in to eqn. (1)

$\Rightarrow y - 2 = \frac{7}{6}(x - 1) \ \ or \ \ 7x - 6y + 5 = 0$

Task

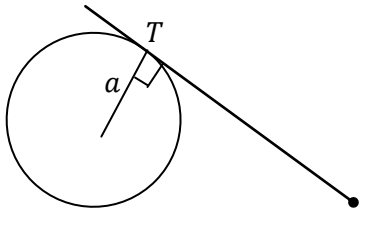
Find the equation of the normal at the given points

c) $x^2 + y^2 = 13$ at $(2, 3)$ **Ans:** $3x - 2y = 0$

d) $x^2 + y^2 - 2x + y = 20$ at $(-2, 3)$ **Ans:** $7x + 6y - 4 = 0$

TANGENTS TO THE CIRCLE FROM A POINT

To find the equation of a tangent to the circle $x^2 + y^2 = a^2$ from a point $P(x_1, y_1)$



Let the line $\overline{PT}$ be the tangent with the equation $y = mx + c$

Obtaining c

Since $P(x_1, y_1)$ lies on this line then;

$y_1 = mx_1 + c$ ..... (1)

Obtaining slope, m

Since $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$, solving the equations simultaneously and applying condition for tangency yields

$a^2(1 + m^2) = c^2$ ..... (2)

Eliminating c from eqn. (1) and eqn. (2)

$a^2(1 + m^2) = (y_1 - mx_1)^2$

$\Rightarrow m^2{x_1}^2 - 2mx_1y_1 + {y_1}^2 = a^2 + a^2m^2$

$$\therefore m^2({x_1}^2 - a^2) - (2mx_1y_1)m + ({y_1}^2 - a^2) = 0$$

, which is a quadratic in m. hence two values of m can be obtained.

NOTE:

Tangents corresponding to these values can be deduced from the quadratic equation.

CASE I: *Real and distinct values of m*

For real and distinct values of m, $B^2 - 4AC > 0$

$\Rightarrow 4{x_1}^2{y_1}^2 > 4({x_1}^2 - a^2)({y_1}^2 - a^2)$

$\Rightarrow {x_1}^2{y_1}^2 > {x_1}^2{y_1}^2 - {x_1}^2a^2 - a^2{y_1}^2 + a^4$

$$\therefore {x_1}^2 + {y_1}^2 > a^2 \ \ or \ \ {x_1}^2 + {y_1}^2 - a^2 > 0$$

$\Rightarrow P(x_1, y_1)$ lies outside the circle. Thus if $P(x_1, y_1)$ lies outside the circle *two* real and distinct tangents cab be drawn form $P(x_1, y_1)$

CASE II: *Coincident/Repeated roots of m*

$$\therefore {x_1}^2 + {y_1}^2 = a^2$$

This clearly shows that $P(x_1, y_1)$ lies on the circle. Thus *two* coincident tangents can be drawn from $P(x_1, y_1)$

CASE III: *Imaginary roots of m*

$$\therefore {x_1}^2 + {y_1}^2 < a^2 \ \ or \ \ {x_1}^2 + {y_1}^2 - a^2 < 0$$

This shows that $P(x_1, y_1)$ lies inside the circle. Hence two imaginary tangents can be drawn from $P(x_1, y_1)$

Examples

1) *Find the equations of the tangents to the circle from the given points*
   a) $x^2 + y^2 + 4x - 6y + 10 = 0\,,\ (0\,,0)$
   b) $x^2 + y^2 + 6x + 10y - 2 = 0\,,\ (2\,,3)$

Solution

Before proceeding to the required solution, firs test the point in the equation of the circle to judge whether the point is/out or on the circle.

a) $x^2 + y^2 + 4x - 6y + 10 = 0\,,\ (0\,,0)$

Checking the point

$\Rightarrow (0)^2 + (0)^2 + 4(0) - 6(y) + 10 \neq 0$ .Hence the point is not on the circle.

Since $10 > 0$, point is outside the circle.

Let the required tangent be $y = mx + c$

Obtaining c

$(0\,,0)$ lies on the line

$\Rightarrow 0 = m(0) + c \ \ \therefore c = 0$

$\therefore y = mx \ \ or \ \ mx - y = 0$ ..... (1)

Now finding slope m

Two methods can be used

Method I

Solving $y = mx$ and $x^2 + y^2 + 4x - 6y + 10 = 0$ simultaneously

$\Rightarrow x^2 + (mx)^2 + 4x - 6(mx) + 10 = 0$

$\Rightarrow x^2(1 + m^2) + x(4 - 6m) + 10 = 0$

For tangency, $B^2 - 4AC = 0$

$\Rightarrow (4 - 6m)^2 - 40(1 + m^2)C = 0$

$\Rightarrow 4(2 - 3m)^2 - 40(1 + m^2)C = 0$

$\Rightarrow 9m^2 - 12m + 4 = 10 + 10m^2$

$\therefore m^2 + 12m + 6 = 0$ ..... (*)

Solving $m = -0.523\,, -11.477$

Thus tangents are:

$y = -0.523x\,, y = -11.477x$

Method II

Eqn. (*) can still be obtained by using the formula

Radius = perpendicular distance of the centre from the tangent

Centre of $x^2 + y^2 + 4x - 6y + 10 = 0$ is $(-2\,, 3)$,

Radius $= \sqrt{(2)^2 + (-3)^2 - 10} = \sqrt{3}$, tangent is $mx - y = 0$

$\Rightarrow \sqrt{3} = \left|\frac{-2m-3}{\sqrt{m^2+1}}\right|$

Squaring both sides

$\Rightarrow 3 = \frac{4m^2+12m+9}{m^2+1}$ or $m^2 + 12m + 6 = 0$, which is as before.

NOTE:

When finding the value of m, method I is hectic instead method II is quicker to the answer.

b) $x^2 + y^2 + 4x + 10y - 2 = 0\,,\ (2\,, 3)$

Let the tangent be $y = mx + c$

Obtaining c

$(2\,, 3)$ lies on the line

$\Rightarrow 3 = m(2) + c\ \ \therefore c = 3 - 2m$ ..... (1)

Now finding slope m

Using

Radius = perpendicular distance of the centre from the tangent

Centre of $x^2 + y^2 + 6x + 10y - 2 = 0$ is $(-3\,, -5)$,

Radius $= \sqrt{(3)^2 + (5)^2 - (-2)} = 6$,

Tangent is $y = mx + c\ or\ mx - y + c = 0$

$\Rightarrow 6 = \left|\frac{-3m+5+c}{\sqrt{m^2+1}}\right|$

Squaring both sides

$\Rightarrow 36 = \frac{(-3m+5+c)^2}{m^2+1}$ or $3\,6m^2 + 36 = (-3m + 5 + c)^2$ ..... (2)

Eqn. (2) in to eqn. (1)

$\Rightarrow 36m^2 + 36 = (-3m + 5 + 3 - 2m\,)^2$

$\Rightarrow 36m^2 + 36 = (-5m + 8)^2$

$\Rightarrow 36m^2 + 36 = 25m^2 - 80m + 64$

$\therefore 11m^2 + 80m - 28 = 0$

Solving $m = 0.335\,, -7.607$

From eqn. (1), $\ c = 3 - 2m$

For $m = 0.335\,, c = 2.33$

For $m = -7.607\,, c = 18.214$

Thus tangents are:

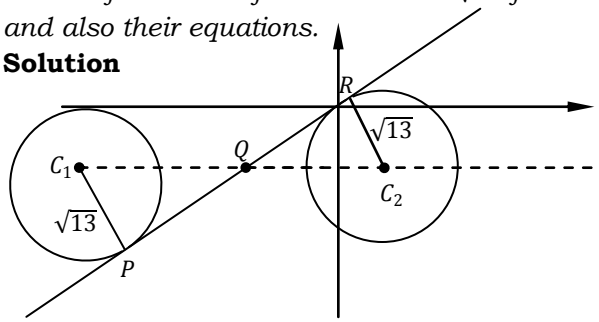
$y = 0..335\,x + 2.33\ \ , y = -7.607x + 18.214$

Task

1. *Find the equations of tangents to the circle* $x^2 + y^2 = 25$ *from point* $(11, -2)$
   Ans: $\mathbf{3x + 4y - 25 = 0}$; $\mathbf{7x - 24y - 125 = 0}$
2. *Find the equations of tangents to the circle* $x^2 + y^2 - 10x + 8y + 5 = 0$ *from point* $(5, 4)$ **Ans:** $\boldsymbol{y = \sqrt{7}x + \left(4 - 5\sqrt{7}\right)}$
3. *Show that the line* $y = x + 1$ *touches the circle* $x^2 + y^2 - 8x - 2y + 9 = 0$. *Find the equation of the other tangent to the circle from the point* $(0, 1)$
   Ans: $\boldsymbol{x + y - 1 = 0}$

2. *Two circles have their centres on the line* $y + 3 = 0$ *and touch the line* $3y - 2x = 0$. *If the radii of the circles are* $\sqrt{13}$ *find the coordinates of the centres and also their equations.*

Solution



Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $(-g, -f)$ lies on $y + 3 = 0$

$\Rightarrow -f + 3 = 0 \ \therefore f = 3$

Using

Radius = perpendicular distance of the centre from the tangent $-2x + 3y = 0$

$\Rightarrow \sqrt{13} = \frac{-2(-g) - 3f}{\sqrt{3^2 + 2^2}} = \frac{2g - 3f}{\sqrt{13}}$

$\Rightarrow 2g - 3f = 13$. But $f = 3 \ \therefore g = 11$

Now finding c

Also *Radius* $= \sqrt{g^2 + f^2 - c}$

$\Rightarrow \sqrt{13} = \sqrt{11^2 + 3^2 - c}$

$\Rightarrow 13 = 130 - c \ \therefore c = 117$

$\therefore$ 1st equation of the circle is $x^2 + y^2 + 2(11)x + 2(3)y + 117 = 0$

$\therefore x^2 + y^2 + 22x + 6y + 117 = 0$

To find the 2nd equation, first get the point of contact of the tangent to the first circle.

Point of contact

Solving $x^2 + y^2 + 22x + 6y + 117 = 0$ and $y = \frac{2x}{3}$ simultaneously

$\Rightarrow x^2 + y^2 + 22x + 6\left(\frac{2x}{3}\right) + 117 = 0$

Simplifying we get

$x^2 + 18x + 81 = 0 \ or \ (x + 9)^2 = 0$

$\therefore x = -9$

From $y = \frac{2x}{3}$, $x = -9 \Rightarrow y = -6$

Point of contact is $(-9\,,-6)$

From the figure above ΔC_1PQ and ΔC_2PQ are similar

$\Rightarrow \frac{\overline{C_1P}}{\overline{C_2P}} = \frac{\overline{PQ}}{\overline{QR}} = \frac{\overline{C_1Q}}{\overline{C_2Q}}$

$\Rightarrow \frac{\sqrt{13}}{\sqrt{13}} = \frac{\sqrt{(-9+9/2\,)^2+(-6+3)^2}}{\overline{QR}} = \frac{\sqrt{(-11+9/2\,)^2+(-3+3)^2}}{\overline{C_2Q}}$

$\Rightarrow 1 = \frac{\sqrt{117}/2}{\overline{QR}} = \frac{13/2}{\overline{C_2Q}}$

$\therefore \overline{C_2Q} = \frac{13}{2}$, $\overline{QR} = \frac{\sqrt{117}}{2}$

Applying Pythagoras theorem on to ΔC_2RQ, let $C_2(-G\,,-3)$

$\Rightarrow \frac{117}{4} + 13 = \left(-G + \frac{9}{2}\right)^2 + (-3+3)^2$

$\Rightarrow \left(-G + \frac{9}{2}\right)^2 = \frac{169}{4} \quad \therefore -G + \frac{9}{2} = \pm\frac{13}{2}$

$\Rightarrow G = 11\,,-2$ taking $G = -2$

Required circle is: $\therefore x^2 + y^2 + 2(-2)x + 2(3)y + k = 0$

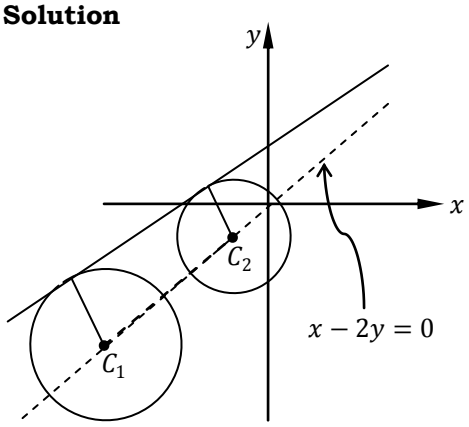
Using Radius $= \sqrt{G^2 + F^2 - k}$

$\Rightarrow \sqrt{13} = \sqrt{(-2)^2 + (-3)^2 - k} \quad \therefore k = 0$

$\therefore$ 2nd equation of the circle is $x^2 + y^2 - 4x + 6y = 0$

4. *Two circles have their centres on the line* $2y = x$ *and touch the line* $3x - 4y + 5 = 0$. *If their radii are* $5cm$ *and* $7cm$ *respectively, find the coordinates of their centres and also their equations*

Solution



Let $C_1\,(-g\,,-f)$ and $C_2(-G\,,-F)$

Centres lie on the line $2y = x$

$\Rightarrow -g = -2f \Rightarrow g = 2f$..... (1)

$\Rightarrow -G = -2F \Rightarrow G = 2F$..... (2)

Also from *Radius = perpendicular distance of the centre from the tangent*

$\Rightarrow \frac{-3g+4f+5}{\sqrt{3^2+4^2}} = 7 \quad \therefore -3g + 4f = 30$ ..... (5)

Also $\frac{-3G+4F+5}{\sqrt{3^2+4^2}} = 5 \quad \therefore -3G + 4F = 20$ ..... (6)

Eqn. (1) and eqn. (2) in to eqn. (5) and (6) respectively

$\Rightarrow -3g + 4f = 30\,, g = 2f \quad \therefore f = -15$ and $g = -30$

$\Rightarrow -3G + 4F = 20\,, G = 2F \quad \therefore F = -10$ and $G = -20$

Centres of the circle are: $(30\,,15)$ and $(20\,,10)$
Thus equations of circle are:
$\therefore (x-30)^2+(y-15)^2=49$
$\therefore (x-20)^2+(y-10)^2=25$

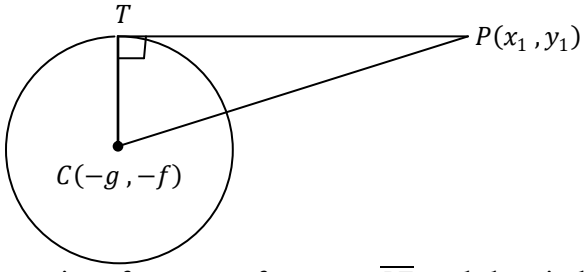
Task

Two circles have radii $5\ cm$ *and* $7\ cm$ *and touches the line* $3x+4y+5=0$. *If the smaller and the larger circle touch respectively the* $y-axis$ *and* $x-axis$, *find the equations of the circles.*

Ans: $(x-2/3)^2+(y-7)^2=49\,,(x-5)^2+(y-5/4)^2=25$

LENGTH OF A TANGENT TO THE CIRCLE FROM A GIVEN POINT $P(x_1\,,y_1)$

Let the circle be $x^2+y^2+2gx+2fy+c=0$ and the point $P(x_1\,,y_1)$ lying outside the circle.



Let T = point of contact of tangent $\overline{PT}$ and the circle
By Pythagoras theorem

$\overline{PT}^2+\overline{CT}^2=\overline{PC}^2 \quad \therefore \overline{PT}=\sqrt{\overline{PC}^2-\overline{CT}^2}$

$\overline{PC}=\sqrt{(x_1+g)^2+(y_1+f)^2}$

$=\sqrt{{x_1}^2+2gx_1+g^2+{y_1}^2+2fy_1+f^2}$

$\overline{CT}=Radius=\sqrt{g^2+f^2-c}$

$\therefore \overline{PT}=\sqrt{{x_1}^2+2gx_1+g^2+{y_1}^2+2fy_1+f^2-(g^2+f^2-c)}$

$$\therefore \overline{PT}=\sqrt{{x_1}^2+{y_1}^2+2gx_1+2fy_1+c}$$

Examples

Find the length of a tangent to the circle from the given point.

a) $x^2+y^2+4x-6y+10=0$, point $(0\,,0)$
b) $3x^2+3y^2+4x+2y+6=0$, point $(2\,,1)$

Solution

a) $x^2+y^2+4x-6y+10=0$, point $(0\,,0)$
Length of the tangent $=\sqrt{(0)^2+(0)^2+4(0)-6(0)+10}=\sqrt{10}\ units$
b) $3x^2+3y^2+4x+2y+6=0$, point $(2\,,1)$
Re writing the equation of the circle(*note this step*)

$\Rightarrow x^2 + y^2 + \frac{4}{3}x + \frac{2}{3}y + 2 = 0$

Length of the tangent

$= \sqrt{(2)^2 + (1)^2 + 4(2) - 6(1) + 2} = 3\,units$

Task

Find the length of a tangent to the circle from the given point.

a) $x^2 + y^2 = 16$, point $(3, -4)$ **Ans: 3**

b) $4x^2 + 4y^2 + 3x - 8y - 2 = 0$, point $(1, 1)$ **Ans: 1**

EQUATION OF A CIRCLE THOUGH THE POINTS OF INTERSECTION OF A GIVEN LINE AND CIRCLE

Consider a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and the line $ax + by + d = 0$

The required equation of a circle through points of intersection is:

$\boxed{x^2 + y^2 + 2gx + 2fy + c + \lambda(ax + by + d) = 0}$ **,** where λ is an arbitrary constant.

The constant λ can be obtained from the given condition in the question.

NOTE:

The equations of a circle and the line must be equated to zero before using the general equation of a circle through the points of intersection.

Example

1. *Find the equation of a circle through the points of intersection of the line* $x - 3y = 0$ *and the circle* $x^2 + y^2 - 10x - 5y + 25 = 0$ *with the centre lying on the line* $x + y = 1$

Solution

Required equation of a circle:

$x^2 + y^2 - 10x - 5y + 25 + \lambda(x - 3y) = 0$

$\Rightarrow x^2 + y^2 + x(\lambda - 10) + (-5 - 3\lambda) + 25 = 0$

Centre of the circle is:

$\left(\frac{\lambda - 10}{-2}, \frac{-5 - 3\lambda}{-2}\right) = \left(\frac{10 - \lambda}{2}, \frac{5 + 3\lambda}{2}\right)$

Since the centre lies on the line $x + y = 1$

$\Rightarrow \frac{10 - \lambda}{2} + \frac{5 + 3\lambda}{2} = 1 \quad \therefore \lambda = -\frac{13}{2}$

Equation of the circle becomes

$\Rightarrow x^2 + y^2 + x\left(-\frac{13}{2} - 10\right) + \left(-5 - 3\left(-\frac{13}{2}\right)\right) + 25 = 0$

$\therefore x^2 + y^2 - \frac{33}{2}x - \frac{49}{2}y + 25 = 0$

Alternatively

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $(-g, -f)$ lies on the line $x + y = 1$

$\Rightarrow -g - f = 1$ or $g + f = -1$..... (1)

Solving the equation of given circle and line simultaneously to obtain points of intersection

$x^2 + y^2 - 10x - 5y + 25 = 0\,, x = 3y$

$\Rightarrow (3y)^2 + y^2 - 10(3y) - 5y + 25 = 0$

$\Rightarrow 2y^2 - 7y + 5 = 0 \quad \therefore y = \frac{5}{2}\,, 1$

Corresponding values of x to y are:

$x = 3\,, y = 1\,, x = \frac{15}{2}\,, y = \frac{5}{2}$

Points of intersection: $(3\,, 1)\,, \left(\frac{15}{2}\,, \frac{5}{2}\right)$

Since these points lie on the circle,

For $(3\,, 1)$: $9 + 1 + 6g + 2f + c = 0$

$\Rightarrow 6g + 2f + c = -10$ ..... (2)

For $\left(\frac{15}{2}\,, \frac{5}{2}\right)$: $\frac{225}{4} + \frac{25}{4} + \frac{30}{2}g + 5f + c = 0$

$\Rightarrow 15g + 5f + c = -\frac{125}{2}$ ..... (3)

Eliminating c from eqn. (1) and eqn. (2)

$6g - f = -35$ ..... (4)

Solving eqn. (1) and eq. (4) simultaneously

$\Rightarrow g = -\frac{33}{4}\,, f = \frac{29}{4}$, from eqn. (2) $c = 25$

Required circle is

$x^2 + y^2 + 2\left(-\frac{33}{4}\right)x + 2\left(\frac{29}{4}\right)y + 25 = 0$

$\therefore x^2 + y^2 - \frac{33}{2}x - \frac{49}{2}y + 25 = 0$

2. *Find the equation of a circle passing through the points of intersection of the circle $x^2 + y^2 = 4$ and the line $x - y = 1$ as a diameter.*

Solution

Required circle: $x^2 + y^2 - 4 + \lambda(x - y - 1) = 0$

$\Rightarrow x^2 + y^2 + \lambda x - \lambda y - \lambda - 4 = 0 = 0$

Centre $\left(-\frac{\lambda}{2}\,, \frac{\lambda}{2}\right)$ lies on the diameter $x - y = 1$

$\Rightarrow -\frac{\lambda}{2} - \frac{\lambda}{2} = 1 \quad \therefore \lambda = -1$

$\Rightarrow x^2 + y^2 - 4 - 1(x - y - 1) = 0$

$\therefore x^2 + y^2 - x + y - 3 = 0$

Task

1. *Find the equation of a circle having the chord $x - y = 1$ as the diameter and passing through the points of intersection of this chord and the circle $x^2 + y^2 - x - 3y = 12.5$*

Ans: $x^2 + y^2 - x - 3y = 12.5$

2. *Find the equation of a circle drawn on the line segment joining points of intersection of the circle $x^2 + y^2 = a^2$ and the line $x\cos\alpha + y\sin\alpha = p$ as the diameter.*

Ans: $x^2 + y^2 - a^2 - 2p(x\cos\alpha + y\sin\alpha - p) = 0$

3. *Show that the equation of the circle passing through the point of intersection of the circle* $x^2 + y^2 = 2ax$ *and the line* $ax - by = 0$ *having its centre on the line* $\frac{x}{a} - \frac{y}{b} = 2$ *is* $x^2 + y^2 - 3ax + by = 0$

4. *Find the equation of the circle containing the circle* $(x-a)^2 + y^2 = a^2$ *and the line* $ax - by = 0$ *as the diameter.*
Ans: $(a^2 + b^2)(x^2 + y^2) = 2ab(bx + ay)$

5. *Find the equation of a circle passing through the points of intersection of the circle* $x^2 + y^2 + 6x - 10y + 7 = 0$ *and the line* $x - 6y = -6$ *as its diameter.*

6. *Find the equation of a circle containing the chord* $ax = by$ *as its diameter and the circle* $x^2 + y^2 + 2by + c^2 = 0$

EQUATION OF A CIRCLE THROUGH THE POINTS OF INTERSECTION OF TWO CIRCLES

Let the given circles be:
$S_1 = x^2 + y^2 + 2gx + 2fy + c_1 = 0$ and $S_2 = x^2 + y^2 + 2Gx + 2Fy + c_2 = 0$
Required equation of the circle is:

$$x^2 + y^2 + 2gx + 2fy + c_1 + k(x^2 + y^2 + 2Gx + 2Fy + c_2) = 0$$

NOTE:

1) The two equations of the circle are in the form $S_1 = 0, S_2 = 0$. Hence the equation of a circle passing through the points of intersection of $S_1 = 0, S_2 = 0$ provided $k \neq -1$ is $\boldsymbol{S_1 + kS_2 = 0}$

2) The arbitrary constant k can be obtained from the given condition

3) If $k = -1$, the 2^0 terms varnish and the equation in this case being 1^0, represents a straight line passing through the points of intersection of two circles. The line is called the COMMON CHORD.

Examples

1. *Find the equation of a circle passing through* $(-1, 1)$ *and the points of intersection of circles* $x^2 + y^2 = 5$ *and* $x^2 + y^2 = 3x + y$
Solution
Required circle: $x^2 + y^2 - 5 + k(x^2 + y^2 - 3x - y) = 0$
This equation satisfies the point $(-1, 1)$
$\Rightarrow 1 + 1 - 5 + k(1 + 1 + 3 - 1) = 0 \quad \therefore k = \frac{3}{4}$
Required circle is:
$x^2 + y^2 - 5 + \frac{3}{4}(x^2 + y^2 - 3x - y) = 0$, simplifying gives
$\therefore 7x^2 + 7y^2 - 9x - 3y - 20 = 0$
Note:

This question can be transformed in to "to find the equation of a circle through the points of intersection of a line and a circle. Here the line is the common chord.

Common chord of intersecting circles:

$x^2 + y^2 = 5$ *and* $x^2 + y^2 = 3x + y$

Subtracting the equations to eliminate x^2 , y^2

$3x + y - 5 = 0$

Required circle:

$x^2 + y^2 - 3x - y + \lambda(3x + y - 5) = 0$

This circle passes through $(-1, 1)$

$\Rightarrow 1 + 1 - 5 + \lambda(-3 + 1 - 5) = 0 \ \therefore \lambda = -\frac{3}{7}$

$\Rightarrow x^2 + y^2 - 3x - y - \frac{3}{7}(3x + y - 5) = 0$, simplifying gives

$\therefore 7x^2 + 7y^2 - 9x - 3y - 20 = 0$ as before

2. *Find the equation of a circle passing through the points of intersection of the circle* $x^2 + y^2 = 2ax$ *and* $x^2 + y^2 = 2bx$ *and having its centre of the line* $\frac{x}{a} - \frac{y}{b} = 2$

Solution

Required circle: $x^2 + \_^2 - 2ax + k(\ x^2 + y^2 - 2bx) = 0$

$\Rightarrow x^2(1 + k) + y^2(1 + k) + x(-2a - 2bk) = 0$

$\Rightarrow x^2 + y^2 + x\left(\frac{-2a-2bk}{1+k}\right) = 0$

$\therefore$ centre is $\left(\frac{a+bk}{1+k}, 0\right)$ which lies on the line $\frac{x}{a} - \frac{y}{b} = 2$

$\Rightarrow \frac{\frac{a+bk}{1+k}}{a} = 2$

$\Rightarrow 2a(1 + k) = a + bk \quad or \quad k = \frac{a}{b-2a}$

$\Rightarrow x^2 + y^2 + x\left(\frac{-2a-2b\left(\frac{a}{b-2a}\right)}{1+\frac{a}{b-2a}}\right) = 0$

$\Rightarrow x^2 + y^2 + x\left(\frac{-2ab+4a^2-2ab}{b-2a+a}\right) = 0$

$\therefore (a + b)(x^2 + y^2) + 4ax(a - b) = 0$

Task

1. *Find the equation of a circle passing through the points of intersection of the circles* $x^2 + y^2 + 6x - 10y + 7 = 0$ *and* $x^2 + y^2 + 4x + 2y - 5 = 0$ *and having the common chord as its diameter.*

Ans: $37x^2 + 37y^2 + 168x - 46y - 65 = 0$

2. *Show that the equation of a circle passing through the points of intersection of the circles* $x^2 + y^2 = 2ax$ *and* $x^2 + y^2 = 2by$ *and having its centre on the line* $\frac{x}{a} - \frac{y}{b} = 2$ *is* $x^2 + y^2 - 3ax + by = 0$

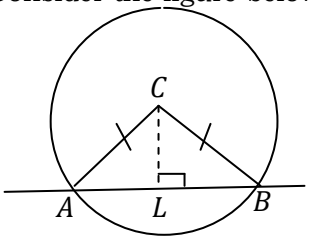
3. *Find the equation of a circle passing through the points of intersection of the circles* $x^2 + y^2 - x - 3y - 12.5 = 0$ *and* $x^2 + y^2 - 2x - 2y - 13.5 = 0$ *and having its centre on the line* $x - y = 1$

Ans: $2x^2 + 2y^2 - 6x - 2y - 21 = 0$

LENGTH OF A CHORD

To find the length of a chord intercepted by a circle on a straight line

Consider the figure below



$\overline{CL}$ bisects AB

From ΔACL, Pythagoras theorem yields

$\overline{AC}^2 = \overline{AL}^2 + \overline{CL}^2$

$\therefore \overline{AL} = \sqrt{\overline{AC}^2 - \overline{CL}^2}$

$\therefore \overline{AB}$ = Length of chord $= 2\sqrt{\overline{AC}^2 - \overline{CL}^2}$

Length of a chord $= 2\sqrt{\{\text{RADIUS}\}^2 - \{\perp \text{ DISTANCE OF CENTRE FROM CHORD}\}^2}$

Centre $(-g, -f)$ and Radius $= \sqrt{g^2 + f^2 - c}$

Let the chord be $ax + by + k = 0$

$$\text{Length of the chord} = 2\sqrt{g^2 + f^2 - c - \left[\frac{ag+bf-k}{\sqrt{a^2+b^2}}\right]^2}$$

Note:

If the line is given in the form $y = mx + c$, first re write it in the form $mx - y + c = 0$, then get the perpendicular distance of the centre from this chord line.

Examples

1. *Find the length of a chord intercepted by:*

i) Circle $x^2 + y^2 - 6x + 4y - 12 = 0$ on the line $4x - 3y + 2 = 0$

ii) Circle $2x^2 + 2y^2 + 3x - 4y = 0$ on the line $x + y - 1 = 0$

Solution

i) $x^2 + y^2 - 6x + 4y - 12 = 0$, centre $(3, -2)$,

Radius $= \sqrt{(-3)^2 + (2)^2 + 12} = 5$

$\perp$ Distance of $(3, -2)$ from $4x - 3y + 2 = 0$ is

$$= \left|\frac{4(3)-3(-2)+2}{\sqrt{4^2+3^2}}\right| = 4$$

$\therefore$ Length of chord $= 2\sqrt{5^2 - 4^2} = 6\ units$

ii) $2x^2 + 2y^2 + 3x - 4y = 0 \Rightarrow x^2 + y^2 + \frac{3}{2}x - 2y = 0$, centre $\left(-\frac{3}{4}, 1\right)$

Radius $= \sqrt{\left(\frac{3}{4}\right)^2 + (1)^2} = \frac{5}{2}$

$\perp$ Distance of $\left(-\frac{3}{4}, 1\right)$ from $x - y - 1 = 0$ is

$$= \left|\frac{-\frac{3}{4}+1-1}{\sqrt{1^2+1^2}}\right| = \frac{3}{4\sqrt{2}}$$

$\therefore$ Length of chord $= 2\sqrt{\frac{25}{4} - \left(\frac{3}{4\sqrt{2}}\right)^2} = 4.886\ units$

Task

Find the length of a chord intercepted by:

Circle $x^2 + y^2 - 5x + 6y + 12 = 0$ on the line $3x + 2y + 5 = 0$

Ans: 0; $x + 2y + 5 = 0$ is a tangent

2. *Find the length of a common chord between two circles*

i) $x^2 + y^2 + 6x - 10y + 7 = 0$ *and* $x^2 + y^2 + 4x + 2y - 5 = 0$

ii) $x^2 + y^2 - 8x + 12 = 0$ *and* $x^2 + y^2 - 4x - 6y - 3 = 0$

Solution

i) Common chord: subtracting the equations to eliminate x^2 , y^2

$\Rightarrow x^2 + y^2 + 6x - 10y + 7 - (x^2 + y^2 + 4x + 2y - 5) = 0$

$\Rightarrow x - 6y + 6 = 0$

Taking $x^2 + y^2 + 6x - 10y + 7 = 0$, centre $(-3\,, 5)$,

Radius $= \sqrt{(3)^2 + (-5)^2 - 7} = \sqrt{27}$

$\perp$ distance of $(-3\,, 5)$ from $x - 6y + 6 = 0$ is

$$= \left|\frac{(-3) - 6(5) + 6}{\sqrt{1^2 + (-6)^2}}\right| = \frac{27}{\sqrt{37}}$$

$\therefore$ Length of chord $= 2\sqrt{27 - \frac{27}{37}} = 10.25$ *units*

ii) Left as a n exercise. **Ans: 2.49**

3. *Prove that the length of a common chord of two circles whose equations are* $(x - a)^2 + (y - b)^2 = c^2$ *and* $(x - b)^2 + (y - a)^2 = c^2$ *is* $\sqrt{4c^2 - 2(a - b)^2}$

Ans: Left as an exercise

3. *Find the equation of a circle with centre* $\left(\frac{5}{2}\,, -3\right)$ *which*

a) *Cuts off a chord of length 6 units from the line* $3x + 2y + 5 = 0$

b) *Cuts off a chord of length 8 units from the line* $3x - 4y - 2 = 0$

Solution

a) Let the required circle be: $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $(-g\,, -f) = \left(\frac{5}{2}\,, -3\right) \Rightarrow g = -\frac{5}{2}\,, f = 3$

The equations becomes

$x^2 + y^2 - 5x + 6y + c = 0$

Length of a chord $= 2\sqrt{\{\text{RADIUS}\}^2 - \{\perp \text{ DISTANCE OF CENTRE FROM CHORD}\}^2}$

Radius $= \sqrt{\left(-\frac{5}{2}\right)^2 + 3^2 - c} = \sqrt{\frac{25}{4} + 9 - c}$

$\perp$ distance of $\left(\frac{5}{2}\,, -3\right)$ from $3x + 2y + 5 = 0$ is

$$= \left|\frac{3(5/2) + 2(-3) + 5}{\sqrt{3^2 + 2^2}}\right| = \frac{\sqrt{13}}{2}$$

$\Rightarrow 6 = 2\sqrt{\frac{61}{4} - c - \frac{13}{4}}$, squaring both sides

$\Rightarrow 9 = \frac{61}{4} - c - \frac{13}{4} \;\; \therefore c = 3$

Thus the equation of the circle is:

$x^2 + y^2 - 5x + 6y + 3 = 0$

b) Left as an exercise. **Ans:** $x^2 + y^2 - 5x + 6y - \frac{61}{25} = 0$

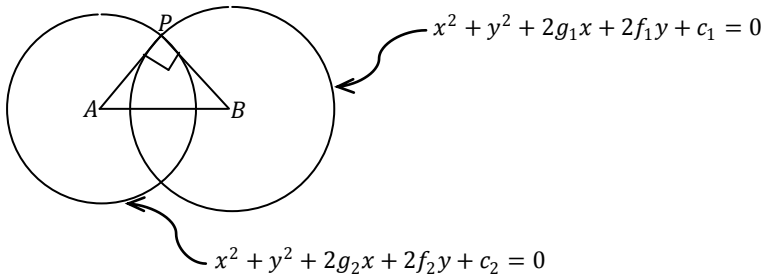
ANGLE OF INTERSECTION OF TWO CIRCLES

When two circles intersect, the angle between them is the angle between the tangents to the circles at the common point of intersection.
If the angle of intersection of two circles is a *right angle.i.e.*90^0, the circles are said to be *ORTHOGANALLY*, and for this case the tangents at the common point to either circle pass through the centre of the other. The line joining their centres subtend a right angle at the point of intersection

ORTHOGONAL CIRCLES

To find the condition that the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ may cut each other orthogonally.

Consider the figure below



From ΔAPB, Pythagoras theorem yields

$\overline{AP}^2 + \overline{PB}^2 = \overline{AB}^2$

$\overline{AP}$ = Radius $= \sqrt{{g_1}^2 + {f_1}^2 - c_1}$

$\overline{PB}$ = Radius $= \sqrt{{g_2}^2 + {f_2}^2 - c_2}$

Centres $A(-g_1, -f_1), B(-g_2, -f_2)$

$\overline{AB} = \sqrt{(-g_1 + g_2)^2 + (-f_1 + f_2)^2}$

$\therefore {g_1}^2 + {f_1}^2 - c_1 + {g_2}^2 + {f_2}^2 - c_2 = (-g_1 + g_2)^2 + (-f_1 + f_2)^2$

Simplifying gives

$-c_1 - c_2 = -2g_1g_2 - 2f_1f_2$

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

This condition can generally stated as

$$\frac{1}{2}\,[product\ of\ coeff.x + product\ of\ coeff.y] = sum\ of\ constants$$

Examples

1. *Show that the circles are orthogonal*

a) $x^2 + y^2 - 6x - 8y + 9 = 0$, $x^2 + y^2 = 9$

b) $x^2 + y^2 - 4x + 2 = 0$, $x^2 + y^2 + 6y - 2 = 0$

c) $x^2 + y^2 - 6y + 8 = 0$, $x^2 + y^2 - 4x + 2y - 14 = 0$

d) $x^2 + y^2 + 6x - 10y + 7 = 0$, $x^2 + y^2 + 4x + 2y - 5 = 0$

Solution

a) $x^2 + y^2 - 6x - 8y + 9 = 0$

Centre $(3\,,4)$, Radius= $\sqrt{(-3)^2 + (-4)^2 - 9} = 4$

$x^2 + y^2 = 9$

Centre $(0\,,0)$, Radius= $\sqrt{(0)^2 + (0)^2 + 9} = 3$

Distance between centres= $\sqrt{(3-0)^2 + (4-0)^2} = 5$

$\therefore (3)^2 + (4)^2 = 5^2$, hence circles are orthogonal

Alternatively

$\frac{1}{2}[-6 \times 0 + 0 \times -8] = -9 + 9 = 0$

b) , (c) and (d) have been left as an exercise to the students.

2. *Find the equation of a circle which passes through the origin and cuts circles orthogonally the circles* $x^2 + y^2 - 8y + 12 = 0$ *and* $x^2 + y^2 - 4x - 6y - 3 = 0$

Solution

$x^2 + y^2 - 8y + 12 = 0$ ..... (1)

$x^2 + y^2 - 4x - 6y - 3 = 0$ ..... (2)

Let the required circle be:

$x^2 + y^2 + 2gx + 2fy + c = 0$ ..... (3)

Circles (1) and (3); (2) and (3) are orthogonal

$\Rightarrow \frac{1}{2}[2g \times 0 + 2f \times (-8)] = 12 + c$, simplifying gives

$\Rightarrow\ -8f - c = 12$ ..... (4)

$\Rightarrow \frac{1}{2}[2g \times (-4) + 2f \times (-6)] = -3 + c$, simplifying gives

$\Rightarrow\ -4g - 6f = -3 + c$ ..... (5)

Circle (3) passes through the origin

$\Rightarrow 0 + 0 + 0 + 0 + c = 0\ \ \therefore c = 0$

Put $c = 0$ in to (4) and (5)

$\Rightarrow -8f = 12\ \ \therefore f = -\frac{3}{2}$

Similarly

$\Rightarrow\ -4g - 6f = -3$, but $f = -\frac{3}{2}$

$\Rightarrow\ -4g - 6\left(-\frac{3}{2}\right) = -3\ \ \therefore g = 3$

Eqn. (3) becomes

$x^2 + y^2 + 2(3)x + 2\left(-\frac{3}{2}\right)y + 0 = 0$

$\therefore x^2 + y^2 + 6x - 3y = 0$

3. *Find the equation of a circle which cuts orthogonally the circle* $x^2 + y^2 - 4x + 6y - 7 = 0$, *passing through the point* $(0, 3)$ *and touches the* $x - axis$

Solution

Let the required circle be:

$x^2 + y^2 + 2gx + 2fy + c = 0$ ..... (1)

Case I

Eqn. (1) cuts $x^2 + y^2 - 4x + 6y - 7 = 0$ orthogonally

$\Rightarrow \frac{1}{2}[-4 \times 2g + 6 \times 2f] = -7 + c$

$\therefore -4g + 6f = c - 7$ ..... (2)

Case II

Eqn. (1) passes through $(0, 3)$

$\Rightarrow 9 + 6f + c = 0$ ..... (3)

Case III

$y = 0$ $(x - axis)$ is a tangent to eqn. (1)

$\Rightarrow x^2 + 2gx + c = 0$

For tangency $B^2 - 4AC = 0$

$\Rightarrow 4g^2 = 4(1)c \quad \therefore g^2 = c$ ..... (4)

Eliminating f from eqn. (2) and eqn. (3)

Eqn. (2)−eqn. (3)

$\Rightarrow 4g - 9 - c = c - 7 \ \ or \ \ c = -2g - 1$ ..... (5)

Eqn. (4) in to eqn. (5)

$\Rightarrow g^2 + 2g + 1 = 0 \quad \therefore g = -1$

Putting $g = -1$ in to eqn. (5)

$\Rightarrow c = 1$

Putting $g = -1, \ c = 1$ in to eqn. (2)

$\Rightarrow -4(-1) + 6f = 1 - 7 \quad \therefore f = -\frac{5}{3}$

Putting $g = -1, c = 1, f = -\frac{5}{3}$ into eqn. (1)

$x^2 + y^2 + 2(-1)x + 2\left(-\frac{5}{3}\right)y + 1 = 0$

$\therefore 3x^2 + 3y^2 - 6x - 10y + 3 = 0$

4. *Find the equation of a circle with centre on the* $y - axis$ *which cuts orthogonally each of the circles* $x^2 + y^2 + 6x + 2y - 9 = 0$ *and* $x^2 + y^2 - 2x - 2y + 1 = 0$

Solution

Let the required circle be:

$x^2 + y^2 + 2gx + 2fy + c = 0$ ..... (1)

Case I

Centre $(-g, -f)$ lies on $y - axis \ . i.e. \ x = 0$

$\Rightarrow -g = 0 \quad \therefore g = 0$ ..... (2)

Circle in eqn. (1) cuts $x^2 + y^2 + 6x + 2y - 9 = 0$ orthogonally

$\Rightarrow \frac{1}{2}[6 \times 2g + 2f \times 2] = c - 9 \quad \therefore 6g + 2f = c - 9$. But $g = 0$

$\therefore 2f = c - 9$ ..... (3)

Case II

Circle in eqn. (1) cuts $x^2 + y^2 - 2x - 2y + 1 = 0$ orthogonally

$\Rightarrow \frac{1}{2}[-2 \times 2g + 2f \times -2] = c + 1 \quad \therefore -2g - 2f = c + 1$. But $g = 0$

$\therefore -2f = c + 1$ ..... (4)

Solving eqn. (3) and eqn. (4) simultaneously gives

$\Rightarrow f = -\frac{5}{2}, \ c = 4$

Required equation of the circle is:

$x^2 + y^2 - 5y + 4 = 0$

Task

1. *The circles* $x^2 + y^2 - 14x + 6by + 42a = 0$ *and* $x^2 + y^2 - 1x + by - 23a = 0$ *are orthogonal. If the first circle passes through the point* $(3, 9)$*, find the values of a and b*
2. *Find the equation of a circle which passes through point* $(1, 2)$*, cuts orthogonally the circle* $x^2 + y^2 + 2x - 8 = 0$ *and touches the line* $3x - 4y + 4 = 0$
5. *Prove that the circles which pass through points* $(0, a)$ *and* $(0, -a)$ *and touch the line* $y = mx + c$ *will cut orthogonally if* $c^2 = a^2(2 + m^2)$

Solution

Let the equation of the circle be: $x^2 + y^2 + 2gx + 2fy + k = 0$

This circle passes through $(0, a)$ *and* $(0, -a)$

$\Rightarrow a^2 + 2fa + k = 0$ ..... (1)

$\Rightarrow a^2 - 2fa + k = 0$ ..... (2)

If the line $y = mx + c$ or $mx - y + c = 0$ is a tangent to the circle $x^2 + y^2 + 2gx + 2fy + k = 0$ then

$\Rightarrow \sqrt{g^2 + f^2 - k} = \left|\frac{m(-g) - (-f) + c}{\sqrt{m^2 + 1}}\right|$, squaring both sides

$\Rightarrow g^2 + f^2 - k = \frac{(-mg + f + c)^2}{m^2 + 1}$ ..... (3)

Since k and f are not required in the proof, solving eqn. (1) and eqn. (2) simultaneously

Eliminating f, adding the equations

$\Rightarrow 2a^2 + 2k = 0 \quad \therefore k = -a^2$ ..... (4)

Putting $k = -a^2$ into eqn. (1)

$\Rightarrow a^2 + 2fa - a^2 = 0 \quad \therefore f = 0$ ..... (5)

Putting $k = -a^2, f = 0$ in to eqn. (3) to solve also for g since it is also unknown

$\Rightarrow g^2 + 0 + a^2 = \frac{(-mg + 0 + c)^2}{m^2 + 1}$

$\Rightarrow g^2m^2 + g^2 + a^2(m^2 + 1) = g^2m^2 - 2mgc + c^2$

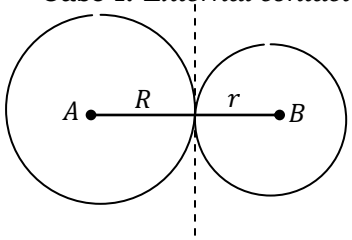
$\therefore g^2 + 2mcg + a^2(m^2 + 1) - c^2 = 0$

From algebra, if g_1 and g_2 are the roots of the equation then
$g_1 + g_2 = -2mc\,,\ g_1.g_2 = a^2(m^2+1) - c^2$ ..... (6)
Now if circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and
$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ cut orthogonally then
$\Rightarrow \frac{1}{2}[2g_1.2g_2 + 2f_1.2f_2] = c_1 + c_2$, but $f = f_1 = 0\,, k = c_2 = c_1 = -a^2$
$\Rightarrow 2g_1 2g_2 = -a^2 - a^2$
$\Rightarrow g_1g_2 = -a^2$ ..... (7)
Eqn. (7) in to eqn. (6) gives
$a^2(m^2+1) - c^2 = -a^2 \quad or \quad c^2 = a^2(m^2+1) + a^2$
$\therefore c^2 = a^2(2 + m^2)$

TOUCHING CIRCLES

Two circles are said to *touch each other* at a point when they have the same tangent (common tangent) at that point called *point of contact* of the circles.
Two cases arise:

Case I: *External contact*

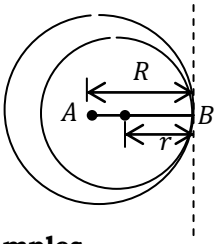


Circles touch each other *externally.*

Condition

$R + r = \overline{AB}$ = Distance between the centres

Case II: *Internal contact*



Circles touch each other *internally.*

Condition

$R - r = \overline{AB}$ = Distance between the centres

Examples

1. *Prove that the circles*

a) $x^2 + y^2 + 3x + y = 0$ and $x^2 + y^2 - 6x - 2y = 0$
b) $x^2 + y^2 + 5(2x+1) = 0$ and $x^2 + y^2 + 5(y+1) = 0$
c) $x^2 + y^2 + 2x - 2y = 7$ and $x^2 + y^2 - 4x - 10y + 25 = 0$

, touch each other. Find the point of contact and the equation of the tangent at that point of contact.

Solution

a) Circle can touch internally or externally
For $x^2 + y^2 + 3x + y = 0$, Centre $\left(-\frac{3}{2}\,, -\frac{1}{2}\right)$

$$\text{Radius} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 0} = \frac{\sqrt{10}}{2} = r$$

For $x^2 + y^2 - 6x - 2y = 0$, Centre $(3\,,1)$

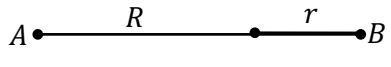
$$\text{Radius} = \sqrt{(-3)^2 + (-1)^2 - 0} = \sqrt{10} = R$$

Distance between two centres$= \sqrt{\left(3 + \frac{3}{2}\right)^2 + \left(1 + \frac{1}{2}\right)^2} = \frac{3\sqrt{10}}{2} = d$

Since $R + r = d$, hence circles touch each other externally

Note:
To find the point of contact, vectors, ratio theorem or solving the equations of the circles simultaneously can be used.
Point of contact

Using Ratio Theorem
Recall: *If* $A(x_1\,, y_1)$ *and* $B(x_2\,, y_2)$ *are end points of the line segment AB and* $P(x\,, y)$ *divide AB internally in the ratio* $m{:}\,n$ *then*
$P(x\,, y) = P\left[\frac{mx_2 + nx_1}{m+n}\,, \frac{my_2 + ny_1}{m+n}\right]$



P divides $\overline{AB}$ in the ratio $\overline{AP}{:}\,\overline{PB} = R{:}\,r$
$\Rightarrow \frac{\overline{AP}}{\overline{PB}} = \frac{R}{r} = \frac{\sqrt{10}}{\sqrt{10}/2} = \frac{2}{1}$
Now coordinates of P are:
$P\left[\frac{2\times -\frac{3}{2} + 1\times 3}{2+1}\,, \frac{2\times -\frac{1}{2} + 1\times 1}{2+1}\right] = P(0\,, 0)$ which is the point of contact

Tangent equation at the point of contact
This can be obtained either by differentiation at the point of contact for any circle of eliminating $x^2,\ y^2$ from the equations.
Eliminating $x^2,\ y^2$ from the equations,
By subtracting the equations gives
$\Rightarrow\ x^2 + y^2 + 3x + y - (\,x^2 + y^2 - 6x - 2y) = 0$
$\Rightarrow 9x + 3y = 0 \quad \therefore 3x + y = 0$

b) Left as an exercise. **Ans:** $(-1\,, -2)$, Tangent $2x - y = 0$
c) Left as an exercise. **Ans:** $(\ ,\ )$, Tangent $3x + 4y = 16$

2. (i) *Show that the circles* $x^2 + y^2 + 2x - 2y = 7$ and $x^2 + y^2 - 4x - 10y + 25 = 0$ touch each other.
(ii) *Find the coordinates of the point of contact of the circles in* (i) *above*
(iii) *Determine the coordinates of the point of intersection of the other common tangents to the circles which do not pass through the point of contact in (ii) above.*

Solution

i) For $x^2 + y^2 - 4x - 10y + 25 = 0$, Centre $C_1(2, 5)$

Radius $= \sqrt{(-2)^2 + (-5)^2 - 25} = 2 = r$

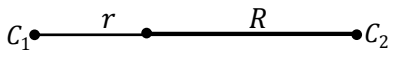
For $x^2 + y^2 + 2x - 2y - 7 = 0$, Centre $C_2(-1, 1)$

Radius $= \sqrt{(1)^2 + (-1)^2 + 7} = 3 = R$

Distance between two centres$= \sqrt{(2+1)^2 + (5-1)^2} = 5 = d$

Since $R + r = d$, hence circles touch each other externally

ii) *Point of contact*



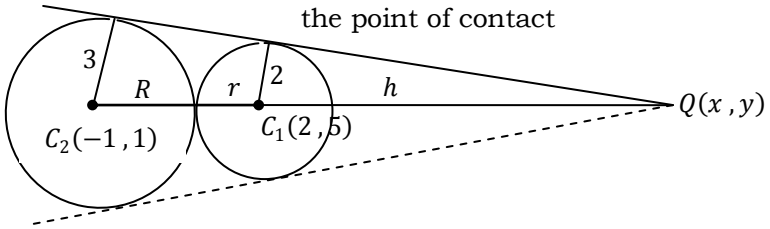
P divides $\overline{C_1C_2}$ in the ratio $\overline{C_1P}:\overline{PC_2} = r:R$

$\Rightarrow \frac{\overline{C_1P}}{\overline{PC_2}} = \frac{r}{R} = \frac{2}{3}$

Now coordinates of P are:

$P\left[\frac{2\times-1+3\times2}{3+2}, \frac{2\times1+3\times5}{3+2}\right] = P\left(\frac{4}{5}, \frac{17}{5}\right)$ which is the point of contact

iii)



Let $h =$ distance of C_2 from Q

$\overline{C_1C_2} = R + r = 3 + 2 = 5$

By similar triangles,

$\frac{3}{2} = \frac{5+h}{h} \quad \therefore h = 10$

Now using coordinate geometry

Q divides $\overline{C_2C_1}$ externally in the ratio $\overline{C_2Q}:\overline{QC_1} = 15:-10 = 3:-2$

Now coordinates of Q are:

$Q\left[\frac{3\times2-2\times-1}{3-2}, \frac{3\times5-2\times1}{3-2}\right] = Q(8, 13)$

Task

Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

PARAMETRIC AND POLAR EQUATION OF A CIRCLE

Parametric Equations of a Circle

The parametric equations of a circle are:

$$\boldsymbol{x = a + c\cos\theta\ , y = b + c\sin\theta}$$

To show that the equations represent a circle

$x = a + c\cos\theta \Rightarrow x - a = c\cos\theta$..... (1)

$y = b + c\sin\theta \Rightarrow y - b = c\sin\theta$..... (2)

Squaring and adding the equations

$(x-a)^2 + (y-b)^2 = c^2\cos^2\theta + c^2\sin^2\theta = c^2(\cos^2\theta + \sin^2\theta)$

$$\therefore \boldsymbol{(x-a)^2 + (y-b)^2 = c^2}$$

Thus the equations $x = a + c\cos\theta\ , y = b + c\sin\theta$ represent an equation of a circle with *Centre* $(a\ , b)$ and *Radius* c

To find the equation of a tangent and normal to the circle defined by $\boldsymbol{x = a + c\cos\theta\ , y = b + c\sin\theta}$

Tangent equation at $(a + c\cos\theta\ ,\ b + c\sin\theta\ )$

$\frac{y-(b+c\sin\theta)}{x-(a+c\cos\theta)} = m_1$..... (1)

$m_1 = \frac{dy}{dx}$ at $(a + c\cos\theta\ ,\ b + c\sin\theta\ )$

$\Rightarrow m_1 = \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$x = a + c\cos\theta\ \Rightarrow \frac{dx}{d\theta} = -c\sin\theta$

$y = b + c\sin\theta \Rightarrow \frac{dy}{d\theta} = c\cos\theta$

$\therefore m_1 = c\cos\theta \times \frac{1}{-c\sin\theta} = \frac{\cos\theta}{-\sin\theta}$ ..... (2)

Eqn. (2) in to eqn. (1)

$\Rightarrow \frac{y-(b+c\sin\theta)}{x-(a+c\cos\theta)} = \frac{\cos\theta}{-\sin\theta}$

$\Rightarrow -y\sin\theta + b\sin\theta + c\sin^2\theta = x\cos\theta - a\cos\theta - c\cos^2\theta$

$\Rightarrow -y\sin\theta + b\sin\theta + c(\sin^2\theta + \cos^2\theta) = x\cos\theta - a\cos\theta$

$$\therefore \boldsymbol{x\cos\theta + y\sin\theta = a\cos\theta + b\sin\theta + c}$$

Equation of the normal at $(a + c\cos\theta\ ,\ b + c\sin\theta\ )$

$\frac{y-(b+c\sin\theta)}{x-(a+c\cos\theta)} = m_2$

$m_2 = -\frac{1}{Slope\ \ of\ \ tangetnt\ \ \ at\ \ (a+c\cos\theta\ ,b+c\sin\theta\ )}$

$= -1/\cos\theta/-\sin\theta = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow \frac{y-(b+c\sin\theta)}{x-(a+c\cos\theta)} = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow y\cos\theta - b\cos\theta - c\sin\theta\cos\theta = x\sin\theta - a\sin\theta - c\cos\theta\sin\theta$

$\Rightarrow -b\cos\theta + a\sin\theta = x\sin\theta - y\cos\theta$

$$\therefore \boldsymbol{x\sin\theta - y\cos\theta = a\sin\theta - b\cos\theta}$$

Polar Equation of a Circle

The three basic equations used in expressing the polar equation of a circle in to Cartesian form and vis versa are:

$\boldsymbol{x = r\cos\theta}$, $\boldsymbol{y = r\sin\theta}$ and $\boldsymbol{x^2 + y^2 = r^2}$, where r =modulus of point $P(x, y)$ from the *origin* and θ = angle made by $\overline{OP}$ and $+ve\ x - axis$.

Example

1. *Show that the given equation represents a circle.*

a) $r = 3\cos\theta$

b) $r = 4\sin\theta$

Solution

a) $r = 3\cos\theta$ multiplying through by r

$\Rightarrow r^2 = 3r\cos\theta$, but $x^2 + y^2 = r^2$ and $x = r\cos\theta$

$\Rightarrow x^2 + y^2 = 3x \ \therefore x^2 + y^2 - 3x = 0$, which is an equation of a circle.

b) $r = 4\sin\theta$ multiplying through by r

$\Rightarrow r^2 = 4r\sin\theta$, but $x^2 + y^2 = r^2$ and $y = r\sin\theta$

$\Rightarrow x^2 + y^2 = 4y \ \therefore x^2 + y^2 - 4y = 0$, which is an equation of a circle.

2. *Express the given equation of a circle in to polar form*

a) $x^2 + y^2 - 5x = 0$

b) $x^2 + y^2 + 2x - 4y = 0$

Solution

a) $x^2 + y^2 - 5x = 0$, using $x^2 + y^2 = r^2$ and $x = r\cos\theta$

$\Rightarrow r^2 - 5r\cos\theta = 0 \ \therefore r = 5\cos\theta$

b) $x^2 + y^2 + 2x - 4y = 0$, using $x^2 + y^2 = r^2$, $x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow r^2 + 2r\cos\theta - 4r\sin\theta = 0 \ \therefore r = 4\sin\theta - 2\cos\theta$

CONIC SECTION

Definition:

A conic section/a conic is the locus of a point that moves in such a way that its distance from a fixed point (FOCUS) bears a constant ratio to its distance from a fixed line (DIRECTRIX) not passing through the focus.

Terms used

A Focus:

Is a fixed point that lies on the axis of a conic.

A Directrix:

Is a line perpendicular to the axis of the conic not passing through the focus.

Eccentricity (e):

Is the constant ratio of distance of a point from a fixed point (focus) to the distance of a same point from the fixed line (Directrix).

Vertex of a conic:

A point (s) where the axis of the conic meets the conic.

Axis of the conic:

Is a line joining the focus and vertices of the conic and is perpendicular to the fixed line (Directrix).

Types of conic sections

- Circle
- Parabola
- Ellipse
- Hyperbola

The nature of locus of any point depends on the value of e, the eccentricity .i.e.

- If $e = 1$, the conic is called parabola
- If $0 < e < 1$, the conic is called ellipse
- If $e > 1$, the conic is called hyperbola

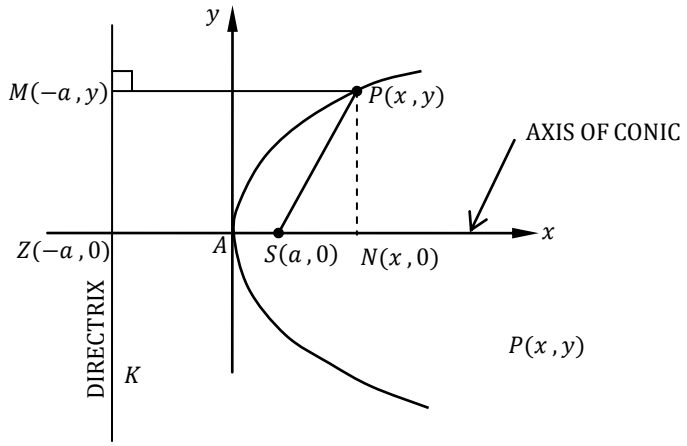
THE PARABOLA

Definition:

Is the locus of a point that moves in such away that its distance from a fixed point (Focus) is equal to its distance from a fixed line (Directrix) not passing through the focus.

Equation of a Parabola in Standard Form

Consider the figure below



Let $S(a\,,0)$ be the focus and ZK be the Directrix.

By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$

$\Rightarrow \overline{PS} = \sqrt{(x-a)^2 + (y-0)^2}$, $\overline{PM} = \sqrt{(x+a)^2 + (y-y)^2}$

$\Rightarrow \sqrt{(x-a)^2 + y^2} = \sqrt{(x+a)^2}$

Squaring both sides

$(x-a)^2 + y^2 = (x+a)^2$

$\Rightarrow x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

$\boxed{y^2 = 4ax}$, is the standard equation of a parabola

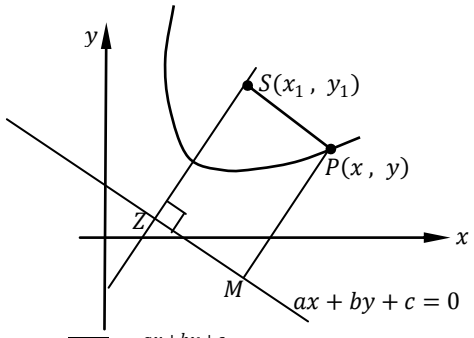
NOTE

1. The parabola in standard form .i.e. $y^2 = 4ax$ has:
- Point $A(0\,,0)$ as the vertex
- Line ZAS as the axis of conic with the equation $y = 0$
- Line MZK as the Directrix with the equation $x = -a$

2. From $y^2 = 4ax$, if $a > 0$, the parabola opens to the right and if $a < 0$, the parabola opens to the left

General Equation of a Parabola

To find the Equation of a Parabola when the Coordinates of the Focus and Equation of the Directrix are given



Let the coordinates of focus be $S(x_1\ ,\ y_1)$, and the equation of the directrix be $ax + by + c = 0$.

Let $P(x\ ,\ y)$ be a point tracing the locus.

By definition

$\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$

$\overline{PS} = \sqrt{(x - x_1)^2 + (y - y_1)^2}$

$\overline{PM} = \perp$ Distance of P from directrix

$\therefore \overline{PM} = \frac{ax+by+c}{\sqrt{a^2+b^2}}$

$\Rightarrow \sqrt{(x - x_1)^2 + (y - y_1)^2} = \frac{ax+by+c}{\sqrt{a^2+b^2}}$

Squaring both sides gives

$$(a^2 + b^2)[(x - x_1)^2 + (y - y_1)^2] = (ax + by + c)^2$$

, which on simplification can be written as:

$$(bx - ay)^2 + 2gx + 2fy + k = 0$$

NOTE

The 2^0 terms of the above equation form a perfect square.

Examples

1) *Find the equation of a parabola with;*

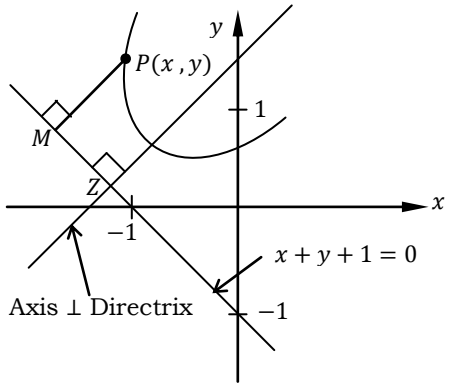
a) *Focus* $(-1\ ,1)$ *and directrix* $x + y + 1 = 0$

b) *Focus* $(0\ ,2)$ *and vertex as the origin*

c) *Focus* $(-7\ ,3)$ *and vertex* $(-2\ ,3)$

Solution

a) Focus: $S(-1\ ,1)$, Directrix: $x + y + 1 = 0$



- Parabola opens to focus
- Directrix ⊥ Axis of the conic
- A, the vertex is the midpoint of $\overline{SZ}$

By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$

$\overline{PS} = \sqrt{(x + 1)^2 + (y - 1)^2}$

$\overline{PM} = \perp$ Distance of P from directrix

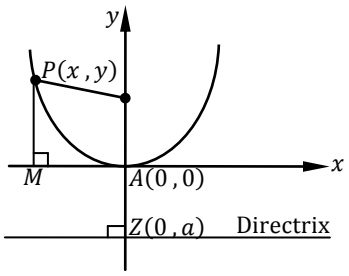
$\overline{PM} = \frac{x+y+1}{\sqrt{1+1}} = \frac{x+y+1}{\sqrt{2}}$

$\Rightarrow \sqrt{(x + 1)^2 + (y - 1)^2} = \frac{x+y+1}{\sqrt{2}}$, squaring

$\Rightarrow (x + 1)^2 + (y - 1)^2 = \frac{[(x+1)+y]^2}{2}$

$\therefore 2(x+1)^2 + 2(y-1)^2 = (x+1)^2 + 2y(x+1) + y^2$
$\Rightarrow (x+1)^2 + 2y^2 - 4y + 2 - y^2 - 2y(x+1) = 0$
$\Rightarrow x^2 + 2x + 1 + y^2 - 4y - 2xy - 2y + 2 = 0$
$\Rightarrow x^2 - 2xy + y^2 + 2x - 6y + 3 = 0$
$\therefore (x-y)^2 + 2x - 6y + 3 = 0$

b) $S(0\,,2)$, $A(0\,,0)$

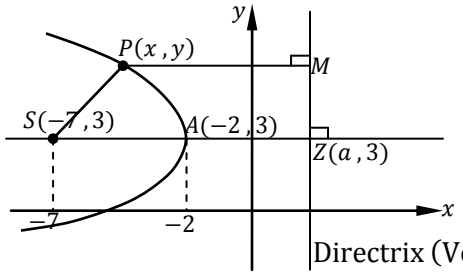


- Directrix ⊥ Axis of the conic
- Parabola opens to focus
- A, the vertex is the midpoint of $\overline{SZ}$

First finding the equation of the directrix
$A\left(\frac{0+0}{2}\,,\ \frac{2+a}{2}\right) \equiv A(0\,,0)$
$\Rightarrow \frac{2+a}{2} = 0\,,\ a = -2$
Equation of the directrix: $y + 2 = 0$

Now equation of parabola:
By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$
$\overline{PS} = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2 + (y-2)^2}$
$\overline{PM}$ =⊥ Distance of P from directrix $y + 2 = 0 \Rightarrow 0x + y + 2 = 0$
$\overline{PM} = \frac{y+2}{\sqrt{0+1}} = y + 2$
$\therefore \sqrt{x^2 + (y-2)^2} = (y+2)$, squaring
$\Rightarrow x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$
$\therefore x^2 = 8y$

c) $S(-7\,,3)$, $A(-2\,,3)$



- Axis passes through SAZ
- Parabola opens to focus
- A, the vertex is the midpoint of $\overline{SZ}$

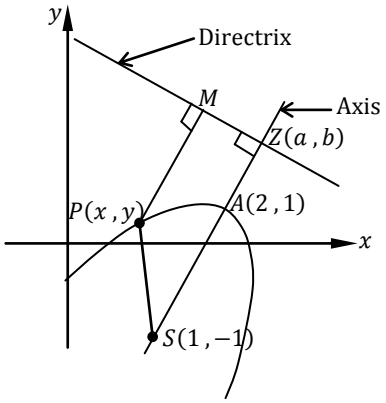
Directrix (Vertical line of the form $x = a$)

Finding the equation of the directrix
$A(-2\,,3) = A\left(\frac{-7+a}{2}\,,\frac{3+3}{2}\right) \Rightarrow -2 = \frac{-7+a}{2} \quad \therefore a = 3$
$\therefore Z(a\,,3) = Z(3\,,3)$
Equation of directrix is $x = 3$ or $x + 0y - 3 = 0$
By definition
$\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$
$\Rightarrow \sqrt{(x+7)^2 + (y-3)^2} = \frac{x-3}{\sqrt{0+1}} = x - 3$, squaring both sides
$\Rightarrow (x+7)^2 + (y-3)^2 = (x-3)^2$
$\Rightarrow x^2 + 14x + 49 + y^2 - 6y + 9 = x^2 - 6x + 9$
$\therefore y^2 - 6y + 20x + 49 = 0$

2) *Find the equation of a parabola with vertex* $(2\,,1)$ *and focus* $(1\,,-1)$.

Solution

Given: *Vertex* $A(2\,,1)$ *and Focus* $S(1\,,-1)$



Finding the equation of the directrix:

$A(2\,,1)$ =Midpoint of $\overline{SZ}$

$A(2\,,1) = A\left(\frac{a+1}{2}, \frac{-1+b}{2}\right) \Rightarrow 2 = \frac{a+1}{2} \quad \therefore a = 3$

$\Rightarrow \frac{-1+b}{2} = 1 \quad \therefore b = 3$

$\therefore Z(3\,,3)$

But the directrix is $\perp$ to axis of conic

From coordinate geometry, $m_1 \times m_2 = -1$

m_1 = Slope of axis of conic = Grad $\overline{AS}$

$= \frac{1+1}{2-1} = 2 \quad \therefore m_2 = -\frac{1}{2}$

Now finding equation of a line (Directrix) passing through a known point $Z(3\,,3)$ and perpendicular to a given line AS is:

$\frac{y-3}{x-3} = -\frac{1}{2} \Rightarrow y - 3 = -\frac{1}{2}(x-3)$ or $x + 2y - 9 = 0$

Now equation of parabola:

By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$

$\overline{PS} = \sqrt{(x-1)^2 + (y+1)^2}$

$\overline{PM}$ =$\perp$ Distance of P from directrix $x + 2y - 9 = 0$

$\overline{PM} = \frac{x+2y-9}{\sqrt{1+4}} = \frac{x+2y-9}{\sqrt{5}}$

$\therefore \sqrt{(x-1)^2 + (y+1)^2} = \frac{x+2y-9}{\sqrt{5}}$, squaring

$\Rightarrow x^2 - 2x + 1 + y^2 + 2y + 1 = \frac{[(x+2y)-9]^2}{5}$

$\Rightarrow 5x^2 - 10x + 5 + 5y^2 + 10y + 5 = (x+2y)^2 - 18(x+2y) + 81$

$\Rightarrow 5x^2 + 5y^2 - 10x + 10y + 10 = x^2 + 4xy + 4y^2 - 18x - 36y + 81$

$\Rightarrow 4x^2 - 4xy + y^2 + 8x + 46y - 71 = 0$

$\Rightarrow (2x - y)^2 + 8x + 46y - 71 = 0$

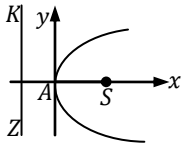
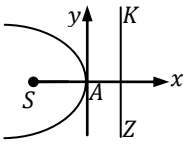
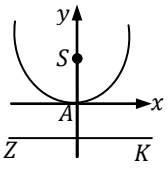
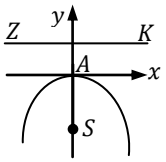
Task

Find the equation of a parabola with:

a) Vertex $(2\,,0)$, directrix $x = 0$ **Ans:** $y^2 = 8x - 16$

b) Vertex $(3\,,1)$, Focus at $(3\,,3)$ **Ans:** $(x-3)^2 = 8(y-1)$

Summary of Standard Equation of a Parabola

Curve	Eqn. of curve	Focus	Directrix	Axis	Vertex	Tangent at vertex
	$y^2 = 4ax$	$(a\,,0)$	$x = -a$	$y = 0$	$A(0\,,0)$	$x = 0$
	$y^2 = -4ax$	$-(a\,,0)$	$x = a$	$y = 0$	$A(0\,,0)$	$x = 0$
	$x^2 = 4ay$	$(0\,,a)$	$y = -a$	$x = 0$	$A(0\,,0)$	$y = 0$
	$x^2 = -4ay$	$(0\,,-a)$	$y = a$	$x = 0$	$A(0\,,0)$	$y = 0$

The above table s used in sketching the conic, determining the vertex, equation of axis, focus and directrix of a parabola from a given equation.

Example

1. *Find the vertex, axis, focus and directrix of a parabola*

a) $y^2 = 5x - 4y - 9$

b) $x^2 = 9x - y - 14$ c) $y^2 - 6y + 20x + 49 = 0$

Solution

a) Expressing the equation in one of the above standard forms of a parabola .i.e. $Y^2 = 4AX$. This form is taken because the perfect square variable is y.

$y^2 = 5x - 4y - 9 \Rightarrow y^2 - 4y = 5x - 9$

Completing squares

$\Rightarrow y^2 - 4y + 4 = 5x - 9 + 4$

$\Rightarrow (y-2)^2 = 5x - 5 = 5(x-1)$, which is of the form $Y^2 = 4AX$

By comparison

$$Y = y + 2\,,\ X = x - 1\,,\ 4A = 5\ \ \therefore A = 5/4$$

But for a parabola $Y^2 = 4AX$,

Vertex $(0\,,0)$, Axis $= 0$, Focus $S(A\,,0)$, Directrix $X = -A$

For vertex, $(0\,,0) = (X,Y)$
$\Rightarrow y+2=0 \;\therefore y=-2\,,\; x-1=0 \;\; \therefore x=1$
$\therefore$ Vertex is $(1\,,-2)$
Axis, $Y=0=y+2 \;\; \therefore y=-2$
Focus, $S(A\,,0)=S(X,Y)$
$\Rightarrow Y=y+2=0 \;\therefore y=-2\,,\; x-1=A=5/4 \;\; \therefore x=9/4$
Focus $S\left(\frac{9}{4},-2\right)$
Directrix, $X=-A \Rightarrow x-1=-\frac{5}{4} \;\; \therefore x=-1/4$

b) $x^2=9x-y-14 \Rightarrow x^2-9x=-y-14$
Completing squares
$\Rightarrow x^2-9x+(-9/2)^2=-y-14+(-9/2)^2$
$\Rightarrow (x-9/2)^2=-y-14+\frac{81}{4}=-y+\frac{25}{4}$
$\therefore (x-9/2)^2=-y+25/4$, which is of the form $X^2=4AY$
For this parabola,
Focus is $S(0\,,A)=S(X,Y)$, Axis is $X=0$, Vertex $A(0\,,0)=A(X,Y)$ and
Directrix $Y=-A$
Comparing $(x-9/2)^2=-y+25/4$ with $X^2=4AY$

$$X=x-9/2\;,\; Y=-y+25/4\,,\; 4A=1 \;\; \therefore A=1/4$$

Focus $X=0=x-\frac{9}{2} \;\; \therefore x=\frac{9}{2};\;\; Y=-y+\frac{25}{4}=A=\frac{1}{4} \;\; \therefore y=6$
Focus $S\left(\frac{9}{2}\,,6\right)$
Axis, $X=x-\frac{9}{2}=0 \;\; \therefore x=\frac{9}{2}$
Vertex, $X=0=x-\frac{9}{2}=0 \;\; \therefore x=\frac{9}{2};\;\; Y=0=-y+\frac{25}{4} \;\; \therefore y=\frac{25}{4}$
Vertex $A\left(\frac{9}{2}\,,\frac{25}{4}\right)$
Directrix, $Y=-A=-y+\frac{25}{4}=-\frac{1}{4} \;\; \therefore y=\frac{13}{2}$ is the directrix

c) $y^2-6y+20x+49=0 \Rightarrow y^2-6y=-20x-49$
Completing squares
$\Rightarrow y^2-6y+9=-20x-49+9$
$\Rightarrow (y-3)^2=-20(x+2)$, which is of the form $Y^2=4AX$
For this parabola
Focus $S(A\,,0\,)=S(X\,,Y) \;\; \therefore X=A,\; Y=0$
Vertex $A(0\,,0)=A(X\,,Y) \;\; \therefore X=0\,,Y=0$
Axis $Y=0$
Directrix $X=-A$
Comparing $(y-3)^2=-20(x+2)$ with $Y^2=4AX$

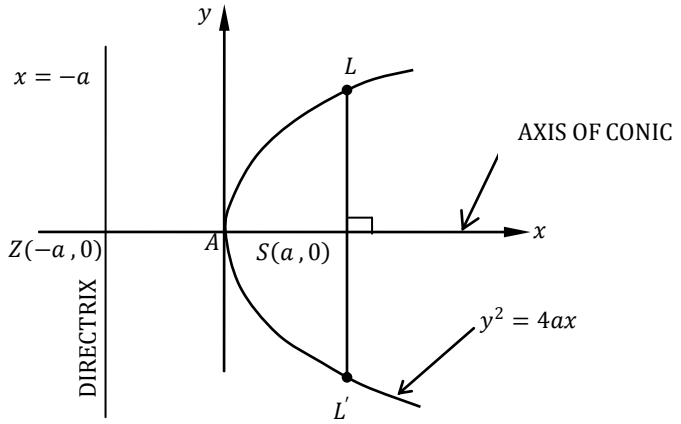
$$Y=y-3\;,\; X=x+2\,,\; 4A=-20 \;\; \therefore A=-5$$

For $X=A=-5=x+2 \;\; \therefore x=-7$

For $Y = 0 = y - 3 \quad \therefore y = 3$
Focus $S(-7\,,3)$
Axis $Y = 0 = y - 3 \quad \therefore y = 3$
Vertex $X = 0 = x + 2 \quad \therefore x = -2$
Vertex $A(-2\,,3)$
Directrix $X = -A \Rightarrow x + 2 = 5 \quad \therefore x = 3$

LATUS RECTUM

The chord of a parabola which passes through the focus and perpendicular to the axis of the parabola is called *Latus Rectum.*



From the figure, LSL' is the rectum.
$LSL' = 2LS = 2L'S$
Since the $x-$ coordinate of $S(a\,,0)$ is $x = a$, then the $y-$ coordinate of ends of latus rectum are: $y^2 = 4a.a \Rightarrow y = \pm 2a$
$\therefore L(a\,,2a)\,,\ L'(a\,,-2a)$, length $\overline{LL'} = 4a$

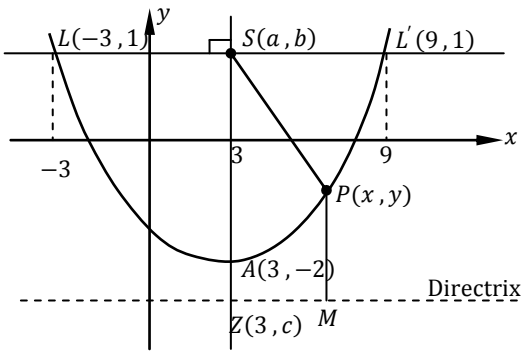
NOTE

Two parabolas are equal if latus rectum is the same.

Example

Find the equation of a parabola with vertex at $(3\,,-2)$*, end points of latus rectum at* $(-3\,,1)$ *and* $(9\,,1)$

Solution



Finding coordinates of focus $S(a\,,b)$

$S(a\,,b)$ is the midpoint of $\overline{LL'}$

$\Rightarrow (a\,,b) = \left(\frac{-3+9}{2}\,,\frac{1+1}{2}\right) = (3\,,1)$

$\therefore S(3\,,1)$

Also $\overline{AS} = \overline{AZ}$, $\overline{PS} = \overline{PM}$. From the definition A is the midpoint of $\overline{SZ}$

$\therefore (3\,,-2) = \left(\frac{a+3}{2}\,,\frac{b+c}{2}\right) = \left(\frac{3+3}{2}\,,\frac{1+c}{2}\right)$

$\Rightarrow \frac{1+c}{2} = -2 \quad \therefore c = -5$

Directrix passes through $Z(3\,,-5)$

Since axis of conic passes through S, V and Z and is vertical, the directrix equation is: $y = -5$ or $y + 5 = 0$

$\therefore \overline{PS}^2 = \overline{PM}^2$ becomes;

$(x-3)^2 + (y-1)^2 = (y+5)^2$

$(x-3)^2 = (y+5)^2 - (y-1)^2 = (y+5+y-1)(y+5-y+1)$

$\therefore (x-3)^2 = 12(y+2)$

A LINE AND A PARABOLA

The following questions must be understood in order to master this sub topic.

a) **To Find the Point of Intersection of the Line $y = mx + c$ and the Parabola $y^2 = 4ax$**

Solving the line and the parabola simultaneously

$y = mx + c$ ..... (1)

$y^2 = 4ax$ ..... (2)

Eqn. (1) in to eqn. (2)

$\Rightarrow (mx + c\,)^2 = 4ax$

$\Rightarrow m^2x^2 + 2mcx + c^2 - 4ax = 0$

$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0$ ..... (3)

Solving this equation gives *two* values of x. Obtain the corresponding values from equation (1), and hence two points of intersection in the form $(x\,,y)$

b) **To Show that a given Line $y = mx + c$ is a Tangent to the Parabola $y^2 = 4ax$.**

Here first solve the line and the parabola simultaneously to obtain eqn. (3) above

$m^2x^2 + (2mc - 4a)x + c^2 = 0$ ..... (3)

Now proceed in two ways

Approach I

Applying condition for tangency .i.e. condition for equal roots of a quadratic equation

For tangency, $B^2 = 4AC$

$\Rightarrow (2mc - 4a)^2 = 4m^2c^2$

$\Rightarrow 4m^2c^2 - 16mca + 16a^2 = 4m^2c^2$

$\boxed{c = \frac{a}{m}}$, which is the condition for a line $y = mx + c$ to be a tangent to $y^2 = 4ax$

Approach II

Solving eqn. (3), two repeated/coincident values of x must be obtained.

Eqn. (3) can have coincident roots if the condition for tangency is used.

Put $c = a/m$ in to eqn. (3)

$\Rightarrow m^2x^2 + \left(2m\frac{a}{m} - 4a\right)x + \left(\frac{a}{m}\right)^2 = 0$

$\Rightarrow m^2x^2 - 2ax + \frac{a^2}{m^2} = 0 \quad \therefore \left(mx - \frac{a}{m}\right)^2 = 0$

$\Rightarrow mx - \frac{a}{m} = 0 \quad \therefore x = \frac{a}{m^2}, \frac{a}{m^2}$

Hence a line is a tangent to the parabola.

c) **To obtain the Point of Contact of a tangent line $y = mx + c$ to the parabola $y^2 = 4ax$**

Proceeding as in approach II above from eqn. (3), $x = \frac{a}{m^2}$, obtain the corresponding $y-$ value from the line.

$y = mx + c$, $x = \frac{a}{m^2}$, $c = a/m$

$\Rightarrow y = m \times \frac{a}{m^2} = \frac{a}{m}$

$\therefore$ Point of contact is $\boldsymbol{P\left(\frac{a}{m^2}, \frac{a}{m}\right)}$

d) **To state the General Point on the parabola $y^2 = 4ax$**

Using the line $y = mx + c$ as a tangent to the parabola $y^2 = 4ax$, obtain the point of contact $P\left(\frac{a}{m^2}, \frac{a}{m}\right)$

Let $t = \frac{1}{m} \Rightarrow P(at^2, 2at)$

$\therefore$ General point on the parabola is $\boldsymbol{T(at^2, 2at)}$

NOTE

From the general point on the parabola, $y^2 = 4ax$, two parametric equations can be deduced.

The parametric equations of $\boldsymbol{y^2 = 4ax}$ are: $\boldsymbol{x = at^2, y = 2at}$

e) **To state the Equation of the Tangent to the parabola in Standard Form**

From the line $y = mx + c$ and condition for tangency $c = \frac{a}{m}$, the tangent in slope form is $\boldsymbol{y = mx + \frac{a}{m}}$

NOTE

In answering questions (b), (c) and (d), you must first obtain eqn. (3) in question (a) above, the approach to finding point of intersection of a line and a parabola.

Example

1. *Find the point of intersection of the line* $x + y + 1 = 0$ *and the parabola* $y^2 = 4(x + 1)$

Solution

$x + y + 1 = 0 \Rightarrow y = -x - 1$ ..... (1)

$y^2 = 4(x + 1)$ ..... (2)

Eqn. (1) in to eqn. (2)

$(-x - 1)^2 = 4(x + 1) \Rightarrow (x + 1)^2 = 4(x + 1)$

$\Rightarrow (x + 1)(x + 1 - 4) = 0 \quad \therefore x = -1, 3$ ..... (3)

Eqn. (3) in to eqn. (1)

For $x = -1,\ y = 0\ ;\ x = 3, y = -4$

Points of intersection are: $(-1, 0)\ ,\ (3, -4)$

2. *Prove that the line* $3y - 6x - 2 = 0$ *touches the parabola* $3y^2 = 16x$. *Hence find the coordinates of point of contact.*

Solution

For a line to be a tangent, it should touch the parabola at only one point, the point of contact.

$3y - 6x - 2 = 0 \Rightarrow y = \frac{6x+2}{3}$ ..... (1)

$3y^2 = 16x$ ..... (2)

Eqn. (2) in to eqn. (1)

$3\left(\frac{6x+2}{3}\right)^2 = 16x \ \Rightarrow 3\left(\frac{36x^2+24x+4}{9}\right) = 16x$

$\Rightarrow 36x^2 + 24x + 4 - 48x = 0$ or $36x^2 - 24x + 4 = 0$

$\Rightarrow 9x^2 - 6x + 1 = 0$ or $(3x - 1)^2 = 0 \quad \therefore x = \frac{1}{3}, \frac{1}{3}$

Hence line touches the parabola at one point since there is only one value of $x = \frac{1}{3}$

Point of contact:

For $x = \frac{1}{3}, y = ?$

Using eqn. (1), $y = \frac{6(1/3)+2}{3} = \frac{4}{3} \quad \therefore \left(\frac{1}{3}, \frac{4}{3}\right)$ is the point of contact

3. *Find the condition that* $lx + my + n = 0$ *may touch the parabola* $y^2 = 4ax$

Solution

$lx + my + n = 0 \Rightarrow y = -\left(\frac{lx+n}{m}\right)$ ..... (1)

$y^2 = 4ax$ ..... (2)

Solving the equations simultaneously

Eqn. (1) in to eqn. (2)

$\left[-\left(\frac{lx+n}{m}\right)\right]^2 = 4ax \Rightarrow (lx+n)^2 = 4am^2x$

$\Rightarrow l^2x^2 + 2nlx + n^2 - 4am^2x = 0$ or $l^2x^2 + (2nl - 4am^2)x + n^2 = 0$

For tangency $B^2 = 4AC$

$\Rightarrow (2nl - 4am^2)^2 = 4l^2n^2$ or $4l^2n^2 - 16nlam^2 + 16a^2m^4 = 4l^2n^2$

$\Rightarrow 16nlam^2 = 16a^2m^4$, dividing through by $16am^2$

$\therefore ln = am^2$

Other questions involving Tangents and Parabolas

4. *A line $x + 2y + k = 0$ is a tangent to the parabola $y^2 = 8x$. Find the possible value(s) of k, hence deduce the point(s) of contact.*

Solution

$x + 2y + k = 0 \Rightarrow x = -(2y + k)$ ..... (1)

$y^2 = 8x$ ..... (2)

Eqn. (2) in to eqn. (1)

$y^2 = 8[-(2y + k)] \Rightarrow y^2 + 16y + 8k = 0$..... (3)

For tangency $B^2 = 4AC$

$\Rightarrow (16)^2 = 4(8k) \quad \therefore k = 8$

$\therefore x + 2y + 8 = 0$ is a tangent.

Point of contact:

From eqn. (3), substituting the value of k,

$\Rightarrow y^2 + 16y + 64 = 0$ or $(y+8)^2 = 0 \quad \therefore y = -8$..... (4)

From eqn. (1), putting $y = -8, k = 8$

$\Rightarrow x = -(2 \times 8 + 8) = -8$

$\therefore$ Point of contact is: $(-8, -8)$

5. *Find the equation of the tangent to the given parabola;*

a) $y^2 = 12x$ *which is parallel to the line* $x - y + 5 = 0$

b) $y^2 = 16x$ *whch is inclined to* $x-$*axis at an angle of* 135^0

c) $y^2 = -24x$ *which is perpendicular to the line* $x + 2y + 10 = 0$

d) $y^2 = 8x$ *which is inclined to* $x -$ *axis at an angle of* $\sin^{-1}\frac{5}{13}$

Solution

a) Let the required tangent be $y = mx + c$..... (1)

In this equation we have a task of finding values of c and m

Obtaining m

Given: $x - y + 5 = 0$ or $y = x + 5$ *is* $\parallel$ *to* $y = mx + c$

For parallel lines, $m = 1$

Eqn. (1) becomes

$y = x + c$ …… (2)

Obtaining c:

Eqn. (2) in to equation of the parabola $y^2 = 12x$

$\Rightarrow (x + c)^2 = 12x$ or $x^2 + (2c - 12)x + c^2 = 0$

For tangency, $B^2 = 4AC$

$\Rightarrow (2c - 12)^2 = 4(1)c^2$ or $4c^2 - 48c + 144 = 4c^2$

$\Rightarrow c = 3$

$\therefore$ Tangent equation is: $y = x + 3$

b) Let the required tangent be $y = mx + c$…… (1)

Finding the values of c and m

Obtaining m

Given: Angle of inclination to x −axis 135^0 $\Rightarrow m = \tan\theta = \tan 135^0 = -1$

Eqn. (1) becomes

$y = -x + c$ …… (2)

Obtaining c:

Eqn. (2) in to equation of the parabola $y^2 = 16x$

$\Rightarrow (-x + c)^2 = 16x$ or $x^2 + (-2c - 16)x + c^2 = 0$

For tangency, $B^2 = 4AC$

$\Rightarrow (-2c - 16)^2 = 4(1)c^2$ or $4c^2 + 64c + 256 = 4c^2$

$\Rightarrow c = -4$

$\therefore$ Tangent equation is: $y = -x - 4$

c) Let the required tangent be $y = mx + c$…… (1)

Finding the values of c and m

Obtaining m

Given: tangent is $\perp\ x + 2y + 10 = 0$ $\Rightarrow m = -1/(-1/2) = 2$

Eqn. (1) becomes

$y = 2x + c$ …… (2)

Obtaining c:

Eqn. (2) in to equation of the parabola $y^2 = -24x$

$\Rightarrow (2x + c)^2 = -24x$ or $4x^2 + (4c + 24)x + c^2 = 0$

For tangency, $B^2 = 4AC$

$\Rightarrow (4c + 24)^2 = 4(4)c^2$ or $16c^2 + 192c + 576 = 16c^2$

$\Rightarrow 192c + 576 = 0$ or $c^2 + 3 = 0$

$\therefore c = -3$

$\therefore$ Tangent equation is: $y = 2x - 3$

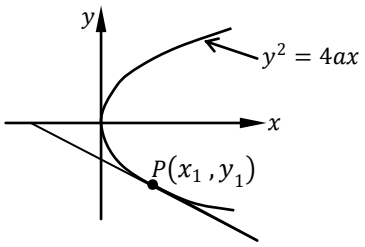
d) Left as an exercise. **Ans:** $65y = 25x + 338$

Task

1. *Prove that the line* $9x - 3y + 1 = 0$ *touches* $y^2 = 4x$
2. *Find the equation of a common tangent to* $y^2 = 4x$ *and* $x^2 = 4y$.
   Ans: $x + y + 1 = 0$
3. *Prove that the line* $6y = 18x + 1$ *touches* $y^2 = 2x$
4. *Find the equation of a straight line which touches both* $y^2 = 8ax$ *and* $x^2 + y^2 = 2a^2$ **Ans:** $y = x + 2a$
5. *Find the equation of the parabola whose focus is* $(0\,,0)$ *and whose directrix is* $2x = y + 1$. *Show also that the line* $2y = 4x - 1$ *touches the parabola.* **Ans:** $(x + 2y)^2 = 2y - 4x + 1$

EQUATION OF A TANGENT AND NORMAL AT ANY POINT $(x_1\,, y_1)$ ON A PARABOLA $y^2 = 4ax$

To Find an Equation of a Tangent at any point $(x_1\,, y_1)$ on the Parabola $y^2 = 4ax$



Required line:

$y - y_1 = m(x - x_1)$ ..... (1)

Now finding slope m of the line

From $y^2 = 4ax$, differentiating w.r.t x

$\Rightarrow 2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$

$\left.\frac{dy}{dx}\right|_{\text{at } P(x_1\,, y_1)} = \frac{2a}{y_1}$

$\therefore y - y_1 = \frac{2a}{y_1}(x - x_1)$ or $yy_1 - {y_1}^2 = 2a(x - x_1)$

$\therefore yy_1 = 2ax + {y_1}^2 - 2ax_1$ ..... (2)

But $(x_1\,, y_1)$ lies on the parabola

$\Rightarrow {y_1}^2 = 4ax_1$

Eqn. (2) becomes

$\therefore yy_1 = 2ax + 4ax_1 - 2ax_1 = 2a(x + x_1)$

$\boxed{yy_1 = 2a(x + x_1)}$, which is the required equation of the tangent at $P(x_1\,, y_1)$ on to the parabola $y^2 = 4ax$.

To Find an Equation of a Tangent at any point $(at^2\,,\ 2at)$ on the Parabola $y^2 = 4ax$

Following the above presentation, and using the equation of a tangent at any point $P(x_1\,, y_1)$ to the parabola $y^2 = 4ax$ which is $yy_1 = 2a(x + x_1)$, the required tangent at $(at^2, 2at)$ is:

Comparing $P(x_1\,, y_1) = (at^2, 2at) \Rightarrow x_1 = at^2\,, y_1 = 2at$

$\Rightarrow y(2at) = 2a(x + at^2)$ or $ty = x + at^2$

$\boxed{ty = x + at^2}$

To Find the Equation of a Tangent to the Parabola $y^2 = 4ax$ from a point $P(x_1, y_1)$ not on the parabola

Let the required tangent be: $y = mx + c$..... (1)

The task is to find the values of m and c

Obtaining the value(s) of m

Solving the equations of tangent and the parabola simultaneously gives:

$m^2x^2 + (2mc - 4a)x + c^2 = 0$

Now applying condition for tangency $B^2 = 4AC$ gives $c = \frac{a}{m}$..... (2)

Thus equation of tangent becomes:

$y = mx + \frac{a}{m}$ ..... (3)

To find c, point $P(x_1, y_1)$ lies on the tangent

$\Rightarrow c = y_1 - mx_1$ ..... (4)

Solving equation (2) and (4) simultaneously gives:

$\frac{a}{m} = y_1 - mx_1$ or $m^2x_1 - my_1 + a = 0$. Hence two values of m are obtained.

From eqn. (3), two tangents can be obtained as:

$y = m_1x + \frac{a}{m_1}$, $y = m_2x + \frac{a}{m_2}$

Alternatively

Considering the given point $P(x_1, y_1)$ to be the general point on the parabola $y^2 = 4ax$ as $T(at^2, 2at)$, the tangent equation at $T(at^2, 2at)$ is

$ty = x + at^2$..... (1)

Now point $P(x_1, y_1)$ is on the tangent

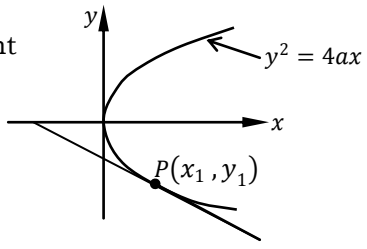
$\Rightarrow ty_1 = x_1 + at^2$ or $at^2 - ty_1 + x_1 = 0$, which is a quadratic in t.

Two values of t can be obtained. Using these values in eqn. (1), two tangents can be deduced.

$t_1y = x + at_1{}^2$ and $t_2y = x + at_2{}^2$

NORMAL AT ANY POINT $P(x_1, y_1)$ TO THE PARABOLA $y^2 = 4ax$

To Find the Equation of the Normal at any point $P(x_1, y_1)$ on the Parabola $y^2 = 4ax$



Using $m_1 \times m_2 = -1$, m_1 = Gradient of tangent , and m_2 = Gradient of the normal

Grad $PT = m_1$ =?

From $y^2 = 4ax$, differentiating w.r.t x

$2y\frac{dy}{dx} = 4a$, $\frac{dy}{dx} = \frac{2a}{y}$

$\therefore \left.\frac{dy}{dx}\right|_{\text{at } P(x_1, y_1)} = \frac{2a}{y_1} = m_1$

$\therefore m_2 = \text{Grad NP} = -\frac{1}{m_1} = -\frac{y_1}{2a}$

Equation of the normal is:

$y - y_1 = m_2(x - x_1)$

$\boxed{y - y_1 = -\frac{y_1}{2a}(x - x_1)}$, which is an equation of the normal at

$P(x_1\,, y_1)$ onto the parabola $y^2 = 4ax$

To Find an Equation of a Normal at any point $T(at^2\,,\ 2at)$ on the Parabola $y^2 = 4ax$

Comparing $P(x_1\,, y_1)$ with $T(at^2\,,\ 2at)$, and using the equation of the normal above at $P(x_1\,, y_1)$ which is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

The required equation of the normal at $T(at^2\,,\ 2at)$ is:

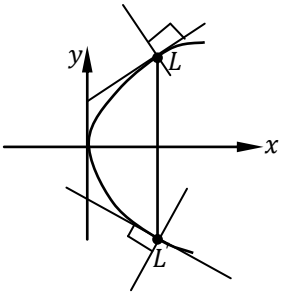
$\Rightarrow y - 2at = -\frac{2at}{2a}(x - at^2) = -tx + at^3$

$$tx + y = at^3 + 2at$$

General Examples

1. *Find the equation of the tangents and normals to the parabola $y^2 = 4ax$ at the ends of latus rectum.*

Solution



$L(a\,, 2a)$, $L'(a\,, -2a)$ are end points of the latus rectum

Equation of tangent at $L(a\,, 2a)$

$y - 2a = m(x - a)$

Obtaining slope m

Differentiating $y^2 = 4ax$ w.r.t x

$\Rightarrow 2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$

$\frac{dy}{dx}$ at P(a ,2a) $= \frac{2a}{2a} = 1$

$\Rightarrow y - 2a = x - a \quad \therefore y = x + a$

Equation of normal at $L(a\,, 2a)$

$y - 2a = m'(x - a)$

Finding slope m'

From $m \times m' = -1\,, m' = -\frac{1}{m} = -\frac{1}{1} = -1$ m =slope of tangent

$\Rightarrow y - 2a = -1(x - a)$

$\therefore y = -x + 3a$

Equation of tangent at $L'(a\,, -2a)$

$y + 2a = m_1(x - a)$

Obtaining slope m_1

Differentiating $y^2 = 4ax$ w.r.t x

$\Rightarrow 2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$

$\frac{dy}{dx}$ at P(a ,– 2a) $= \frac{2a}{-2a} = -1$

$\Rightarrow y + 2a = -1(x - a)$

$\therefore y = -x - a$

Equation of normal at $L'(a\,, -2a)$

$y + 2a = m_2(x - a)$

Finding slope m_2

But $m_2 = -\frac{1}{m_1} = -\frac{1}{-1} = 1$

$\Rightarrow y + 2a = (x - a)$

$\therefore y = x - 3a$

2. *Show that the equation of the tangent to the curve $y^2 = 4ax$ at the ppoint $(at^2, 2at)$ is $ty = x + at^2$ and the equation of the tangent to the curve $x^2 = 4by$ at the point $(2bp, bp^2)$ is $y = px - bp^2$.*

The curves $y^2 = 32x$ and $x^2 = 4y$ intersect at the origin and at A. find the equation of the common tangent to these curves and the coordinates of the points of contact B and C between the tangent and the curves. Calculate area of triangle ABC.

Solution

$y^2 = 4ax$, required tangent: $ty = x + at^2$

Tangent equation at $(at^2, 2at)$

$y - 2at = m(x - at^2)$ ..... (1)

Obtaining slope, m

For $y^2 = 4ax$, differentiating w.r.t x

$2y\frac{dy}{dx} = 4a$, $\frac{dy}{dx} = \frac{2a}{y}$

Now $\left.\frac{dy}{dx}\right|_{\text{at } (at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t} = m$..... (2)

Eqn. (2) in to eqn. (1)

$\Rightarrow y - 2at = \frac{1}{t}.(x - at^2)$

$\Rightarrow ty - 2at^2 = x - at^2 \quad \therefore ty = x + at^2$

For $x^2 = 4by$, differentiating w.r.t x

$2x\frac{dy}{dx} = 4b$, $\frac{dy}{dx} = \frac{x}{2b}$

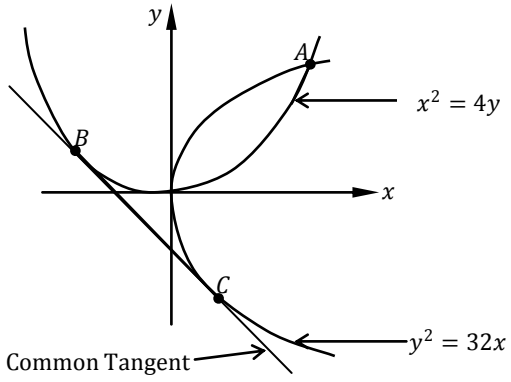
Now $\frac{dy}{dx}_{\text{at}(2bp, bp^2)} = \frac{2bp}{2b} = p = m'$

Tangent equation at $(2bp, bp^2)$

$y - bp^2 = m'(x - 2bp) \Rightarrow y - bp^2 = p(x - 2bp)$

$\Rightarrow y = px - 2bp^2 + bp^2$

$\therefore y = px - bp^2$



Comparing curves $y^2 = 32x$ and $y^2 = 4ax \Rightarrow 32 = 4a \quad \therefore a = 8$

Point on $y^2 = 4ax$ is $C(at^2, 2at)$

$\therefore C(8t^2, 16t)$

Now equation of tangent to $y^2 = 4ax$ at $(at^2, 2at)$ is $ty = x + at^2$

$\Rightarrow$ Equation of tangent at $C(8t^2, 16t)$ to the curve $y^2 = 32x$ is

$ty = x + 8t^2$..... (1)

Eqn. (1) meets $x^2 = 4y$ at B
From curve $x^2 = 4by$ and $x^2 = 4y$, by comparison, $4b = 4 \quad \therefore b = 1$
Equation of the tangent to the curve $x^2 = 4y$ at $B(2bp, bp^2)$ is $y = px - bp^2$
Now equation of the tangent to the curve $x^2 = 4y$ at $B(2p, p^2)$ is:
$y = px - p^2$ ..... (2)
Since the tangent at B and the tangent at C are the same, comparing the two equations,
$ty = x + 8t^2 \Rightarrow y = \frac{1}{t}x + 8t \; ; \; y = px - p^2$
$\Rightarrow \frac{1}{t} = p$, and $8t = -p^2$, solving the two equations
$\Rightarrow 8t = -\frac{1}{t^2} \quad \therefore 8t^3 + 1 = 0$
$\Rightarrow (2t + 1)(4t^2 - 2t + 1) = 0 \quad \therefore \; t = -\frac{1}{2}$
$\Rightarrow t = -\frac{1}{2}, p = -2$
Using eqn. (2) and $p = -2$
Equation of the common tangent is:
$y = -2x - 4$

Alternatively
Tangent at B: $y = px - p^2$ is also a tangent to the curve $y^2 = 32x$
Solving the two equations simultaneously and applying the condition for tangency $B^2 = 4AC$
$(px - p^2)^2 = 32x$
$p^2x^2 - 2p^3x + p^4 - 32x = 0 \Rightarrow p^2x^2 - (2p^3 + 32)x + p^4 = 0$
For tangency $B^2 = 4AC$
$\Rightarrow [-(2p^3 + 32)]^2 = 4p^2.p^4$
$\Rightarrow 4p^6 + 128p^3 + 1024 = 4p^6$ or $p^3 + 8 = 0$
$\Rightarrow (p + 2)(p^2 - 2p + 4) = 0 \quad \therefore p = -2$
From eqn. (2) $y = px - p^2$, $p = -2$
Equation of the common tangent is:
$y = -2x - 4$
Now obtaining the coordinates of B and C
$B(2p\,,\; p^2) \equiv B(-4\,, 4)$
To find $C(8t^2\,, 16t)$, lies on the tangent $y = -2x - 4$
$\Rightarrow 16t = -16t^2 - 4$ or $4t^2 + 4t + 1 = 0$, $t = -\frac{1}{2}$
$\therefore C(8t^2\,, 16t) \equiv C(2\,, -8)$

3. *Find the equation of a tangent to the parabola from the given points:*
a) *Parabola:* $y^2 = 8x$, *point* $(-10\,, -8)$
b) *Parabola:* $y^2 = -32x$, *point* $(1\,, -12)$
c) *Parabola:* $y^2 = 4ax$, *point* $(-4a\,, -3a)$
d) *Parabola:* $y^2 = -16ax$, *point* $(4a\,, 2a)$

Solution

a) **Approach I:**

Let the required tangent be: $y = mx + c$

Obtaining value of c

$(-10, -8)$ lies on this tangent

$\Rightarrow c = -8 + 10m$ ..... (1)

Obtaining the slope, m

Solving the equation of the tangent and parabola simultaneously

$(mx + c)^2 = 8x \Rightarrow m^2x^2 + (2mc - 8)x + c^2 = 0$

For tangency, $B^2 = 4AC$

$\Rightarrow (2mc - 8)^2 = 4m^2c^2$

$\Rightarrow 4m^2c^2 - 32mc + 64 = 4m^2c^2$

$\therefore mc = 2$..... (2)

Eqn. (1) into eqn. (2)

$m(-8 + 10m) = 2$ or $5m^2 - 4m - 1 = 0 \quad \therefore m = 1, -\frac{1}{5}$

For $m = 1, c = -8 + 10(1) = 2$

For $m = -\frac{1}{5}, c = -8 + 10\left(-\frac{1}{5}\right) = -10$

$\therefore y = x + 2, \; y = -\frac{1}{5}x - 10 \Rightarrow -5y = x + 50$

Approach II:

The equation of the tangent to the parabola $y^2 = 4ax$ at the point $T(at^2, 2at)$ is $ty = x + at^2$

Now comparing $y^2 = 4ax$ with $y^2 = 8x$, $a = 2$

Thus the general point on the given parabola is: $P(2t^2, 4t)$

The tangent at $P(2t^2, 4t)$ passes through $(-10, -8)$

Tangent is $ty = x + 2t^2$..... (1)

$\Rightarrow -8t = -10 + 2t^2$ or $t^2 + 4t - 5 = 0 \quad \therefore t = 1, -5$

From equation (1), the tangents are:

For $t = 1, \; y = x + 2$

For $t = -5, \; -5y = x + 50$

Approach III:

The general tangent to the parabola $y^2 = 4ax$ with slope m is: $y = mx + \frac{a}{m}$

This tangent passes through $(-10, -8)$.

Comparing $y^2 = 4ax$ with $y^2 = 8x$, $a = 2$

Required tangent is:

$y = mx + \frac{2}{m}$ ..... (1)

Eqn. (1) is satisfied by the point $(-10, -8)$

$\Rightarrow -8 = -10m + \frac{2}{m}$ or $5m^2 - 4m - 1 = 0 \quad \therefore m = 1, -\frac{1}{5}$

Tangent equations are:

$y = x + 2, y = -\frac{1}{5}x - 10 \Rightarrow -5y = x + 50$

b) Left as an exercise. **Ans:** $y = 2x - 4, 3y = 18x - 4$

c) Using approach III, let the tangent be:

$y = mx + \frac{a}{m}$ ..... (1)

This tangent is the general tangent on $y^2 = 4ax$

Point $(-4a\,, -3a)$ lies on this tangent

$\Rightarrow -3a = -4am + \frac{a}{m}$ or $4m^2 - 3m - 1 = 0 \quad \therefore m = 1\,, -1/4$

Tangent equations are:

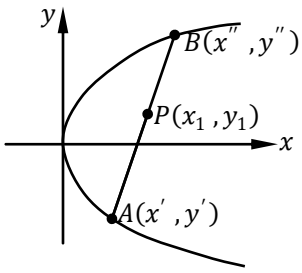
$y = x + a\,, y = -\frac{1}{4}x - 4a \Rightarrow -4y = x + 4a$

d) Left as an exercise. **Ans:** $y = 2x - 2a\,, 2y = -x + 8a$

CHORD WITH GIVEN MIDPOINT

To find the equation of a chord of the parabola $y^2 = 4ax$ having $M(x_1\,, y_1)$ as its midpoint

Consider the figure below.



Equation of chord is

$y - y_1 = m(x - x_1)$ ..... (1)

Finding slope of the chord

$P(x_1\,, y_1) = P\left[\frac{x' + x''}{2}\,, \frac{y' + y''}{2}\right]$

$\therefore 2x_1 = x' + x''$ ..... (2)

$\therefore 2y_1 = y' + y''$ ..... (3)

Also A and B lie on the parabola

$\Rightarrow y'^2 = 4ax'$ ..... (3)

$\Rightarrow y''^2 = 4ax''$ ..... (4)

Eqn. (3) − eqn. (4)

$y'^2 - y''^2 = 4a(x' - x'')$

$(y' + y'')(y' - y'') = 4a(x' - x'')$

$\Rightarrow \text{Grad APB} = \frac{y' - y''}{x' - x''} = \frac{4a}{y' + y''}$ ..... (5)

Eqn. (3) into eqn. (5)

$\text{Grad APB} = \frac{4a}{2y_1} = \frac{2a}{y_1}$

From eqn. (1), $y - y_1 = m(x - x_1)$ becomes

$yy_1 - y_1^2 = 2a(x - x_1)$

$\boxed{yy_1 - y_1^2 = 2a(x - x_1)}$, which is the required equation

Rewriting this equation

$yy_1 = y_1^2 - 4ax_1 - 2ax + 2ax_1$

$\boxed{yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1}$

To Find the Equation of a Chord joining points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lying of the Parabola $y^2 = 4ax$

This is generally given as

$$(y - y_1)(y - y_2) = y^2 - 4ax$$

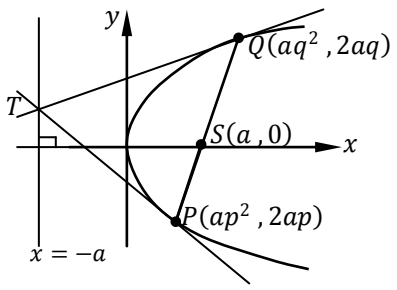
NOTE

A line joining two points on parabola and passing through the focus is called a *Focal Chord.*

Example

1. *Prove that the tangent at the extremities of the focal chord of a parabola $y^2 = 4ax$ intersects at right angles on the directrix. Take coordinates of extremities to be $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$.*

Solution



Slope of the tangent to the curve $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$

$\frac{dy}{dx}_{\text{at } P} = \frac{2a}{2ap} = \frac{1}{p}$, $\frac{dy}{dx}_{\text{at } Q} = \frac{2a}{2aq} = \frac{1}{q}$

Equation of tangent at P

$y - 2ap = \frac{1}{p}(x - ap^2)$ ..... (1)

Equation of tangent at Q

$y - 2aq = \frac{1}{q}(x - aq^2)$..... (2)

Solving for point of intersectic

Eqn. (1) − eqn. (2)

$2aq - 2ap = \frac{x - ap^2}{p} - \frac{x - aq^2}{q}$ ..... $\times pq$

$pq(2aq - 2ap) = qx - ap^2q - px + apq^2$

$2apq(q - p) = (q - p)x - apq(q - p)$ or $2apq = x - apq \quad \therefore x = apq$

For the focal chord, Grad $\overline{PS}$ = Grad $\overline{SQ}$

$\Rightarrow \frac{2ap - 0}{ap^2 - a} = \frac{2aq - 0}{aq^2 - a}$ or $\frac{2p}{p^2 - 1} = \frac{2q}{p^2 - 1}$

$\Rightarrow pq^2 - p = p^2q - q$ or $pq(q - p) = p - q \quad \therefore pq = -1$

eqn. (3) − eqn. (4)

$x = -a$, which is the $x-$ coordinate of point of intersection, and is the equation of directrix.

Now for intersection at right angles,

Grad $\overline{PT} \times$ Grad $\overline{QT} = -1$

Grad $\overline{PT} = \frac{1}{p}$, Grad $\overline{QT} = \frac{1}{q}$

$\Rightarrow \frac{1}{p} \times \frac{1}{q} = \frac{1}{pq} = \frac{1}{-1} = -1 =$ RHS

2. *Prove that the tangent to the parabola* $y^2 = 4ax$ *at points* $P(ap^2, 2ap)$ *and* $Q(aq^2, 2aq)$ *meet at* $T\big(apq, a(p+q)\big)$.
If M is the midpoint of PQ, prove that TM is bisected by the parabola.

Solution

First case, see solution above in example (1)

For TM to be bisected by the parabola, midpoint of TM must lie on the parabola.

$\Rightarrow M\left(\frac{ap^2+aq^2}{2}, \frac{2ap+2aq}{2}\right) = M\left(\frac{a(p^2+q^2)}{2}, a(p+q)\right)$

$T\big(apq, a(p+q)\big),\ M\left(\frac{a(p^2+q^2)}{2}, a(p+q)\right)$

Let N = midpoint of TM

$\Rightarrow N\left(\frac{apq+\frac{a(p^2+q^2)}{2}}{2}, \frac{a(p+q)+a(p+q)}{2}\right) = N\left(\frac{a(p^2+2pq+q^2)}{4},\ a(p+q)\right)$

$\Rightarrow N\left(\frac{a(p+q)^2}{4}, a(p+q)\right)$

Let $t = p+q \Rightarrow N\left(\frac{at^2}{4}, at\right)$, $y^2 = 4ax$

$\Rightarrow\ (at)^2 = 4a\left(\frac{at^2}{4}\right) = a^2t^2$

Hence point N lies on parabola, which is the midpoint of TM. i.e. parabola bisects TM.

Uneb 2011

a) *Find the equation of a tangent to the parabola* $y^2 = \frac{x}{16}$ *at the point* $\left(t^2, \frac{t}{4}\right)$

b) *If the tangents to the parabola in (a) above at points* $P\left(p^2, \frac{p}{4}\right)$ *and* $Q\left(q^2, \frac{q}{4}\right)$ *meet on the line* $y = 2$

i) *Show that* $p + q = 16$

ii) *Deduce the midpoint of PQ lies on the line* $y = 2$

Solution

a) Required tangent: $y - \frac{t}{4} = m(x - t^2)$..... (1)

Finding the slope of the tangent at $\left(t^2, \frac{t}{4}\right)$

Differentiating $y^2 = \frac{x}{16}$ w.r.t x

$2y\frac{dy}{dx} = \frac{1}{16} \quad \therefore \frac{dy}{dx} = \frac{1}{32y}$

$M = \frac{dy}{dx}_{\text{at}\ \left(t^2, \frac{t}{4}\right)} = \frac{1}{32\times\frac{t}{4}} = \frac{1}{8t}$

$\Rightarrow y - \frac{t}{4} = \frac{1}{8t}(x - t^2)$ or $8ty - 2t^2 = x - t^2$

$\therefore x - 8ty + t^2 = 0$

b) (i) Tangent at P is $x - 8py + p^2 = 0$..... (2)

Tangent at Q is $x - 8qy + q^2 = 0$..... (3)
Solving for point of intersection
Eqn. (2) − eqn. (3)
$(8q - 8p)y + (p^2 - q^2) = 0$
$y = \frac{q^2-p^2}{8(q-p)} = \frac{(q+p)(q-p)}{8(q-p)} = \frac{p+q}{8}$
But y − coordinate of point of intersection is $y = 2$
$\Rightarrow 2 = \frac{p+q}{8} \quad \therefore p + q = 16$

ii) Midpoint of PQ is $\left(\frac{q^2+p^2}{2}, \frac{p+q}{8}\right)$
For this point to lie on $y = 2$, y − coordinate must be equal to 2.
$\Rightarrow \frac{p+q}{8} = \frac{16}{8} = 2$, as required

3. a) *Show that the equation of the normal to the parabola* $y^2 = 4ax$ *at the point* $P(at^2, 2at)$ *is* $y + tx = 2at + at^3$
b) *the normal at P meets* $x-$*axis*
at G and the midpoint of PG is N. Find the coordinates of G and N

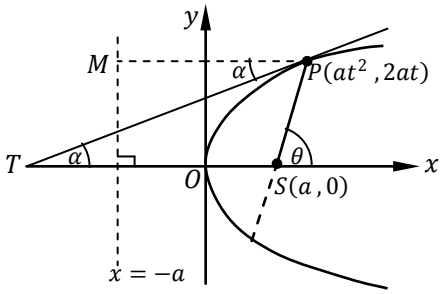
Solution

a) Required normal:
$y - 2at = m_1(x - at^2)$..... (1)
Finding slope of the normal m_1
$m_1 \times m_2 = -1$, m_2 = Slope of the tangent at P
For $y^2 = 4ax$, differentiating w.r.t x
$2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$
$m_2 = \left.\frac{dy}{dx}\right|_{\text{at } P(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$
$\therefore m_1 = -t \quad [\because m_1 \times m_2 = -1]$
$\Rightarrow y - 2at = -t(x - at^2) \quad \therefore y + tx = 2at + at^3$, as required.

b) For x − intercept, $y = 0$
$\Rightarrow 0 + tx = 2at + at^3 \quad \therefore x = 2a + at^2$
$\therefore G(2at + at^3, 0)$
Midpoint of PN: $P(at^2, 2at)$, $G(2a + at^2, 0)$
$\Rightarrow N\left(\frac{at^2+at^2+2a}{2}, \frac{2at+0}{2}\right) = N[a(t^2 + 1), at]$

4. *Show that the tangent at any point P on the parabola* $y^2 = 4ax$ *bisects the angle between the focal chord through P and perpendicular from P onto the directrix.*

Solution



From the figure coordinates of $M(-a\,,2at)$
Let m_1 =Slope of $\overline{PS}$, m_2 =Slope of $\overline{PT}$

$m_1 = \frac{2at-0}{at^2-a} = \frac{2t}{t^2-1} = \tan\theta$

$m_2 = \frac{dy}{dx}\Big|_{\text{at } P(at^2,2at)} = \frac{2a}{2at} = \frac{1}{t} = \tan\alpha$

$< T\hat{P}S = \theta - \alpha$

Introducing tangents

$$\tan < T\hat{P}S = \tan(\theta-\alpha) = \frac{\tan\theta - \tan\alpha}{1+\tan\theta\tan\alpha}$$

$$= \frac{\frac{2t}{t^2-1} - \frac{1}{t}}{1+\frac{2t}{t^2-1}\cdot\frac{1}{t}} = \frac{2t^2-t^2+1}{(t^2-1+2)\,t}$$

$$= \frac{t^2+1}{t(t^2+1)} = \frac{1}{t}$$

$\Rightarrow \tan < T\hat{P}S = \frac{1}{t} = \tan\alpha \quad \therefore < T\hat{P}S = \alpha$

Hence tangent bisects $< M\hat{P}S$

LOCUS PROBLEMS IN PARABOLA

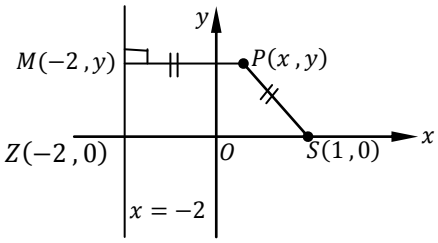
1. *Find the locus of a point $P(x\,,y)$ which moves in a such a way that:*

a) *It is equidistant from the point $S(1\,,0)$ and the line $x = -2$*
b) *It is equidistant from point $S(-1\,,1)$ and the line $y = -3$*
c) *It is equidistant from $S(2\,,-1)$ and the line $3x - 4y = 0$*

Hence describe and sketch the locus in each case above.

Solution

a) Using a very clear diagram,



Given: $\overline{PS} = \overline{PM}$

$\Rightarrow \sqrt{(x--2)^2+(y-y)^2} = \sqrt{(x-1)^2+(y-0)^2}$

Squaring both sides

$\Rightarrow x^2+4x+4 = x^2-2x+1+y^2$

$y^2 = 6x+3 = 3(2x+1)$

$\therefore y^2 = 3(2x+1)$, is the locus of P.

The locus is a parabola with focus $S(1\,,0)$ and equation of directrix $x = -2$

Sketching $y^2 = 3(2x+1)$

Obtaining the vertex and axis of conic

Axis of the conic is the line through the focus and perpendicular to the directrix, hence axis is $x - axis$. The parabola is symmetrical about the axis

Vertex is the midpoint of $\overline{SZ}$

$= A\left(\frac{-2+1}{2}, \frac{0+0}{2}\right) = A\left(-\frac{1}{2}, 0\right)$

Obtaining intercepts:

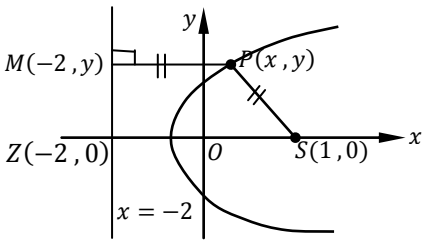
For $x = 0 \Rightarrow y^2 = 3(2(0)+1) \quad \therefore y = \pm\sqrt{3}$

$\therefore B(0, +\sqrt{3}),\ B'(0, -\sqrt{3})$, are $y-$ intercepts.

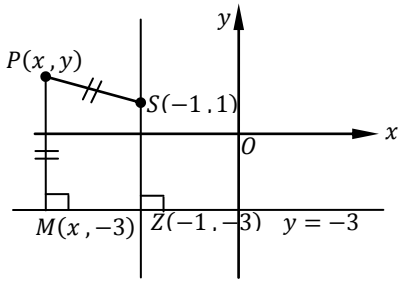
For $y = 0 \Rightarrow 0 = 3(2x+1) \quad \therefore x = -\frac{1}{2}$

$A\left(-\frac{1}{2}, 0\right)$, is an $x-$ intercept.

Note: the $x-$ intercept is the vertex



b) Representing the information on a clearer diagram.



Given: $\overline{PS} = \overline{PM}$

$\Rightarrow \sqrt{(x - -1)^2 + (y-1)^2} = \sqrt{(x - x)^2 + (y+3)^2}$

Squaring both sides

$\Rightarrow x^2 + 2x + 1 + y^2 - 2y + 1 = y^2 + 6y + 9$

$x^2 + 2x - 6y - 7 = 0$

$\therefore x^2 + 2x - 6y - 7 = 0$, is the locus of P.

The locus is a parabola with focus $S(-1, 1)$ and directrix $y = -3$

<u>Sketching $x^2 + 2x - 6y - 7 = 0$</u>

Obtaining the vertex and axis of conic

Axis of the conic is the line through the focus and perpendicular to the directrix, hence axis is $x = -1$. The parabola is symmetrical about the axis

Vertex is the midpoint of $\overline{SZ}$

$= A\left(\frac{-1-1}{2}, \frac{1+-3}{2}\right) = A(-1, -1)$

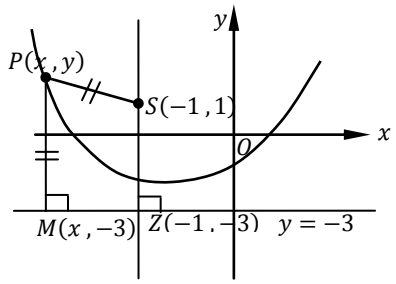
Obtaining intercepts:

For $x = 0$, $\frac{7}{6}$

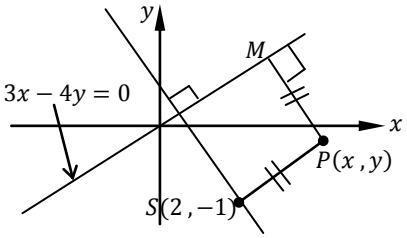
$\therefore A(0, +7/6)$, is the $y-$ intercepts.

For $y = 0 \Rightarrow x^2 + 2x - 7 = 0$

$\therefore B(1.828, 0),\ B'(-1.828, 0)$ are $x-$intercepts



c) Using a clearer diagram,



Given: $\overline{PS} = \overline{PM}$

$\Rightarrow \overline{PS} = \sqrt{(x-2)^2 + (y+1)^2}$

$\overline{PM}$ = Perpendicular distance of $P(x, y)$ from the line $3x - 4y = 0$

$\Rightarrow \overline{PM} = \left|\frac{3x-4y}{\sqrt{3^2+4^2}}\right| = \frac{1}{5}|3x - 4y|$

Now $\sqrt{(x-2)^2 + (y+1)^2} = \frac{1}{5}|3x - 4y|$, squaring both sides

$\Rightarrow x^2 - 4x + 4 + y^2 + 2y + 1 = \frac{9x^2 - 24xy + 16y^2}{25}$

$\Rightarrow 16x^2 + 9y^2 + 24xy - 100x + 125 + 50y = 0$

$\Rightarrow (4x + 3y)^2 - 100x + 50y + 125 = 0$, which is the required locus of P.

The locus is a parabola with focus $S(2, -1)$ and directrix $3x - 4y = 0$

Sketching $(4x + 3y)^2 - 100x + 50y + 125 = 0$

Obtaining the vertex and axis of conic

Axis of the conic is the line through the focus and perpendicular to the directrix.

Since the directrix is perpendicular to the axis, slope of the axis is:

$$= -\frac{1}{3/4} = -\frac{4}{3}$$

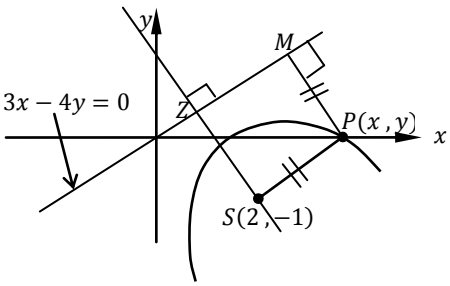
Equation of the axis must pass through the focus

$\Rightarrow y + 1 = -\frac{4}{3}(x - 2) \quad \therefore 4x + 3y - 5 = 0$

Obtaining the point of intersection of the axis and the directrix in order to find the vertex

Solving $4x + 3y - 5 = 0$ and $3x - 4y = 0$, point of intersection is $Z\left(\frac{4}{5}, \frac{3}{5}\right)$

Vertex is the midpoint of $\overline{SZ}$



$= A\left(\frac{2+4/5}{2}, \frac{-1+3/5}{2}\right) = A\left(\frac{7}{5}, -\frac{1}{5}\right)$

Obtaining intercepts:

For $x = 0 \Rightarrow 9y^2 + 50y + 125 = 0$, hence no real values of y.

No $y-$ intercept

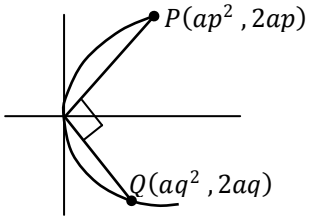
For $y = 0 \Rightarrow 16x^2 - 100x + 125 = 0$

$\therefore x = 4.52, 1.73$

$\therefore B(4.52, 0)$, $B'(1.73, 0)$, are intercepts

2. *P and Q are two points on the parabola* $y^2 = 4ax$ *whose coordinates are are* $(ap^2, 2ap)$ *and* $(aq^2, 2aq)$. *O is the origin and* $\overline{OP} \perp \overline{OQ}$, *show that* $pq + 4 = 0$.

Solution



Let $m_1 = \text{Grad OP}$, $m_2 = \text{Grad OQ}$

$m_1 = \frac{2ap}{ap^2} = \frac{2}{p}$, $m_2 = \frac{2aq}{aq^2} = \frac{2}{q}$

But $m_1 \times m_2 = -1$

$\Rightarrow \frac{2}{p} \times \frac{2}{q} = -1$

$\Rightarrow 4 = -pq$

$\therefore pq + 4 = 0$

3. a) *Show that the equation of a chord joining points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on a parabola $y^2 = 4ax$ is given as $y(p+q) = 2(x + apq)$*

b) *A variable focal chord PQ of a parabola $y^2 = 4ax$ is moving with its midpoint R. prove that the locus of point R defines a parabola of length of latus rectum $2a$. Find the focus and the equation of the directrix of locus of R.*

Solution

a) General equation of the chord joining any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the parabola is $(y - y_1)(y - y_2) = y^2 - 4ax$

Equation of the required chord is:

$(y - 2ap)(y - 2aq) = y^2 - 4ax$

$\Rightarrow y^2 - 2aqy - 2apy + 4a^2pq = y^2 - 4ax$

$\Rightarrow y(-2ap - 2aq) = -4ax - 4a^2pq$

$\Rightarrow y(p + q) = 2(x + apq)$

b) Since PQ is a variable focal chord, then PQ passes through the focus $S(a, 0)$

$y(p + q) = 2(x + apq) \Rightarrow 0 = 2(a + apq)$

$\Rightarrow 1 + pq = 0 \quad \therefore pq = -1$

$\therefore$ Equation of the focal chord is $y(p + q) = 2(x - a)$..... (1)

But R is the midpoint of PQ

$\Rightarrow R\left[\frac{ap^2 + aq^2}{2}, \frac{2ap + 2aq}{2}\right] = R\left[\frac{a(p^2 + q^2)}{2}, a(p + q)\right]$

Let the coordinates of R be $R(X, Y)$

$R(X, Y) = R\left[\frac{a(p^2 + q^2)}{2}, a(p + q)\right]$

$\Rightarrow X = \frac{a(p^2 + q^2)}{2}$..... (2)

$\Rightarrow Y = a(p + q)$..... (3)

Solving eqn. (2) and eqn. (3) simultaneously, eliminating the variables p and q in the coordinates,

From eqn. (2)

$\frac{2X}{a} = p^2 + q^2 = (p + q)^2 - 2pq$

$\Rightarrow \frac{2X}{a} = (p + q)^2 + 2 \quad [\because pq = -1]$ ..... (4)

Eqn. (3) into eqn. (4) to eliminate p and q

$\Rightarrow \frac{2X}{a} = \left(\frac{Y}{a}\right)^2 + 2$ or $2aX = Y^2 + 2a^2$

$\Rightarrow Y^2 = 2a(X - a)$

$\therefore$ the locus of R is a parabola $y^2 = 2a(x - a)$

NOTE

When handling numbers of locus, the following points are worth noting.

(a) When mention of any point, line or any angle is made, you must first look for what has been mentioned in case it is not given in the question.

(b) The locus of a point needed is obtained by first letting the point to be (X, Y)

(c) Formulate two equations from step (b) above.

(d) Solve simultaneously the two formulated equations eliminating the variables/parameter involved in the coordinates given.

(e) Write the locus in terms of the coordinate variables on the curve. Usually write it in terms of variables x and y.

4. a) *Find the equation of a line parallel to the normal to the parabola* $y^2 = 4ax$ *at the point* $(p^2\ \ 2p)$ *passing through* $(1, 0)$.

b) *The tangent to* $y^2 = 4ax$ *at the point* $P(at^2, 2at)$ *meets the* $y-$*axis at G and the normal at P meets the* $x-$ *axis at H*

i) *Show that the midpoint M of HG has coordinates* $\left(a + \frac{at^2}{2}, \frac{at}{2}\right)$

ii) *Find the locus of M as P moves on the parabola.*

Solution

For parallel lines, gradients are the same. *i.e.* $m_1 = m_2$

Let the required parallel line with slope m_1 be:

$y - 0 = m_1(x - 1)$ or $y = m_1(x - 1)$..... (1)

Obtaining the slope, m_2 *of the normal at* $(p^2\ \ 2p)$

Given: $y^2 = 4ax$, differentiating w.r.t x to obtain the slope of the tangent

$\Rightarrow 2y\frac{dy}{dx} = 4a$ or $\frac{dy}{dx} = \frac{2a}{y}$

$\therefore m' = \frac{dy}{dx}\Big|_{\text{at } (p^2\ \ 2p)} = \frac{2a}{2p} = \frac{a}{p}$ = Slope of the tangent at $(p^2\ \ 2p)$

$\Rightarrow m_2 = -\frac{1}{m'} = -\frac{p}{a}$

$\therefore$ Slope of the required lie is $m_1 = m_2 = -\frac{p}{a}$

$\therefore$ Required equation of the line is: $y = -\frac{p}{a}(x - 1)$or $\quad px + ay = p$

b) *Obtaining the tangent equation at* $P(at^2, 2at)$

$y - 2at = m(x - at^2)$ ..... (1)

Obtaining slope m

$m = \frac{dy}{dx}\Big|_{\text{at } P(at^2, 2at)}$, differentiating $y^2 = 4ax$ w.r.t x

$\Rightarrow 2y\frac{dy}{dx} = 4a$ or $\frac{dy}{dx} = \frac{2a}{y}$

$\therefore m = \frac{dy}{dx}\Big|_{\text{at } P(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$

From eqn. (1), the tangent equation is:

$y - 2at = \frac{1}{t}(x - at^2) \Rightarrow ty - 2at^2 = x - at^2$

$\therefore ty = x + at^2$, now we have the tangent.

This line meets $y-$ axis at G. This is the $y-$ intercept which occurs when $x = 0$

$\Rightarrow ty = at^2$ or $y = at$

$\therefore G(0\,, at)$, now we have G

Obtaining the equation of the Normal at $P(at^2\,, 2at)$

$y - 2at = m'(x - at^2)$ ..... (2)

Obtaining slope m

$m' = -\frac{1}{\text{Slope of the tangent at P}} = -\frac{1}{1/t} - t$

From eqn. (2)

$y - 2at = -t(x - at^2)$ or $y = -tx + 2at + at^3$

$\therefore y = -tx + 2at + at^3$, now we have the normal

This line meets the $x-$axis at H. this is the $x-$intercept which occurs when $y = 0$

$\Rightarrow 0 = -tx + 2at + at^3 \quad \therefore x = 2a + at^2$

$\therefore H(2a + at^2\,, 0)$, now we have also H

Midpoint of HG= $\left(\frac{0+2a+at^2}{2}\,, \frac{at+0}{2}\right) = \left(a + \frac{at^2}{2}\,, \frac{at}{2}\right)$

Obtaining the locus of M

Let $M(X\,, Y) = \left(a + \frac{at^2}{2}\,, \frac{at}{2}\right)$

$\Rightarrow X = a + \frac{at^2}{2}$ or $2X = 2a + at^2$..... (1)

$\Rightarrow Y = \frac{at}{2}$ or $t = \frac{2Y}{a}$..... (2)

Eliminating t from eqn. (1) and eqn. (2), eqn. (2) into eqn. (1)

$2X = 2a + a\left(\frac{2Y}{a}\right)^2 \Rightarrow 2aX = 2a^2 + 4Y^2$

$\therefore Y^2 = \frac{a}{2}(X - a)$

$\therefore$ The locus of M is $\therefore y^2 = \frac{a}{2}(x - a)$

5. *The normal to the parabola* $y^2 = 4ax$ *at* $P(at^2\,, 2at)$ *meets the* $x-$ *axis at G and the midpoint of PG is N.*

i) *Find the locus of N as P moves on the parabola*

ii) *If the focus of the parabola is S, prove that* $SN \perp PG$

iii) *If* ΔSRG *is equilateral, find the coordinates of R*

iv) *If* $P(at^2\,, 2at)$, $Q\left(\frac{a}{t^2}\,, -\frac{2a}{t}\right)$ *and* $S(a\,, 0)$ *respectively are points in* $x - y$ *plane, show that* $\frac{1}{SP} + \frac{1}{SQ}$ *is independent of the parameter t*

Solution

i) Obtaining the equation of the normal to the curve $y^2 = 4ax$ at $P(at^2\,, 2at)$

See solution in 4(b) above.

$\therefore y = -tx + 2at + at^3$, is now the equation of the normal.

This normal meets the x −axis at G (the x −intercept) when $y = 0$

$\Rightarrow 0 = -tx + 2at + at^3 \quad \therefore x = 2a + at^2$

$\therefore G(2a + at^2\,, 0)$, is now known

Obtaining the midpoint of PG . $i.e.$ N

$N\left(\frac{2a+at^2+at^2}{2}\,, \frac{0+2at}{2}\right) = N(at^2 + a\,, at)$, is also now known

To obtain the locus of N, let $N(X, Y)$

$\Rightarrow N(X, Y) = N(at^2 + a\,, at)$

$X = at^2 + a$ ..... (1)

$Y = at \Rightarrow t = \frac{Y}{a}$ ..... (2)

Eqn. (2) into eqn. (1)

$X = a\left(\frac{Y}{a}\right)^2 + a \Rightarrow aX = Y^2 + a^2$ or $Y^2 = a(X - a)$

$\therefore$ Locus of N is: $y^2 = a(x - a)$, which is a parabola.

ii) To prove that $SN \perp PG\,, S(a\,, 0)\,, N(at^2 + a\,, at)$

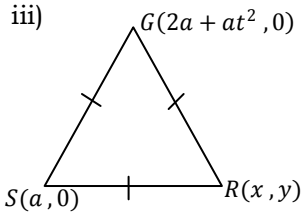
For $SN \perp PG$, Grad $\overline{SN} \times$ Grad $\overline{PG} = -1$

Grad $\overline{SN} = \frac{at-0}{at^2+a-a} = \frac{1}{t}$

Grad $\overline{PG} = \frac{2at-0}{at^2-(2a+at^2)} = \frac{2at}{-2a} = -t$

$\therefore \; SN \perp PG$ if $\frac{1}{t} \times -t = -1 \qquad [m_1 \times m_2 = -1]$

iii)



By coordinate geometry, $\overline{SR}^2 = \overline{SG}^2 = \overline{GR}^2$

For $\overline{SR}^2 = \overline{GR}^2$

$(x - a)^2 + y^2 = [x - (2a + at^2)]^2 + y^2$

$x^2 - 2ax + a^2 + y^2 = x^2 - 2x(2a + at^2) + (2a + at^2)^2 + y^2$

$-2ax + a^2 = -2x(2a + at^2) + (2a + at^2)^2$

$\Rightarrow 2x(2a + at^2 - a) = (2a + at^2)^2 - a^2 = (3a + at^2)(a + at^2)$

$\Rightarrow x = \frac{(3a+at^2)(a+at^2)}{2(a+at^2)} = \frac{a(3+t^2)}{2}$

For $\overline{SR}^2 = \overline{SG}^2$

$\Rightarrow (x - a)^2 + y^2 = [(2a + at^2) - a]^2 + 0$

$\Rightarrow (x - a)^2 + y^2 = (a + at^2)^2 = a^2(1 + t^2)^2$

But $x = \frac{a(3+t^2)}{2}$

$\Rightarrow \left(\frac{a(3+t^2)}{2} - a\right)^2 + y^2 = (a + at^2)^2 = a^2(1 + t^2)^2$

$\Rightarrow y^2 = a^2(1 + t^2)^2 - \frac{a^2(1+t^2)^2}{4}$

$\Rightarrow y^2 = \frac{3}{4}a^2(1 + t^2)^2 \quad \therefore y = \frac{a(1+t^2)\sqrt{3}}{2}$

$\therefore R\left(\frac{a(3+t^2)}{2}\,, \frac{a(1+t^2)\sqrt{3}}{2}\right)$

NOTE

The x −coordinate of R can still be obtained from the midpoint of SG since these two lie on the same horizontal line.

$x = \frac{a+2a+at^2}{2} = \frac{a(3+t^2)}{2}$

To find the y − coordinate, use the formula for angle between two lines, say $\overline{SG}$ and $\overline{SR}$

$\tan\theta = \frac{m_2-m_1}{1+m_2.m_1}$

Let $m_2 = \text{Grad}\,\overline{SR} = \frac{y-0}{x-a} = \frac{y}{x-a}$

$m_1 = \text{Grad}\,\overline{SG} = \frac{0-0}{2a+at^2-a} = 0$

$\therefore \tan\theta = \frac{m_2-m_1}{1+m_2.m_1} = m_2$, since $m_1 = 0$

$\therefore \tan 60^0 = \frac{y}{x-a} \Rightarrow y = \sqrt{3}.(x-a)$

$y = \sqrt{3}.\left[\frac{a(3+t^2)}{2} - a\right] = \frac{a(1+t^2)\sqrt{3}}{2}$

$\therefore R\left(\frac{a(3+t^2)}{2}, \frac{a(1+t^2)\sqrt{3}}{2}\right)$

iv) $P(at^2, 2at)$, $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$, $S(a, 0)$

$\frac{1}{\overline{SP}} + \frac{1}{\overline{SQ}} = ?$

$\overline{SP} = \sqrt{(a-at^2)^2 + (2at-0)^2} = \sqrt{a^2(1-2t^2+t^4+4t^2)}$

$= a\sqrt{(t^4+2t^2+1)}$

$= a\sqrt{(t^2+1)^2} = a(t^2+1)$

$\overline{SQ} = \sqrt{\left(a-\frac{a}{t^2}\right)^2 + \left(0--\frac{2a}{t}\right)^2} = \sqrt{a^2\left(1-\frac{2}{t^2}+\frac{1}{t^4}+\frac{4}{t^2}\right)}$

$= \frac{a}{t^2}\sqrt{(t^4+2t^2+1)}$

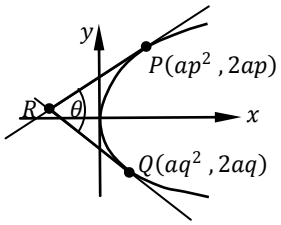
$= \frac{a}{t^2}\sqrt{(t^2+1)^2} = \frac{a}{t^2}(t^2+1)$

$\therefore \frac{1}{\overline{SP}} + \frac{1}{\overline{SQ}} = \frac{1}{a(t^2+1)} + \frac{1}{\frac{a}{t^2}(t^2+1)}$

$= \frac{(t^2+1)}{a(t^2+1)} = \frac{1}{a}$, which is independent of t

6. *Find the coordinates of the point of intersection R of the tangents to the parabola $y^2 = 4ax$ at the points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$.*
If the tangents at P, Q are inclined to each other at an angle of 45^0, show that the locus of R is the curve $y^2 = x^2 + 6ax + a^2$.

Solution



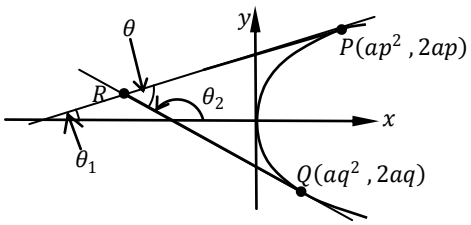
Tangents at P and Q can be obtained as $py = x + ap^2$ and $qy = x + aq^2$ respectively.
Solving for point of intersection we obtain:
$R(apq, a(p+q))$ (Left for student as an exercise)
Given: $\theta = 45^0$

Let $m_1 = \text{Grad}\,\overline{PR}$ and $m_2 = \text{Grad}\,\overline{QR}$ where $m_1 > m_2$

$\Rightarrow m_1 = \frac{1}{p}\ ,\ m_2 = \frac{1}{q}$

Let $\overline{PR}$ be inclined to the $x-$axis at an angle θ_1 and ,

$\overline{QR}$ be inclined to the $x-$axis at an angle θ_2



From the figure

$\theta_1 + 180^0 - \theta_2 = \theta \ \Rightarrow \theta - 180^0 = (\theta_1 - \theta_2)$

Introduce tangent on both sides

$\frac{\tan\theta - \tan 180^0}{1+\tan\theta.\tan 180^0} = \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1.\tan\theta_2}$

$\tan\theta = \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1.\tan\theta_2} = \frac{m_1 - m_2}{1+m_1.m_2}$

$\tan\theta = \frac{\frac{1}{p}-\frac{1}{q}}{1+\frac{1}{p}.\frac{1}{q}} = -\frac{(p-q)}{pq+1}$

$\Rightarrow \tan 45^0 = -\frac{(p-q)}{pq+1}$ or $pq + 1 = -(p - q)$……..

(1)

Let $R(X,Y) = R\big(apq\ , a(p+q)\big)$

$\Rightarrow X = apq$ or $pq = \frac{X}{a}$…… (2)

$\Rightarrow Y = a(p+q)$…… (3)

Expressing eqn. (1) in terms of $p+q$ and pq which are in equations (2) and (3)

$pq + 1 = -(p - q)$, squaring both sides

$\Rightarrow (pq+1)^2 = p^2 - 2pq + q^2$

$\Rightarrow (pq+1)^2 = (p+q)^2 - 2pq - 2pq$

$\therefore (pq+1)^2 = (p+q)^2 - 4pq$…….. (4)

Eqn. (2), eqn. (3) into eqn. (4)

$(p+q)^2 = (pq+1)^2 + 4pq$

$\Rightarrow \left(\frac{Y}{a}\right)^2 = \left(\frac{X}{a}+1\right)^2 + \frac{4X}{a}$…….. $(\times a^2)$

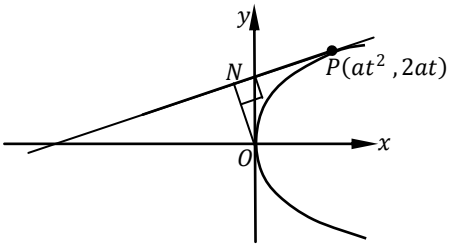
$\Rightarrow Y^2 = (X+a)^2 + 4aX$

$\Rightarrow Y^2 = X^2 + 2aX + a^2 + 4aX$ or $Y^2 = X^2 + 6aX + a^2$

$\therefore$ Locus of R is $y^2 = x^2 + 6ax + a^2$

7. *$P(at^2\ , 2at)$is a point on the parabola $y^2 = 4ax$. N is the foot of the perpendicular drawn from the origin to the tangent at P. show that as P varies the locus of N is the curve $x(x^2+y^2) + ay^2 = 0$*

Solution



Equation of the tangent at P can be obtained as:
$ty = x + at^2$ ..... (1)
Obtaining the equation of the normal through the origin and perpendicular to the tangent
$y - 0 = m(x - 0)$ or $y = m_1 x$
But for perpendicular lines, $m_1 \times m_2 = -1$
$\therefore m_1 = -\frac{1}{1/t} = -t$
Equation of normal: $y = -tx$..... (2)

Solving for the coordinates of N
Eqn. (1) into eqn. (2)
$-t^2x = x + at^2 \quad \Rightarrow x = -\frac{at^2}{t^2+1}$ ..... (3)
Eqn. (3) into eqn. (2)
$\Rightarrow y = \frac{at^3}{t^2+1}$
$\therefore$ *Coordinates of* $N\left(-\frac{at^2}{t^2+1}, \frac{at^3}{t^2+1}\right)$
Let $N(X, Y) = N\left(-\frac{at^2}{t^2+1}, \frac{at^3}{t^2+1}\right)$
$\Rightarrow X = -\frac{at^2}{t^2+1}$..... (1)
$\Rightarrow Y = \frac{at^3}{t^2+1}$..... (2)
Dividing the equations to simplify in eliminating parameter t
$\Rightarrow \frac{X}{Y} = -\frac{at^2}{t^2+1} \times \frac{t^2+1}{at^3} = -\frac{1}{t}$
$\Rightarrow t = -\frac{Y}{X}$ ..... (3)
Eqn. (3) into eqn. (1)
$\Rightarrow X = -\frac{a\left(-\frac{Y}{X}\right)^2}{\left(-\frac{Y}{X}\right)^2+1} = -\frac{aY^2}{Y^2+X^2}$ or $X(X^2 + Y^2) = -aY^2$
$\Rightarrow X(X^2 + Y^2) + aY^2 = 0$
$\therefore$ Required locus is $x(x^2 + y^2) + ay^2 = 0$

Task

1. *A variable chord through the focus of the parabola* $y^2 = 4ax$ *cuts the curve at P and Q. the straight line joining P to the point* $(0, 0)$ *cuts the line joining Q to the point* $(-a, 0)$ *at R. show that the equation of the locus of point R is* $y^2 + 8x^2 + 4ax = 0$

2. (i) *Show that the locus of the midpoint of the line joining the parabola* $y^2 = 8x$ *and the point* $(8, 0)$ *is a parabola.*
Hint: $y^2 = 8x$ *can be compared with* $y^2 = 4ax \ \therefore a = 2$. *The general point on the parabola is* $(at^2, 2at) = (2t^2, 4t)$

ii) *Determine the points at which lines from the new focus are perpendicular to the parabola* $y^2 = 8x$

iii) *Find the $y-$coordinate of the point at which the tangent at one of the points meet the $y-$axis.*

4. *A tangent from the point $T(t^2, 2t)$ touches $y^2 = 4x$. Find:*
   i) *The equation of the tangent*
   ii) *The equation of the line L parallel to the normal at $T(t^2, 2t)$ and passing through $(1, 0)$*
   iii) *The point of intersection X of line L and tangent.*
   *(**Note:** point T lies on parabola)*

5. *Find the locus of the point of intersection of tangents to $y^2 = 4ax$ which are perpendicular to each other.* ***Ans:*** $x + a = 0$

6. *The normal at P to $y^2 = 4ax$ cuts the $x-$axis at G. A line through the midpoint of PG perpendicular to $0x$ cuts the curve at Q and $0x$ at N. prove that $QN = PG$*

7. *The normal to $y^2 = 4ax$ at P meets the $x-$axis in G. GP is produced so that $PQ = PG$. Find the locus of Q.* ***Ans:*** $y^2 = 16a(x + 2a)$

8. *P is any point on $y^2 = 4ax$. O is the origin and M is a point on $0x$ so that $MPO = 90^0$. If PN is the ordinate of P prove that $MN = 4a$*

9. *The focus of the parabola $y^2 = 4ax$ is S. Any tangent to the curve meets the directrix in M and the latus rectum in L. prove that $SM = SL$*

10. *The normal at the point $P(4a, 4a)$ to $y^2 = 4ax$ cuts the curve again at Q. if S is the focus prove that angle $PSQ = 90^0$*

11. *P is any point on the parabola $y^2 = 4ax$. The normal at P cuts the $x-$axis in Q. find the locus of the midpoint of PQ.* ***Ans:*** $y^2 = a(x - a)$

12. *The normal at P to the parabola $y^2 = 4ax$ cuts the curve again at Q. O is the origin and S is the focus. A line through P parallel to OQ meets the $x-$axis in R. prove that $OR = 2PS$*

13. *The vertex of the parabola $y^2 = 4ax$ is O. OP and OQ are two chords of the parabola and angle $POQ = 90^0$. Show that PQ cuts the $x-$axis in a fixed point.*

14. *Chords of the parabola $y^2 = 4ax$ are drawn so that the chords all subtend 90^0 at $O(0, 0)$. Prove that the midpoint of the chords all lie on the parabola $y^2 = 2a(x - 4a)$*

15. *The normal at P to the parabola $y^2 = 4ax$ cuts the $x-$axis in G. the normals at Q and R pass through the midpoint of PG. Prove that QR cuts the $x-$axis at $(-a, 0)$*

16. *PQ is a focal chord of $y^2 = 4ax$. O is the origin and PO cuts the directrix at M. prove that MQ is parallel to $y = 0$*

17. *P is the point $(at^2, 2at)$ on $y^2 = 4ax$. P is joined to point $S(a, 0)$ and the line PS cuts the curve again at Q. the tangents at P and Q cut at R. Prove that $RP.RQ = a^2\left(\frac{t^2+1}{t}\right)^3$*

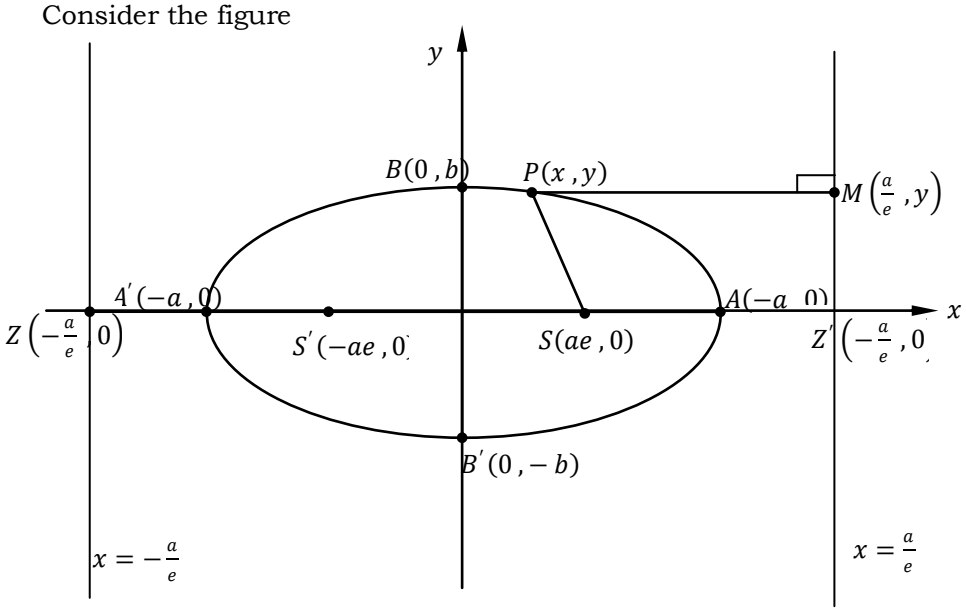
ELLIPSE

Definition:

An ellipse is the locus of a point which moves in such a way that its distance from a fixed point (Focus) bears a constant ratio $e, (0 < e < 1)$ to its distance from a fixed line (Directrix) not passing through the focus.

Equation of an Ellipse in Standard Form

Consider the figure



Point A divides ZS internally and externally in the ratio $1:e$

From the definition

$\overline{PS} = e\overline{PM}$ or $\overline{AS} = e\overline{AZ}$

$\overline{PS} = \sqrt{(x-ae)^2 + (y-0)^2}$, $\overline{PM} = \sqrt{\left(x - \frac{a}{e}\right)^2 + (y-y)^2}$

$\Rightarrow \overline{PS}^2 = e\overline{PM}^2$

$\Rightarrow (x-ae)^2 + (y-0)^2 = e^2\left(x - \frac{a}{e}\right)^2$

$\Rightarrow x^2 - 2aex + a^2e^2 + y^2 = (xe - a)^2$

$\Rightarrow x^2 - 2aex + a^2e^2 + y^2 = x^2e^2 - 2aex + a^2$

$\Rightarrow x^2(1-e^2) + y^2 = a^2(1-e^2)$ ..... $\div a^2(1-e^2)$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$, let $b^2 = a^2(1-e^2)$

$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ b^2 = a^2(1-e^2)}$, which is the standard equation of an ellipse

Definitions

1) AA' is called the *Major Axis* of the ellipse and its length is $2a$

2) BB' is called the *Minor Axis* of the ellipse and its length is $2b$

3) The points A and A' are *Vertices* of the ellipse

4) *Eccentricity* of the ellipse is given by the equation $b^2 = a^2(1 - e^2)$

5) *Intercepts* of the ellipse are: $A(a\,,0)\,, A'(-a\,,0)\,, B(0\,,b)\,,\ B'(0\,,-b)$

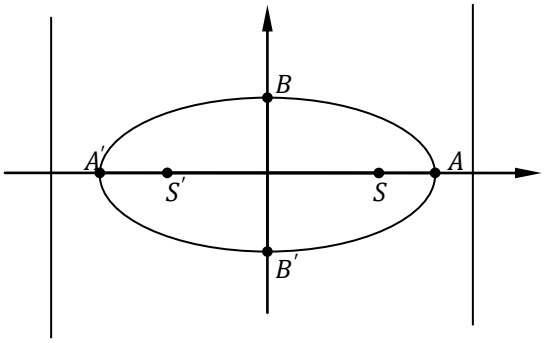
Note:

The points A and A' are $x-$ intercepts and the points B and B' are $y-$ intercepts.

The major axis is also called Transverse axis.

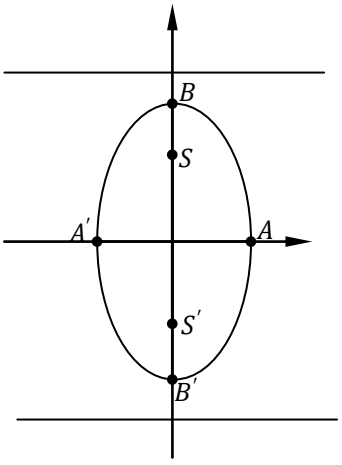
Summary of Standard Equation of an Ellipse

Case I: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$ **where** $b^2 = a^2(1 - e^2)$



Foci	$(\pm ae\,,0)$
Directrices	$x = \pm\frac{a}{e}$
Vertices	$(\pm a\,,0)\,,(0\,,\pm b)$
Major axis	$x-$axis or $y = 0$
Minor axis	$y-$axis or $x = 0$

Case II: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a < b$ **where** $a^2 = b^2(1 - e^2)$



Foci	$(0\,,\pm be\,)$
Directrices	$y = \pm\frac{b}{e}$
Vertices	$(\pm a\,,0)\,,(0\,,\pm b)$
Major axis	$y-$axis or $y = 0$
Minor axis	$x-$axis or $x = 0$

NOTE:

1. If the transverse axis (Major axis) is the y −axis rather than the x −axis, the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$ where $a^2=b^2(1-e^2)$

2. The table above is used in stating the focus, equation of Directrices, vertices, eccentricity and equation of the ellipse.

Example

1. *Find the equation of the ellipse with:*

i) Foci at $(\pm3\,,0)$, vertices $(\pm5\,,0)$

ii) Foci at $(0\,,\pm4)$, vertices $(0\,,\pm7)$

iii) Foci at $(\pm2\,,0)$, minor axis of length 3 *units*

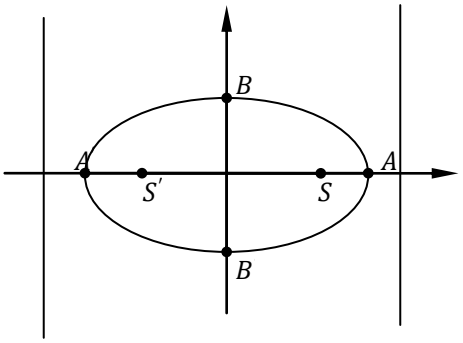
iv) Foci at $(0\,,\pm5)$, major axis of length 24 *units*

v) Vertices $(\pm8\,,0)$, contains $\left(4\,,\sqrt{3}\right)$

Solution

i) Foci at $(\pm3\,,0)$, vertices $(\pm5\,,0)$

$\Rightarrow S(3\,,0)\,,S'(-3\,,0)\,,\ A(5\,,0)\,,\ A'(5\,,0)$



Since the major axis is the x −axis, minor axis is the y −axis

The equation of the ellipse in of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$

Vertices: $(\pm a\,,0)=(\pm5\,,0)\ \ \therefore a=5$

Foci: $(\pm ae\,,0)=(\pm3\,,0)\ \ \therefore ae=3$

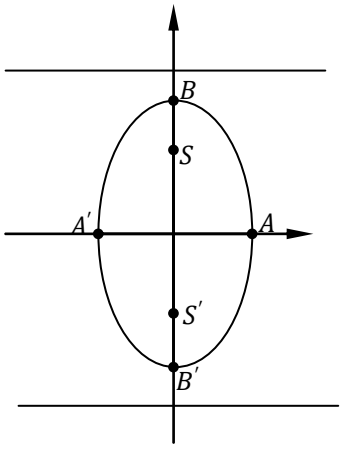
$\Rightarrow e=\frac{3}{5}$

Also $b^2=a^2(1-e^2)=25\left(1-\frac{9}{25}\right)=16$

$\therefore \frac{x^2}{25}+\frac{y^2}{16}=1$, is the required ellipse.

ii) Foci at $(0\,,\pm4)$, vertices $(0\,,\pm7)$

$\Rightarrow S(0\,,4)\,,S'(0\,,-4)\,,\ B(0\,,7)\,,\ B'(0\,,-7)$



Since the major axis is the y −axis, minor axis is the x −axis

The equation of the ellipse in of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$

Vertices: $(0\,,\pm b)=(0\,,\pm7)\ \ \therefore b=7$

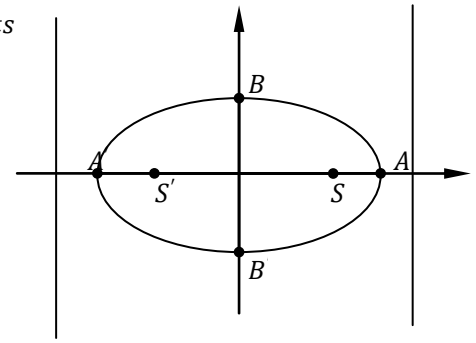
Foci: $(0\,,\pm be)=(0\,,\pm4)\ \ \therefore be=4$

$\Rightarrow e=\frac{4}{7}$

Also $a^2=b^2(1-e^2)=49\left(1-\frac{16}{49}\right)=33$

$\therefore \frac{x^2}{33}+\frac{y^2}{49}=1$

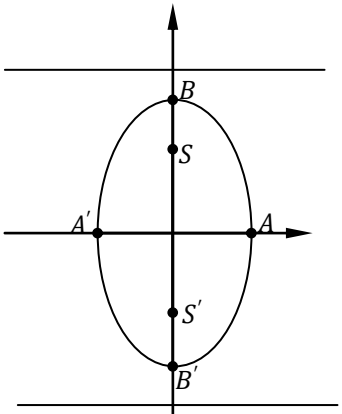
iii) Foci at $(\pm 2\,,0)$, minor axis of length 3 *units*



Since the major axis is the x −axis,
minor axis is the y −axis
The equation of the ellipse in of the form
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$
Foci: $(\pm ae\,,0)=(\pm 2\,,0)\quad \therefore ae=2$

Length of minor axis $=2b=3\quad \therefore b=\frac{3}{2}$
Since $a>b$, using $b^2=a^2(1-e^2)$
$\Rightarrow \frac{9}{4}=a^2(1-e^2)=a^2-a^2e^2$
$\Rightarrow \frac{9}{4}=a^2-4\quad [\because ae=2\Rightarrow a^2e^2=4]$
$\therefore a^2=\frac{25}{4}$
Required ellipse: $\frac{x^2}{25/4}+\frac{y^2}{9/4}=1$ or $\frac{4x^2}{25}+\frac{4y^2}{9}=1$

iv) Foci at $(0\,,\pm 5)$, major axis of length 24 *units*



Since the major axis is the y −axis,
minor axis is the x −axis
The equation of the ellipse in of the form
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$
Foci: $(0\,,\pm be)=(0\,,\pm 5)\quad \therefore be=5$
Length of Major axis $=24=2b\quad \therefore b=12$
$\Rightarrow e=5/12$
Also $a^2=b^2(1-e^2)=144\left(1-\frac{25}{144}\right)=119$
$\therefore \frac{x^2}{119}+\frac{y^2}{144}=1$

v) Vertices $(\pm 8\,,0)$, contains $\left(4\,,\sqrt{3}\right)$
Vertices $(\pm 8\,,0)=(\pm a\,,0)\ \Rightarrow a=8$, since the vertices are on the x −axis, it is the major axis.
Using $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$
$\Rightarrow \frac{x^2}{64}+\frac{y^2}{b^2}=1$, point $\left(4\,,\sqrt{3}\right)$ is on the curve
$\Rightarrow \frac{4^2}{64}+\frac{\left(\sqrt{3}\right)^2}{b^2}=1$ or $1-\frac{1}{4}=\frac{3}{b^2}\quad \therefore b^2=4$
$\therefore \frac{x^2}{64}+\frac{y^2}{4}=1$, is the ellipse

2. *Find the major axis, length of the major axis and semi minor axis and coordinates of the foci*

i) $\frac{x^2}{9}+\frac{y^2}{25}=1$
ii) $9x^2+25y^2=225$
iii) $32x^2+16y^2=64$
iv) $162x^2+128y^2=10368$

Solution

i) $\frac{x^2}{9}+\frac{y^2}{25}=1$, which is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$, major axis is $y-$ axis
Foci: $(0\ ,\pm be)\ ;\ a^2=b^2(1-e^2)$
Now $a^2=9\Rightarrow a=\pm3\ ,\ b^2=25\ \Rightarrow b=\pm5$
$\Rightarrow 9=25(1-e^2)\quad \therefore e=+4/5$
Major axis: $x=0\ \ (y-\text{axis})$
Length of major axis: $2b=10\ units$
Length of semi minor axis: $a=3\ units$
Foci: $\left(0\ ,\pm5\times\frac{4}{5}\right)=(0\ ,\pm4)$

ii) $9x^2+25y^2=225$……$(\div 225)$
$\frac{x^2}{25}+\frac{y^2}{9}=1$, which is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$
$\Rightarrow a^2=25\ \ \therefore a=\pm5;\ \ b^2=9\ \ \therefore b=\pm3$
Using $b^2=a^2(1-e^2)$
$\Rightarrow 9=25(1-e^2)\ \ \therefore e=+4/5$
Transverse axis: $y=0\ \ (x-\text{axis})$
Length of major axis: $2a=10\ units$
Semi minor axis: $b=3\ units$
Foci: $\left(\pm5\times\frac{4}{5}\ ,0\right)=(\pm4\ ,0)$

iii) Left as an exercise

iv) Left as an exercise

3. *Find the eccentricity, foci and Directrices of the ellipse;*

a) $\frac{x^2}{25}+\frac{y^2}{9}=1$ c) $4x^2+y^2=1$

b) $16x^2+25y^2=100$ d) $\frac{x^2}{4}+y^2=1$ e) $\frac{x^2}{2}+\frac{2y^2}{9}=2$

Solution

a) $\frac{x^2}{25}+\frac{y^2}{9}=1$ is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$
$a^2=25\ ,\ b^2=9$, Foci: $(\pm ae\ ,0)$, Directrices: $x=\pm\frac{a}{e}\ ,\ b^2=a^2(1-e^2)$
$\Rightarrow 25=9(1-e^2)\ \ \therefore e=+4/5$
Foci: $\left(\pm5\times\frac{4}{5}\ ,0\right)=(\pm4\ ,0)$
Directrices: $x=\pm\frac{5}{4/5}=\pm\frac{25}{4}$

b) $16x^2+25y^2=100\Rightarrow\frac{x^2}{25/4}+\frac{y^2}{4}=1$, which of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$
$a^2=25/4\ ,\ b^2=4$, Foci: $(\pm ae\ ,0)$, Directrices: $x=\pm\frac{a}{e}\ ,\ b^2=a^2(1-e^2)$
$\Rightarrow\frac{25}{4}=4(1-e^2)\ \ \therefore e=+3/5$
Foci: $\left(\pm\frac{5}{2}\times\frac{3}{5}\ ,0\right)=\left(\pm\frac{3}{2}\ ,0\right)$

Directrices: $x = \pm\frac{5/2}{3/5} = \pm\frac{25}{6}$

c) $4x^2 + y^2 = 1 \Rightarrow \frac{x^2}{1/4} + \frac{y^2}{1} = 1$ which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a < b$

$a^2 = 1/4\,,\ b^2 = 1$, Foci: $(0\,, \pm be)$, Directrices: $y = \pm\frac{b}{e}$, $a^2 = b^2(1 - e^2)$

$\Rightarrow \frac{1}{4} = 1(1 - e^2)\ \ \therefore e = +\sqrt{3}/2$

Foci: $\left(0\,, \pm 1 \times \frac{\sqrt{3}}{2}\right) = \left(0\,, \pm\frac{\sqrt{3}}{2}\right)$

Directrices: $y = \pm\frac{1}{+\sqrt{3}/2} = \pm\frac{2}{\sqrt{3}}$

d) Left as an exercise. **Ans:** $e = \frac{\sqrt{3}}{2}, \left(\pm\sqrt{3}\,,\ 0\right), x = \pm\frac{4\sqrt{3}}{3}$

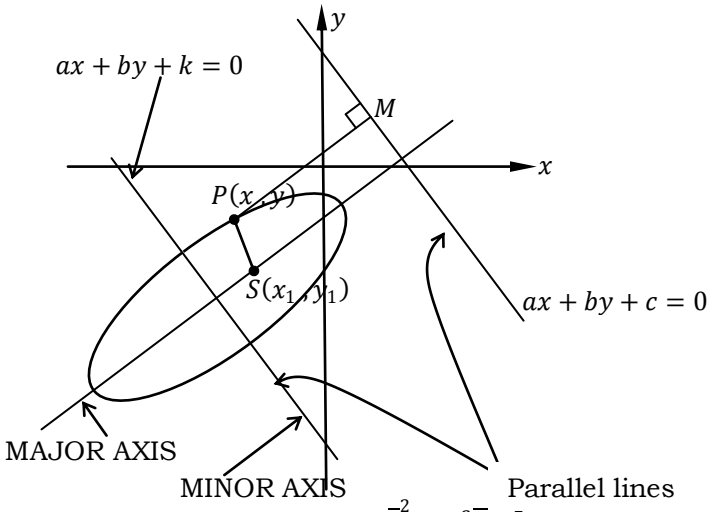
e) Left as an exercise. **Ans:** $e = \frac{\sqrt{5}}{3}\,, \left(0\,, \pm\sqrt{5}\right), y = \frac{9}{5}\sqrt{5}$

GENERAL EQUATION OF AN ELLIPSE

To Find the Equation of an Ellipse given coordinates of Focus and Equation of Directrix

Let the focus have coordinates $S(x_1\,, y_1)$ and the equation of the directrix $ax + by + c = 0$

Let $P(x\,, y)$ be a general point on ellipse.



By definition, $PS = ePM$ or $\overline{PS}^2 = e^2\overline{PM}^-$

$\overline{PM}$ =⊥ Distance of P from the directrix, $ax + by + c = 0$

$\overline{PM} = \pm\frac{ax+by+c}{\sqrt{a^2+b^2}}$

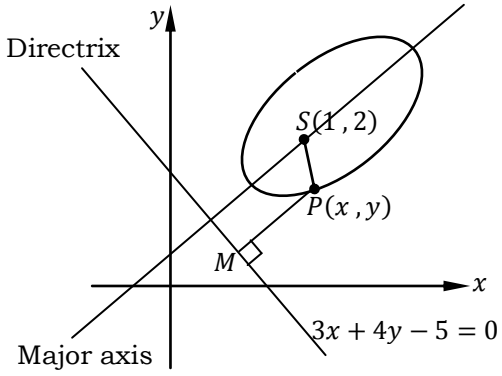
$\overline{PS} = \sqrt{(x - x_1)^2 + (y - y_1)^2}$

$$(x - x_1)^2 + (y - y_1)^2 = \frac{(ax+by+c)^2}{\sqrt{a^2+b^2}}$$

Example

Find the equation of the ellipse with focus $(1\,,2)$*, equation of directrix* $3x+4y=5$ *and eccentricity* $e=1/2$.

Solution



Using $\overline{PS}=e\overline{PM}$ or $\overline{PS}^2=e^2\overline{PM}^2$

$\overline{PM}=\perp$ Distance of P from the directrix,

$$3x+4y-5=0$$

$$\overline{PM}=\pm\frac{3x+4y-5}{\sqrt{3^2+4^2}}=\pm\frac{3x+4y-5}{5}$$

$$\overline{PS}=\sqrt{(x-1)^2+(y-2)^2}$$

$$\therefore (x-1)^2+(y-2)^2=\frac{1}{4}.\frac{(3x+4y-5)}{25}$$

$$100x^2+100y^2-200x-400y+500=9x^2+16y^2+24xy-30x-40y$$

$$\therefore 91x^2+84y^2-24xy-170x-360y+475=0$$

USING STANDARD EQUATION OF AN ELIPSE TO DEDUCE THE CENTRE, LENGTH OF AXES, ECCENTRICITY, FOCI AND VERTICES OF A GIVEN ELLIPSE

Example

Find the centre, length of the axes, eccentricity, foci and vertices of the ellipse

i) $x^2+4y^2-4x+24y+31=0$

ii) $12x^2+4y^2+24x-16y+25=0$

iii) $4x^2+9y^2-48x+72y+144=0$

iv) $x^2+4y^2-10x-8y+13=0$

Solution

i) Rewriting the equation in the form $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$

Completing squares

$x^2-4x+4-4+4(y^2+6y+9-9)+31=0$

$(x-2)^2+4(y+3)^2-4-36+31=0$

$(x-2)^2+4(y+3)^2=9$

$\frac{(x-2)^2}{9}+\frac{(y+3)^2}{9/4}=9$, which is of the form $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1\,,a>b$

By comparison

$X=x-2\,,Y=y+3\,,a^2=9\,,\ b^2=9/4$

Centre: $(0\,,0)=(X\,,\ Y)$

$\Rightarrow x-2=0\ \therefore x=2$

$\Rightarrow y+3=0\ \therefore y=-3$

$\therefore$ Centre: $(2\,,-3)$

Using $b^2=a^2(1-e^2)\Rightarrow\frac{9}{4}=9(1-e^2)\ \ \therefore e=+\frac{\sqrt{3}}{2}$

$\therefore$Eccentricity: $e = +\frac{\sqrt{3}}{2}$

Vertices: $(\pm a\,, 0) = (X\,,\ Y)$ and $(0\,, \pm b) = (X\,,\ Y)$

$\Rightarrow x - 2 = \pm 3\ \ \therefore x = 5\,, -1$

$\Rightarrow y + 3 = 0\ \ \therefore y = -3$

$\therefore$ Vertices: $(5\,, -3)\,, (-1\,, -3)$

$\Rightarrow x - 2 = 0\ \ \therefore x = 2$

$\Rightarrow y + 3 = \pm\frac{3}{2}\ \ \therefore y = -\frac{3}{2}\,, -\frac{9}{2}$

$\therefore$ Vertices: $\left(2\,, -\frac{3}{2}\right)\,, \left(2\,, -\frac{9}{2}\right)$

Foci: $(\pm ae\,, 0) = (X\,,\ Y)$

$\Rightarrow x - 2 = \pm 3 \times \frac{\sqrt{3}}{2}\ \ \therefore x = 2 \pm \frac{3\sqrt{3}}{2}$

$\Rightarrow y + 3 = 0\ \ \therefore y = -3$

$\therefore$ Foci: $\left(2 \pm \frac{3\sqrt{3}}{2}\,, -3\right)$

Length of major axis: $2a = 2 \times 3 = 6\ units$

Length of minor axis: $2b = 2 \times \frac{3}{2} = 3\ units$

ii) Rewriting the equation in the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

Completing squares

$12(x^2 + 2x + 1 - 1) + 4(y^2 - 4y + 4 - 4) + 25 = 0$

$12(x + 1)^2 + 4(y - 2)^2 - 12 - 16 + 25 = 0$

$12(x + 1)^2 + 4(y - 2)^2 = 3$

$\frac{(x+1)^2}{1/4} + \frac{(y-2)^2}{3/4} = 1$, which is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1\,, a < b$

By comparison

$X = x + 1\,, Y = y - 2\,, a^2 = 1/4\,,\ b^2 = 3/4$

Centre: $(0\,, 0) = (X\,,\ Y)$

$\Rightarrow x + 1 = 0\ \ \therefore x = -1$

$\Rightarrow y - 2 = 0\ \ \therefore y = 2$

$\therefore$ Centre: $(-1\,, 2)$

Using $a^2 = b^2(1 - e^2) \Rightarrow \frac{1}{4} = \frac{3}{4}(1 - e^2)\ \ \therefore e = +\frac{\sqrt{6}}{3}$

$\therefore$Eccentricity: $e = +\frac{\sqrt{6}}{3}$

Vertices: $(\pm a\,, 0) = (X\,,\ Y)$ and $(0\,, \pm b) = (X\,,\ Y)$

$\Rightarrow x + 1 = \pm\frac{1}{2}\ \ \therefore x = -\frac{1}{2}\,, -\frac{3}{2}$

$\Rightarrow y - 2 = 0\quad \therefore y = 2$

$\therefore$ Vertices: $\left(-\frac{1}{2}\,, 2\right)\,, \left(-\frac{3}{2}\,, 2\right)$

$\Rightarrow x + 1 = 0\ \ \therefore x = -1$

$\Rightarrow y - 2 = \pm\frac{\sqrt{3}}{2}\ \ \therefore y = 2 \pm \frac{\sqrt{3}}{2}$

$\therefore$ Vertices: $\left(-1\,, 2 \pm \frac{\sqrt{3}}{2}\right)$

Foci: $(0\,, \pm be) = (X\,,\ Y)$

$\Rightarrow x+1=0 \quad \therefore x=-1$

$\Rightarrow y-2=\pm\frac{\sqrt{3}}{2}\times\frac{\sqrt{6}}{3}=\pm\frac{\sqrt{2}}{2} \quad \therefore y=2\pm\frac{\sqrt{2}}{2}$

$\therefore$ Foci: $\left(-1\,,2\pm\frac{\sqrt{2}}{2}\right)$

Length of major axis: $2b=2\times\frac{\sqrt{3}}{2}=\sqrt{3}\,units$

Length of minor axis: $2a=2\times\frac{1}{2}=1\,unit$

iii) Left as an exercise. **Ans:** centre $(2\,,0)$, Foci $\left(2\,,\pm2\sqrt{3}\right),e=\sqrt{3}/2$

iv) Left as an exercise. **Ans:** centre $(5\,,1)$, Foci $\left(5\pm2\sqrt{3}\,,\ 1\right),e=\sqrt{3}/2$

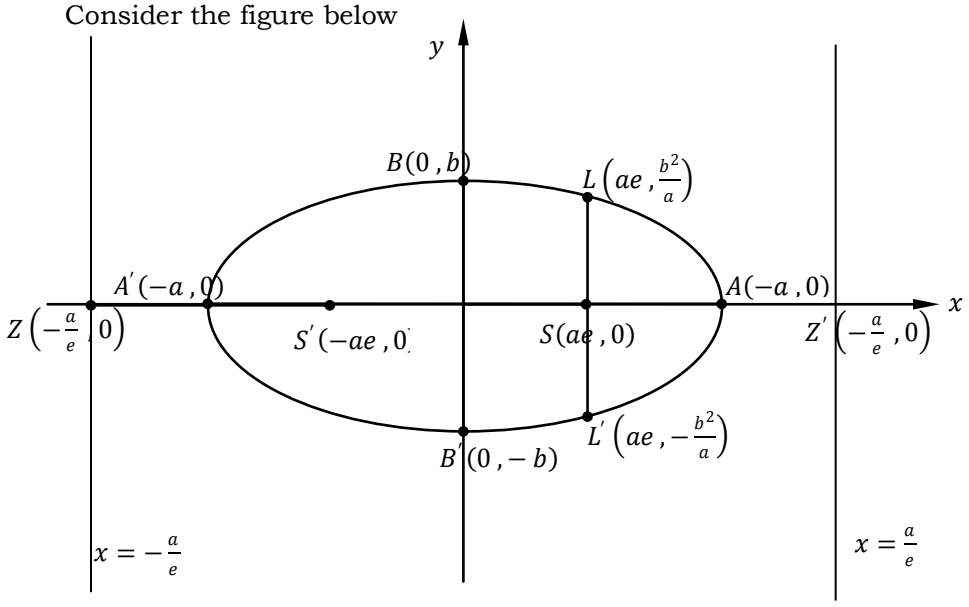
A CHORD, FOCAL CHORD AND LATUS RECTUM OF $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Chord: is any segment whose endpoints are on the ellipse

Focal Chord: is a chord containing the focus

Latus Rectum: the focal chord which is perpendicular to the transverse axis (Major axis)

Consider the figure below



The line LSL' is latus rectum (latus recta in plural).

Finding the coordinates of L' and L

The $x-$coordinate of L' and L is $x=+ae$

$\frac{(ae)^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow\frac{y^2}{b^2}=1-\frac{a^2e^2}{a^2}=1-e^2$

But from $b^2=a^2(1-e^2)\Rightarrow 1-e^2=\frac{b^2}{a^2}$

$\Rightarrow\frac{y^2}{b^2}=\frac{b^2}{a^2} \quad \therefore y=\pm\frac{b^2}{a}$

$\therefore L\left(ae\,,\frac{b^2}{a}\right),\ L'\left(ae\,,-\frac{b^2}{a}\right)$

Length of latus rectum is $\frac{2b^2}{a}$

Example

Find the equation of the ellipse in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,, a>b$ *given that:*

i) *Distance between the foci is 6 and distance between the Directrices is 24*

ii) *Latus rectum is* $10\frac{2}{3}$ *and distance between the Directrices is* 36

Solution

i) Foci: $(\pm ae\,, 0)$ $\therefore$ Distance= $2ae=6$

$\therefore ae=3$ ..... (1)

Distance between Directrices $=\frac{2a}{e}=24$ $\therefore a=12e$..... (2)

Eqn. (1) into eqn. (2)

$\Rightarrow 12e^2=3$ $\therefore e=+\frac{1}{2}$

From eqn. (2), $a=12\times\frac{1}{2}=6$ $\Rightarrow a^2=36$

Using $b^2=a^2(1-e^2)$

$\Rightarrow b^2=36(1-(1/2)^2)$ $\therefore b^2=27$

$\therefore$ Required ellipse is $\frac{x^2}{36}+\frac{y^2}{27}=1$

ii) Length of latus rectum is $\frac{2b^2}{a}=10\frac{2}{3}$ $\therefore b^2=\frac{16}{3}a$..... (1)

Distance between Directrices $=\frac{2a}{e}=36$ $\therefore a=18e$..... (2)

Using $b^2=a^2(1-e^2)$

$\Rightarrow \frac{16}{3}a=a^2\left(1-\frac{a^2}{324}\right)$ or $1728=324a-a^3$

$\Rightarrow a^3-324a+1728=0$ $\therefore a=14.2337, -20.2337\,,\ 6$

Taking the most accurate value, the whole number, $a=6$

$\Rightarrow a^2=36\,,\ b^2=\frac{16}{3}\times 36=32$

$\therefore$ Required ellipse: $\frac{x^2}{36}+\frac{y^2}{32}=1$

A LINE AND AN ELLIPSE

Just like in the parabola, the following questions must be well understood in mastering this sub topic.

f) **To Find the Point of Intersection of the Line $y = mx + c$ and the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**

Solving the equations simultaneously

$y = mx + c$ ..... (1)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ..... (2)

Eqn. (1) into eqn. (2)

$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$\Rightarrow b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$

$\Rightarrow (b^2 + a^2m^2)x^2 + (2mca^2)x + a^2c^2 - a^2b^2 = 0$ ..... (3)

Solving this equation gives *two* values of x. Obtain the corresponding values from equation (1), and hence two points of intersection in the form (x, y)

NOTE

Points of intersection can be deduced from the roots of the equation above, which either can be

- Real and distinct
- Coincident/Repeated or
- Imaginary

, depending on the nature of the discriminant $B^2 - 4AC$

For real and distinct roots, $B^2 - 4AC > 0$

For Repeated roots, $B^2 - 4AC = 0$

For Imaginary roots, $B^2 - 4AC < 0$

g) **To Show that a given Line $y = mx + c$ is a Tangent to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**

Solving the equations of the line and the ellipse simultaneously, obtain eqn. (3) above as

$(b^2 + a^2m^2)x^2 + (2mca^2)x + (a^2c^2 - a^2b^2) = 0$ ..... (3)

Proceed in two ways

Approach I:

Applying condition for tangency .i.e. condition for equal roots of a quadratic equation

For tangency, $B^2 = 4AC$

$\Rightarrow (2mca^2)^2 = 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2)$

$\Rightarrow 4m^2c^2a^4 = 4a^2\ (b^2 + a^2m^2)(c^2 - b^2)$..... $(\div 4a^2)$

$\Rightarrow m^2c^2a^2 = b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2$

$\Rightarrow b^2c^2 - b^4 - a^2m^2b^2 = 0$ ..... $(\div b^2)$

$\Rightarrow c^2 - b^2 - a^2m^2 = 0 \quad \therefore c^2 = b^2 + a^2m^2$

$$\boxed{c^2 = a^2m^2 + b^2 \text{ or } c = \pm\sqrt{a^2m^2 + b^2}}$$

, which is the condition for a line $y = mx + c$ to be a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Approach II

Solving eqn. (3), two repeated/coincident values of x must be obtained. Eqn. (3) can have coincident roots if the condition for tangency is used.

Put $c = \pm\sqrt{a^2m^2 + b^2}$ in to eqn. (3)

$(b^2 + a^2m^2)x^2 \pm \left(2ma^2\sqrt{a^2m^2 + b^2}\right)x + [a^2(a^2m^2 + b^2) - a^2b^2] = 0$

$(b^2 + a^2m^2)x^2 \pm \left(2ma^2\sqrt{a^2m^2 + b^2}\right)x + (a^4m^2 + a^2b^2 - a^2b^2) = 0$

$(b^2 + a^2m^2)x^2 \pm \left(2ma^2\sqrt{a^2m^2 + b^2}\right)x + a^4m^2 = 0$ ..... (4)

$\Rightarrow \left(x\sqrt{a^2m^2 + b^2} \pm ma^2\right)^2 = 0 \quad \therefore x = \pm\frac{ma^2}{\sqrt{a^2m^2+b^2}}$, which will be a repeated value. Hence the line is a tangent to the ellipse.

h) **To obtain the Point of Contact of a tangent line $y = mx + c$ to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**

Proceeding as in approach II above, from eqn. (4), $x = \pm\frac{ma^2}{\sqrt{a^2m^2+b^2}}$, obtain the corresponding $y-$ value from the line.

Taking $x = +\frac{ma^2}{\sqrt{a^2m^2+b^2}}$, $c = -\sqrt{a^2m^2 + b^2}$

$\Rightarrow y = m \times \frac{ma^2}{\sqrt{a^2m^2+b^2}} - \sqrt{a^2m^2 + b^2} = \frac{a^2m^2 - a^2m^2 - b^2}{\sqrt{a^2m^2+b^2}} = -\frac{b^2}{\sqrt{a^2m^2+b^2}}$

$\therefore$ Point of contact is $\boldsymbol{P}\left(\frac{\boldsymbol{ma^2}}{\sqrt{\boldsymbol{a^2m^2+b^2}}} , -\frac{\boldsymbol{b^2}}{\sqrt{\boldsymbol{a^2m^2+b^2}}}\right)$

Similarly, taking $x = -\frac{ma^2}{\sqrt{a^2m^2+b^2}}$, $c = +\sqrt{a^2m^2 + b^2}$

$\Rightarrow y = -m \times \frac{ma^2}{\sqrt{a^2m^2+b^2}} + \sqrt{a^2m^2 + b^2} = \frac{-a^2m^2 + a^2m^2 + b^2}{\sqrt{a^2m^2+b^2}} = \frac{b^2}{\sqrt{a^2m^2+b^2}}$

$\therefore$ Point of contact is $\boldsymbol{P'}\left(-\frac{\boldsymbol{ma^2}}{\sqrt{\boldsymbol{a^2m^2+b^2}}} , \frac{\boldsymbol{b^2}}{\sqrt{\boldsymbol{a^2m^2+b^2}}}\right)$

i) **To state the Equation of the Tangent to the parabola in Standard Form**

From the line $y = mx + c$ and condition for tangency $c = \pm\sqrt{a^2m^2 + b^2}$, the tangent in slope form is $\boldsymbol{y = mx \pm \sqrt{a^2m^2 + b^2}}$

NOTE

In answering questions (b) and (c), you must first obtain eqn. (3) in question (a) above, the approach to finding point of intersection of a line and a parabola.

Example

1. *Find the point of intersection of the line and the ellipse*
Line: $y = x$ *and the ellipse* $4x^2 + 5y^2 = 49$

Solution

Solving the equations simultaneously
$4x^2 + 5y^2 = 49$, $y = x$
$\Rightarrow 4x^2 + 5x^2 = 49 \quad \therefore x = \pm 7$
From the equation of the line, $y = x = \pm 7$
$\therefore (7, 7)$, $(-7, -7)$ are the points of intersection

2. *Prove that* $y = 2x + 3$ *touches the ellipse* $\frac{x^2}{2} + y^2 = 1$ *and that the point of contact is* $\left(-\frac{4}{3}, \frac{1}{3}\right)$

Solution

$y = 2x + 3$ ..... (1)
$\frac{x^2}{2} + y^2 = 1 \Rightarrow x^2 + 2y^2 = 2$ ..... (2)
Solving the equations simultaneously
Eqn. (1) into eqn. (2)
$\Rightarrow x^2 + 2(2x + 3)^2 = 2$
$\Rightarrow x^2 + 2(4x^2 + 12x + 9) = 2$
$\Rightarrow 9x^2 + 24x + 16 = 0$ or $(3x + 4)^2 = 0$
$\therefore x = -\frac{4}{3}, -\frac{4}{3}$, which is a repeated value. Since there is only one value of $x = -\frac{4}{3}$, then there is only one point of contact. Hence the line is a tangent to the ellipse.
Point of contact
For $x = -\frac{4}{3}$, y =?
From eqn. (1), $y = 2 \times -\frac{4}{3} + 3 = \frac{1}{3}$
$\therefore$ Point of contact: $\left(-\frac{4}{3}, \frac{1}{3}\right)$

Task

1) *Prove that* $y = 2x + 5$ *touches* $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ *and find the point of contact.*
Ans: $\left(-\frac{8}{5}, \frac{9}{5}\right)$

2) *Prove that* $y = x - 5$ *touches* $\frac{1}{16}x^2 + \frac{1}{9}y^2 = 1$ *and find the point of contact.*
Ans: $\left(\frac{16}{5}, -\frac{9}{5}\right)$

3) *Show that the line* $x - 2y + 10 = 0$ *is a tangent to the ellipse* $\frac{x^2}{64} + \frac{y^2}{9} = 1$.
Hence deduce the point of contact. **(Uneb 2000)** **Ans:** $\left(-\frac{32}{4}, \frac{9}{5}\right)$

Other questions involving Tangents and Ellipse

3. *Find the equations of tangents to an ellipse;*

a) $3x^2 + 4y^2 = 12$, which are parallel to the line $x - 2y + 1 = 0$. Hence find the points of contact of the tangents to the ellipse.

b) $\frac{x^2}{16} + y^2 = 1$, which make an angle of 60^0 with $x-$ axis

Solution

a) The equation of tangent line parallel to $x - 2y + 1 = 0$ is given as:

$x - 2y + k = 0$ or $x = 2y - k$…… (1)

For point of intersection of the tangent line and ellipse, solve the two equations simultaneously

Putting eqn. (1) into $3x^2 + 4y^2 = 12$

$\Rightarrow 3(2y - k)^2 + 4y^2 = 12$

$\Rightarrow 3(4y^2 - 4ky + k^2) + 4y^2 = 12$ or $16y^2 - 12k\,y + (3k^2 - 12) = 0$…… (2)

For tangency, $B^2 = 4AC$

$\Rightarrow (-12k\,)^2 = 4(16)(3k^2 - 12)$

$\Rightarrow 144k^2 = 192k^2 - 768 \quad \therefore k = \pm 4$

$\therefore$ Equations of tangents are: $x - 2y \pm 4 = 0$

Point of contact

Putting $k = \pm 4$ into eqn. (2)

$16y^2 - 12k\,y + (3k^2 - 12) = 0\,,\ k = +4$

$\Rightarrow 16y^2 - 48y + (48 - 12) = 0$ or $\Rightarrow 16y^2 - 48y + 36 = 0 \quad \therefore y = \frac{3}{2}$

From eqn. (1), for $y = \frac{3}{2}\,,\ x = 2 \times \frac{3}{2} - 4 = -1$

$\therefore$ Point of contact: $\left(-1\,, \frac{3}{2}\right)$

$16y^2 - 12k\,y + (3k^2 - 12) = 0\,,\ k = -4$

$\Rightarrow 16y^2 + 48y + (48 - 12) = 0$ or $\Rightarrow 16y^2 + 48y + 36 = 0 \quad \therefore y = -\frac{3}{2}$

From eqn. (1), for $y = -\frac{3}{2}\,,\ x = 2 \times -\frac{3}{2} + 4 = 1$

$\therefore$ Point of contact: $\left(1\,, -\frac{3}{2}\right)$

b) Let the equation of the tangent be: $y = mx + c$

Given: $m = \tan 60^0 = \sqrt{3}$

$y = \sqrt{3}x + c$ …… (1)

Solving the equation of tangent and ellipse simultaneously

$\frac{x^2}{16} + y^2 = 1$ …… (2)

Eqn. (1) into eqn. (2)

$\frac{x^2}{16} + \left(\sqrt{3}x + c\,\right)^2 = 1$ or $x^2 + 16\left(3x^2 + 2\sqrt{3}x + c^2\right) - 16 = 0$

$49x^2 + \left(32\sqrt{3}c\right)x + 16(c^2 - 1)$

For tangency, $B^2 = 4AC$

$\Rightarrow \left(32\sqrt{3}c\,\right)^2 = 4(49)(16)(c^2 - 1)$

$\Rightarrow 3072c^2 = 3136c^2 - 3136$ or $64c^2 = 3136$

$\Rightarrow c^2 = 49 \quad \therefore c = \pm 7$

$\therefore$ Equations of tangents are: $y = \sqrt{3}x \pm 7$

Task

Find the equations of tangents to the ellipse $4x^2 + 9y^2 = 8$ *which are parallel to* $2x - 3y = 0$ **Ans:** $3y = 2x \pm 4$

4. *Show that if the line* $y = mx + c$ *touches the curve* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *the* $c^2 = a^2m^2 + b^2$. *Hence find the equation of tangents to the ellipse;*

a) $4x^2 + 9y^2 = 1$ *which are perpendicular to* $y = 2x + 3$

b) $\frac{x^2}{64} + \frac{y^2}{9} = 1$, *that pass through* $(-2, 4)$

c) $\frac{x^2}{4} + \frac{y^2}{9} = 1$ *wwhich are inclined to* $x -$ *axis at an angle* $\sin^{-1}\left(\frac{2}{5}\sqrt{5}\right)$

d) $\frac{x^2}{3} + \frac{y^2}{4} = 1$, *that pass through* $\left(0, \sqrt{31}\right)$

e) $x^2 + 2y^2 = 8$ *which are parallel to* $y = 2x$

Solution

The proof of a line being a tangent to the ellipse is left as an exercise.

a) Required tangent: $y = mx \pm \sqrt{a^2m^2 + b^2}$

Comparing $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $4x^2 + 9y^2 = 1$ or $\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$

$a^2 = \frac{1}{4}$, $b^2 = \frac{1}{9}$

$\Rightarrow y = mx \pm \sqrt{\frac{m^2}{4} + \frac{1}{9}}$, $m =$ slope of the tangent which must be $-\frac{1}{\text{Slope of the given line}} \quad \therefore m = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}x \pm \sqrt{\frac{1}{16} + \frac{1}{9}}$ or $y = -\frac{1}{2}x \pm \frac{5}{12}$

b) Required tangent: $y = mx \pm \sqrt{a^2m^2 + b^2}$

Comparing $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $\frac{x^2}{64} + \frac{y^2}{9} = 1$

$a^2 = 64$, $b^2 = 9$

$\Rightarrow y = mx \pm \sqrt{64m^2 + 9}$, $m =$ slope of the tangent which must be obtained

Point $(-2, 4)$ lies on this line

$\Rightarrow 4 = -2m \pm \sqrt{64m^2 + 9}$

$\Rightarrow (4 + 2m) = \pm\sqrt{64m^2 + 9}$, squaring both sides

$\Rightarrow 16 + 16m + 4m^2 = 64m^2 + 9$ or $60m^2 - 16m - 7 = 0 \quad \therefore m = \frac{1}{2}, -\frac{7}{30}$

$\therefore$ Tangents are obtained from: $y = mx \pm \sqrt{64m^2 + 9}$

For $m = \frac{1}{2}$, $y = \frac{1}{2}x \pm 5$

For $m = -\frac{7}{30}$, $y = -\frac{7}{30}x \pm \frac{53}{15}$

c) Left as an exercise. **Ans:** $y = 2x \pm 5$

d) Left as an exercise. **Ans:** $y = 3x - \sqrt{31}$, $y = -3x + \sqrt{31}$

EQUATION OF A TANGENT AND NORMAL AT ANY POINT (x_1, y_1) ON AN ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

To Find an Equation of a Tangent at any point $P(x_1, y_1)$ on the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Required tangent: $y-y_1=m(x-x_1)$..... (1)

Finding slope of the Tangent, m

From $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, differentiating w.r.t x

$\left.\frac{dy}{dx}\right|_{\text{at } P(x_1, y_1)}=m$

$\Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$, multiplying through by $\frac{a^2b^2}{2}$

$\Rightarrow b^2x+a^2y\frac{dy}{dx}=0 \quad \therefore \frac{dy}{dx}=-\frac{b^2x}{a^2y}$

$\therefore \left.\frac{dy}{dx}\right|_{\text{at } P(x_1, y_1)}=m=-\frac{b^2x_1}{a^2y_1}$ ..... (2)

Eqn. (2) into eqn. (1)

$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$

$a^2y_1y-a^2{y_1}^2=-b^2x_1x+b^2{x_1}^2$

$\Rightarrow a^2y_1y+b^2x_1x=a^2{y_1}^2+b^2{x_1}^2$ ..... $(\div a^2b^2)$

$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{b^2}=1$

Since point $P(x_1, y_1)$ lies on the ellipse, $\frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{b^2}=1$

$\boxed{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$, which is the equation of the tangent at $P(x_1, y_1)$

To Find an Equation of the Normal at any point $P(x_1, y_1)$ on the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Required tangent: $y-y_1=m'(x-x_1)$..... (1)

Finding slope of the Normal, m'

Slope of tangent, $m=-\frac{b^2x_1}{a^2y_1}$

$\therefore$ Slope of Normal, $m'=\frac{a^2y_1}{b^2x_1} \quad [\because m\times m'=-1]$

$\Rightarrow y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$

$\boxed{\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{y-x_1}{\frac{x_1}{b^2}} \text{ or } \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2}$

Example

Find the equation of tangent and normal to an ellipse at the given point;

i) $\frac{x^2}{9}+\frac{y^2}{16}=1$ *at* $(0, 4)$

ii) $\frac{x^2}{9}+\frac{y^2}{4}=1$ *at* $(3, 0)$

iii) $4x^2 + 5y^2 = 120 \ at \ (-5\,, -2)$

iv) $9x^2 + 16y^2 = 40 \ at \left(2\,, \frac{1}{2}\right)$

Solution

i) Point $(0\,, 4)$ lies on the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Required Tangent

$y - 4 = m(x - 0) = mx$ or $y = mx + 4$

Finding the slope, m

$m = \frac{dy}{dx}\Big|_{at\ (0\,,4)}$

Differentiating the curve w.r.t x

$\frac{2x}{9} + \frac{2y}{16}\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{16x}{9y}$

$\Rightarrow m = \frac{dy}{dx}\Big|_{at\ (0\,,4)} = -\frac{0}{72} = 0$

$\therefore y = 4$

Required equation of the normal

$y - 4 = m'(x - 0) = m'x$ or $y = m'x + 4$

Finding the slope, m'

$m' = -\frac{1}{\text{Slope of the tangent at } (0\,,4)} = -\frac{1}{m} = -\frac{1}{0}$

Rewriting the equation of the normal as:

$\frac{y-4}{(x-0)} = m' = -\frac{1}{0}$

$\Rightarrow x - 0 = 0 \quad \therefore x = 0$

ii) Left as an exercise. **Ans:** Tangent: $x = 3$, Normal: $y = 0$

iii) $x^2 + 5y^2 = 120$, $(-5\,, -2)$

Required Tangent

$y + 2 = m(x + 5)$

Finding the slope, m

$m = \frac{dy}{dx}\Big|_{at\ (-5\,,-2)}$

Differentiating the curve w.r.t x

$2x + 10y\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{4x}{5y}$

$\Rightarrow m = \frac{dy}{dx}\Big|_{at\ (-5\,,-2)} = \frac{-4\times-5}{5\times-2} = -2$

$\therefore y + 2 = -2(x + 5)$ or $2x + y + 12 = 0$

Required equation of the normal

$y + 2 = m'(x + 5)$

Finding the slope, m'

$m' = -\frac{1}{\text{Slope of the tangent at } (0\,,4)} = -\frac{1}{m} = -\frac{1}{-2} = \frac{1}{2}$

Rewriting the equation of the normal as:

$y + 2 = \frac{1}{2}(x + 5)$

$\Rightarrow 2y + 4 = x + 5 \quad \therefore x - 2y + 1 = 0$

iv) Left as an exercise. **Ans:** Tangent: $9x + 4y = 20$, Normal: $8x - 18y - 7 = 0$

Task

a) *Show that the line* $5y = 4x + 25$ *is a tangent to the ellipse* $\frac{x^2}{25} + \frac{y^2}{9} = 1$

b) *Find the equation of the normal to the ellipse at the point of contact Determine the eccentricity of the ellipse.* (**Uneb 2004**) **Ans:** $\left(-4, \frac{9}{5}\right)$, $e = \frac{4}{5}$

To Find the Equation of a Tangent to the Ellipse from a Point $P(x_1, y_1)$ not on the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

1. *Find the equation of tangent(s) to the ellipse:*

a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ *passing through the point* $\left(-8, -\frac{7}{5}\right)$

b) $\frac{x^2}{64} + \frac{y^2}{9} = 1$ *from the point* $(10, 15)$

Solution

To attempt this question, two approaches can be used.

Approach I:

Let the equation of the tangent be:

$y = mx + c$ ..... (1)

The task is to find the values of m and c

Obtaining c

The point $\left(-8, -\frac{7}{5}\right)$ lies on the tangent

$\Rightarrow -\frac{7}{5} = -8m + c \quad \therefore c = 8m - \frac{7}{5}$ ..... (1)

Obtaining the slope, m

Solve the equation of the tangent and the ellipse simultaneously

$\frac{x^2}{25} + \frac{y^2}{9} = 1$ ..... (2)

Eqn. (1) into eqn. (2)

$\frac{x^2}{25} + \frac{(mx+c)^2}{9} = 1$ ..... $(\times 225)$

$9x^2 + 25(m^2x^2 + 2mcx + c^2) = 225$

$(9 + 25m^2)x^2 + 50mc\,x + (25c^2 - 225) = 0$

Applying the condition for repeated roots $B^2 = 4AC$

$(50mc)^2 = 4(9 + 25m^2)(25c^2 - 225) = (36 + 100m^2)(25c^2 - 225)$

$2500m^2c^2 = 900c^2 - 8100 + 2500m^2c^2 - 22500m^2$

$0 = 900c^2 - 8100 - 22500m^2$ ..... $(\div 900)$

$c^2 - 9 - 25m^2 = 0$..... (3)

Eqn. (1) into eqn. (3)

$\left(8m - \frac{7}{5}\right)^2 - 9 - 25m^2 = 0$ or $(40m - 7)^2 - 225 - 625m^2 = 0$

$1600m^2 - 560m + 49 - 225 - 625m^2 = 0$

$975m^2 - 560m - 176 = 0 \quad \therefore m = \frac{4}{5}, -\frac{44}{195}$

From eqn. (1),

For $m=\frac{4}{5}$, $c=8\times\frac{4}{5}-\frac{7}{5}=5$ $\quad\therefore y=\frac{4}{5}x+5$

For $m=-\frac{44}{195}$, $c=8\times-\frac{44}{195}-\frac{7}{5}=-\frac{125}{39}$ $\quad\therefore y=-\frac{44}{195}x-\frac{125}{39}$

Approach II:

Using the general tangent in slope form to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let the required tangent be:

$y=mx\pm\sqrt{a^2m^2+b^2}$

Comparing $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $\frac{x^2}{25}+\frac{y^2}{9}=1\Rightarrow a^2=25\,,b^2=9$

Tangent equation becomes:

$y=mx\pm\sqrt{25m^2+9}$ ..... (1)

Point $\left(-8\,,-\frac{7}{5}\right)$ lies on this tangent

$\Rightarrow-\frac{7}{5}=-8m\pm\sqrt{25m^2+9}$

$\Rightarrow 40m-7=\pm5\sqrt{25m^2+9}$, squaring both sides

$\Rightarrow 1600m^2-560m+49=25(25m^2+9)$

$\Rightarrow 1600m^2-560m+49=625m^2+225$

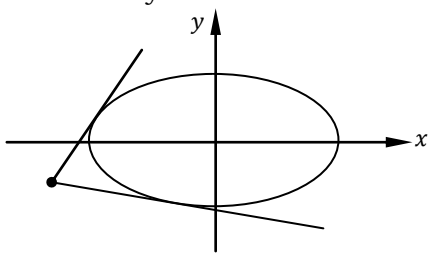
$\Rightarrow 975m^2-560m-176=0\quad\therefore m=\frac{4}{5}\,,-\frac{44}{195}$ ..... (2)

Eqn. (2) into eqn. (1)

For $m=\frac{4}{5}$, $y=\frac{4}{5}x\pm\sqrt{25\left(\frac{4}{5}\right)^2+9}\quad\Rightarrow y=\frac{4}{5}x\pm5$

For $m=-\frac{44}{195}$, $y=-\frac{44}{195}x\pm\sqrt{25\left(-\frac{44}{195}\right)^2+9}\quad\Rightarrow y=-\frac{44}{195}x\pm\frac{125}{39}$

It appears that we have four solutions. From a point, two tangents are normally drawn. So let us take the only two tangents required.



The ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ has its major axis as the $x-$axis. Plot of the point suggests that two tangents can be drawn to the ellipse.

A tangent with $m=\frac{4}{5}$ has a $+ve\ y-$ intercept and a tangent with $m=-\frac{44}{195}$ has a $-ve\ y-$ intercept.

Thus the required tangents are: $\therefore y=\frac{4}{5}x+5\,,\ y=-\frac{44}{195}x-\frac{125}{39}$

The other tangents do not pass through the point $\left(-8\,,-\frac{7}{5}\right)$

2. *The line $y=mx+c$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.*

a) *Obtain an expression for c an terms of $a\,,b$ and m*

b) *Calculate the gradients of the tangents to the ellipse through the point $\left(\sqrt{a^2+b^2}\,,0\right)$* (**Uneb 2018**)

Solution

a) Solving the equation of the line and ellipse simultaneously, and then applying the condition for repeated roots of a quadratic equation

$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$ ……$(\times a^2b^2)$

$a^2(m^2x^2+2mcx+c^2)+b^2x^2-a^2b^2=0$

$(a^2m^2+b^2)x^2+2mca^2\,x+(a^2c^2-a^2b^2)=0$

For tangency, $B^2=4AC$

$(2mca^2)^2=4(a^2m^2+b^2)(a^2c^2-a^2b^2)$

$4m^2c^2a^4=4a^2\ (b^2+a^2m^2)(c^2-b^2)$…… $(\div 4a^2)$

$m^2c^2a^2=b^2c^2-b^4+a^2m^2c^2-a^2m^2b^2$

$b^2c^2-b^4-a^2m^2b^2=0$ …… $(\div b^2)$

$\Rightarrow c^2-b^2-a^2m^2=0$

$\therefore c^2=b^2+a^2m^2$

b) The point $\left(\sqrt{a^2+b^2}\,,0\right)$ does not lie on the ellipse. It is regarded to be a point not on the ellipse.

Tangent in slope form is:

$y=mx\pm\sqrt{b^2+a^2m^2}$

The point $\left(\sqrt{a^2+b^2}\,,0\right)$ lies on this tangent

$\Rightarrow 0=m\sqrt{a^2+b^2}\pm\sqrt{b^2+a^2m^2}$

$\Rightarrow -m\sqrt{a^2+b^2}=\pm\sqrt{b^2+a^2m^2}$, squaring both sides

$\Rightarrow m^2(a^2+b^2)=b^2+a^2m^2$

$\Rightarrow m^2(a^2+b^2-a^2)=b^2$ or $m^2=1\quad\therefore m=\pm1$

$\therefore$ Gradients of the tangents are: ±1

Task

1. *Show that the $x-$coordinates of any point of intersection of the line $y=mx+c$ and the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ are given by the solutions of the quadratic equation $(4+9m^2)x^2+18mc\,x+(9c^2-36)=0$*
If the line $y=mx+c$ is a tangent of the ellipse, prove that $c^2=4+9m^2$.
The line $y=mx+c$ passes through the point $(2\,,3)$ write down a second equation connecting m and c, hence prove that m must satisfy the equation $5m^2+12m-5=0$.
Prove that the two tangents drawn from the point $(2\,,3)$ to the ellipse are perpendicular to each other.

2. *Find the equation of the tangent and normal to $ax^2+by^2=c$ at the point $(x_1\,,y_1)$.* **Ans:** $axx_1+byy_1=c;\ bxy_1-ax_1y=(b-a)x_1y_1$

PARAMETRIC COORDINATES

The general coordinates on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $(a\cos\theta\ , b\sin\theta)$ or generally denoted as θ.

The parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are:

$$\boldsymbol{x=a\cos\theta\ ,\ y=b\sin\theta}$$

To Find the Equation of a Tangent and Normal to an Ellipse

$\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ **at** $\boldsymbol{(a\cos\theta\ , b\sin\theta)}$

Tangent Equation

$y-\ b\sin\theta=m_1(x-a\cos\theta)$ ..... (1)

Obtaining slope, m_1

Method I:

$m_1=\frac{dy}{dx}\Big|_{at\ (a\cos\ \theta\ ,b\sin\ \theta)}$

Differentiating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ w.r.t x

$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \quad \therefore \frac{dy}{dx}=-\frac{b^2x}{a^2y}$

$\Rightarrow m_1=\frac{dy}{dx}\Big|_{at\ (a\cos\ \theta\ ,b\sin\ \theta)}=-\frac{b^2a\cos\ \theta}{a^2b\sin\ \theta}=-\frac{b\cos\ \theta}{a\sin\ \theta}.$

$\therefore m_1=-\frac{b\cos\ \theta}{a\sin\ \theta}$

Method II:

Formulating parametric equations

$x=a\cos\theta\ ,\ y=b\sin\theta$

$m_1=\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$

$\frac{dx}{d\theta}=-a\sin\theta\ ,\frac{dy}{d\theta}=b\cos\theta$

$\therefore m_1=b\cos\theta\times\frac{1}{-a\sin\ \theta}=-\frac{b\cos\ \theta}{a\sin\ \theta}$

Tangent equation becomes:

$y-\ b\sin\theta=-\frac{b\cos\ \theta}{a\sin\ \theta}(x-a\cos\theta)$

$\Rightarrow ya\sin\theta-ab\sin^2\theta=-b\cos\theta\,x+ab\cos^2\theta$

$\Rightarrow ya\sin\theta+b\cos\theta\,x=ab(\cos^2\theta+\sin^2\theta)$

$\therefore ay\sin\theta+bx\cos\theta=ab \qquad [\because \cos^2\theta+\sin^2\theta]$ ..... $(\div ab)$

$\therefore \frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

Equation of the Normal

$y-\ b\sin\theta=m_2(x-a\cos\theta)$ ..... (2)

Obtaining slope, m_2

$m_2=-\frac{1}{\text{Slope of the tangent at }(a\cos\ \theta\ ,b\sin\ \theta)}=-\frac{1}{m_1}=\frac{a\sin\ \theta}{b\cos\ \theta}$

$y-\ b\sin\theta=\frac{a\sin\ \theta}{b\cos\ \theta}(x-a\cos\theta)$

$yb\cos\theta - b^2\cos\theta\sin\theta = a\sin\theta\, x - a^2\cos\theta\sin\theta$

$\therefore a\sin\theta\, x - by\cos\theta = (a^2 - b^2)\cos\theta\sin\theta$ ..... $(\div \cos\theta\sin\theta)$

$\Rightarrow a\frac{\sin\theta}{\cos\theta\sin\theta}x - by\frac{\cos\theta}{\cos\theta\sin\theta} = a^2 - b^2$

$\therefore ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2$

To show that the parametric equations $x = a\cos\theta$, $y = b\sin\theta$ represent an equation of an ellipse

$x = a\cos\theta \quad \Rightarrow \cos\theta = \frac{x}{a}$

$y = b\sin\theta \quad \Rightarrow \sin\theta = \frac{y}{b}$

Using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ to eliminate parameter θ

$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \therefore \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which is an ellipse.

General Examples

1. a) *A conic section is given by $x = 4\cos\theta$ and $y = 3\sin\theta$. Show that the conic is an ellipse. Determine its eccentricity*

b) *Given that the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that $c^2 = a^2m^2 + b^2$. Hence determine the equations of the tangents to at the point $(-3, 3)$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$*

Solution

a) $x = 4\cos\theta \quad \Rightarrow \cos\theta = \frac{x}{4}$

$y = 3\sin\theta \Rightarrow \sin\theta = \frac{y}{3}$

From $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \quad \therefore \ \frac{x^2}{16} + \frac{y^2}{9} = 1$, which is an ellipse.

Eccentricity

Using $b^2 = a^2(1 - e^2)$

$\Rightarrow 9 = 16(1 - e^2)$

$\Rightarrow e^2 = +\frac{\sqrt{7}}{4}$

b) Left as an exercise. See notes and the equation of tangents to the ellipse.
Ans: $y = -3, 7y = 18x - 25$

2. a) *Find the equation of the tangent and normal to the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ at the point $P(2\cos\theta\ , \sin\theta)$*

b) *if the tangent in (a) cuts the $y-$ axis at point A and the $x-$ axis at point B, and the normal cuts the $x-$ axis at point C, find the coordinates of A, B and C.*

Solution

a) **Tangent Equation**

$y - \sin\theta = m_1(x - 2\cos\theta)$ ..... (1)

Obtaining slope, m_1

$m_1 = \frac{dy}{dx}\Big|_{at\ (2\cos\theta, \sin\theta)}$

Differentiating $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ w.r.t x

$\frac{2x}{4} + 2y\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{x}{4y}$

$\Rightarrow m_1 = \frac{dy}{dx}\Big|_{at\ (2\cos\theta, \sin\theta)} = -\frac{2\cos\theta}{4\sin\theta} = -\frac{\cos\theta}{2\sin\theta}.$

$\therefore m_1 = -\frac{\cos\theta}{2\sin\theta}$

Tangent equation becomes:

$y - \sin\theta = -\frac{\cos\theta}{2\sin\theta}(x - 2\cos\theta)$

$\Rightarrow 2y\sin\theta - \sin^2\theta = -b\cos\theta\, x + 2\cos^2\theta$

$\Rightarrow 2y\sin\theta + x\cos\theta = 2(\cos^2\theta + \sin^2\theta)$

$\therefore 2y\sin\theta + x\cos\theta = 2 \qquad [\because \cos^2\theta + \sin^2\theta]$

Equation of the Normal

$y - \sin\theta = m_2(x - 2\cos\theta)$ ..... (2)

Obtaining slope, m_2

$m_2 = -\frac{1}{\text{Slope of the tangent at } (2\cos\theta, \sin\theta)} = -\frac{1}{m_1} = \frac{2\sin\theta}{\cos\theta}$

$y - \sin\theta = \frac{2\sin\theta}{\cos\theta}(x - 2\cos\theta)$

$y\cos\theta - \cos\theta\sin\theta = 2\sin\theta\, x - 4\cos\theta\sin\theta$

$\therefore 2\sin\theta\, x - y\cos\theta = 3\cos\theta\sin\theta$ ..... $(\div \cos\theta\sin\theta)$

$\Rightarrow a\frac{\sin\theta}{\cos\theta\sin\theta}x - by\frac{\cos\theta}{\cos\theta\sin\theta} = a^2 - b^2$

$\therefore 2x\sec\theta - y\,\text{cosec}\,\theta = 3$

b) $2y\sin\theta + x\cos\theta = 2$, $y-$ intercept $x = 0$

$\Rightarrow 2y\sin\theta = 1 \quad \therefore y = \frac{1}{2\sin\theta} = \frac{1}{2}\text{cosec}\,\theta$

$\therefore A\left(0, \frac{1}{2}\text{cosec}\,\theta\right)$

$x-$ Intercept, $y = 0$

$\Rightarrow x\cos\theta = 1 \quad \therefore x = \sec\theta$

$\therefore B(\sec\theta, 0)$

Normal: $2x\sec\theta - y\,\text{cosec}\,\theta = 3$

$x-$ Intercept, $y = 0$

$\Rightarrow 2x\sec\theta = 3 \quad \therefore x = \frac{3}{2\sec\theta} = \frac{3\cos\theta}{2}$

$\therefore C\left(\frac{3\cos\theta}{2}, 0\right)$

3. *The coordinates of a point* $P(x, y)$ *on the curve is given parametrically by the equations* $x = a\cos\theta$, $y = b\sin\theta$ *where a and b are constants and* θ *is a parameter. Find:*

i) *The Cartesian equation of the curve and identify the curve*

ii) *The equation of the tangent to the curve at the point at which the parameter* $\theta = \varphi$

iii) *The relation between φ_1 and φ_2 if the tangents at the points $(a\cos\varphi_1\ , b\sin\varphi_1)$ and $(a\cos\varphi_2\ , b\sin\varphi_2)$ are at right angles to one another.*

Solution

i) $x = a\cos\theta \quad \Rightarrow \cos\theta = \frac{x}{a}$

$y = b\sin\theta \quad \Rightarrow \sin\theta = \frac{y}{b}$

Using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ to eliminate parameter θ

$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \therefore \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which is an ellipse.

ii) Obtaining the tangent at $(a\cos\varphi\ , b\sin\varphi)$ on to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, see notes. Try to obtain the equation as:

$ay\sin\varphi + bx\cos\varphi = ab$

iii) For tangents to intersect at right angles, then $m_1 \times m_2 = -1$

$m_1 = \left.\frac{dy}{dx}\right|_{\text{at } (a\cos\varphi_1\ , b\sin\varphi_1)} = -\frac{b\cos\varphi_1}{a\sin\varphi_1}$

$m_2 = \left.\frac{dy}{dx}\right|_{\text{at } (a\cos\varphi_2\ , b\sin\varphi_2)} = -\frac{b\cos\varphi_2}{a\sin\varphi_2}$

$\Rightarrow m_1 \times m_2 = -\frac{b\cos\varphi_1}{a\sin\varphi_1} \times -\frac{b\cos\varphi_2}{a\sin\varphi_2} = -1$

$\Rightarrow \frac{b^2}{a^2}.\frac{\cos\varphi_1\cos\varphi_2}{\sin\varphi_1\sin\varphi_2} = -1$ or $a^2\sin\varphi_1\sin\varphi_2 + b^2\cos\varphi_1\cos\varphi_2 = 0$

4. *Prove that the equation of the tangent at the point $(x_1\ , y_1)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$*

The tangent at the point $\left(2\cos\theta\ , \sqrt{3}\sin\theta\right)$ on the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ passes through the point $P(2\ , 1)$. Show that $\sqrt{3}\cos\theta + \sin\theta = \sqrt{3}$

Without using tables/calculator, find all the solutions to this equation which are in the range $0^0 \le \theta \le 360^0$. Hence obtain the coordinates of the point of contact Q and R, of the tangents to the ellipse from P.

Verify that the line through the origin and the point P passes through the midpoint of the line QR.

Solution

The student should derive the tangent equation $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ as in notes.

Let $(x_1\ , y_1) = \left(2\cos\theta\ , \sqrt{3}\sin\theta\right)$

$\Rightarrow \ \frac{x(2\cos\theta)}{a^2} + \frac{y(\sqrt{3}\sin\theta)}{b^2} = 1$, but $a^2 = 4\ , b^2 = 3$

$\therefore \frac{2x\cos\theta}{4} + \frac{y\sqrt{3}\sin\theta}{3} = 1$ or $\therefore \frac{x}{2}\cos\theta + \frac{y\sqrt{3}}{3}\sin\theta = 1$ is the tangent at $\left(2\cos\theta\ , \sqrt{3}\sin\theta\right)$

Point $P(2\ , 1)$lies on the tangent

$\Rightarrow \frac{2}{2}\cos\theta + \frac{\sqrt{3}}{3}\sin\theta = 1$ or $\cos\theta + \frac{\sqrt{3}}{3}\sin\theta = 1$ .....$(\times\sqrt{3})$

$\therefore \sqrt{3}\cos\theta + \sin\theta = \sqrt{3}$, as required.

Let $\sqrt{3}\cos\theta + \sin\theta = R\sin(\theta + \alpha)$

$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

Comparing

$\Rightarrow \sqrt{3} = R\ \sin\alpha\ , 1 = R\cos\alpha$

Dividing the equations

$\sqrt{3} = \tan\alpha \quad \therefore \alpha = 60^0$

Squaring and adding the equations

$3 + 1 = R^2(\sin^2\alpha + \cos^2\alpha) \quad \therefore R = 2$

$\Rightarrow \sqrt{3} = 2\ \sin(\theta + 60^0)$ or $\frac{\sqrt{3}}{2} = \sin(\theta + 60^0)$

$\Rightarrow \theta + 60^0 = 60^0\ , 120^0\ , 420^0$

$\therefore \theta = 0^0\ , 60^0\ , 360^0$

Taking $\theta = 0^0\ , 60^0$ and from the tangent equation $\frac{x}{2}\cos\theta + \frac{y\sqrt{3}}{3}\sin\theta = 1$

For $\theta = 0^0$, $\frac{x}{2} = 1 \quad \therefore x = 2$

For $\theta = 60^0$, $\frac{x}{2}\times\frac{1}{2} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{3}y = 1 \quad \therefore x + 2y = 4$

From the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$, for $x = 2$, $\frac{y^2}{3} = 0\ \therefore y = 0$

$\therefore$ Point of contact is $(2\ , 0)$

From $x + 2y = 4$, $\frac{x^2}{4} + \frac{y^2}{3} = 1$, solving the equations simultaneously

$x = 4 - 2y$ into eqn. $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$\Rightarrow \frac{(4-2y)^2}{4} + \frac{y^2}{3} = 1$, simplifying we get

$4y^2 - 12y + 9 = 0 \quad \therefore y = 3/2$

From $x + 2y = 4\ , y = \frac{3}{2}, x = 4 - 3 = 1$

$\therefore$ Point of contact is $\left(1\ , \frac{3}{2}\right)$

To prove that $\overline{OP}$ passes through midpoint of $\overline{QR}$

$O(0\ , 0)\ , P(2\ , 1)\ , Q\left(1\ , \frac{3}{2}\right)\ , R(2\ , 0)$

Equation of the line through $\overline{OP}$

$\frac{y-0}{x-0} = \frac{1-0}{2-0} \quad \therefore y = \frac{1}{2}x$

Midpoint of $\overline{QR} = \left(\frac{1+2}{2}\ , \frac{\frac{3}{2}+0}{2}\right) = \left(\frac{3}{2}\ , \frac{3}{4}\right)$, this point must satisfy the line

$y = \frac{1}{2}x$

For $x = \frac{3}{2}\ , y = \frac{3}{4} = y -$ coordinate. Hence proved as required.

Task

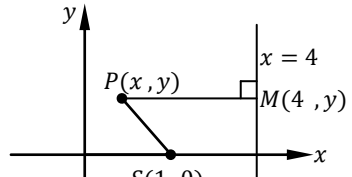
1. *Find the parametric coordinates of any point on* $4x^2 + 9y^2 = 16$. *Find also the equation of the tangent at this point.*
   Ans: $\left(2\cos\theta\ , \frac{4}{5}\sin\theta\right)$, $2x\cos\theta + 3y\sin\theta = 4$
2. *Show that the equation of the tangent to* $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ *at* $(2\cos\theta\ , 3\sin\theta)$ *is* $3x\cos\theta + 2y\sin\theta = 6$
3. *The normal at any point P to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *meets the axes in A and B. prove AB is divided externally at P in a constant ratio.*
4. S *and* S_1 *are foci of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *P is a point on the ellipse so that PS is parallel to the* $y-$*axis. The tangent at P meets the* $y-$*axis in M. prove* MS_1 *is parallel to the normal at P*
5. *The tangent at any point on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *cuts the minor axis at Q and the tangent at one end of the major axis in R. if S is the focus prove* $QR = QS$

LOCUS PROBLEMS IN ELLIPSE

1. a) *Find the locus of a point* $P(x\ , y)$ *such that twice its distance from* $S(1\ , 0)$ *is equal to its distance from the line* $x = 4$. *Identify the locus.*
   b) *Find the locus of a point* $P(x\ , y)$ *such that* $3PS = PM$ *where* $S(-3\ , 0)$ *and M is a point on the line* $x = -27$ *such that PM is parallel to* $x-$*a xis. Identify the locus.*

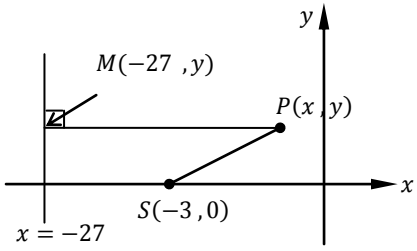
Solution

a) Using a clear diagram



Given: $2\overline{PS} = \overline{PM}$ or $4\overline{PS}^2 = \overline{PM}^2$
$\Rightarrow 4[(x-1)^2 + (y-0)^2] = (x-4)^2 + (y-y)^2$
$\Rightarrow 4x^2 - 8x + 4 + 4y^2 = x^2 - 8x + 16$
$\Rightarrow 3x^2 + 4y^2 = 12$ or $\frac{x^2}{4} + \frac{y^2}{3} = 1$, which is an ellipse

b) Using a clear diagram,



Given: $3\overline{PS} = \overline{PM}$ or $9\overline{PS}^2 = \overline{PM}^2$
$\Rightarrow 9[(x+3)^2 + (y-0)^2] = (x+27)^2 + (y-y)^2$
$\Rightarrow 9x^2 + 54x + 81 + 9y^2 = x^2 + 54x + 729$
$\Rightarrow 8x^2 + 9y^2 = 648$ or $\frac{x^2}{81} + \frac{y^2}{72} = 1$, which is an ellipse

2. *The normal at* $P(2\cos\theta\ , \sin\theta)$ *of an ellipse meets* $x-$*axis in Q and* $y-$*axis in R. A point S is taken on QR produced such that* $QS = RQ$, *prove that the locus is a circle whose equation is* $x^2 + y^2 = 9$

Solution

$P(2\cos\theta, \sin\theta)$ lies on ellipse

$x = 2\cos\theta \Rightarrow \cos\theta = \frac{x}{2}, y = \sin\theta$

Using $\cos^2\theta + \sin^2\theta = 1$

The Cartesian equation of the curve is: $\frac{x^2}{4} + \frac{y^2}{1} = 1$, which is an ellipse

Obtaining the equation of the normal.

$y - \sin\theta = m_1(x - 2\cos\theta)$ ..... (1)

$m_1 = -\frac{1}{\frac{dy}{dx}\Big|_{\text{at } P(2\cos\theta, \sin\theta)}}$

By chain rule, $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

For $x = 2\cos\theta \Rightarrow \frac{dx}{d\theta} = -2\sin\theta$

For $y = \sin\theta \Rightarrow \frac{dy}{d\theta} = \cos\theta$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \cos\theta \times \frac{1}{-2\sin\theta}$

$\Rightarrow m_1 = -\frac{1}{\frac{dy}{dx}\Big|_{\text{at } P(2\cos\theta, \sin\theta)}} = \frac{2\sin\theta}{\cos\theta}$

$\Rightarrow y - \sin\theta = \frac{2\sin\theta}{\cos\theta}(x - 2\cos\theta)$

$\Rightarrow y\cos\theta - \sin\theta\cos\theta = 2x\sin\theta - 2\sin\theta\cos\theta$

$\therefore 2x\sin\theta - y\cos\theta = 3\sin\theta\cos\theta$, which is the normal at P

For x −intercept, $y = 0$

$2x\sin\theta = 3\sin\theta\cos\theta \Rightarrow x = \frac{3}{2}\cos\theta$

$\therefore Q\left(\frac{3}{2}\cos\theta, 0\right)$

For y −intercept, $x = 0$

$-y\cos\theta = 3\sin\theta\cos\theta \Rightarrow y = -3\sin\theta$

$\therefore R(0, -3\sin\theta)$

$R(0, -3\sin\theta)$ $\quad Q\left(\frac{3}{2}\cos\theta, 0\right)$ $\quad S(X, Y)$

Q is the midpoint of RS

$\Rightarrow \left(\frac{3}{2}\cos\theta, 0\right) = \left(\frac{X+0}{2}, \frac{-3\sin\theta + Y}{2}\right)$

$\Rightarrow \frac{X}{2} = \frac{3}{2}\cos\theta \quad \therefore \cos\theta = \frac{X}{3}$ ..... (1)

$\Rightarrow \frac{-3\sin\theta + Y}{2} = 0 \quad \therefore \sin\theta = \frac{Y}{3}$ ..... (2)

Using $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{9} = 1 \quad \therefore x^2 + y^2 = 9$

3. *The normal at the point $P(5\cos\theta, 4\sin\theta)$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ meets the major and minor axes at L and M respectively. Show that the locus of R, the midpoint of LM is an ellipse having the same eccentricity as the given ellipse.*

Solution

The equation of the normal *at the point* $P(5\cos\theta, 4\sin\theta)$ *is:*

$5x\sec\theta - 4y\,\text{cosec}\,\theta = 9$ ..... (1)

Let the student look for this equation.

The major axis is the x −axis. So L is the x − intercept

For $y = 0$, $5x\sec\theta = 9 \quad \therefore x = \frac{9\cos\theta}{5}$

$\therefore L\left(\frac{9\cos\theta}{5}, 0\right)$

The minor axis is the y −axis. So M is the y − intercept

For $x = 0$, $-4y\,\text{cosec}\,\theta = 9 \quad \therefore y = -\frac{9\sin\theta}{4}$

$\therefore M\left(0, -\frac{9\sin\theta}{4}\right)$

Let $R(X, Y)$ be the midpoint of LM

$R(X, Y) = R\left(\frac{\frac{9\cos\theta}{5}+0}{2}, \frac{0-\frac{9\sin\theta}{4}}{2}\right)$

$\therefore X = \frac{9\cos\theta}{10} \Rightarrow \cos\theta = \frac{10X}{9}$ ..... (2)

$\therefore Y = -\frac{9\sin\theta}{8} \Rightarrow \sin\theta = -\frac{8Y}{9}$ ..... (3)

Using $\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{10X}{9}\right)^2 + \left(-\frac{8Y}{9}\right)^2 = 1$ or $\frac{X^2}{81/100} + \frac{Y^2}{81/64} = 1$

Required locus is: $\frac{x^2}{81/100} + \frac{y^2}{81/64} = 1$

The locus is an ellipse with its major axis as the y − axis

Calculating Eccentricity for $\frac{x^2}{81/100} + \frac{y^2}{81/64} = 1$

Using $a^2 = b^2(1-e^2)$

$\Rightarrow \frac{81}{100} = \frac{81}{64}(1-e^2)$

$\Rightarrow e^2 = \frac{36}{100} \quad \therefore e = +\frac{3}{5}$

Calculating Eccentricity for $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Using $b^2 = a^2(1-e^2)$

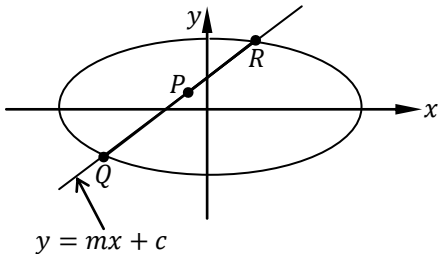
$\Rightarrow 16 = 25(1-e^2)$

$\Rightarrow e^2 = \frac{9}{16} \quad \therefore e = +\frac{3}{5}$

The eccentricity for the required locus is the same as that of the given locus. i.e. $e = +\frac{3}{5}$

3. *A variable straight line with constant gradient m meets the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *at* Q, R. *find the locus of P the midpoint of QR.*

Solution



Let $Q(x_1, y_1)$ and $R(x_2, y_2)$ be the points of intersection of the line and the ellipse.
Let $y = mx + c$ be the line.
(c is unkown, m is known)
Solving the equations simultaneously,

$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$, simplifying

$x^2(a^2m^2 + b^2) + (2mca^2)x + a^2(c^2 - b^2) = 0$ ..... (1)

This equation is quadratic in x, which gives the $x-$ coordinates of points of intersection for Q and R

Using sum of roots

$x_1 + x_2 = -\frac{2mc\,a^2}{a^2m^2+b^2}$ ..... (2)

Let the midpoint of QR be $P(X, Y)$

$P(X, Y) = P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$\therefore X = -\frac{2mc\,a^2}{a^2m^2+b^2} \times \frac{1}{2} = -\frac{mc\,a^2}{a^2m^2+b^2}$ ..... (3)

In a similar way, solving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $y = mx + c$ simultaneously we obtain

$\frac{\left(\frac{y-c}{m}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$, simplifying

$y^2(a^2m^2 + b^2) - (2cb^2)y + b^2(c^2 - a^2m^2) = 0$ ..... (4)

Using sum of roots

$y_1 + y_2 = \frac{cb^2}{a^2m^2+b^2}$ ..... (5)

$\therefore Y = \frac{2cb^2}{a^2m^2+b^2} \times \frac{1}{2} = \frac{cb^2}{a^2m^2+b^2}$ ..... (6)

Since c is unknown eliminating it form equations (3) and (6) to obtain the locus of P

$\Rightarrow \frac{X}{Y} = -\frac{mc\,a^2}{a^2m^2+b^2} \div \frac{cb^2}{a^2m^2+b^2} = \frac{-ma^2}{b^2}$

$\Rightarrow b^2X = -ma^2Y$ or $b^2X + ma^2Y = 0$

$\therefore$ Required locus is: $b^2x + ma^2y = 0$

4. *A variable tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at $T(4\cos\theta, 2\sin\theta)$ meets the parabola $y^2 = 4ax$ at L and M. Find the locus of the midpoint of LM.*

Solution

Tangent at $T(4\cos\theta, 2\sin\theta)$ is $x\cos\theta + 2y\sin\theta = 4$ ..... (1)
(Student, look for this equation)

Solving this equation and parabola $y^2 = 4ax$ simultaneously to obtain points of intersection L and M

$x = \frac{2y \sin\theta - 4}{\cos\theta}$ into $y^2 = 4ax$

$\Rightarrow y^2 = 4a\left(\frac{2y \sin\theta - 4}{\cos\theta}\right)$

$y^2 \cos\theta + 8a \sin\theta\ y - 16a = 0$..... (2)

This equation is quadratic in y, which gives the $y-$ coordinates of points of intersection for L and M

Using sum of roots

$y_1 + y_2 = -\frac{8a \sin\theta}{\cos\theta}$ ..... (3)

Let the midpoint of QR be $P(X, Y)$

In a similar way, solving $x\cos\theta + 2y\sin\theta = 4$ and $y^2 = 4ax$ simultaneously we obtain:

$y = \frac{4 - x\cos\theta}{2\sin\theta}$

$\Rightarrow \left(\frac{4 - x\cos\theta}{2\sin\theta}\right)^2 = 4ax$ or $16 - 8x\cos\theta + x^2\cos^2\theta = 16a\sin^2\theta\ x$

$x^2\cos^2\theta - x(8\cos\theta + 16a\sin^2\theta) + 16 = 0$

$x_1 + x_2 = \frac{8\cos\theta + 16a\sin^2\theta}{\cos^2\theta}$ ..... (4)

Let the points of intersection be:

$L(x_1, y_1)$ and $M(x_2, y_2)$

$\Rightarrow P(X, Y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$\Rightarrow 2X = \frac{8\cos\theta + 16a\sin^2\theta}{\cos^2\theta} = 8\sec\theta + 16a\tan^2\theta$

$\therefore 2X = 8\sec\theta + 16a\tan^2\theta$ ..... (5)

$\therefore 2Y = -8a\tan\theta$ ..... (6)

From eqn. (5) $2X = 8\sec\theta + 16a\tan^2\theta$

$\Rightarrow (X - 8a\tan^2\theta) = 4\sec\theta$, squaring both sides

$\Rightarrow (X - 8a\tan^2\theta)^2 = 16\sec^2\theta = 16(1 + \tan^2\theta)$..... (7)

From eqn. (6) $Y = -4a\tan\theta \Rightarrow \tan\theta = \frac{Y}{-4a}$..... (8)

Eqn. (8) into eqn. (7)

$\Rightarrow \left(X - 8a\left(\frac{Y}{-4a}\right)^2\right)^2 = 16\left(1 + \left(\frac{Y}{-4a}\right)^2\right)$

$\Rightarrow (16a^2X - 8aY^2)^2 = 16(16a^2 + Y^2)$

$\therefore 4a^2(2aX - Y^2)^2 = 16a^2 + Y^2$

Required locus: $4a^2(2ax - y^2)^2 = 16a^2 + y^2$

Task

1) *The tangent at any point P on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *cuts the minor axis at Q and the tangent at one end of the major axis in R. is S is a focus prove that* $QR = QS$

2) *A perpendicular OY is drawn from the origin to any tangent to* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. *Prove the locus of Y is* $x^2+y^2=a^2x^2+b^2y^2$

3) *The normal at point P on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *cuts the major axis at G. show that the locus of the midpoint of PG is an ellipse.*

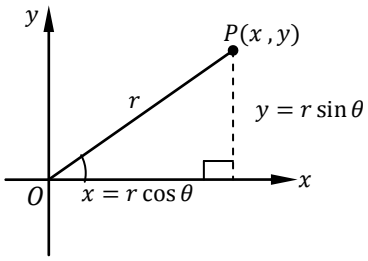
Ans: $\frac{4a^2x^2}{(2a^2-b^2)^2}+\frac{4y^2}{b^2}=1$

4) *The tangent at any point P on* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *cuts Ox in T. if PN is the ordinate of P prove that* $ON.OT=a^2$

5) *The normal at the point* $P(a\cos\theta\ ,b\sin\theta)$ *on* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *meets the* $y-$*axis at Q. if S is either focus of the ellipse, prove* $\frac{SQ}{PQ}$ *is independent of* θ

POLAR COORDINATES

These specify the position of a point in a plane.
Consider a point $P(x\ ,y)$ having coordinates in vector form. i.e.
Rectangular Coordinates



$OP=r$
The ordered pair of numbers $(r\ ,\theta)$ is called the *Polar Coordinate of P*
The equations $x=r\cos\theta$ and $y=r\sin\theta$ help in conversion of polar coordinates to rectangular coordinates.
Given $r\ ,\theta$ the $x\ ,y$ can be found using

$$x=r\cos\theta\ ,y=r\sin\theta$$

Conversely if x,y are given, then

$$r^2=x^2+y^2,\ \tan\theta=\frac{y}{x}$$

Angle θ is determined for $r=+\sqrt{x^2+y^2}$ and should lie in the interval $-\pi<\theta<\pi$

NOTE

If $r=+\sqrt{x^2+y^2}$, then
$\theta=\tan^{-1}\left(\frac{y}{x}\right)\ ;x>0$
$\theta=\tan^{-1}\left(\frac{y}{x}\right)+\pi\ ;\ x<0$

Examples

1) a) *Find the rectangular coordinates of* $(3\,,150^0)$

b) Find the polar coordinates of $(2\,,-2)$ *for* $r>0$

Solution

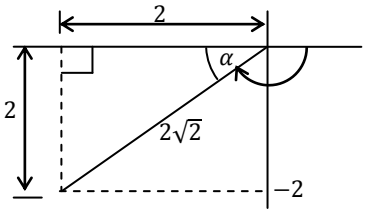
a) Polar coordinates: $(r\,,\theta)=(3\,,150^0)$

$\Rightarrow r=3\,,\ \theta=150^0$

Using $x=r\cos\theta\ ,\ y=r\sin\theta$

$x=3\cos 150^0=\left(-\frac{3\sqrt{3}}{2}\,,\frac{3}{2}\right)$

b) Polar coordinates: $(r\,,\theta)$ =?



$\tan\alpha=\frac{2}{2}=1 \quad \therefore \alpha=45^0$

$\therefore \theta=-135^0$

$r=\sqrt{x^2+y^2}=\sqrt{2^2+2^2}$

$=2\sqrt{2}$

$\therefore (r\,,\theta)=\left(2\sqrt{2}\,,-135^0\right)$

2. Find

i) The Cartesian coordinates of the point whose polar coordinates are $\left(5\,,\frac{\pi}{4}\right)$

ii) The polar coordinates whose Cartesian coordinates are $(-1\,,1)$

Solution

a) Cartesian coordinates: $x=r\cos\theta\ ,y=r\sin\theta$

Polar coordinates $\left(5\,,\frac{\pi}{4}\right)=(r\,,\theta)$

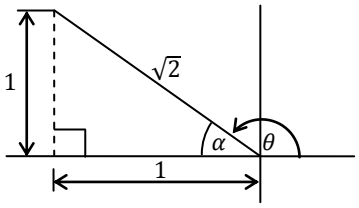
$r=5\,,\ \theta=\frac{\pi}{4}$

$\therefore$ Cartesian coordinates:

$x=5\cos\frac{\pi}{4}=\frac{5}{\sqrt{2}}\,,\ y=5\sin\frac{\pi}{4}=\frac{5}{\sqrt{2}}$

$\therefore \left(\frac{5}{\sqrt{2}}\,,\frac{5}{\sqrt{2}}\right)$

b)



$\tan\alpha=\frac{1}{1}=1 \quad \therefore \alpha=45^0=\frac{\pi}{4}$

$\therefore \theta=180^0-45^0=135^0=\frac{3\pi}{4}$

Polar coordinates:

$(r\,,\theta)=\left(\sqrt{2}\,,\frac{3\pi}{4}\right)$

θ

Expressing Equations in terms of Polar Coordinates

Expressing the equations in polar coordinates

a) $x = 3$ c) $x^2 + y^2 = 16$ e) $y^2 = 4(x + 3)$

b) $2x - y = 3$ d) $\frac{x^2}{4} + y^2 = 1$

Solution

Using $x = r\cos\theta$, $y = r\sin\theta$

a) $x = 3 \Rightarrow r\cos\theta = 3$

b) $2x - y = 3 \Rightarrow 2r\cos\theta - r\sin\theta = 3$

$\therefore r(2\cos\theta - \sin\theta) = 3$

c) $x^2 + y^2 = 16 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 16$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 16 \quad \therefore r = 4$

d) $\frac{x^2}{4} + y^2 = 1 \Rightarrow \frac{r^2\cos^2\theta}{4} + r^2\sin^2\theta = 1$

$\therefore r^2(\cos^2\theta + 4\sin^2\theta) = 4$

e) $y^2 = 4(x + 3) \Rightarrow r^2\sin^2\theta = 4(r\cos\theta + 3)$

$r^2\sin^2\theta = 4(r\cos\theta + 3)$

Task

1. (I*) Find the coordinates of the points where the ellipse* $\frac{x^2}{4} + \frac{y^2}{9} = 1$ *cuts the axes*

i) Express the equation in poplar form

ii) *If the line* $y = mx + c$ *is a tangent to the ellipse, show that* $c^2 = 4m^2 + 9$

2. *Find the Cartesian equations corresponding to*

(i) $r = a\sec\theta$ (ii) $r^2\sin 2\theta = a^2$ **Ans:** (i) $x = a$ (ii) $2xy = a^2$

3. *Find the polar equations corresponding to:*

(i) $(x^2 + y^2)^2 = 2a^2xy$ (ii) $(x^2 + y^2)^2 = a^2(x^2 + y^2)$

(iii) $x(x^2 + y^2)ay^2$ (iv) $y^2(2a - x) = x^3$

Ans: (i) $r^2 = a^2\cos 2\theta$ (ii) $r = a\sin\theta\tan\theta$ (iii) $r^2 = a^2\sin 2\theta$

(iv) $r = 2a\sin^2\theta\sec\theta$

THE HYPERBOLA

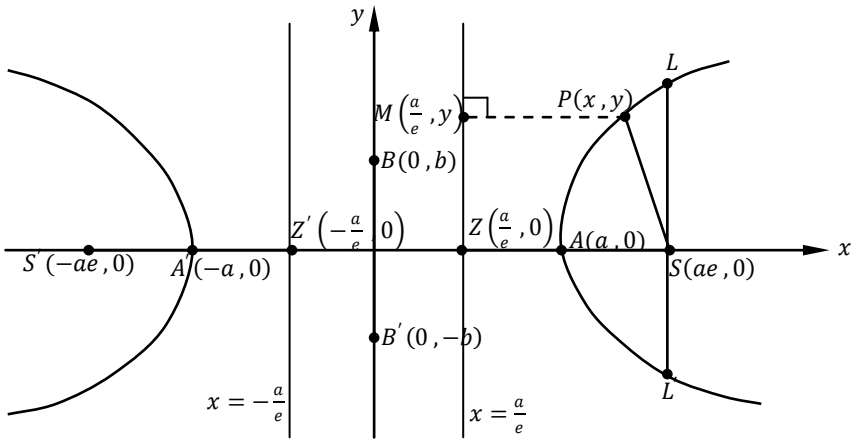
Definition:

Is the locus of a point which moves in such a way that its distance from a fixed point (Focus) bears a constant ratio, $e > 1$ to its distance from a fixed line (Directrix)

Standard Equation of a Hyperbola

To Find the equation of a Hyperbola in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Consider the figure below.



A and A' divide $\overline{SZ}$ internally and externally in the ratio $\overline{SA}:\overline{AZ} = e:1$

From the definition,

$\overline{PS}:\overline{PM} = e:1 \quad \therefore \overline{PS} = e\,\overline{PM}$ or $\overline{PS}^2 = e^2\,\overline{PM}^2$ or $\overline{AS}:\overline{AZ} = e:1$

$\overline{PS} = \sqrt{(x-ae)^2 + (y-0)^2} = \sqrt{x^2 - 2ae\,x + a^2e^2e^2}$

$\overline{PM} = \sqrt{\left(x - \frac{a}{e}\right)^2 + (y-y)^2} = \sqrt{x^2 - \frac{2a}{e}\,x + \frac{a^2}{e^2}}$

$\Rightarrow x^2 - 2ae\,x + a^2e^2 = e^2\left(x^2 - \frac{2a}{e}\,x + \frac{a^2}{e^2} + y^2\right)$

$\Rightarrow x^2 - 2ae\,x + a^2e^2 + y^2 = e^2x^2 - 2ae\,x + a^2$

$\Rightarrow x^2(e^2-1) - y^2 = a^2(e^2-1)$, dividing through by $a^2(e^2-1)$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$

Let $b^2 = a^2(e^2-1)$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ , } b^2 = a^2(e^2-1)}$$

Important Observations about the Standard Equation of a Hyperbola

1. $A(a\,,0)$ and $A'(-a\,,0)$ are its vertices and line $\overline{AA'}$ is called the *Transverse Axis* of a hyperbola.
2. Line $\overline{BB'}$ .i.e. $B(0\,,b)$ and $B'(0\,,-b)$ is called the *Conjugate Axis*, thus $\overline{AA'}$ and $\overline{BB'}$ are axes of a hyperbola.
3. Foci are: $S(ae\,,0)$ and $S'(-ae\,,0)$
4. Equation of Directrices are: $x = \pm\frac{a}{e}$
5. If $(x_1\,,y_1)$ lies on the curve, so is the point $(-x_1\,,-y_1)$. But $(x_1\,,y_1)$ and $(-x_1\,,-y_1)$ are points on the line through the origin. Thus every chord through the origin is bisected at O. thus O is the *Centre* of the curve.
6. Since the powers of x and y are even, the curve is symmetrical about the coordinate axes.
7. When $x \to \infty$ also $y \to \infty$
8. The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be written as $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$. When $x < a$, y is imaginary hence no part of the curve lies between the lines $x = \pm a$
9. The chord through any focus and perpendicular to the transverse axis is called *Latus Rectum.*
   From the figure, line LSL' is latus rectum.

   Obtaining the coordinates at the end points of latus rectum

   From $x = ae$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ becomes

   $\frac{a^2e^2}{a^2} - \frac{y^2}{b^2} = 1 \;\Rightarrow y^2 = b^2(e^2 - 1)$

   $\therefore y = \pm\sqrt{b^2(e^2 - 1)}$

   But $b^2 = a^2(e^2 - 1)$ or $e^2 - 1 = \frac{b^2}{a^2}$

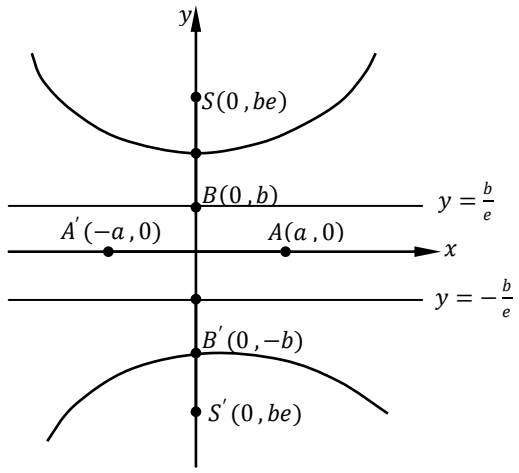
   $\Rightarrow y = \pm\sqrt{\frac{b^2}{a^2}.b^2} = \pm\frac{b^2}{a}$

   $\therefore L\left(ae\,,\frac{b^2}{a}\right), S(ae\,,0), L'\left(ae\,,-\frac{b^2}{a}\right)$

   Length of Latus Rectum $\overline{LL'} = \frac{2b^2}{a}$
10. Equations of latus recta are $x = \pm ae$

CONJUGATE HYPERBOLA

The equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is another standard equation of a hyperbola



Observations

1. $\overline{AA'}$ is the conjugate axis, $\overline{BB'}$ is the transverse axis
2. Vertices are: $(0\,,\pm b)$
3. Directrices are: $y=\pm\frac{b}{e}$
4. Foci are: $(0\,,\pm be)$
5. Length of Latus rectum= $\frac{2a^2}{b}$
6. Equations of latus recta: $y=\pm be$
7. For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$, $a^2=b^2(e^2-1)$

Illustrative Examples

1. *Find the equation of the hyperbola with:*

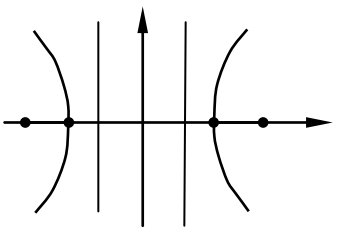
i) Foci $(\pm5\,,0)$ and equation of Directrices $x=\pm\frac{9}{5}$

ii) Foci $(0\,,\pm5)$ and vertices are $(0,\pm4)$

iii) Foci $(\pm6\,,0)$ and vertices are $(\pm3,0)$

Solution

i)



Since the foci are on the $x-$axis, then $y=0$ is a transverse axis of hyperbola. Hence using $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Foci: $(\pm ae\,,0)=(\pm5\,,0)\ \therefore ae=5$..... (1)

Directrices: $x=\pm\frac{a}{e}=\pm\frac{9}{5}\ \therefore\frac{a}{e}=\frac{9}{5}$..... (2)

Solving simultaneously eqn. (1) × eqn. (2)

$ae\times\frac{a}{e}=\frac{9}{5}\times5\qquad\therefore a^2=9$

Also using $b^2=a^2(e^2-1)$

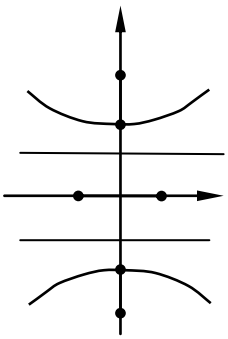
$\Rightarrow b^2=9(e^2-1)$

From eqn. (1), $e=\frac{5}{a}=\frac{5}{3}$

$\therefore b^2=9\left(\frac{25}{9}-1\right)=16$

∴Required equation of hyperbola is: $\frac{x^2}{9}-\frac{y^2}{16}=1$

ii)



Since the transverse axis is through the foci points, the equation of the required hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

Given: Foci: $(0\,,\pm5)=(0\,,\pm be)\ \therefore be=5$..... (1)

Vertices: $(0,\pm4)=(0\,,\pm b)\ \ \therefore b=4$..... (2)

Solving eqn. (1) and eqn. (2) simultaneously $e=\frac{5}{4}$

Using $a^2=b^2(e^2-1)$

$a^2=b^2(e^2-1)=16\left(\frac{25}{16}-1\right)=9$

$\therefore$Required equation of hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=-1$

iii) Left as an exercise. **Ans:** $\frac{x^2}{9}-\frac{y^2}{27}=-1$

2. *Find the eccentricity, foci, equation of Directrices, centre of the hyperbola*

i) $16x^2-9y^2=144$ iii) $9x^2-16y^2+18x+32y-151=0$

ii) $x^2-y^2=32$ iv) $9x^2-16y^2+144=0$

Solution

i) $16x^2-9y^2=144$, rewriting the equation in standard form $(\div 144)$

$\Rightarrow \frac{x^2}{9}-\frac{y^2}{16}=1\ \ \ \therefore a^2=9\,,\ b^2=16\,,\ b^2=a^2(e^2-1)$

$\Rightarrow 16=9(e^2-1)\ \ \therefore e=5/3$

Foci: $(\pm ae\,,0)=\left(\pm3\times\frac{5}{3}\,,0\right)=(\pm5\,,0)$

Directrices: $x=\pm\frac{a}{e}=\pm\frac{3}{5/3}=\pm\frac{9}{5}\qquad\therefore x=\pm\frac{9}{5}$

Centre: $(0\,,0)$

ii) Left as an exercise.

Ans: Foci: $(\pm8\,,0)$, Directrices: $x=\pm4$, centre: $(0\,,0)$

iii) $9x^2-16y^2+18x+32y-151=0$

$(9x^2+18x)+(-16y^2+32y)=151$, completing squares,

$9(x^2+2x+1)-16(y^2-2y+1)=151+9-16$

$9(x+1)^2-16(y-1)^2=144$, dividing through by 144

$\frac{(x+1)^2}{16}-\frac{(y-1)^2}{9}=1$, which is of the form $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$

$\therefore X=x+1\,,Y=y-1\,,\ a^2=16\,,\ b^2=9$

But for hyperbola $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$

Foci: $(\pm Ae\,,\ 0)=(X\,,Y)$

$\Rightarrow Ae=X=x+1\,,\ Y=0=y-1\ \ \therefore y=1$

Also $b^2=a^2(e^2-1)$

$9=16(e^2-1)\ \ \therefore e=\frac{5}{4}$

$\therefore Ae=4\times\frac{5}{4}=x+1\ \Rightarrow x=4$

$\therefore$Foci: $(\pm 4\,,1)$

Directrices: $X = \pm\frac{A}{e} = \pm\frac{4}{5/4} = \pm\frac{16}{5}$

$\Rightarrow x + 1 = \pm\frac{16}{5}$ or $x = -\frac{21}{5}\,,\frac{11}{5}$

$\therefore$ Directrices are $5x - 11 = 0\,,\ 5x + 21 = 0$

Centre: $(0\,,0) = (X\,,Y)$

$X = x + 1 = 0 \ \ \therefore x = -1$

$Y = 0 = y - 1 \ \ \therefore y = 1$

$\therefore$ Centre: $(-1\,,1)$

iv) Left as an exercise. **Ans:** Foci: $(0\,,\pm 5)$, Directrices: $y = \pm\frac{9}{5}$, centre: $(0\,,0)$

NOTE

Most results of ellipse obtained earlier for the standard equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ also hold pretty well for the hyperbola in standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

When b^2 is changed into $-b^2$, the method of proof being exactly the same, the results are as follows:

A LINE AND A HYPERBOLA

j) **To Find the Point of Intersection of the Line $y = mx + c$ and the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$**

Solving the two equations simultaneously, obtain

$(b^2 - a^2m^2)x^2 - (2mca^2)x - (a^2c^2 - a^2b^2) = 0$…… (1)

This equation gives two values of x, hence two points of intersection.

k) **To Show that a given Line $y = mx + c$ is a Tangent to the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$**

Approach I:

First solve the two equations simultaneously to obtain

$(b^2 - a^2m^2)x^2 - (2mca^2)x - (a^2c^2 - a^2b^2) = 0$…… (1)

Applying the condition for tangency $B^2 = 4AC$ to obtain

$c^2 = a^2m^2 - b^2$ or $c = \pm\sqrt{a^2m^2 - b^2}$…… (2)

Approach II:

Solving eqn. (1) for point of contact

Eqn. (2) into eqn. (1) to obtain

$(a^2m^2 - b^2)x^2 \pm \left(2ma^2\sqrt{a^2m^2 - b^2}\right)x + a^4m^2 = 0$

$\left(x\sqrt{a^2m^2 - b^2} \pm ma^2\right)^2 = 0 \ \ \therefore x = \pm\frac{ma^2}{\sqrt{a^2m^2 - b^2}}$, which will be a repeated value. Hence the line is a tangent to the ellipse

l) **To obtain the Point of Contact of a tangent line $y = mx + c$ to the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$**

Using $x = \pm\frac{ma^2}{\sqrt{a^2m^2 - b^2}}$ into $y = mx + c$, obtain point of contact as:

∴Point of contact is $\boldsymbol{P}\left(\frac{\boldsymbol{ma^2}}{\sqrt{\boldsymbol{a^2m^2 - b^2}}}, -\frac{\boldsymbol{b^2}}{\sqrt{\boldsymbol{a^2m^2 - b^2}}}\right)$ for $x = +\frac{ma^2}{\sqrt{a^2m^2 - b^2}}$, $c = -\sqrt{a^2m^2 - b^2}$

∴ Point of contact is $\boldsymbol{P'}\left(-\frac{\boldsymbol{ma^2}}{\sqrt{\boldsymbol{a^2m^2 - b^2}}}, \frac{\boldsymbol{b^2}}{\sqrt{\boldsymbol{a^2m^2 - b^2}}}\right)$ for $x = -\frac{ma^2}{\sqrt{a^2m^2 - b^2}}$, $c = +\sqrt{a^2m^2 - b^2}$

m) **To state the Equation of the Tangent to the parabola in Standard Form**

From the line $y = mx + c$ and condition for tangency $c = \pm\sqrt{a^2m^2 - b^2}$, the tangent in slope form is $\boldsymbol{y = mx \pm \sqrt{a^2m^2 - b^2}}$

EQUATION OF A TANGENT AND NORMAL AT ANY POINT (x_1, y_1) ON A HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

To Find an Equation of a Tangent at any point $P(x_1, y_1)$ on the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Proceeding as in ellipse, the tangent equation is:

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

To Find an Equation of the Normal at any point $P(x_1, y_1)$ on the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Proceeding as in ellipse obtain

$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

To Find an Equation of a Chord with the Midpoint $P(x_1, y_1)$ on the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Follow the steps in ellipse for this question to obtain

$\frac{x^2x_1}{a^2} - \frac{y^2y_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

General Examples

Find the equation of the tangent and the normal to the

a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at $(0, 3)$

b) $4x^2 - 9y^2 = 36$ at $\left(2, \frac{4\sqrt{5}}{3}\right)$

Solution

a) Required tangent: $y-3=m(x-0)=mx$..... (1)

$m=\left.\frac{dy}{dx}\right|_{at\ (0,3)}$

$\frac{2x}{16}-\frac{2y}{9}\frac{dy}{dx}=0 \quad \Rightarrow \frac{dy}{dx}=\frac{9x}{16y}$

$\therefore m=\frac{0}{48}=0$

∴Required tangent is $y-3=0$

Required normal: $y-3=m'(x-0)=m'x$..... (2)

$m'=-\frac{1}{m}=-\frac{1}{0}$

Rewriting the required normal

$\frac{y-3}{x}=m'=-\frac{1}{0}$

∴Required tangent is $x=0$

b) Left as an exercise. **Ans:** $2x-3\sqrt{5}y+16=0\,;9\sqrt{5}x+6y+10\sqrt{5}=0$

ASSYMPTOTES

Definition:

Are lines which meet the curve in two points at infinity but does not wholly lie at infinity.

To Find the Asymptotes of Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Two methods can be used.

Method I:

Let $y=mx+c$ be the asymptote.

Solving $y=mx+c$ and the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ obtain:

$(b^2-a^2m^2)x^2-(2mca^2)x-(a^2c^2-a^2b^2)=0$

Equating coefficients of higher powers of x to zero

For x^2: $b^2-a^2m^2=0 \ \Rightarrow m=\pm\frac{b}{a}$

For x: $2mca^2=0 \quad \Rightarrow c=0$

Equation of asymptotes: $\boxed{y=\pm\frac{b}{a}x}$

Method II:

From $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$..... $(\times\ b^2x^2)$

$\frac{b^2}{a^2}-\frac{y^2}{x^2}=\frac{b^2}{x^2}$ or $\frac{y^2}{x^2}=\frac{b^2}{a^2}-\frac{b^2}{x^2}$

As $x\to\pm\infty$, $\frac{y^2}{x^2}\to\frac{b^2}{a^2}$ or $\frac{y}{x}\to\pm\frac{b}{a}$

$\boxed{\therefore y=\pm\frac{b}{a}x}$

General Equation of Asymptotes of Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

From $y=\pm\frac{b}{a}x$ or $\frac{y}{b}=\pm\frac{x}{a}$

$\Rightarrow \frac{x}{a}-\frac{y}{b}=0$, $\frac{x}{a}+\frac{y}{b}=0$

Multiplying he equations

$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=0$

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=0}$$

Relationship between Equation of Hyperbola and its Asymptotes

The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ differs from its asymptotes $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ by a constant. i.e. 1, and also the equation of a conjugate hyperbola differ from its asymptotes by the same constant.

Generally if $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then the equation of asymptotes combined is:

$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)=0$

Then the equation of the hyperbola which differs from that of asymptotes by a constant k is:

$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)=k$

Example

1. *Find the equation of the hyperbola passing through the point* $(4\,,6)$ *and whose asymptotes are* $y=\pm\sqrt{3}x$

Solution

$y=\pm\sqrt{3}x$ or $\sqrt{3}x-y=0$ and $\sqrt{3}x+y=0$

Combined equation of asymptotes:

$\left(\sqrt{3}x+y\right)\left(\sqrt{3}x-y\right)=0 \;\Rightarrow 3x^2-y^2=0$

Equation of hyperbola differs from that of asymptotes by a constant

$\Rightarrow 3x^2-y^2=k$

Now $(4\,,6)$ lies on the hyperbola

$\Rightarrow 48-36=k \;\; \therefore k=12$

$\therefore 3x^2-y^2=12$

2. *Find the points of intersection of the tangent to hyperbola* $\frac{x^2}{16}-\frac{y^2}{25}=1$ *at* $(0\,,5)$ *and its asymptotes.*

Solution

Equation of asymptotes to $\frac{x^2}{16}-\frac{y^2}{25}=1$ are: $\frac{x^2}{16}-\frac{y^2}{25}=0$ or $\left(\frac{x}{4}+\frac{y}{5}\right)\left(\frac{x}{4}-\frac{y}{5}\right)=0$

Considering $\frac{x}{4}+\frac{y}{5}=0$ or $x=-\frac{4}{5}y$..... (1)

Equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(x_1\,,y_1)$ is

$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

Thus the tangent at $(0\,,5)$ to the hyperbola $\frac{x^2}{16}-\frac{y^2}{25}=1$ is:

$\frac{x(0)}{16}-\frac{y(5)}{25}=1 \quad \therefore y=5$ ..... (2)

Considering $y=5$ and $\frac{x}{4}+\frac{y}{5}=0$

$\Rightarrow \frac{x}{4}+1=0 \quad \therefore x=-4$

$\therefore (-4\,,5)$

Considering $y=5$ and $\frac{x}{4}-\frac{y}{5}=0$

$\Rightarrow \frac{x}{4}-1=0 \quad \therefore x=4$

$\therefore (4\,,5)$

3. *Show that the ellipse* $\frac{x^2}{16}+\frac{y^2}{7}=1$ *and* $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ *have the same foci. Find the point of intersection of the curves.*

Solution

For the ellipse $\frac{x^2}{16}+\frac{y^2}{7}=1\,,a>b$

Using $b^2=a^2(1-e^2)$

$7=16(1-e^2) \quad \therefore e=\frac{3}{4}$

Foci: $(\pm ae\,,0)=\left(\pm 4\times\frac{3}{4}\,,0\right)=(\pm 3\,,0)$

For the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ or $\frac{x^2}{144/25}-\frac{y^2}{81/25}=1$

Using $b^2=a^2(1-e^2)$

$\frac{81}{25}=\frac{144}{25}(1-e^2) \quad \therefore e=\frac{5}{4}$

Foci: $(\pm ae\,,0)=\left(\pm\frac{12}{5}\times\frac{5}{4}\,,0\right)=(\pm 3\,,0)$

Hence foci are the same.

Point of intersection of the curves

$\frac{x^2}{16}+\frac{y^2}{7}=1\Rightarrow 7x^2+16y^2=112$ ..... (1)

$\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}\Rightarrow 2025x^2-3600y^2=11664$ ..... (2)

Solving the equations, $x^2=10.24\,,\ y^2=2.52$

$\Rightarrow x=\pm\frac{16}{5}\ ,y=\pm\frac{3\sqrt{7}}{5}$

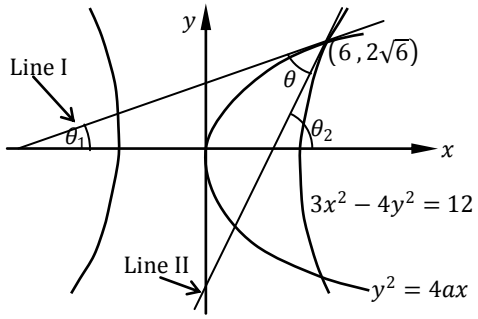
$\therefore$Points of intersection are: $\left(\pm\frac{16}{5}\ ,\ \pm\frac{3\sqrt{7}}{5}\right)$

4. *Find the point of intersection in the first quadrant of the hyperbola* $3x^2-4y^2=12$ *and the parabola* $y^2=4x$*, and also find the angle at which the curves intersect.*

Solution

Point of intersection is: $\left(6\,,2\sqrt{6}\right)$

The student should find this point. Left as an exercise



From the figure $\theta = \theta_2 - \theta_1$

Finding slope of Line I $= m_1 = \tan\theta_1$

For $y^2 = 4x\,, 2y\frac{dy}{dx} = 4 \quad \therefore \frac{dy}{dx} = \frac{2}{y}$

$\therefore m_1 = \frac{dy}{dx}\Big|_{at\ (6,2\sqrt{6})} = \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$

Finding slope of Line I $= m_1 = \tan\theta_1$

For $3x^2 - 4y^2 = 12\,, 6x - 8y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{3x}{4y}$

$\therefore m_2 = \frac{dy}{dx}\Big|_{at\ (6,2\sqrt{6})} = \frac{3\times6}{4\times2\sqrt{6}} = \frac{9}{4\sqrt{6}}$

From $\theta = \theta_2 - \theta_1$

Introducing tangents on both sides

$$\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_2\tan\theta_1} = \frac{\frac{9}{4\sqrt{6}} - \frac{1}{\sqrt{6}}}{1+\frac{9}{4\sqrt{6}}\times\frac{1}{\sqrt{6}}}$$

$$= \frac{\frac{1}{\sqrt{6}}\left(\frac{9}{4}-1\right)}{1+9/24} = \frac{10}{11\sqrt{6}}$$

$$\therefore \theta = \tan^{-1}\left(\frac{10}{11\sqrt{6}}\right)$$

PARAMETRIC EQUATIONS OF THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The general point $(a\sec\theta\ , b\tan\theta)$ lies on the hyperbola. Thus parametric equations are:

$$x = a\sec\theta\ ,\ y = b\tan\theta$$

To Show that the Parametric Equations represent a Hyperbola

For $x = a\sec\theta \quad \Rightarrow \sec\theta = \frac{x}{a}$

$y = b\tan\theta \qquad \Rightarrow \tan\theta = \frac{y}{b}$

Using $\sec^2\theta = 1 + \tan^2\theta \ \Rightarrow \sec^2\theta - \tan^2\theta = 1$

$$\therefore \ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

To Find the Equation of a Tangent and Normal to a Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta\ , b\tan\theta)$

Tangent Equation

$y - b\tan\theta = m_1(x - a\sec\theta)$ ..... (1)

Obtaining slope, m_1

Method I:

$m_1 = \left.\frac{dy}{dx}\right|_{at\ (a\sec\theta\ ,b\tan\theta)}$

Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ w.r.t x

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{b^2x}{a^2y}$

$\Rightarrow m_1 = \left.\frac{dy}{dx}\right|_{at\ (a\sec\theta\ ,b\tan\theta)} = -\frac{b^2a\sec\theta}{a^2b\tan\theta} = \frac{b\sec\theta}{a\tan\theta}.$

$\therefore m_1 = \frac{b\sec\theta}{a\tan\theta}$

Method II:

Formulating parametric equations

$x = a\sec\theta\ ,\ y = b\tan\theta$

$m_1 = \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta\ ,\frac{dy}{d\theta} = b\sec^2\theta$

$\therefore m_1 = b\sec^2\theta \times \frac{1}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$

Tangent equation becomes:

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$\Rightarrow ya\tan\theta - ab\tan^2\theta = b\sec\theta\, x - ab\sec^2\theta$

$\Rightarrow b\sec\theta\, x - ya\tan\theta = ab(\sec^2\theta - \tan^2\theta)$

$\therefore ay\tan\theta - bx\sec\theta = ab \qquad [\because \sec^2\theta - \tan^2\theta = 1]$ ..... $(\div ab)$

$\therefore \boldsymbol{\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1}$

Equation of the Normal

$y - b\tan\theta = m_2(x - a\sec\theta)$ ..... (2)

Obtaining slope, m_2

$m_2 = -\frac{1}{\text{Slope of the tangent at } (a\cos\theta\ ,b\sin\theta)} = -\frac{1}{m_1} = -\frac{a\tan\theta}{b\sec\theta}$

$y - b\tan\theta = -\frac{a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

$yb\sec\theta - b^2\tan\theta\sec\theta = -a\tan\theta\, x + a^2\tan\theta\sec\theta$

$\therefore a\tan\theta\, x + by\sec\theta = (a^2 + b^2)\tan\theta\sec\theta$ ..... $(\div \cos\theta\sin\theta)$

$\Rightarrow a\frac{\tan\theta}{\tan\theta\sec\theta}x + by\frac{\sec\theta}{\tan\theta\sec\theta} = a^2 + b^2$

$\therefore \boldsymbol{\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2}$

Examples

1. (i) *Show that the equation of the tangent to the hyperbola* $(a\sec\theta\ ,b\tan\theta)$ *is* $bx - ay\sin\theta - ab\cos\theta = 0$

ii) *Find the equations of tangents to* $\frac{x^2}{4} - \frac{y^2}{9} = 1$ *at the points where* $\theta = 45^0$ *and where* $\theta = -135^0$

Solution

i) Required equation:

$y - b\tan\theta = m(x - a\sec\theta)$ ..... (1)

Obtaining slope, m

Method:

$m = \frac{dy}{dx}\Big|_{at\ (a\sec\theta\ ,b\tan\theta)}$

Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ w.r.t x

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{b^2x}{a^2y}$

$\Rightarrow m = \frac{dy}{dx}\Big|_{at\ (a\sec\theta\ ,b\tan\theta)} = -\frac{b^2a\sec\theta}{a^2b\tan\theta} = \frac{b\sec\theta}{a\tan\theta}.$

$\therefore m = \frac{b\sec\theta}{a\tan\theta}$

$\Rightarrow y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$\Rightarrow ya\tan\theta - ab\tan^2\theta = b\sec\theta\, x - ab\sec^2\theta$

$\Rightarrow b\sec\theta\, x - ya\tan\theta = ab(\sec^2\theta - \tan^2\theta)$

$\therefore bx\sec\theta - ay\tan\theta = ab \qquad [\because \sec^2\theta - \tan^2\theta = 1]$ ..... $(\div ab)$

Expressing the equation in terms of $\sin\theta$ and $\cos\theta$

$\Rightarrow \frac{bx}{\cos\theta} - \frac{ay\sin\theta}{\cos\theta} = ab$ ..... $(\times \cos\theta)$

$\therefore bx - ay\sin\theta - ab\cos\theta = 0$

ii) Equation of tangent at $(a\sec\theta\ , b\tan\theta)$ is $bx - ay\sin\theta - ab\cos\theta = 0$

Now at $\theta = 45^0$

$bx - ay\sin 45^0 - ab\cos 45^0 = 0 \quad \Rightarrow bx - \frac{ay}{\sqrt{2}} - \frac{ab}{\sqrt{2}} = 0$

$\therefore 3\sqrt{2}x - 2y - 6 = 0$

Now at $\theta = -135^0$

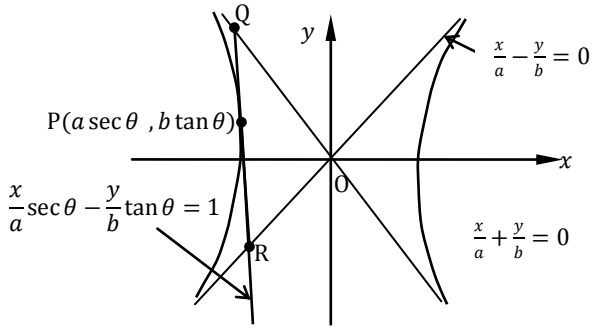
$bx - ay\sin(-135^0) - ab\cos(-135^0) = 0 \quad \Rightarrow bx + \frac{ay}{\sqrt{2}} + \frac{ab}{\sqrt{2}} = 0$

$\therefore 3\sqrt{2}x + 2y + 6 = 0$

2. *Prove that the portion of a tangent to the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *intercepted between asymptotes is bisected at the point of contact and the area of the triangle formed by the asymptotes and the tangent is a constant.*

Solution

Equation of asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ or $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$



Let the point of contact on the hyperbola be P$(a\sec\theta\ , b\tan\theta)$

Obtaining the tangent at P gives

$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$…… (1)

Let the student look for this tangent

Solving for the point of intersection Q and R

$\frac{x}{a}+\frac{y}{b}=0$ or $x=-\frac{ay}{b}$…… (2)

eqn. (2) Into eqn. (1)

$\Rightarrow -\frac{ay}{b}.\ \frac{1}{a}\sec\theta-\frac{y}{b}\tan\theta=1$ or $-\frac{y}{b}\sec\theta-\frac{y}{b}\tan\theta=1$

$\Rightarrow y=-\frac{b}{\sec\theta+\tan\theta}$ …… (3)

Eqn. (3) into eqn. (2)

$x=-\frac{a}{b}\times-\frac{b}{\sec\theta+\tan\theta}=\frac{a}{\sec\theta+\tan\theta}$

$\therefore Q\left(\frac{a}{\sec\theta+\tan\theta}\ ,\ -\frac{b}{\sec\theta+\tan\theta}\right)$ or $Q[\,a\,(\sec\theta-\tan\theta)\,,\ -b(\sec\theta-\tan\theta)]$

Now finding R

$\frac{x}{a}-\frac{y}{b}=0$ or $x=\frac{ay}{b}$…… (4)

eqn. (4) into eqn. (1)

$\Rightarrow \frac{ay}{b}.\ \frac{1}{a}\sec\theta-\frac{y}{b}\tan\theta=1$ or $\frac{y}{b}\sec\theta-\frac{y}{b}\tan\theta=1$

$\Rightarrow y=\frac{b}{\sec\theta-\tan\theta}=b(\sec\theta+\tan\theta)$ …… (5)

Eqn. (5) into eqn. (4)

$x=\frac{a}{b}\times b(\sec\theta+\tan\theta)=\ a(\sec\theta+\tan\theta)$

$\therefore R[\,a(\sec\theta+\tan\theta)\,,\ \ b(\sec\theta+\tan\theta)]$

Midpoint of QR

$\left(\frac{a\,(\sec\theta+\tan\theta)+a\,(\sec\theta-\tan\theta)}{2}\ ,\frac{-b(\sec\theta-\tan\theta)+b(\sec\theta+\tan\theta)}{2}\right)=(a\ \sec\theta\ ,b\ \tan\theta)$,

which lies on a hyperbola

Area of the triangle QOR $=\frac{1}{2}\times\overline{QR}\times$ Height

$H=\perp$ Distance of $(0\,,0)$ from QR

$H=\frac{1}{\sqrt{\left(\frac{\sec\theta}{a}\right)^2+\left(\frac{\tan\theta}{b}\right)^2}}=\frac{ab}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}$

$\therefore \overline{QR}=\sqrt{(2a\ \tan\theta)^2+(2b\ \sec\theta)^2}=2\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}$

$\therefore$Area of $\Delta QOR=\frac{1}{2}\times 2\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}\times\frac{ab}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}=ab$ (Constant)

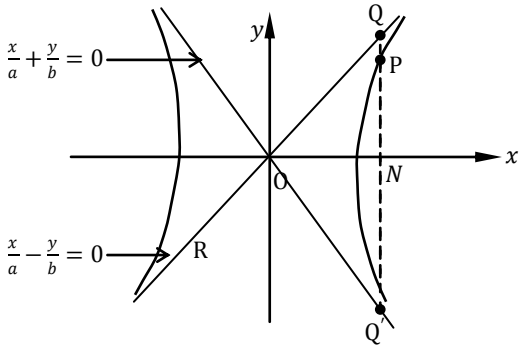
3. *If the ordinate of the point* $P(a\ \sec\theta\ ,b\ \tan\theta)$ *on the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *be produced to meet the asymptotes in Q and* Q'*, prove that* $PQ.PQ'=b^2$

Solution

In the figure below, the x −coordinate (abscissa) of P which is $x=a\ \sec\theta$ is the same as that of Q and Q'.

The $y-$ coordinate of Q is:

$\frac{a\ \sec\theta}{a}-\frac{y}{b}=0\ \ \therefore y=b\ \sec\theta$



$\therefore Q(a\ \sec\theta\ ,\ b\ \sec\theta)$

The $y-$ coordinate of Q' is:

$\frac{a\ \sec\theta}{a}+\frac{y}{b}=0\ \ \therefore y=-b\ \sec\theta$

$\therefore Q'(a\ \sec\theta\ ,\ b\ \sec\theta)$

$\overline{PQ}=b\ \sec\theta-b\ \tan\theta=b(\sec\theta-\tan\theta)$

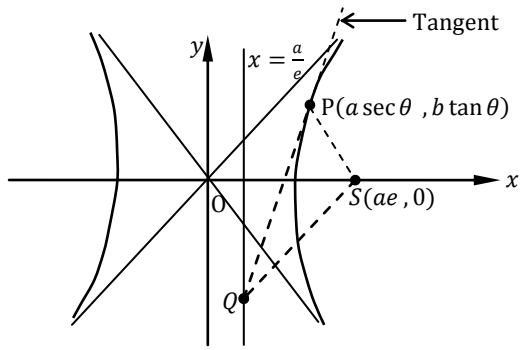
$\overline{PQ'}=b\ \tan\theta+b\ \tan\theta=b(\tan\theta+\tan\theta)$

$\therefore \overline{PQ}.\overline{PQ'}=b^2(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)$

$=b^2(\sec^2\theta-\tan^2\theta)=b^2$

4. *A tangent to a hyperbola at $P(a\ \sec\theta\ ,b\ \tan\theta)$ meets the directrix at Q. if S is the corresponding focus, prove that PQ subtends a right angle at S*

Solution



Tangent at P is:

$bx-ay\sin\theta-ab\cos\theta=0$

Finding the coordinates of Q

When $x=\frac{a}{e}\ ,y=?$

$\frac{ba}{e}-ay\sin\theta-ab\cos\theta=0$

$\therefore y=\frac{b(1-e\cos\theta)}{e\ \sin\theta}$

$\therefore Q\left(\frac{a}{e}\ ,\frac{b(1-e\cos\theta)}{e\ \sin\theta}\right)$

Slope of $QS=m_1=\frac{\frac{b(1-e\cos\theta)}{e\ \sin\theta}}{\frac{a}{e}-ae}=\frac{b(1-e\cos\theta)}{a(1-e^2)\cos\theta}$

Slope of $PS=m_2=\frac{b\tan\theta}{a\sec\theta-ae}=\frac{b\sin\theta}{a(1-e\cos\theta)}$

Now $m_1\times m_2=\frac{b(1-e\cos\theta)}{a(1-e^2)\cos\theta}\times\frac{b\sin\theta}{a(1-e\cos\theta)}=\frac{b^2}{a^2(1-e^2)}$

But $b^2=a^2(e^2-1)\ \Rightarrow -b^2=a^2(1-e^2)$

$\therefore m_1\times m_2=\frac{b^2}{-b^2}=-1$

Thus $SQ\perp SP$, so $\overline{PQ}$ subtends $< 90^0$ at the focus.

LOCUS PROBLEMS IN HYPERBOLA

1. *P is a variable point given by the parametric equations* $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$, $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$. *Show that the locus of P is* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. *State the asymptotes of the curve.*

Solution

$x = \frac{a}{2}\left(t + \frac{1}{t}\right) \Rightarrow t + \frac{1}{t} = \frac{2x}{a}$ ..... (1)

$y = \frac{b}{2}\left(t - \frac{1}{t}\right) \Rightarrow t - \frac{1}{t} = \frac{2y}{b}$ ..... (2)

Squaring eqn. (1) and eqn. (2) to create the squares on x and y

$\Rightarrow \left(t + \frac{1}{t}\right)^2 = \frac{4x^2}{a^2}$ or $\frac{4x^2}{a^2} = t^2 + \frac{1}{t^2} + 2$..... (3)

$\Rightarrow \left(t - \frac{1}{t}\right)^2 = \frac{4y^2}{b^2}$ or $\frac{4y^2}{b^2} = t^2 + \frac{1}{t^2} - 2$..... (4)

Eqn. (3)− eqn. (4)

$\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4 \quad \therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is the required locus.

Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

2. *P is any point on the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *and Q is the point* (a, b). *Find the locus of the point dividing PQ in the ratio* $2:1$.

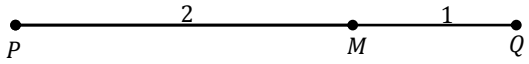
Solution

Let $M(X, Y)$ be the point dividing PQ in the ratio $2:1$

Ratio of division; $PM:MQ = 2:1$

Let $P(a\sec\theta, b\tan\theta)$, obtaining the coordinates of M in terms of a and b

From coordinate theorem, applying the ratio theorem



$\Rightarrow M(X, Y) = \left(\frac{1\times a\sec\theta + 2.a}{2+1}, \frac{1\times b\tan\theta + 2.b}{2+1}\right)$

$\Rightarrow X = \frac{a}{3}(\sec\theta + 2) \quad \therefore \sec\theta = \frac{3X}{a} - 2$

$\Rightarrow Y = \frac{b}{3}(\tan\theta + 2) \quad \therefore \tan\theta = \frac{3Y}{b} - 2$

Using $1 + \tan^2\theta = \sec^2\theta$

$\Rightarrow 1 + \left(\frac{3Y}{b} - 2\right)^2 = \left(\frac{3X}{a} - 2\right)^2$

$\Rightarrow 1 + \frac{9Y^2}{b^2} - \frac{12Y}{b} + 4 = \frac{9X^2}{a^2} - \frac{12Y}{a} + 4$ ..... $(\times a^2b^2)$

$\therefore 9b^2X^2 - 9a^2Y^2 = 12ab^2X + 12a^2bY - a^2b^2 = 0$

Thus the locus is:

$9b^2x^2 - 9a^2y^2 = 12ab^2x + 12a^2by - a^2b^2 = 0$

3. *The normal at* $P(a\sec\theta, b\tan\theta)$ *to the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *meets the axes at E and F. find the locus of the midpoint of EF.*

Solution

Equation of the normal: $ax\sin\theta + by - (a^2+b^2)\tan\theta = 0$

The student should derive the equation

Finding the x and $y-$ intercepts

When $x = 0$, $by = (a^2+b^2)\tan\theta$ $\therefore E\left(0, \frac{(a^2+b^2)\tan\theta}{b}\right)$

When $y = 0$, $ax\sin\theta - (a^2+b^2)\tan\theta = 0$ $\therefore F\left(\frac{(a^2+b^2)\tan\theta}{a\sin\theta}, 0\right)$

Let $M(X, Y)$ be the midpoint

$\Rightarrow X = \frac{(a^2+b^2)\tan\theta}{2a\sin\theta} = \frac{(a^2+b^2)}{2\cos\theta}$ $\therefore X = \frac{(a^2+b^2)\sec\theta}{2a}$ ..... (1)

$\Rightarrow Y = \frac{(a^2+b^2)\tan\theta}{2b}$ ..... (2)

From eqn. (1)

$\sec\theta = \frac{2aX}{(a^2+b^2)}$ $\Rightarrow \sec^2\theta = \frac{2a^2X^2}{(a^2+b^2)^2}$

From eqn. (2)

$\tan\theta = \frac{2bY}{(a^2+b^2)}$ $\Rightarrow \tan^2\theta = \frac{4b^2Y^2}{(a^2+b^2)^2}$

From $1 + \tan^2\theta = \sec^2\theta$

$\Rightarrow 1 + \frac{4b^2Y^2}{(a^2+b^2)^2} = \frac{2a^2X^2}{(a^2+b^2)^2}$ or $(a^2+b^2)^2 = 4(a^2X^2 - b^2Y^2)$

$\therefore (a^2+b^2)^2 = 4(a^2x^2 - b^2y^2)$

Task

1. The normal at any point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the $x-$axis at G. Q is a point on either asymptote such that PQ is parallel to the $y-$axis. Show that GQ is perpendicular to the asymptote on which Q lies.

2. Find the eccentricity, the foci and the directrices of:

a) $\frac{1}{9}x^2 - \frac{1}{16}y^2 = 1$ **Ans:** $\frac{5}{3}, (\pm 5, 0), 5x \pm 9 = 0$

b) $5x^2 - 4y^2 = 20$ **Ans:** $\frac{3}{2}, (\pm 3, 0), 3x \pm 4 = 0$

c) $\frac{1}{4}x^2 - \frac{1}{9}y^2 = 1$ **Ans:** $\frac{1}{2}\sqrt{13}, \left(\pm\sqrt{13}, 0\right), \sqrt{13}x \pm 4 = 0$

3. Prove that the locus of a point which moves so that the difference of its distances from the points $(\pm ae, 0)$ is $2a$ is $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$

4. Find the equation of the tangent and normal to $\frac{1}{25}x^2 - \frac{1}{16}y^2 = 1$ at the point $\left(6\frac{1}{4}, 3\right)$. **Ans:** $4x - 3y = 16, 16y + 12x - 123 = 0$

5. Find the equation of the tangent to $x = a\sec\varphi$, $y = b\tan\varphi$ at the point $\varphi = 1/4\pi$ **Ans:** $\frac{x\sqrt{2}}{a} - \frac{y}{b} = 1$

6. Find the equation of the hyperbola whose focus is $-1, 1$ and corresponding directrix $x = y$, and whose eccentricity is 2.

Ans: $x^2 + y^2 - 4xy + 2y - 2x - 2 = 0$

7. Find the equation of a hyperbola whose focus is $(1\,,0)$, directrix $x = 3$, and eccentricity $\frac{4}{3}$. **Ans:** $7x^2 - 9y^2 - 78x + 135 = 0$

8. Prove that the line is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2l^2 - b^2m^2 = n^2$

9. P is any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at P cuts $\boldsymbol{Ox}$ in T and PN is the ordinate of P. prove that $ON.OT = a^2$

10. ON is drawn perpendicular to the tangent at any point P on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from the origin **O**. prove that the locus of N is $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$

THE RECTANGULAR HYPERBOLA

Definition: If the asymptotes of the hyperbola are perpendicular to each other, it is called a *rectangular hyperbola.*

i) Condition for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to be a Rectangular Hyperbola

Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

$\Rightarrow \frac{x}{a} + \frac{y}{b} = 0$ or $y = -\frac{b}{a}x \quad \therefore m_1 = -\frac{b}{a}$

$\Rightarrow \frac{x}{a} - \frac{y}{b} = 0$ or $y = \frac{b}{a}x \quad \therefore m_2 = \frac{b}{a}$

For lines to be perpendicular to each other, $m_1 \times m_2 = -1$

$\Rightarrow -\frac{b}{a} \times \frac{b}{a} = -1$ or $a^2 = b^2$

$\boxed{\therefore a = b}$

Thus the equation of a rectangular hyperbola is:

$\boxed{x^2 - y^2 = a^2}$

ii) Eccentricity of a Rectangular Hyperbola $x^2 - y^2 = 0$

From $b^2 = a^2(e^2 - 1)\,,\ a = b$

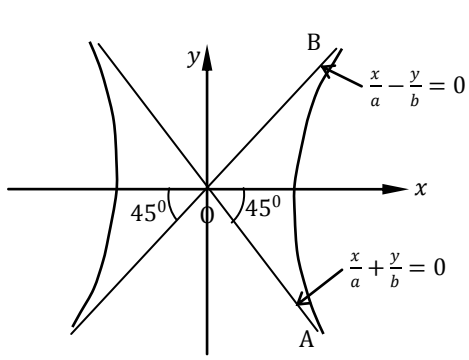
$\Rightarrow a^2 = a^2(e^2 - 1) \qquad \therefore e = \sqrt{2}$

Eccentricity of a Rectangular Hyperbola $x^2 - y^2 = a^2$ is $\boldsymbol{e = \sqrt{2}}$

iii) Equation of a Rectangular Hyperbola referred to its Asymptotes as Coordinate Axes is $xy = c^2$

Equation of the rectangular hyperbola is $x^2 - y^2 = a^2$

Asymptotes: $x^2 - y^2 = 0$ or $(x + y)(x - y) = 0$



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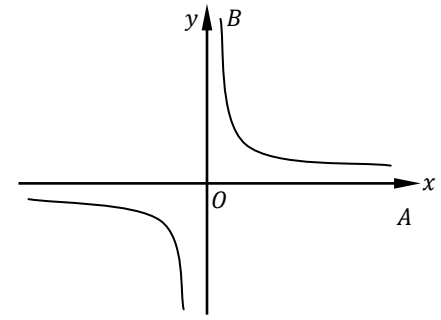


Rotation od aymptotes about O through $+45^0$

Asymptotes: $x^2 - y^2 = 0$

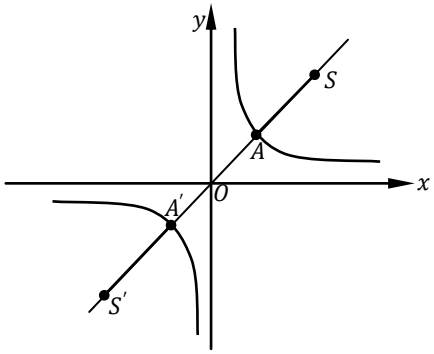
Asymptotes: $x.y = 0$

Joint equation of asymptotes after rotation through $+45^0$ about the origin is

$x.y = 0$

Since a hyperbola and its asymptotes differ by a constant, say c^2, the required equation of a rectangular hyperbola referred to its asymptotes as axes is: $\boxed{x.y = c^2}$

iv) Staing the Vertices, coordinates of the Foci and Equations of Directrices of $x.y = c^2$



Comparing $x.y = c^2$ with $xy = \frac{1}{2}a^2$ (Standard equation of rectangular hyperbola)

$\Rightarrow c^2 = \frac{1}{2}a^2 \quad \therefore a = \pm c\sqrt{2}$

From the *figure*, $\overline{OA} = \overline{0A'} = c\sqrt{2}$; $\overline{OS} = \overline{0S'} = ae$

Since the hyperbola is rectangular $e = \sqrt{2}$

$\therefore ae = 2c = \overline{OS} = \overline{0S'}$

Now the axis $SAOA'S'$ is inclined to the axes at 45^0

$x-$ coordinate of point A $= \overline{OA}\cos 45^0 = c\sqrt{2} \times \frac{1}{\sqrt{2}} = c = \overline{0A'}$

$y-$ coordinate of point A $= \overline{OA}\sin 45^0 = c\sqrt{2} \times \frac{1}{\sqrt{2}} = c$

$\therefore$ Vertices are: $(\pm \boldsymbol{c}, \pm \boldsymbol{c})$

$x-$ coordinate of point S $= \overline{OS}\cos 45^0 = 2c \times \frac{\sqrt{2}}{2} = c\sqrt{2}$

$y-$ coordinate of point S $= \overline{OS}\sin 45^0 = 2c \times \frac{\sqrt{2}}{2} = c\sqrt{2}$

$\therefore$ Coordinates of foci are: $\left(\pm \boldsymbol{c\sqrt{2}}, \pm \boldsymbol{c\sqrt{2}}\right)$

Since directrices are perpendicular to the axis, let their distance from the origin be $\overline{OZ} = \frac{a}{e} = \frac{c\sqrt{2}}{\sqrt{2}} = c$

$x-$ coordinate of point Z $= \overline{OZ}\cos 45^0 = c \times \frac{\sqrt{2}}{2} = \frac{c\sqrt{2}}{2}$

$y-$ coordinate of point Z $= \overline{OZ}\sin 45^0 = c\times\frac{\sqrt{2}}{2} = \frac{c\sqrt{2}}{2}$

$\therefore$ Directrices pass through points $\left(\pm\frac{c\sqrt{2}}{2}, \pm\frac{c\sqrt{2}}{2}\right)$, and have a slope of $m=-1$

Taking $\left(\frac{c\sqrt{2}}{2}, \frac{c\sqrt{2}}{2}\right)$

$y-\frac{c\sqrt{2}}{2} = -\left(x-\frac{c\sqrt{2}}{2}\right) \quad \therefore x+y=c\sqrt{2}$

$\therefore$ Equationof directrices are: $\boldsymbol{x+y=\pm c\sqrt{2}}$

Task

Find the coordinates of the vertices, foci and equations of directrices of the hyperbola $xy=18$

Ans: $\left(\pm3\sqrt{2}, \pm3\sqrt{2}\right), (\pm6, \pm6), x+y=\pm6$

v) Parametric Equations of Rectangular Hyperbola $x.y=c^2$

The general point on the rectangular hyperbola $x.y=c^2$ is $\boldsymbol{P\left(ct, \frac{c}{t}\right)}$. Thus the parametric equations are:

$$\boxed{x=ct \ , \ y=\frac{c}{t}}$$

vi) Equation of the Tangent to Hyperbola $x.y=c^2$ at $P\left(ct, \frac{c}{t}\right)$

Required Tangent:

$y-\frac{c}{t} = m(x-ct)$..... (1)

Obtaining slope, m

Approach I:

$m = \left.\frac{dy}{dx}\right|_{at\ P\left(ct, \frac{c}{t}\right)}$

Differentiating the hyperbola $x.y=c^2$

$\Rightarrow x\frac{dy}{dx}+y=0 \quad \therefore \frac{dy}{dx} = -\frac{y}{x}$

$\Rightarrow m = \left.\frac{dy}{dx}\right|_{at\ P\left(ct, \frac{c}{t}\right)} = -\frac{c/t}{ct} = -\frac{1}{t^2}$

Approach II:

$m = \frac{dy}{dt}\times\frac{dt}{dx}$

From $x=ct \ , \ y=\frac{c}{t}$

$\frac{dx}{dt}=c \ , \frac{dy}{dt} = -\frac{c}{t^2}$

$\Rightarrow m = \frac{dy}{dt}\times\frac{dt}{dx} = -\frac{c}{t^2}\times\frac{1}{c} = -\frac{1}{t^2}$

Eqn. (1) becomes

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$..... $(\times t^2)$

$t^2y - ct = -x + ct$

$$\boxed{x + t^2y = 2\,ct}$$

vii) Equation of the Normal to Hyperbola $x.y = c^2$ at $P\left(ct, \frac{c}{t}\right)$

Required Tangent:

$y - \frac{c}{t} = m'(x - ct)$..... (1)

Obtaining slope, m'

$m' = -\frac{1}{\text{Slope of the Tangent at P}} = -\frac{1}{-1/t^2} = t^2$

$\therefore y - \frac{c}{t} = t^2(x - ct)$ or $t^3x - ty = c(t^4 - 1)$

$$\boxed{t^3x - ty = c(1 + t^4)}$$

viii) Equation of a Tangent and Normal to $x.y = c^2$ at $P(x_1, y_1)$

This point is on the curve $x.y = c^2$

Tangent equation:

From equation of tangent at $P\left(ct, \frac{c}{t}\right)$

$x + t^2y = 2\,ct$..... $(\div ct)$

$\frac{x}{ct} + \frac{t^2y}{ct} = 2$, $P\left(ct, \frac{c}{t}\right) = P(x_1, y_1)$

$\Rightarrow \frac{x}{x_1} + \frac{y}{y_1} = 2$

$\therefore$ Equation of the tangent to $x.y = c^2$ at $P(x_1, y_1)$ is:

$$\boxed{\frac{x}{x_1} + \frac{y}{y_1} = 2}$$

From equation of the Normal at $P\left(ct, \frac{c}{t}\right)$

$y - \frac{c}{t} = t^2(x - ct)$, but $P\left(ct, \frac{c}{t}\right) = P(x_1, y_1)$ and $t^2 = \frac{ct}{\frac{c}{t}} = \frac{x_1}{y_1}$

$\Rightarrow y - y_1 = \frac{x_1}{y_1}(x - x_1)$

$$\boxed{x.x_1 - yy_1 = {x_1}^2 - {y_1}^2}$$

ix) To Find the Equation of a Tangent to the curve $x.y = c^2$ from a point $Q(x_1, y_1)$

This point is not on the curve.

First obtaining the equation of the tangent to the curve at $P\left(ct, \frac{c}{t}\right)$.

This gives:

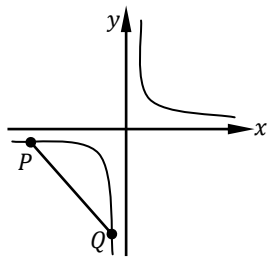
$x + t^2y = 2\,ct$ ..... (1)

The point $Q(x_1, y_1)$ lies on this tangent

$\Rightarrow x_1 + t^2y_1 = 2\,ct$ or $t^2y_1 - 2\,ct + x_1 = 0$..... (2)

Eqn. (2) gives two values of t hence two tangents from point Q.

x) To Find the Equation of a Chord joining $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ on the rectangular hyperbola $x.y = c^2$



Required equation of the chord is:

$$y - \frac{c}{p} = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq}(x - cp)$$

i.e. $y - \frac{c}{p} = -\frac{1}{pq}(x - cp)$

i.e. $y + \frac{x}{pq} = c\left(\frac{1}{p} + \frac{1}{q}\right)$

i.e. $\boldsymbol{pq\,y + x = c(p + q)}$

To Find the Equation of a Chord of the Rectangular Hyperbola $x.y = c^2$ having $M(x_1, y_1)$ as its midpoint

Let the chord have end points as $P(x', y')$ and $Q(x'', y'')$

Required chord:

$y - \boldsymbol{y_1} = m(x - \boldsymbol{x_1})$ ..... (1)

Obtaining the slope of the chord

$$M(x_1, y_1) = P\left[\frac{x' + x''}{2}, \frac{y' + y''}{2}\right]$$

$2x_1 = x' + x''$ ..... (2)

$\therefore 2y_1 = y' + y''$ ..... (3)

Also P and Q lie on the hyperbola

$\Rightarrow x'.y' = c^2$..... (4)

$\Rightarrow x''.y'' = c^2$..... (5)

Eqn. (3) + eqn. (4)

$\Rightarrow x' + x'' = \frac{c^2}{y'} + \frac{c^2}{y''}$

$\Rightarrow x' + x'' = c^2\left(\frac{y'' + y'}{y'y''}\right) \quad \therefore y'y'' = \frac{c^2.2y_1}{2x_1} = \frac{c^2.y_1}{x_1}$ ..... (6)

$$m = \frac{y'' - y'}{x'' - x'} = \frac{y'' - y'}{\frac{c^2}{y''} - \frac{c^2}{y'}}$$

$$= \frac{y'' - y'}{-c^2\left(\frac{y'' - y'}{y'y''}\right)} = -\frac{y'y''}{c^2}$$

$$= -\frac{y_1}{x_1}$$

Eqn. (1) becomes

$y - \boldsymbol{y_1} = -\frac{y_1}{x_1}(x - \boldsymbol{x_1}) \Rightarrow x_1 y - x_1\boldsymbol{y_1} = -xy_1 + x_1\boldsymbol{y_1}$

$\Rightarrow x_1 y + xy_1 = x_1\boldsymbol{y_1} + x_1\boldsymbol{y_1} = 2x_1\boldsymbol{y_1}$ ..... $(\div x_1\boldsymbol{y_1})$

$\therefore \frac{x}{\boldsymbol{x_1}} + \frac{y}{\boldsymbol{y_1}} = 2$

Examples

1. *Find the equation of a chord joining points* $\left(ct_1, \frac{c}{t_1}\right)$ *to* $\left(ct_2, \frac{c}{t_2}\right)$ *on a hyperbola. Hence deduce the equation of the tangent at* $\left(ct, \frac{c}{t}\right)$*. Find the equations of the tangents to the hyperbola* $x = 4t \;\; y =, \frac{4}{t}$ *which pass through the point* $(4, 3)$.

Solution

Let the required chord be:

$y - \frac{c}{t_1} = m(x - ct_1)$ ..... (1)

$m = \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} = \frac{t_1 - t_2}{(t_2 - t_1)t_1 t_2} = -\frac{1}{t_1 t_2}$

$\therefore \; y - \frac{c}{t_1} = -\frac{1}{t_1 t_2}(x - ct_1)$.....$(\times t_1 t_2)$

$\therefore \; yt_1 t_2 - ct_2 = -x + ct_1$

$\therefore x + \; yt_1 t_2 = ct_2 + ct_1 = c(t_1 + t_2)$

$\therefore x + \; yt_1 t_2 = c(t_1 + t_2)$

For a line (chord) to be a tangent to the hyperbola, $t_1 \to t_2$ i.e. at $\left(ct_2, \frac{c}{t_2}\right)$

$\therefore x + \; yt_1 t_2 = c(t_1 + t_2)$, becomes

$x + \; yt_2 t_2 = c(t_2 + t_2)$

$x + \; yt_2{}^2 = 2ct_2$, but at $\left(ct, \frac{c}{t}\right), t = t_2$

$\therefore \; x + \; yt^2 = 2ct$, which is the tangent at $\left(ct, \frac{c}{t}\right)$

Tangents to the hyperbola $x = 4t \;\; y =, \frac{4}{t}$ *through* $(4, 3)$

From $x + \; yt^2 = 2ct$ and $\left(ct, \frac{c}{t}\right)$, comparing $\left(ct, \frac{c}{t}\right)$ with $\left(4t, \frac{4}{t}\right)$

$c = 4$

$\therefore$ Tangent at $\left(4t, \frac{4}{t}\right)$ is: $x + \; yt^2 = 8t$

This tangent pass through $(4, 3)$

$\Rightarrow 4 + \; 3t^2 = 8t$ or $3t^2 - 8t + 4 = 0$ from which $t = 2, \frac{2}{3}$

$\therefore$ Tangents are:

$x + 4y = 16, \;\; x + \frac{4y}{9} = \frac{16}{3}$ or $x + 4y = 16, \; 9x + 4y = 48$

2. *The normal to the rectangular hyperbola* $xy = 8$ *at the point* $(4, 2)$ *meets the asymptotes at M and N. find the length MN.*

Solution

Asymptotes are: $xy = 0 \;\; \Rightarrow x = 0, y = 0$

Finding the equation of the normal at $(4, 2)$

$y - 2 = m(x - 4)$ ..... (1)

$m = -\frac{1}{\left.\frac{dy}{dx}\right|_{at\ (4,2)}}$

$x.y = 8$, differentiating w.r.t x

$y + x\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{y}{x}$

$\therefore m = -\frac{1}{-2/4} = 2$

$\therefore\ y - 2 = 2(x - 4)$ or $y = 2x - 6$

For $x = 0\,,\ y = -6 \quad \therefore M(0\,,-6)$

For $y = 0\,,\ x = 3 \quad \therefore N(3\,,0)$

$\overline{MN} = \sqrt{(3-0)^2 + (0+6)^2} = \sqrt{45} = 3\sqrt{5}\ units$

3. *Find the equation of the normal at* $(3\,,4)$ *to the rectangular hyperbola* $x.y = 12$ *and the coordinates of its second point of intersection with the curve.*

Solution

Required normal: $y - 4 = m(x - 3)$..... (1)

$m = -\frac{1}{\left.\frac{dy}{dx}\right|_{at\ (4\,,2)}}$

$x.y = 8$, differentiating w.r.t x

$y + x\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{y}{x}$

$\therefore m = -\frac{1}{-4/3} = \frac{3}{4}$

$\therefore\ y - 4 = \frac{3}{4}(x - 3)$ or $3x - 4y + 7 = 0$..... (2)

From $x.y = 12\ \Rightarrow x = \frac{12}{y}$, putting into eqn. (2) for points of intersection

$\frac{36}{y} - 4y + 7 = 0$ or $4y^2 - 7y - 36 = 0 \quad \therefore y = 4\,, -\frac{9}{4}$

From $x = \frac{12}{y}$, for $y = -\frac{9}{4}\ \ , x = -\frac{16}{3}$

$\therefore \left(-\frac{16}{3}\,, -\frac{9}{4}\right)$

Alternatively

Comparing $xy = 12$ with $xy = c^2\,,\ c = \sqrt{12} = 2\sqrt{3}$

General point on rectangular hyperbola $xy = c^2$ is $\left(ct\,, \frac{c}{t}\right)$

Thus the general point on $xy = 12$ is $\left(2\sqrt{3}t\,, \frac{2\sqrt{3}}{t}\right)$

Obtaining the equation of the normal at $\left(2\sqrt{3}t\,, \frac{2\sqrt{3}}{t}\right)$

$y - \frac{2\sqrt{3}}{t} = m\left(x - 2\sqrt{3}t\right)$..... (1)

$m = \frac{dy}{dt} \times \frac{dt}{dx}$

For $x = 2\sqrt{3}t\ \Rightarrow \frac{dx}{dt} = 2\sqrt{3}$

For $y = \frac{2\sqrt{3}}{t}\ \Rightarrow \frac{dy}{dt} = -\frac{2\sqrt{3}}{t^2}$

$\therefore m = -\frac{2\sqrt{3}}{t^2} \times \frac{1}{2\sqrt{3}} = -\frac{1}{t^2}$

$\therefore y - \frac{2\sqrt{3}}{t} = -\frac{1}{t^2}\left(x - 2\sqrt{3}t\right)$

Comparing $\left(2\sqrt{3}t\,,\frac{2\sqrt{3}}{t}\right)$ with $(3\,,4)$ since they represent the same point on the curve

$\Rightarrow 2\sqrt{3}t = 3 \quad \therefore t = \frac{1}{2}\sqrt{3}$

$\therefore y - \frac{2\sqrt{3}}{t} = -\frac{1}{t^2}\left(x - 2\sqrt{3}t\right)$, becomes

$y - \frac{2\sqrt{3}}{\sqrt{3}/2} = -\frac{1}{(\sqrt{3}/2)^2}\left(x - 2\sqrt{3}\left(\frac{1}{2}\sqrt{3}\right)\right)$

$\therefore y - 4 = -\frac{4}{3}(x-3)$, which is eqn. (2) in approach (1)

$\therefore 3x - 4y + 7 = 0$ is the normal.

But $\left(2\sqrt{3}t\,,\frac{2\sqrt{3}}{t}\right)$ lies on this normal

$3\left(2\sqrt{3}t\right) - 4\left(\frac{2\sqrt{3}}{t}\right) + 7 = 0$ or $6\sqrt{3}t^2 + 7t - 8\sqrt{3} = 0$

$\therefore t = \frac{-7\pm\sqrt{7^2-4(6\sqrt{3})(-8\sqrt{3})}}{12\sqrt{3}} = \frac{-7\pm25}{12\sqrt{3}}$

$\therefore t = t = \frac{1}{2}\sqrt{3}\,,\ t = -\frac{8}{9}\sqrt{3}$

Putting $t = -\frac{8}{9}\sqrt{3}$ into $\left(2\sqrt{3}t\,,\frac{2\sqrt{3}}{t}\right)$

$\left(2\sqrt{3}\times\left(-\frac{8}{9}\sqrt{3}\right)\,,\frac{2\sqrt{3}}{-\frac{8}{9}\sqrt{3}}\right) = \left(-\frac{16}{3}\,,-\frac{9}{4}\right)$

Task

1. Find the equation of the tangent and normal to $xy = 4$ at the point $(1\,,4)$.
   Ans: $4x + y = 8\,,4y = x + 15$
2. Find c if the line $y = mx + c$ is a tangent to $xy = a^2$ for all values of m.
   Ans: $c = \pm 2a\sqrt{(-m)}$
3. Find the coordinates of the vertices, coordinates of the foci and directrices of $xy = c^2$. **Ans:** $(\pm c\,,\pm c)\,,\left(\pm c\sqrt{2}\,,\pm c\sqrt{2}\right),x + y = \pm c\sqrt{2}$
4. The normal at $P\left(ct\,,\frac{c}{t}\right)$ cuts the curve again at Q. prove that $PQ = \frac{c\left(1+t^4\right)^2}{t^3}$
5. Prove that the equation of a chord of $xy = c^2$ whose midpoint is $(h\,,k)$ is $\frac{x}{h} + \frac{y}{k} = 2$
6. Show that the equation of the chord joining the points $(a\sec\theta\,,b\tan\theta)$ and $(a\sec\varphi\,,b\tan\varphi)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
   Is $\frac{x}{a}\cos\frac{1}{2}(\theta-\varphi) - \frac{y}{b}\sin\frac{1}{2}(\theta+\varphi) = \cos\frac{1}{2}(\theta+\varphi)$
   Deduce the equation of the tangent at the point $(a\sec\theta\,,b\tan\theta)$.
7. Find the condition for the line $lx + my = n$ to touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
   By writing $+b^2$ in place of $-b^2$ deduce the condition for the same line to meet the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
   Ans: $a^2l^2 - b^2m^2 = n^2\,;\ a^2l^2 + b^2m^2 = n^2$

8. Show that the point $\left[\frac{a}{2}\left(t+\frac{1}{t}\right), \frac{b}{2}\left(t-\frac{1}{t}\right)\right]$ always lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$for all values of t

9. Find the equations and the points of contact of the tangents to the hyperbola $2x^2 - 3y^2 = 5$ which are parallel to $8x = 9y$.
Ans: $8x - 9y = \pm 5$; $(\pm 4, \pm 3)$

10. Prove that the straight line $lx + my = n$ touches the rectangular hyperbola $xy = c^2$ if $n^2 = 4lmc^2$. Find the co-ordinates of the point of contact. **Ans:** $\left(\frac{n}{2l}, \frac{2c^2 l}{n}\right)$

11. Show that if the line $y = mx + c$ touches the hyperbola $x^2 - 3y^2 = 1$, then $3m^2 = 3c^2 + 1$.
Obtain an equation for the gradients of the two tangents to the hyperbola from the point $P(x_0, y_0)$. Show that if these tangents are perpendicular then P lies on the circle $x^2 + y^2 = \frac{2}{3}$
Ans: $m^2({x_0}^2 - 1) - 2x_0 y_0 m + {y_0}^2 + \frac{1}{3} = 0$

12. Show that if the line $y = mx + c$ is a tangent to the rectangular hyperbola $x^2 - y^2 = a^2$ then $c^2 = a^2(m^2 - 1)$ and the co-ordinates of the point of contact T are $\left(-\frac{ma^2}{c}, -\frac{a^2}{c}\right)$
If the line meets the asymptotes at P, Q, show that T is the midpoint of PQ.

11. Find the equation of the tangents to the hyperbola $x^2 - y^2 = 7$ which are parallel to $3y = 4x$ and find their points of contact. Find the area of the triangle which one of these tangents makes with the asymptotes.
Ans: $4x - 3y = \pm 7, (4, 3)$ and $(-4, -3)$. 7 square units

LOCUS PROBLEMS ON RECTANGULAR HYPERBOLA $x.y = c^2$

1. *The tangent at any point $P(ct, c/t)$ on the hyperbola $x.y = c^2$ meets x and $y-$ axis at A and B respectively.* ***O*** *is the origin. Prove that:*

i) $\overline{AP} = \overline{PB}$ ii) *The area of ΔAOB is a constant*

Solution

Find the tangent equation as:

$x + t^2 y = 2\,ct$ ..... (1)

Obtaining the $x, y-$ intercept

For $y-$ intercept, $x = 0$

$\Rightarrow t^2 y = 2\,ct$ or $y = \frac{2c}{t}$ $\therefore B(0, 2c/t)$

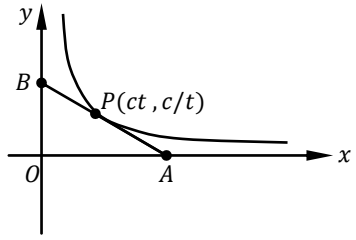
For $x-$ intercept, $y = 0$

$\Rightarrow x = 2\,ct$ or $\therefore A(2ct, 0)$

Obtaining $\overline{AP} = \sqrt{(2ct - ct)^2 + (0 - c/t)^2} = c\sqrt{t^2 + 1/t^2}$

Obtaining $\overline{PB} = \sqrt{(0 - ct)^2 + \left(\frac{c}{t} - \frac{2c}{t}\right)^2} = c\sqrt{t^2 + 1/t^2}$

$\therefore \overline{AP} = \overline{PB} = c\sqrt{t^2 + 1/t^2}$



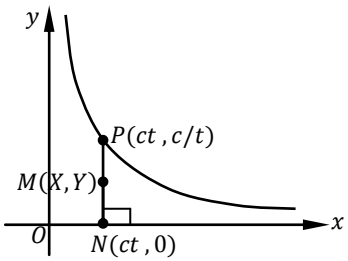
Area of $\Delta AOB = \frac{1}{2} \times \overline{OA} \times \overline{OB}$

$= \frac{1}{2} \times 2ct \times \frac{2c}{t} = 2c^2$

Hence there is no variable t in the expression of area.

2. *PN is the perpendicular to an asymptote from a point on a rectangular hyperbola $x.y = c^2$. Prove that the locus of the midpoint of PN is a rectangular hyperbola with the same axes.*

Solution



Midpoint of $PN = M(X , Y) = M\left(\frac{ct+ct}{2} , \frac{\frac{c}{t}+0}{2}\right)$

$\therefore X = ct \ , \ Y = \frac{c}{2t}$

Multiplying the equations

$X.Y = ct \times \frac{c}{2t} = \frac{c^2}{2}$

$\therefore xy = \frac{c^2}{2}$, which is a rectangular hyperbola with $x , y -$ axes

3. *PQ is a chord of the rectangular hyperbola $x.y = c^2$ and R is the midpoint of PQ. If PQ has a constant length k, find the locus of R.*

Solution

Let $P(cp , c/p) , Q(cq , c/q)$ and $R(X , Y)$ be the given points

$\Rightarrow X = \frac{1}{2}c(p+q)$..... (1)

$\Rightarrow Y = \frac{1}{2}c\left(\frac{p+q}{pq}\right)$..... (2)

But $\overline{PQ} = k$ or $\overline{PQ}^2 = k^2$

$\Rightarrow \overline{PQ}^2 = k^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2$

$\Rightarrow k^2 = c^2(p-q)^2 + c^2\left(\frac{1}{p} - \frac{1}{q}\right)^2$

$\Rightarrow k^2 = c^2(p-q)^2 + c^2\left(\frac{q-p}{pq}\right)^2$

$\Rightarrow k^2 = c^2(p-q)^2\left[1 + \frac{1}{p^2q^2}\right]$

But $(p-q)^2 = p^2 + q^2 - 2pq = (p+q)^2 - 4pq$

$\therefore k^2 = c^2[(p+q)^2 - 4pq]\left[1 + \frac{1}{p^2q^2}\right]$..... (3)

The task is to eliminate variables p , q from the formulated equations

From eqn. (1)

$p + q = \frac{2X}{c}$..... (4)

Eqn. (1) ÷ eqn. (2)

$\frac{X}{Y} = pq$..... (5)

Eqn. (4), eqn. (5) into eqn. (3)

$\Rightarrow k^2 = c^2\left[\left(\frac{2X}{c}\right)^2 - 4\left(\frac{X}{Y}\right)\right]\left[1 + \frac{1}{(X/Y)^2}\right]$

$\Rightarrow k^2 = c^2\left[\frac{4X^2}{c^2} - \frac{4X}{Y}\right]\left[\frac{Y^2+X^2}{X^2}\right]$

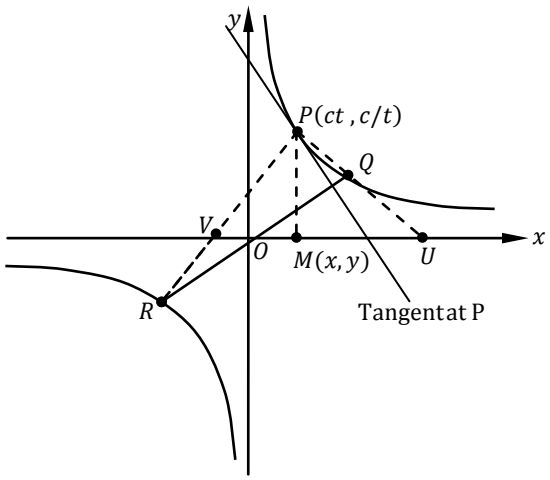
$\Rightarrow X^2Yk^2 = (4X^2Y - 4Xc^2)(X^2 + Y^2)$, reducing both sides by X

$\Rightarrow XYk^2 = (4XY - 4c^2)(X^2 + Y^2)$

$\therefore 4(X^2 + Y^2)(xy - c^2) = xyk^2$, is the required locus

4. *The perpendicular from the origin to the tangent at a point P on the rectangular hyperbola $x.y = c^2$ meets the curve at Q and R. the chords PQ and PR meet the $x-$axis at u and v. prove that the midpoint of uv is the foot of the perpendicular from P to the $x-$ axis*

Solution



To prove: $M(x, y)$ is the midpoint of uv.

Let $P(ct\,, c/t)$ be any point on $x.y = c^2$

$x = ct\,,\ y = c/t$

Obtaining the tangent at P

$x + t^2y = 2\,ct$..... (1)

Obtaining the equation of a line through origin and perpendicular to the tangent

$y - 0 = m'(x - ct) \Rightarrow y = m'x$

But $m' = -\frac{1}{\frac{dy}{dx}\big|_{at\ P}} = -\frac{1}{-1/t^2} = t^2$

$\therefore y = t^2x$

Obtaining the points of intersection of $xy = c^2$ and $y = t^2x$

$\Rightarrow t^2x^2 = c^2 \qquad \therefore x = \pm\frac{c}{t}$

For $x = \frac{c}{t}\,, y = t^2 \times \frac{c}{t} = ct \qquad \therefore Q\left(\frac{c}{t}\,, ct\right)$

For $x = -\frac{c}{t}\,, y = t^2 \times -\frac{c}{t} = -ct \qquad \therefore Q\left(-\frac{c}{t}\,, -\,ct\right)$

Obtaining the equation of the chord PQ

$y - \frac{c}{t} = m_1(x - ct)$, but $m_1 = \frac{\frac{c}{t} - ct}{ct - c/t} = -1$

$\Rightarrow y - \frac{c}{t} = -1(x - ct)$ or $y = -x + ct + c/t$

Finding coordinates of u, $y = 0$

$0 = -x + ct + \frac{c}{t} \qquad \therefore x = ct + c/t$

$\therefore u\left(ct + \frac{c}{t}, 0\right)$

Obtaining the equation of the chord PR

$y - \frac{c}{t} = m_2(x - ct)$, but $m_1 = \frac{\frac{c}{t}+ct}{ct+c/t} = 1$

$\Rightarrow y - \frac{c}{t} = (x - ct)$ or $y = x - ct + c/t$

Finding coordinates of v, $y = 0$

$0 = x - ct + \frac{c}{t} \quad \therefore x = ct - c/t$

$\therefore u\left(ct - \frac{c}{t}, 0\right)$

Now midpoint of $uv = M(x, y) = \left(\frac{ct+\frac{c}{t}+ct-\frac{c}{t}}{2}, \frac{0+0}{2}\right) = (ct, 0)$

But abscissa of P is $x = ct$

$\therefore$ Foot of perpendicular from P to $x-$ axis is $(ct, 0)$

Task

1. A chord PQ of the rectangular hyperbola $x.y = c^2$ subtends a right angle at another point R on the curve. Prove PQ parallel to the normal at R.
2. The normal at a point P on $x.y = c^2$ meets the asymptotes in Q and R and the transverse axis in G. prove that $PG^2 = PQ.PR$
3. N is the foot of the perpendicular from the point $(0, 0)$ on to the tangent at point P on $x.y = c^2$. Prove that the locus of N is $(x^2 + y^2)^2 = 4c^2xy$
4. If chords of a constant length $2d$ are drawn in the curve $x.y = c^2$ prove the locus of their midpoint is $(x^2 + y^2)(xy - c^2) = d^2xy$
5. Show that the equation of the line joining points $(cp, c/p)$, $(cq, c/q)$ on the rectangular hyperbola $xy = c^2$ is $x + pqy = c(p + q)$. Deduce the equation of the tangent at the point P. **Ans:** $x + p^2y = 2cp$
6. Show that the normal to the rectangular hyperbola $xy = c^2$ at the point $P(ct, c/t)$ meets the curve again at $Q(ct_1, c/t_1)$ such that $t^3t_1 = -1$
7. Show that the locus of the midpoint of the normal chords of a rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$

Conclusion

When handling numbers involving conic section,

- Knowledge of Coordinate Geometry I is relevant
- Try to interpret the question with an aid of a clear and illustrative diagram
- Knowledge of Algebra when handling equations involving parameters is also necessary
- Your ability to handle and interpret the question carefully determines how quick can you reach to the required answer
- DO NOT ASSUME THAT CONIC SECTION IS HARD.

MACLAURIN'S THEOREM

This is used in polynomial approximations for a function, estimating of values of constants etc.

Under certain conditions, a function $f(x)$ is expressed as an infinite series of ascending powers of x.

Derivation of Maclaurin's Theorem

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + a_rx^r + \cdots \ldots\ldots\ldots\ldots$

Differentiating with respect to x

$f'(x) = a_1 + a_2x + 2a_3x^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + ra_rx^{r-1} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f''(x) = 2a_2 + 6a_3x + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + r(r-1)a_rx^{r-2} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots$

$f'''(x) = 6a_3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + r(r-1)(r-2)a_rx^{r-2} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots$

$f^r(x) = r(r-1)(r-2) \ldots\ldots\ldots\ldots 3 \times 2 \times a_r + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Substituting $x = 0$

$f(0) = a_0$ $\quad\therefore a_0 = f(0)$

$f'(0) = a_1$ $\quad\therefore a_1 = f'(0)$

$f''(0) = 2a_2$ $\quad\therefore a_2 = \frac{f''(0)}{2}$

$f'''(0) = 3.2a_3 = 6a_3$ $\quad\therefore a_3 = \frac{f'''(0)}{3!}$

And $f^r(0) = r.(r-1)(r-2) \ldots\ldots\ldots 3 \times 2a_2a_r$ $\quad\therefore a_r = \frac{f^r(0)}{r!}$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \ldots\ldots\ldots\ldots + \frac{f^r(0)}{r!}x^r$$

NOTE

- The Maclaurin's series for a function $f(x)$ can be obtained if the derivatives of $f'(0), f''(0), \ldots\ldots\ldots\ldots\ldots$ can be determined.
- For some functions, Maclaurin's theorem holds for only within restricted range of values of x.

Derivation of Various Important Expressions by Maclaurin's Theorem

a) **Binomial Theorem**

Let $f(x) = (1+x)^n$, n is a real number

$f'(x) = n(1+x)^{n-1}$

$f''(x) = n(n-1)(1+x)^{n-2}$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f^r(x) = n(n-1)(n-2) \ldots\ldots\ldots\ldots\ldots\ldots (n-r+1)(1+x)^{n-r}$

$\therefore f(0) = 1, f'(0) = n, f''(0) = n(n-1), f'''(0) = n(n-1)(n-2)$, and

$f^r(0) = n(n-1)(n-2) \ldots\ldots\ldots\ldots\ldots\ldots (n-r+1)$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \ldots \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

, which is valid for $|x| < 1$ or $-1 < x < 1$

b) **Exponential Series, e^x**

Let $f(x) = e^x$, then $f(0) = e^0 = 1$

$f'(x) = e^x$, then $f'(0) = e^0 = 1$

$f''(x) = e^x$, then $f''(0) = e^0 = 1$

……

$f^r(x) = e^x$, then $f^r(0) = e^0 = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots + \frac{x^r}{r!} + \cdots \ldots ..$$

, which valid for all values of x

c) **Logarithm Series, $\ln(1+x)$**

Let $f(x) = \ln(1+x)$, then $f(0) = \ln(1) = 0$

$f'(x) = (1+x)^{-1}$, then $f'(0) = 1$

$f''(x) = -(1+x)^{-2}$, then $f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$, then $f''(0) = 2$

$f^4(x) = -6(1+x)^{-4}$, then $f''(0) = -6$

……

$f^r(x) = (-1)^{r+1}(r-1)!(1+x)^{-r}$, then $f^r(0) = (-1)^{r+1}(r-1)!$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ldots \ldots \ldots \ldots \ldots + (-1)^{r+1}\frac{x^r}{r} + \cdots \ldots ..$$

NOTE

- The function $\ln x$ is not defined at $x = 0$ and thus no Maclaurin's series
- The expansion of $\ln(1+x)$ is valid for $-1 < x \leq 1$

d) **Expansion of $\sin x$**

Let $f(x) = \sin x$, then $f(0) = \sin 0 = 0$

$f'(x) = \cos x$, then $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$, then $f''(0) = \sin 0 = 0$

$f'''(x) = -\cos x$, then $f''(0) = -\cos 0 = -1$

$f^4(x) = \sin x$, then $f''(0) = \sin 0 = 0$

$f^5(x) = \cos x$, then $f^5(0) = \cos 0 = 1$

……

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

e) **Expansion of $\cos x$**

In a similar way

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The expansion of $\sin x$ and $\cos x$ are valid for all values of x (in radians)

General considerations for Validity of an expansion

Expansion	**Condition**
$(1+x)^n$	$\lvert x\rvert < 1$ or $-1 < x < 1$
$\ln(1+x)$	$-1 < x \leq 1$
$\tan^{-1} x$	$-1 \leq x \leq 1$
e^x , $\sin x$, $\cos x$	valid for all values of x

Examples

1. *Assuming that x is so small that terms in x^3 and higher powers may be neglected, find a quadratic approximation to* $\sqrt{\frac{1-x}{1+2x}}$

Solution

$$\sqrt{\frac{1-x}{1+2x}} = (1-x)^{1/2}(1+2x)^{-1/2}$$

$$(1-x)^{1/2} = 1 + \frac{1}{2}(-x) + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-x)^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(1+2x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(2x) + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(2x)^2 + \cdots \ldots\ldots\ldots$$

$$= 1 - x + \frac{3}{2}x^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Neglecting terms in x^3 and higher power terms

$$\sqrt{\frac{1-x}{1+2x}} \simeq \left(1 - \frac{1}{2}x - \frac{1}{8}x^2\right)\left(1 - x + \frac{3}{2}x^2\right)$$

$$\simeq 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{8}x^2$$

$$\therefore \sqrt{\frac{1-x}{1+2x}} \simeq 1 - \frac{3}{2}x + \frac{15}{8}x^2$$

2. *Expand $\frac{1}{1+x+2x^2}$ in ascending powers of x up to and including the term in x^3*

Solution

Let $y = x + 2x^2$

$$\frac{1}{1+y} = (1+y)^{-1} = 1 + (-1)y + \frac{(-1)(-2)}{2!}y^2 + \frac{(-1)(-2)(-3)}{3!}y^3 + \cdots \ldots\ldots\ldots\ldots$$

$$= 1 - y + y^2 - y^3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Substituting $y = x + 2x^2$

$\frac{1}{1+x+2x^2} = 1-(x+2x^2)+(x+2x^2)^2-(x+2x^2)^3+\cdots\ldots\ldots\ldots\ldots\ldots\ldots$
$= 1-x-2x^2+x^2+4x^3-x^3+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$= 1-x-x^2+3x^3+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

3. *Expand* $\frac{1}{(1-x)(1+2x)}$ *in ascending powers of* x

Solution

$\frac{1}{(1-x)(1+2x)} = (1+x-2x^2)^{-1} = 1-(x-2x^2)+(x-2x^2)^2+\cdots\ldots\ldots\ldots$

Alternatively

$\frac{1}{(1-x)(1+2x)} = (1-x)^{-1}(1+2x)^{-1}$
$= (1+x+x^2+\cdots\ldots\ldots\ldots\ldots)(1-2x+4x^2+\cdots\ldots\ldots\ldots\ldots)$

NOTE

If the general formula for the term in x^r is required, a method of partial fractions can be used.

4. *Expand* $\frac{1}{(1-x)(1+2x)}$ *in ascending powers of* x *giving the first three terms and the term in* x^r*. State the values of* x *for which the expansion is valid.*

Solution

$\frac{1}{1-x} = (1-x)^{-1} = 1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\cdots\ldots+\frac{(-1)(-2)\ldots\ldots\ldots(-r)}{2!}(-x)^r$
$\therefore \frac{1}{1-x} = 1+x+x^2+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+x^r+\cdots\ldots\ldots\ldots\ldots\ldots\ldots$
Similarly
$\frac{1}{1+2x} = (1+2x)^{-1} = 1-2x+4x^2+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+(-2)^r.x^r+\cdots\ldots\ldots$
Let $\frac{1}{(1-x)(1+2x)} = \frac{A}{1-x}+\frac{B}{1+2x}$
$\Rightarrow 1 \equiv A(1+2x)+B(1-x)$
Let $x=1$, $1=3A \quad \therefore A=\frac{1}{3}$
$x=-\frac{1}{2}$, $1=\frac{3}{2}B \quad \therefore B=\frac{2}{3}$
$\therefore \frac{1}{(1-x)(1+2x)} = \frac{1}{3}(1-x)^{-1}+\frac{2}{3}(1+2x)^{-1}$
$= \frac{1}{3}(1+x+x^2+\cdots+x^r+\cdots)+\frac{2}{3}(1-2x+4x^2+\cdots+(-2)^r.x^r+\cdots)$
$= 1-x+3x^2+\cdots\ldots\ldots\left[\frac{1}{3}+\frac{2}{3}(-2)^r\right]x^r+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$= 1-x+3x^2+\cdots\ldots\ldots\frac{1}{3}[1-(-2)^{r+1}]x^r+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
The expansion of $(1-x)^{-1}$ is valid for $|x|<1$ and the expansion $(1+2x)^{-1}$ is valid for $|2x|<1$.i.e. $|x|<\frac{1}{2}$
Combining the two validities $|x|<\frac{1}{2}$ is common. So the expansion $\frac{1}{(1-x)(1+2x)}$ is valid for $|x|<\frac{1}{2}$

5. *Expand* e^{2x} *in ascending powers of* x *giving the first three terms and term in* x^r*.*

Solution

Using $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{x^r}{r!} + \cdots \ldots\ldots$, for all values of x

$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{(2x)^r}{r!} + \cdots \ldots\ldots$

$e^{2x} = 1 + 2x + 2x^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + \frac{2^r}{r!}x^r + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

6. *Find the first three terms in the expansion of* e^{x+x^2} *in ascending powers of* x

Solution

$e^{x+x^2} = 1 + (x + x^2) + \frac{(x+x^2)}{2!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= 1 + x + x^2 + \frac{1}{2}x^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= 1 + x + \frac{3}{2}x^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

7. *Expand* e^{2-x} *in ascending powers of* x *as far as term in* x^3

Solution

$e^{2-x} = e^2 \times e^{-x} = e^2\left[1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots + \frac{x^r}{r!} + \cdots \ldots\ldots\right]$

$= e^2\left[1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{x^r}{r!} + \cdots \ldots\ldots\right]$

8. *Expand* $\ln(1 - x)$ *in ascending powers of* x*. Hence state the validity of the expansion.*

Solution

Using $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ldots\ldots\ldots\ldots\ldots + (-1)^{r+1}\frac{x^r}{r} + \cdots \ldots\ldots$

,where $-1 < x \leq 1$

Replacing x by $-x$

$\Rightarrow \ln(1 - x) = x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \cdots \ldots\ldots\ldots + (-1)^{r+1}\frac{(-x)^r}{r} + \cdots \ldots\ldots$

$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - \frac{x^r}{r} - \cdots \ldots\ldots\ldots\ldots$,

where $-1 < -x \leq 1$ or $1 > x \geq -1$

$\therefore$ Validity is $-1 \leq x < 1$

9. *Expand* $\ln(2 + 3x)$ *in ascending powers of* x *as far as the term in* x^3*. State the values of* x *for which the expansion is valid.*

Solution

$\ln(2 + 3x) = \ln 2\left(1 + \frac{3}{2}x\right) = \ln 2 + \ln\left(1 + \frac{3}{2}x\right)$

$= \ln 2 + \left(\frac{3}{2}x\right) - \frac{1}{2}\left(\frac{3}{2}x\right)^2 + \frac{1}{3}\left(\frac{3}{2}x\right)^3 - \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= \ln 2 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{9}{8}x^3 - \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

The expansion for validity is $-1 < \frac{3}{2}x \leq 1$ or $-\frac{2}{3} < x \leq \frac{2}{3}$

10. *Expand* $\ln\left(\frac{1+x}{\sqrt{1-2x}}\right)$ *in ascending powers of* x *as far as the term in* x^3*, stating the values of* x *for which the expansion is valid.*

Solution

$\ln\left(\frac{1+x}{\sqrt{1-2x}}\right) = \ln(1+x) - \frac{1}{2}\ln(1-2x)$

$= \left\{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \ldots\ldots\right\} - \frac{1}{2}\left\{(-2x) - \frac{1}{2}(-2x)^2 + \frac{1}{3}(-2x)^3 - \cdots \ldots.\right\}$

$= \left\{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \ldots\ldots\ldots\right\} + \left\{x + x^2 + \frac{4}{3}x^3 - \cdots \ldots\ldots\ldots\ldots..\right\}$

$= 2x + \frac{1}{2}x^2 + \frac{5}{3}x^3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots$

The expansion of $\ln(1+x)$ is valid for $-1 < x \leq 1$, and the expansion of $\ln(1-2x)$ is valid for $-1 < -2x \leq 1$ or $-\frac{1}{2} \leq x < \frac{1}{2}$

11. *Assuming that x is so small that x^3 and higher power terms may be neglected, obtain an equivalent approximation to the function* $\frac{1+e^x}{1+\ln(1+x)}$

Solution

$1 + e^x = 1 + \left(1 + x + \frac{x^2}{2!} + \cdots \ldots\ldots\ldots\ldots\ldots.\right) \simeq 2 + x + \frac{1}{2}x^2$

$1 + \ln(1+x) = 1 + \left(x - \frac{x^2}{2} + \cdots \ldots\ldots\ldots\ldots\right) \simeq 1 + x - \frac{1}{2}x^2$

Using Binomial expansion

$(1+y)^{-1} = 1 + (-1)y + \frac{(-1)(-2)}{2!}y^2 + \cdots \ldots\ldots\ldots\ldots\ldots = 1 - y + y^2 - \cdots \ldots.\ldots$

Substituting $y = x - \frac{1}{2}x^2$

$\Rightarrow \frac{1}{1+\ln(1+x)} \simeq \left[1 + \left(x - \frac{1}{2}x^2\right)\right]^{-1}$

$= 1 - \left(x - \frac{1}{2}x^2\right) + \left(x - \frac{1}{2}x^2\right)^2 - \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

$= 1 - x + \frac{1}{2}x^2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore \frac{1+e^x}{1+\ln(1+x)} \simeq \left(2 + x + \frac{1}{2}x^2\right).\left(-x + \frac{1}{2}x^2\right)$

$= 2 - 2x + 3x^2 + x - x^2 + \frac{1}{2}x^2$

$= 2 - x + \frac{5}{2}x^2$

12. *Expand $e^{\sin x}$ in ascending powers of x, neglecting terms in x^4 and higher power terms of x.*

Solution

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = x - \frac{1}{6}x^3 + \cdots \ldots\ldots\ldots..$

$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots..= 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \cdots \ldots\ldots\ldots\ldots....$

Substituting $y = x - \frac{1}{6}x^3$

$e^{\sin x} = 1 + \left(x - \frac{1}{6}x^3\right) + \frac{1}{2}\left(x - \frac{1}{6}x^3\right)^2 + \frac{1}{6}\left(x - \frac{1}{6}x^3\right)^3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots$

$= 1 + x - \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= 1 + x + \frac{1}{2}x^2$

13. *Expand* $\sec x$ *in ascending powers of x,* as far term in x^4

Solution

$\sec x = (\cos x)^{-1} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \ldots\ldots\ldots\ldots\ldots\right)^{-1}$

$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - .\ldots\ldots\ldots\ldots\ldots\ldots\right)^{-1}$

Using Binomial expansion

$(1+y)^{-1} = 1 + (-1)y + \frac{(-1)(-2)}{2!}y^2 + \cdots \ldots\ldots\ldots\ldots\ldots = 1 - y + y^2 - \cdots \ldots\ldots\ldots$

Thus substituting $y = -\frac{x^2}{2} + \frac{x^4}{24}$

$\sec x = 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 - \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

14. *Find the expansion in ascending powers of x for* $\ln(1 - x - 6x^2)$*, giving the terms up to and including that in* x^4*, and state the values of x for which the expansion is valid.*

Solution

$\ln(1 - x - 6x^2) = \ln[(1 - 3x)(1 + 2x)] = \ln(1 - 3x) + \ln(1 + 2x)$

Using $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\Rightarrow \ln(1 - x - 6x^2) = \left(-3x - \frac{(-3x)^2}{2} + \frac{(-3x)^3}{3} - \frac{(-3x)^4}{4} - \cdots \ldots\ldots\ldots\ldots\ldots\right) +$

$+ \left(2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} - \cdots \ldots\ldots\ldots\ldots\ldots\right)$

$\therefore \ln(1 - x - 6x^2) = -x - \frac{13}{2}x^2 - \frac{19}{3}x^3 - \frac{97}{4}x^4 - \cdots \ldots\ldots\ldots\ldots\ldots$

The expansion for $\ln(1 - 3x)$ is valid for $-1 < -3x \leq 1$. i.e. $-\frac{1}{3} \leq x < \frac{1}{3}$

The expansion for $\ln(1 + 2x)$ is valid for $-1 < 2x \leq 1$. i.e. $-\frac{1}{2} \leq x < \frac{1}{2}$

As both expansions have been used, the expansion is valid for $-\frac{1}{3} \leq x < \frac{1}{3}$

15. *Use Maclaurin's theorem to find the first three terms non- zero terms in the expansion of sin x. hence find* $\sin(0.1)\,rad$ *correct to 7 d.p*

Solution

From the notes above a student must show that

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$, is the required expansion.

Thus $\sin(0.1)\,rad \simeq 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!}$

$= 0.09983341$

$\therefore \sin(0.1)\,rad = 0.09983341$ Correct to 7 d.p

Task

1. Obtain the series of the expansion of $\frac{\ln(1+x)}{2+x}$ up to and including the term in x^4 and state the values of x for which the expansion is valid.

Ans: $\frac{1}{2}x - \frac{1}{2}x^2 + \frac{5}{12}x^3 - \frac{1}{3}x^4$; $-1 < x \leq 1$

2. Obtain the expansion in ascending powers of x of the function $f(x) = \ln\left\{\frac{(1-3x)^2}{(1-2x)}\right\}$. Give the values of the coefficients of the powers of x up to and including x^3, and give an expression for the coefficient of x^n. For what range of values of x is the expansion valid?

Ans: $1 - x + \frac{5}{2}x^2 - \frac{29}{6}x^3$, $|x| < \frac{1}{2}$

INEQUALITIES IN ONE VARIABLE

To Solve and Graph Inequalities in One Variable

Properties of Inequalities

1. If $a < b$, then $a + c < b + c$. Here the symbol does not *reverse* when you add/subtract a real number on both sides of the inequality.

2. If $a < b$ and $c > 0$, then $ac < bc$ or $\frac{a}{c} < \frac{b}{c}$. Here the inequality symbol does not *reverse*; when you multiply/divide an inequality by a positive number.

3. If $a < b$ and $c < 0$, then $ac > bc$ or $\frac{a}{c} > \frac{b}{c}$. Here the inequality symbol *reverses*; when you multiply/divide an inequality by a negative number.

Examples

Solve the inequality

a) $3x - 4 \leq 10 + x$ b) $\frac{8-2x}{5} > 6$

Solution

a) $3x - 4 \leq 10 + x \Rightarrow 2x \leq 14 \quad \therefore x \leq 7$

b) $\frac{8-2x}{5} > 6 \Rightarrow 8 - 2x > 30$

$\Rightarrow -2x > 22 \quad \therefore x < -11$ (Symbol reverses)

Graphing Linear Inequalities

Inequalities containing symbols ≤ or ≥ represent end points of inequalities to be included in the graph. Usually it is represented as (●), solid dot and if < or > are used, the end points are not included in the graph. Usually it is denoted as (◯)

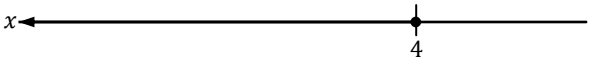
Example

1. *Represent the inequalities on a graph*

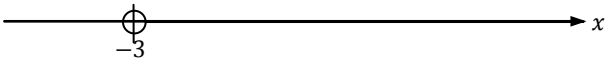
a) $x \leq 4$ b) $x > -3$ c) $-3 < x \leq 4$

Solution

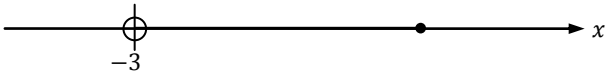
a) For $x \leq 4$



b) For $x > -3$



c) For $-3 < x \leq 4$



2. *Solve the inequality and graph its solution*

a) $\frac{4}{3}\left(x - \frac{1}{2}\right) + \frac{1}{2}x \geq \frac{2}{3}\left(2x - \frac{5}{2}\right)$ b) $\frac{x+2}{4} - \frac{2-x}{3} < 4 - \frac{4x-5}{6}$

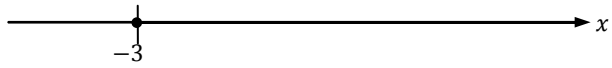
Solution

a) $\frac{4}{3}\left(x - \frac{1}{2}\right) + \frac{1}{2}x \geq \frac{2}{3}\left(2x - \frac{5}{2}\right) \Rightarrow \frac{4x}{3} - \frac{2}{3} + \frac{1}{2}x \geq \frac{4x}{3} - \frac{5}{3}$

Multiply through by 6 (L.C.M of 2 and 3)

$\frac{24x}{3} - \frac{12}{3} + \frac{6x}{3} \geq \frac{24x}{3} - \frac{30}{3}$

$8x - 4 + 2x \geq 8x - 10 \Rightarrow 2x \geq -6 \quad \therefore x \geq -3$



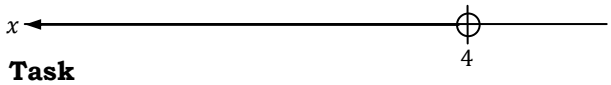
b) $\frac{x+2}{4} - \frac{2-x}{3} < 4 - \frac{4x-5}{6}$

Multiply through by 12 (L.C.M of 3 , 4 and 6)

$\frac{12(x+2)}{4} - \frac{12(2-x)}{3} < 4(12) - \frac{12(4x-5)}{6}$, re arranging

$3x + 6 - 8 + 4x + 8x - 10 < 48$

$15x - 12 < 48 \quad \therefore x < 4$



Task

Solve and graph the inequality

a) $\frac{15-6x}{3} > 5$ **Ans:** $x < 0$

b) $8x + 3(x + 1) > 5x - (9x - 6)$ **Ans:** $x > \frac{1}{5}$

POLYNOMIAL INEQUALITIES IN ONE VARIABLE

To Solve and Graph Polynomial Inequalities in One variable

Let $P(x)$ be any polynomial, then $P(x) < 0$ and $P(x) > 0$ are called *Polynomial Inequalities*

Two ways of solving polynomial inequalities are:

- Use a sign graph of $P(x)$
- Analyze the graph of $P(x)$

Note:

- If $P(x) > 0$, the graph $y = P(x)$ is said to be *Above the $x-$ axis*
- If $P(x) < 0$, the graph $y = P(x)$ is said to be *Below the $x-$ axis*

If $P(x)$ is factorisable, then the easiest method to be used is sign graph method.

Example

1. *Solve the inequality and hence sketch $P(x)$*

a) $x^3 - 2x^2 - 3x < 0 \;\; ; P(x) < 0$

b) $(x + 1)(x - 1)(x - 2) > 0 \;\; ; P(x) > 0$

c) $x^3 + 5x^2 + 3x \leq 9 \;\; ; P(x) \leq 0$

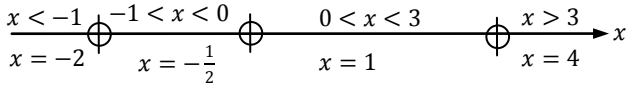
Solution

a) $x^3 - 2x^2 - 3x < 0 \, ; \, P(x) = x^3 - 2x^2 - 3x$

Finding the zeroes of $P(x) = x^3 - 2x^2 - 3x$

$P(x) = x^3 - 2x^2 - 3x = 0 \quad \therefore x = 0 \, , -1 \, , 3$

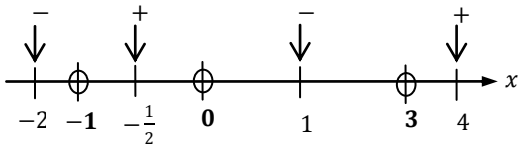
Plotting the zeroes on a number line



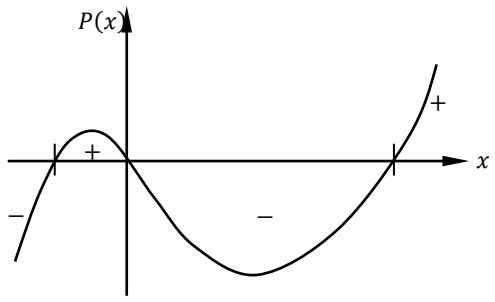
Use open dots since $P(x)$ is *less than* zero.

$P(x)$ in each interval: $P(x) = x(x + 1)(x - 3)$

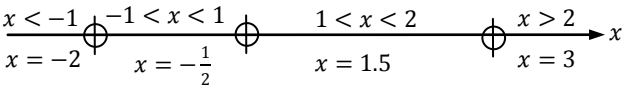
$P(-2) = -2(-2+1)(-2-3) = -$
$P\left(-\frac{1}{2}\right) = -\frac{1}{2}\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}-3\right) = +$
$P(1) = 1(1+1)(1-3) = -$
$P(4) = 4(4+1)(4-3) = +$
$\therefore$ The solution of $P(x) < 0$ is
$x < -1$ or $0 < x < 3$



Now sketching $P(x)$

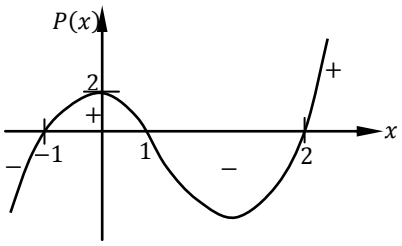


b) $(x+1)(x-1)(x-2) > 0$; $P(x) = (x+1)(x-1)(x-2)$
Finding the zeroes of $P(x) = (x+1)(x-1)(x-2)$ $\therefore x = \pm 1, 2$
Plotting the zeroes on a number line



Use open dots since $P(x)$ is *greater than* zero.
Testing $P(x)$ for each interval
$P(x) = (x+1)(x-1)(x-2)$
$P(-2) = (-2+1)(-2-1)(-2-2) = -$
$P(-0.5) = (-0.5+1)(-0.5-1)(-0.5-2) = +$
$P(1.5) = (1.5+1)(1.5-1)(1.5-2) = -$
$P(2) = (2+1)(2-1)(2-2) = +$
Since $P(x) > 0$, then the solution is:
$-1 < x < 1$ and $x > 2$

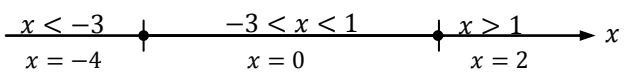
Now sketching $P(x)$



c) $x^3 + 5x^2 + 3x \le 9$; $P(x) = x^3 + 5x^2 + 3x - 9$
Let the zeroes of $P(x)$ be $x = \frac{p}{q}$ where p divides -9 and q divides 1
Possible values of p: $\pm 1, \pm 3, \pm 9$
For $P(1) = 1 + 5 + 3 - 9 = 0$ $(x+1)$ is a factor
For $P(-3) = (-3)^3 + 5(-3)^2 + 3(-3) - 9 = 0$ $(x+3)$ is a factor
$\therefore (x-1)(x+3) = x^2 + 2x - 3$ is a factor of $P(x)$

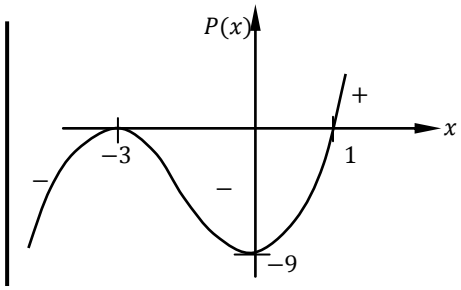
$$
\begin{array}{r}
x+3 \\
x^2+2x-3 \overline{)\ x^3+5x^2+3x-9} \\
\underline{x^3+2x^2-3x} \\
3x^2+6x^2-9 \\
\underline{3x^2+6x^2-9} \\
0
\end{array}
$$

$\therefore P(x) = (x+3)(x+3)(x-1)$
$\therefore$ Zeroes of $P(x)$ are: $x = -3, 1$
On number line



Test value :

Solid circles are used.
Testing $P(x)$ for each interval
$P(-4) = (-4+3)(-4+3)(-4-1) = -$
$P(0) = (0+3)(0+3)(0-1) = -$
$P(2) = (2+3)(2+3)(2-1) = +$

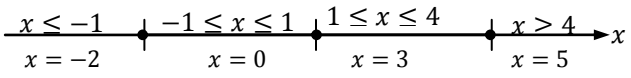


2. *Solve:*
a) $(x^2-1)(x-4)^2 \geq 0$
b) $x^4 - 3x^2 - 10 > 0$

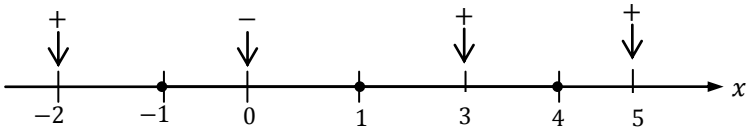
Solution

a) Let $P(x) = (x^2-1)(x-4)^2 = (x+1)(x-1)(x-4)^2 \;\; ; P(x) \geq 0$
Finding zeroes of $P(x) = (x^2-1)(x-4)^2$
$(x+1)(x-1)(x-4)^2 = 0 \quad \therefore x = \pm 1, 4$
Plotting the zeroes on a number line

Test value :



Testing $P(x)$ for each interval
$P(-2) = (-2+1)(-2-1)(-2-4)^2 = +$
$P(0) = (0+1)(0-1)(0-4)^2 = -$
$P(3) = (3+1)(3-1)(3-4)^2 = +$
$P(5) = (5+1)(5-1)(5-4)^2 = +$



$\therefore P(x) \geq 0$ has a solution $x \leq -1, x \geq 1$
The above solution can be simplified in to a tabular solution as below.

Inequality	$x \leq -1$	$-1 \leq x \leq 1$	$1 \leq x \leq 4$	$x \geq 4$
Test value	-2	0	3	5
$x+1$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
$(x-4)^2$	$+$	$+$	$+$	$+$
$(x+1)(x-1)(x-4)^2$	$+$	$-$	$+$	$+$

$\therefore P(x) \geq 0$ has a solution $x \leq -1, x \geq 1$

b) Left as an exercise. **Ans:** $x < \sqrt{5}, x > \sqrt{5}$
[**HINT**: *Let* $x^2 = y \quad \Rightarrow y^2 - 3y - 10 > 0$]

Task

Solve the inequality

a) $4x^4 - 4x^3 - 3x^2 + 4x - 1 > 0$ **Ans:** $x < -1, x > 1$
b) $(x-1)(x-2)^2(x-3)^2 < 0$ **Ans:** $x < 4, x \neq 2, x \neq 3$
c) $n^3 - 7n + 6 < 0$ **Ans:** $n < -3, 1 < n < 2$

SOLVING INEQUALITIES INVOLVING FUNCTION OF THE FORM $\frac{P(x)}{Q(x)}$ WHERE $P(x)$ AND $Q(x)$ ARE POLYNOMIALS/ PRODUCTS OF POLYNOMILAS

Example

Solve:

a) $\frac{(x+2)(x-5)^2}{x-4} \leq 0$

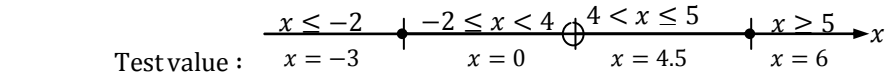
b) $\frac{(x-3)(x-4)}{(x-5)(x-6)^2} < 0$ c) $\frac{x-1}{x-2} > \frac{x-2}{x+3}$

Solution

a) $\frac{(x+2)(x-5)^2}{x-4} \leq 0$, let $P(x) = \frac{(x+2)(x-5)^2}{x-4}$

Plotting the zeroes of all linear factors occurring in Numerator and Denominator

Zeroes of $P(x) = \frac{(x+2)(x-5)^2}{x-4}$ are: $x = -2\,,4\,,5$



Use solid dots/circles for -2 and 5 since $P(x)$ can equal to zero; and open dot/circle for 4, to remind yourself to omit it from the solution since the denominator of $P(x)$ can never be equal to zero.

Do a sign analysis of $P(x) = \frac{(x+2)(x-5)^2}{x-4}$

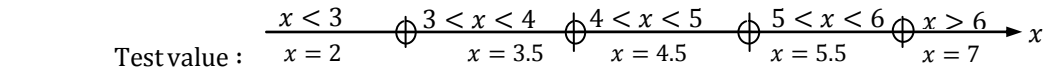
Inequality	$x \leq -2$	$-2 \leq x < 4$	$4 < x \leq 5$	$x \geq 5$
$x+2$	$-$	$+$	$+$	$+$
$(x-5)^2$	$+$	$+$	$+$	$+$
$x-4$	$-$	$-$	$+$	$+$
$P(x) = \frac{(x+2)(x-5)^2}{x-4}$	$+$	$-$	$+$	$+$

$\therefore$ The solution of $P(x) \leq 0$ is: $-2 \leq x < 4$

b) $\frac{(x-3)(x-4)}{(x-5)(x-6)^2} < 0$, let $P(x) = \frac{(x-3)(x-4)}{(x-5)(x-6)^2}$

Plotting the zeroes of all linear factors occurring in Numerator and Denominator

Zeroes of $P(x) = \frac{(x-3)(x-4)}{(x-5)(x-6)^2}$ are: $x = 3\,,4\,,5\,,6$



Use open dots/circles for 3 and 4 since $P(x)$ is less than zero; and open dot/circle for 5 and 6 since the denominator of $P(x)$ can never be equal to zero.

Do a sign analysis of $P(x) = \frac{(x-3)(x-4)}{(x-5)(x-6)^2}$

Inequality	$x < 3$	$3 < x < 4$	$4 < x < 5$	$5 < x < 6$	$x > 6$
$x - 3$	$-$	$+$	$+$	$+$	$+$
$x - 4$	$-$	$+$	$+$	$+$	$+$
$x - 5$	$-$	$-$	$+$	$+$	$+$
$(x-6)^2$	$+$	$-$	$+$	$+$	$+$
$P(x) = \frac{(x-3)(x-4)}{(x-5)(x-6)^2}$	$-$	$+$	$-$	$+$	$+$

$\therefore$ The solution of $P(x) < 0$ is: $x < 3$ and $4 < x < 5$

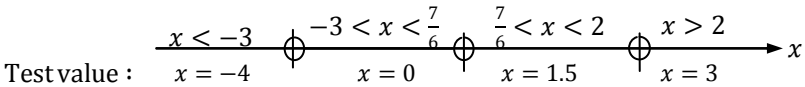
c) $\frac{x-1}{x-2} > \frac{x-2}{x+3}$

$\frac{x-1}{x-2} - \frac{x-2}{x+3} > 0$

$\frac{(x-1)(x+3)-(x-2)(x-2)}{(x-2)(x+3)} > 0$ or $\frac{x^2+2x-3-(x^2-4x+4)}{(x-2)(x+3)} > 0$

$\frac{6x-7}{(x-2)(x+3)} > 0$, let $P(x) = \frac{6x-7}{(x-2)(x+3)}$

Finding the zeroes of $P(x) = \frac{6x-7}{(x-2)(x+3)}$ both for the numerator and denominator, zeroes are: $x = \frac{7}{6}, 2, -3$



Inequality	$x < -3$	$-3 < x < \frac{7}{6}$	$\frac{7}{6} < x < 2$	$x > 2$
$6x - 7$	$-$	$-$	$+$	$+$
$x - 2$	$-$	$-$	$-$	$+$
$x + 3$	$-$	$+$	$+$	$+$
$P(x) = \frac{6x-7}{(x-2)(x+3)}$	$-$	$+$	$-$	$+$

The solution of $P(x) > 0$ is: $x > 2$, $-3 < x < \frac{7}{6}$

STATING THE RANGE OF VALUES OF A QUADRATIC RATIONAL POLYINOMIAL FUNCTION

Example

If x is a real number, prove that the expression $\frac{x^2+2x-11}{2(x-3)}$ doe not lie strictly between 2 and 6.

Solution

Let $y = \frac{x^2+2x-11}{2(x-3)}$ $\Rightarrow x^2 + 2x - 11 = 2xy - 6y$

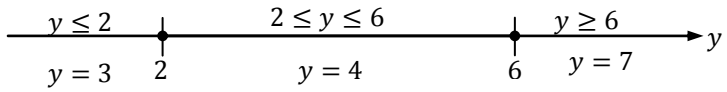
$\Rightarrow x^2 + x(2-2y) + 6y - 11 = 0$

For real values of x, $B^2 - 4AC \geq 0$

$(2-2y)^2 - 4(6y-11) \geq 0$

$4 - 8y + 4y^2 - 24y + 44 \geq 0$ or $4y^2 - 32y + 48 \geq 0$

$\therefore y^2 - 8y + 12 \geq 0$ or $(y-6)(y-2) \geq 0$, let $P(y) = (y-6)(y-2)$
Finding the zeroes of $P(y)$
$\Rightarrow y = 2, 6$



Inequality	$y \leq 2$	$2 \leq y \leq 6$	$y \geq 6$
$y-2$	$-$	$+$	$+$
$y-6$	$-$	$-$	$+$
$P(y) = (y-6)(y-2)$	$+$	$-$	$+$

Since $P(y) \geq 0$, the interval $2 \leq y \leq 6$ does not solve the inequality.
Thus the expression $\frac{x^2+2x-11}{2(x-3)}$ does not strictly lie between 2 and 6.

Task

1. *If x is a real number, prove that the expression $\frac{x}{x^2-5x+9}$ lies between $-\frac{1}{11}$ and 1*
2. *If x is real, prove that $\frac{x^2+34x-71}{x^2+2x-7}$ cannot lie strictly between 5 and 9*
3. *Show that for real x, $-\frac{1}{4} \leq \frac{x}{x^2+4} \leq \frac{1}{4}$*

SOLVING INEQUALITIES INVOLVING MODULUS

Example

Find the set of values of x for which:

a) $|2x-3| > 5$
b) $|4x+2| < x+8$
c) $|x+1| > |x-3|$
d) $3|x+5| > |x+3|$
e) $\left|\frac{2x-4}{x+1}\right| < 4$
f) $\left|\frac{x}{x-3}\right| < x$
g) $\left|\frac{2x+5}{x^2-4}\right| \geq \frac{1}{5}$
h) $|2x-3| > 4|x|$

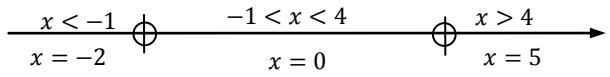
Solution

a) $|2x-3| > 5$

Note

$|2x-3| = \sqrt{(2x-3)^2}$
$\Rightarrow \sqrt{(2x-3)^2} > 5$
Squaring both sides
$\Rightarrow (2x-3)^2 > 25$ or $(2x-3)^2 - 25 > 0$
$\Rightarrow (2x-3+5)(2x-3-5) > 0$
$\Rightarrow (2x+2)(2x-8) > 0$ or $(x+1)(x-4) > 0$
Let $P(x) = (x+1)(x-4)$

$\Rightarrow (x+1)(x-4)=0 \quad \therefore x=-1\,,\ x=4$



Inequality	$x<-1$	$-1<x<4$	$x>4$
Test value	-2	0	5
$x+1$	$-$	$+$	$+$
$x-4$	$-$	$-$	$+$
$(x+1)(x-4)$	$+$	$-$	$+$

$\therefore x<-1\,, x>4$

b) Left as an exercise. **Ans:** $-2<x<2$

c) Left as an exercise. **Ans:** $x>1$

d) $3|x+5|>|x+3|$

Squaring both sides to eliminate the modulus symbols

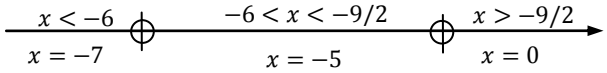
$\Rightarrow 9(x+5)^2>(x+3)^2$ or $9(x+5)^2-(x+3)^2>0$

$\Rightarrow (3x+15+x+3)(3x+15-x-3)>0$

$\Rightarrow (4x+18)(2x+12)>0$ or $(2x+9)(x+6)>0$

Let $P(x)=(2x+9)(x+6)$

$\Rightarrow (2x+9)(x+6)=0 \quad \therefore x=-\frac{9}{2}\,,\ x=-6$



Inequality	$x<-6$	$-6<x<-9/2$	$x>-9/2$
Test value	-7	-5	0
$2x+9$	$-$	$-$	$+$
$x+6$	$-$	$+$	$+$
$(x+1)(x-4)$	$+$	$-$	$+$

$\therefore x<-6\,, x>-\frac{9}{2}$

e) Left as an exercise. **Ans:** $x<-4\,, x>0$

f) Left as an exercise. **Ans:** $0<x<2\,;\ x>4$

g) Left as an exercise. **Ans:** $-7\le x\le -3\,, -2.35\le x<-2\,, 2<x\le 12.35$

h) Left as an exercise. **Ans:** $-\frac{3}{2}<x<\frac{1}{2}$

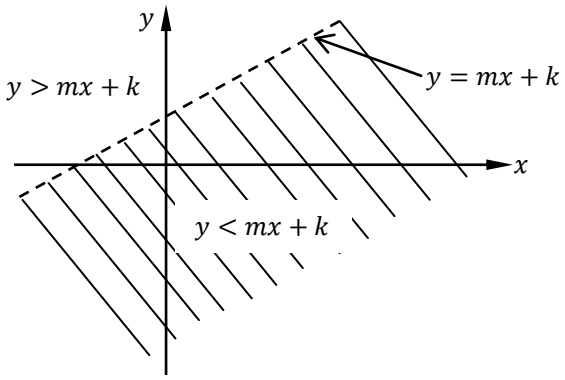
INEQUALITITES IN TWO VARIABLES

Polynomial Inequalities in Two variables and to Graph the solution set of a system of inequalities.

The solution to inequalities in two variables is an ordered pair of numbers. Thus graphs of inequalities in two variables consist of points in the plane.

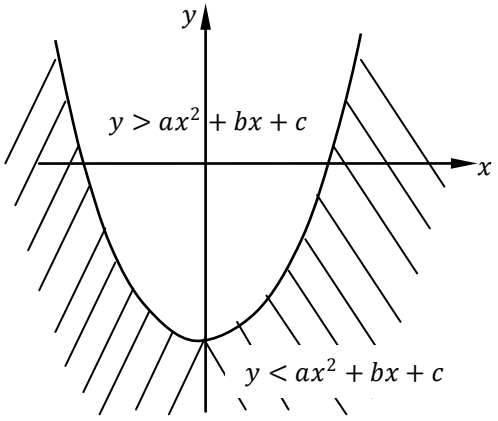
Graphing $y > mx + k$ and $y < mx + k$

Start by graphing the line $y = mx + k$ (Dotted if symbols < or > are used, and solid if symbols ≤ or ≥ are used)



Any point (x, y) below the line $y = mx + k$ satisfies the inequality $y < mx + k$ and above the line $y = mx + k$ satisfies the inequality $y > mx + k$

Graphing $y > ax^2 + bx + c$ and $y < ax^2 + bx + c$



Any point (x, y) below (Shaded) the curve $y = ax^2 + bx + c$ satisfies the inequality $y < ax^2 + bx + c$ and above (Not shaded) the curve $y = ax^2 + bx + c$ satisfies the inequality $y > ax^2 + bx + c$

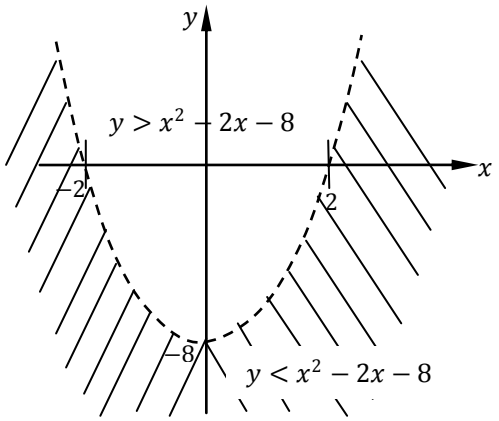
Example

1. *Sketch the graph;*

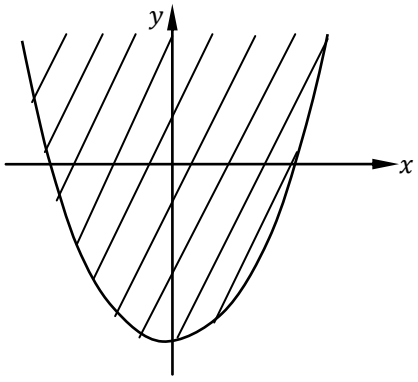
a) $y > x^2 - 2x - 8$ b) $y \leq x^2$

Solution

a) Sketching the curve $y = x^2 - 2x - 8$ (Dotted curve boundary)



b) Sketching the curve $y \leq x^2$ (Solid boundary)



Un Shaded region is $y \leq x^2$

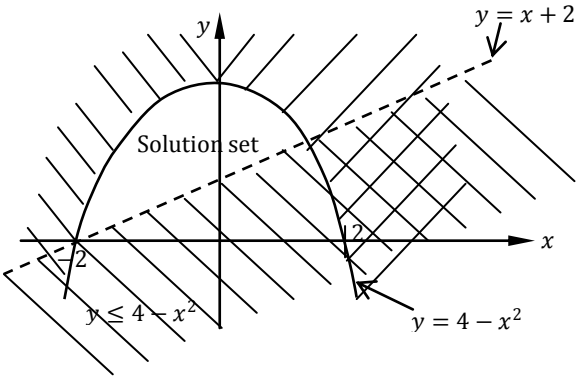
2. *Graph the solution set of the inequality*

a) $y \leq 4 - x^2$ and $y > x + 2$ b) $y \leq x^2 + x - 2$ and $y \leq x^2 + x - 2$

Solution

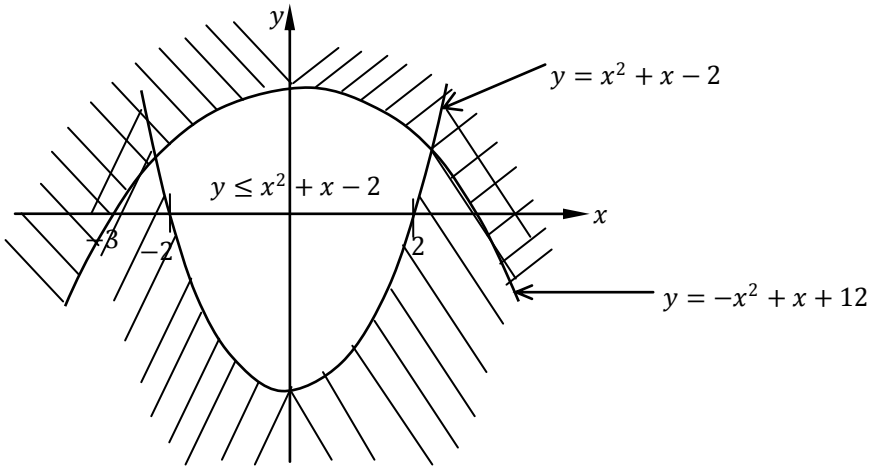
Inequality	**Test point**	**Result**
$y \leq 4 - x^2$	$(0\,,0)$	$0 \leq 4$, True (Required)
$y > x + 2$	$(0\,,0)$	$0 < 4$, False (Not required)

The unshaded region is the solution set.



$y = x^2 + x - 2$

b)



Inequality	**Test point**	**Result**
$y \leq x^2 + x - 2$	$(0\,,0)$	$0 \not\leq -2$, False
$y \leq -x^2 + x - 2$	$(0\,,0)$	$0 \leq 12$, True